# Vacuum Polarization and Casimir Effect within (3+1)D Maxwell-Chern-Simons Electrodynamics with Lorentz violation

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# Outline

# Standard Model Extension (SME [Kostelecký])

- Elaborated for studying the manifestation of the 'New Physics' (Strings, Extra Dimensions, Quantum Gravity,...) at low energies $E \ll m_{\rm Pl} \sim 10^{19}$ГэВ
- Introduces a set of correction terms to the Lagrangian of SM (no new fields!), that maintain some 'natural' features of SM:
  - observer Lorentz invariance (although the vacuum is not Lorentz-invariant)
  - unitarity
  - microcausality
  - $SU(3)_C \times SU(2)_I \times U(1)_Y$ gauge invariance
  - power-counting renormalizablilty (for the minimal SME)
- When $E \ll m_W \sim$, the SME results in the extended QED with $U(1)_{em}$ gauge invariance typical for SM

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# (3+1)D Maxwell-Chern-Simons electrodynamics

A particular case of extended QED with the Chern-Simons term:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\eta^{\mu}\epsilon_{\mu\nu\alpha\beta}A^{\nu}F^{\alpha\beta} + \bar{\psi}(i\gamma^{\mu}D_{\mu} - m)\psi.$$

$\eta^{\mu}$ is a constant 4-vector, breaks $\mathrm{CPT}$ and Lorentz invariance.

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# The model we use

(3+1)D Maxwell-Chern-Simons electrodyanamics, $\eta^\mu = \{\eta, \mathbf{0}\}$

- Photon sector: $\psi, \bar{\psi} = 0$
- Two infinite parallel superconductor plates separated by $D = 2a$
- Gauge: $A^0 = 0$, $\operatorname{div} \mathbf{A} = 0$
- Equations of motion: $\Box \mathbf{A} = 2\eta \operatorname{rot} \mathbf{A}$
- $T^{00} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{H}^2) - \eta \mathbf{A} \cdot \mathbf{H}$

When $a < \pi/4|\eta|$, the theory is stable [1]!

Vacuum energy and Casimir force (per unit area):

$$E_{vac} = \int \frac{d^3x}{L^2} \langle T^{00}(x) \rangle = \sum_n \frac{\omega_n(D)}{2L^2}, \qquad f_{\text{Casimir}} = \frac{1}{L^2} \frac{\partial E_{vac}}{\partial D},$$

$n$ is a complete set of quantum numbers, $L \to \infty$ is linear plate size.

The force is gauge-invariant, athough the energy is not [1]

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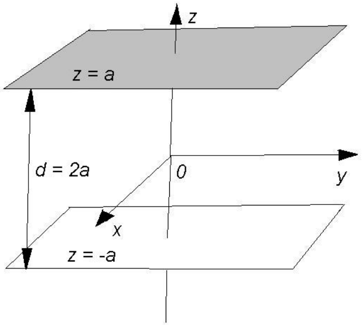
# One-photon eigenstates [1]




$$\mathbf{A}(\mathbf{x}, t) = N\, e^{\mp i\omega t + i\mathbf{kx}}\, \mathbf{f}(z), \quad \mathbf{k} = \{k_x, k_y, 0\}.$$

$$(\nabla^2 + 2\eta\, \mathrm{rot} + \omega^2)\mathbf{A} = 0, \quad \mathrm{div}\, \mathbf{A} = 0,$$

$A_x = A_y = 0$ at $z = \pm a$ (boundary conditions at the conductor)

# One-photon eigenstates [2]

Ansatz:

$$\mathbf{A}_{\epsilon,\mathbf{k},\Pi,n_z}(\mathbf{x},t) = N\,e^{-i\epsilon\omega t + i\mathbf{kx}}(f_z\mathbf{e}_z + f_k\hat{\mathbf{k}} + f_{zk}[\mathbf{e}_z\hat{\mathbf{k}}]), \quad \mathbf{k} = \{k_x, k_y, 0\}.$$

Transversality implies: $f_k = \frac{i}{k}\partial_z f_z$,

Parity $\Pi = \pm 1$: $f_k(-z) = -\Pi f_k(z)$, $f_{zk,z}(-z) = \Pi f_{zk,z}(z)$.

Equations for $f_{kz,z}$:

$$\begin{aligned}(\omega^2 - k^2 + \partial_z^2)kf_y &= 2i\eta(k^2 - \partial_z^2)f_z,\\ (\omega^2 - k^2 + \partial_z^2)f_z &= -2i\eta k f_y,\\ f_y(a) = 0, &\qquad \partial_z f_z(a) = 0\end{aligned}$$

The existence of nontrivial solutions implies that:

$$g_\Pi(\omega^2) \equiv \varphi_\Pi(\varkappa_+ a)\varphi_{-\Pi}(\varkappa_- a)\sin\theta_- + \varphi_\Pi(\varkappa_- a)\varphi_{-\Pi}(\varkappa_+ a)\sin\theta_+ = 0,$$

$\varkappa_\pm = \sqrt{K_\pm - k^2}$, $K_\pm = \mp\eta + \sqrt{\omega^2 + \eta^2}$, $\sin\theta_\pm = \varkappa_\pm / K_\pm$; $\varphi_{\pm 1}(x) \equiv \begin{cases}\cos x\\ \sin x\end{cases}$

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Renormalized vacuum energy (Casimir energy) per 1cm$^2$:

$$E_{vac} = \sum_n \frac{\omega_n}{2L^2} = \frac{1}{2}\zeta(-1/2), \quad f_{Casimir} = \frac{\partial E_{vac}}{\partial D},$$

$$\zeta(s) = \frac{1}{L^2}\sum_n (\omega_n^2)^{-s} = \int_0^\infty \frac{kdk}{2\pi} \sum_{\Pi=\pm1} \sum_{n_z} (\omega^2_{k,\Pi,n_z})^{-s}.$$

For sufficiently large $\mathrm{Re}\, s$, the series for $\zeta(s)$ is convergent; for other $s \in \mathbb{C}$, it is analytically continued.

When $\eta = 0$:

$$\zeta(s)|_{\eta=0} = \int_0^\infty \frac{kdk}{2\pi} \sum_{\Pi=\pm1} \sum_{n_z=1}^{\infty} \left(k^2 + \left(\frac{\pi n_z}{2a}\right)^2\right)^{-s} = \left(\frac{D}{\pi}\right)^{2s-2} \frac{\zeta_R(2s-2)}{2\pi(s-1)},$$

$$f_{Casimir}|_{\eta=0} = \frac{\pi^2}{240 D^4} \quad \text{(attraction)},$$

$\zeta_R(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann zeta function.

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# The $\eta$-correction to the zeta function [1]

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$$\omega_n|_{\eta=0} \equiv \omega_{0n} = \sqrt{k^2 + \left(\frac{\pi n_z}{2a}\right)^2}.$$

Note that $\omega_n$ are the roots of the equation $g_\Pi(\omega^2) = 0$, which is even with respect to changing the sign of $\eta$, then $\omega_n = \omega_n(\eta^2)$.

$$\sum_{\Pi=\pm 1} \frac{\partial \omega_n}{\partial(\eta^2)}\bigg|_{\eta=0} = -\sum_{\Pi=\pm 1} \frac{\partial^2 g_\Pi/\partial \eta^2}{2\partial g_\Pi/\partial \omega}\bigg|_{\eta=0,\omega=\omega_{0n}} = -\frac{1}{\omega_{0n}} + \frac{4a^2k^2}{n_z^2\pi^2\omega_{0n}},$$

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$$= \frac{s-2}{2\pi(s-1)}\left(\frac{2a}{\pi}\right)^{2s}\zeta_R(2s)$$

$$\zeta(s) = \frac{1}{2\pi(s-1)}\left(\frac{D}{\pi}\right)^{2s-2}\left(\zeta_R(2s-2)+(s-2)\left(\frac{\eta D}{\pi}\right)^2\zeta_R(2s)+\mathcal{O}(\eta^4)\right)$$

# The correction to the Casimir force

$$f_{Casimir} = \frac{\partial}{\partial D}\frac{\zeta(-1/2)}{2} = \frac{\pi^2}{240 D^4}\left(1 + \frac{25}{3\pi^2}(\eta D)^2 + \mathcal{O}((\eta D)^4)\right), \ |\eta| D \ll 1.$$

Discussion:

- The correction is attractive, contrary to the recent result obtained by [Frank, Turan, 2006]
- The difference from the Maxwell value is considerable for comparatively large $D$
- Experimental data [Mohideen et al., $D \sim 500$nm, $L \sim 1$cm, 1%accuracy] gives the constraint:

$$|\eta| \lesssim 5 \cdot 10^{-3}\text{eV}.$$

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# Sum → complex plane integral [1]

$$f_{Casimir} = \frac{1}{L^2}\frac{\partial}{\partial D}\sum_{\omega_n\in\mathbb{R}^+}\frac{\omega_n(D)}{2} = \frac{1}{2}\frac{\partial}{\partial D}\int_0^\infty \frac{kdk}{2\pi}D(S_+ + S_-),$$

Smooth cutoff regularization:

$$S_\Pi = \frac{1}{D}\sum_{\omega_{k,\Pi,n_z}\in\mathbb{R}^+}\omega_{k,\Pi,n_z}e^{-\omega_{k,\Pi,n_z}/\sqrt{k\Lambda}}, \quad \Lambda\to+\infty.$$

Instead of $g_\Pi(\omega^2)$ whose zeros are the one-photon energy eigenvalues, we will use the meromorphic (analytical, except for the numerable set of poles; in particular, with no branch points) function

$$\tilde{g}_\Pi(K_+) \equiv \frac{g_\Pi(\omega)}{\varphi_\Pi(\varkappa_+ a)\varphi_\Pi(\varkappa_- a)} = \tan^\Pi \varkappa_+ a\sin\theta_+ + \tan^\Pi \varkappa_- a\sin\theta_-,$$

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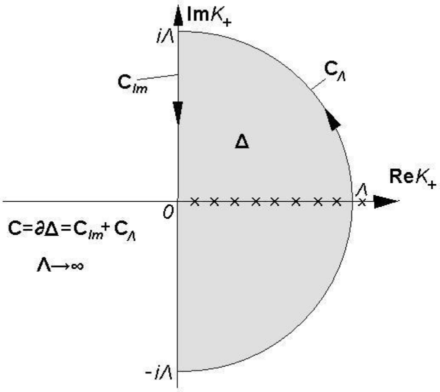


Residue theorem (we assume $\eta \geq 0$, since the spectrum depends on $|\eta|$):

$$\oint_C \frac{dK_+}{2\pi i}\omega\frac{\partial\tilde{g}_\Pi/\partial K_+}{\tilde{g}_\Pi} = S_\Pi D + \sum_{\bar{\omega}_n}\bar{\omega}_n\frac{\mathrm{Res}\left[\partial\tilde{g}_\Pi/\partial K_+, K_+ = \bar{\omega}_n\right]}{\tilde{g}_\Pi(\bar{\omega}_n)},$$

where $\bar{\omega}_n$ are the poles of function $\partial\tilde{g}_\Pi/\partial K_+$ within $\Delta$.

# Sum → complex plane integral [3]

Transforming the pole residue term back into an integral, we obtain:

$$\tilde{g}_\Pi(K_+) \equiv \tan^\Pi \varkappa_+ a \sin\theta_+ + \tan^\Pi \varkappa_- a \sin\theta_-,$$

$$S_\Pi = \frac{\Pi}{2} \oint\limits_C \frac{\omega dK_+}{2\pi i \tilde{g}_\Pi(K_+)} \left\{ 2 - \tan^\Pi \varkappa_+ a \tan^\Pi \varkappa_- a \left( \frac{\sin\theta_-}{\sin\theta_+} + \right.\right.$$

$$\left.\left. + \frac{\sin\theta_+}{\sin\theta_-} \right) + \frac{\Pi \tan^\Pi \varkappa_+ a}{\varkappa_+ a} + \frac{\Pi \tan^\Pi \varkappa_- a}{\varkappa_- a} \right\}$$

The integral over the semicircle $C_\Lambda$ does not depend on $a$, when $\Lambda \to \infty$, within any finite order in $a$, thus it is cancelled when renormalized.

Renormalization:

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# Sum $\to$ complex plane integral [3]

Transforming the pole residue term back into an integral, we obtain:

$$\tilde{g}_\Pi(K_+) \equiv \tan^\Pi \varkappa_+ a \sin\theta_+ + \tan^\Pi \varkappa_- a \sin\theta_-,$$

$$S_\Pi = \frac{\Pi}{2} \oint\limits_C \frac{\omega dK_+}{2\pi i \tilde{g}_\Pi(K_+)} \left\{ 2 - \tan^\Pi \varkappa_+ a \tan^\Pi \varkappa_- a \left( \frac{\sin\theta_-}{\sin\theta_+} + \right.\right.$$

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# After renormalization and $\Lambda \to \infty$

Let us redefine $K_+ \to -iK_+$, and make all momentum quantities dimensionless multiplying them by $a$, then we obtain:

$$f_{Casimir} = \frac{1}{2}\frac{\partial}{\partial D}\left(D\frac{(\tilde{S}_+ + \tilde{S}_-)}{a^4}\right),$$

$$\tilde{S}_\Pi = -\frac{1}{2}\int\limits_0^\infty \frac{kdk}{2\pi}\int\limits_\infty^{+\infty}\frac{dK_+}{2\pi}\frac{\operatorname{sgn} K_+\sqrt{K_+K_-}}{\tanh^\Pi \varkappa_+ \cosh\theta_+ + \tanh^\Pi \varkappa_- \cosh\theta_-}\Sigma_\Pi,$$

$$\Sigma_\Pi = 1 + \tanh^\Pi \varkappa_+ \tanh^\Pi \varkappa_- \frac{\cosh\theta_+}{\cosh\theta_-} - \left(1 + \frac{\cosh\theta_+}{\cosh\theta_-}\right)\tan^\Pi \varkappa_+ +$$

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# The results of the calculation

After the expansion with respect to $\eta D$ and taking the integrals, we obtain:

$$\tilde{S}_+ + \tilde{S}_- = -\frac{\pi^2}{5760} - \frac{5(\eta D)^2}{1152} + \mathcal{O}((\eta D)^4),$$

$$f_{Casimir} = \frac{1}{2}\frac{\partial}{\partial D}\left(D\frac{\tilde{S}_+ + \tilde{S}_-}{a^4}\right) = \frac{\pi^2}{240D^4}\left(1 + \frac{25}{3\pi^2}(\eta D)^2 + \mathcal{O}((\eta D)^4)\right),$$

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# Main results

- The eigenstates and energy eigenvalues for the Maxwell-Chern-Simons photon between the conducting plates
- The vacuum is stable when $D|\eta| < \pi/2$ [1]
- The leading correction to the Casimir force, which is quadratic in $\eta$
- Constraint on $\eta$

References:

[1] V.Ch.Zhukovsky and O.G.Kharlanov, *Casimir effect within (3+1)D Maxwell-Chern-Simons electrodynamics*, arXiv/0905.3689[hep-th] .

# The Model

Another special case of the Extended QED; fermion sector:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_\psi,$$
$$\mathcal{L}_\psi[\bar\psi, \psi, A, b] = \bar\psi(i\hat\partial - \hat A + \hat b\gamma^5 - m)\psi, \quad \hat\xi \equiv \gamma^\mu\xi_\mu.$$

where $b_\mu$ is a constant 4-vector, that violates CPT and Lorentz invariance of the theory.

Experimental constraints on $b^\mu$ for the electron:

$$|b_0| \lesssim 10^{-2}\text{eV}, \quad |\mathbf{b}| \lesssim 10^{-19}\text{eV}.$$

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# Induced Chern-Simons term in the constant field [1]

$F_{\mu\nu} = const$

Effective action in the proper time representation:

$$i\Gamma^{\rm eff}[A, b] = -\frac{1}{2}\int_0^\infty \frac{dz}{z}\,\mathrm{Tr}\, e^{-zH},$$

$$H = -\pi^\mu\pi_\mu - 2i\sigma^{\mu\nu} b_\mu \pi_\nu \gamma^5 + \frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu} + b_\mu b^\mu + m^2,$$

where $\pi^\mu = i\partial_\mu - A_\mu$, $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$.

In this expression, we apply the Baker-Hausdorf formula to the exponent to find its expansion into a series with respect to $b_\mu$:

$$\exp(\tau(A+B)) = \exp(\tau A)\cdot\exp(\tau B)\cdot L^{-1}(\tau),\ \frac{d\ln L}{d\tau} = B - e^{-\tau B} f(\tau) e^{\tau B}$$

$$A \equiv z\pi^\mu\pi_\mu - \frac{1}{2}z\sigma^{\mu\nu}F_{\mu\nu},\ B \equiv 2iz\sigma^{\mu\nu}b_\mu\pi_\nu\gamma^5,\ f(\tau) = e^{-\tau A}Be^{\tau A}.$$

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where $R^{\mu\alpha}$ is the combination of the field tensor $F^{\mu\nu}$ and the metric $g^{\mu\nu}$.

The matrix element $\langle x|\ldots|x\rangle$ can be transformed into the form:

$$\int d^4x\,\langle x|\,\pi_{\mu}e^{(z(\pi_{\nu}\pi^{\nu}))}|x\rangle = P_{\mu}^{\rho}\frac{\partial}{\partial\lambda^{\rho}}\int d^4x\,\langle x|\,e^{-\frac{1}{4}z\lambda^2}e^{(z(\pi^{\nu}+\lambda^{\nu})^2)}|x\rangle\Big|_{\lambda=0},$$

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# Induced Chern-Simons term in the constant field [3]

In the following papers, the Chern-Simons-like contribution of the form $\beta \tilde{F}^{\mu\nu} A_\mu b_\nu$ to the effective action of QED was calculated:

- R. Jackiw and V.A. Kostelecky, *Phys. Rev. Lett* **82**, 3572 (1999).
  The earliest publication, the coefficient $\beta = \frac{3}{16\pi^2}$
- M.B. Hott, J.L.Tomazelli, *Induced Lorentz and PCT symmetry Breaking in External Electromagnetic Field*, `arXiv/hep-th/9912251`.
  $\beta \neq 0$, though depends on the regularization scheme.
- Y.A. Sitenko, K.Y. Rulik *On the effective lagrangian in spinor electrodynamics with added violation of Lorentz and CPT symmetries*, `arXiv/hep-th/0212007`.
- ...and others

Our calculations show that no Chern-Simons term (linear in $b_\mu$) is induced by a fermion loop in the framework of the extended QED

# Quadratic contribution: Magnetic field [1]

Constant homogeneous magnetic field: $\mathrm{F}^{12} = -\mathrm{F}^{21} = -\mathrm{H}$, $b_\mu = \{b_0, 0, 0, 0\}$.

Then the exponent $e^{\tau(A+B)} = e^{\tau A} e^{\tau B}$ is "decoupled" in the expression for the effective action, since

$$[A, B] = 4z^2 \gamma^5 b^\mu \Pi_\nu \sigma^{\nu\alpha} F_{\alpha\mu} = 0.$$

The $b_0^2$-contribution to the effective action reads:

$$i\Gamma^{\mathrm{eff}}_{b_0^2}(H, b) = -2 \int_0^\infty \frac{dz}{z}$$

$$\times \left( A_0 + \frac{A_1}{z} \frac{\partial}{\partial \alpha} \right) \int d^4x \, \langle x | \, e^{-z((\pi_4)^2 + \alpha(\boldsymbol{\pi}_\perp^2 + \pi_\parallel^2))} \, | x \rangle \Big|_{\alpha=1},$$

where $A_0$, $A_1$ are certain field combinations.
The matrix element in the integrand expression

$$\int d^4x \, \langle x | \, e^{-z(\boldsymbol{\pi})^2_E} \, | x \rangle = \frac{H}{16\pi^2 z \sqrt{\alpha}} \frac{1}{\mathrm{sh}(\alpha z H)}.$$

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Finally we find the $b_0^2$-contribution to the effective action:

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and, after the renormalization,

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This integral leads to the Euler psi-function, taking the magnetic field strength $H$ exactly into consideration

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# Quadratic contribution: Electric field

Constant homogeneous electric field: $\mathrm{F}^{03} = -\mathrm{F}^{30} = \mathrm{E}$,
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# Effective action asymptotics: Magnetic field

Consider the asymptotics of the effective action $i\Gamma^{\text{eff}}_{b_0^2}(H,b)$.

In the weak field, $H \ll H_0$ (here $H_0 = m^2/e = 4.41 \times 10^{13}$ Gs):

$$i\Gamma^{\text{eff}}_{b_0^2}(H,b) \sim b_0^2 \frac{H^2}{m^2} + \mathcal{O}\left((H^2 b_0/m^3)^2\right), \quad \Gamma^{\text{eff}}_{H-E}(H) \sim \frac{H^4}{m^4},$$

$$\frac{i\Gamma^{\text{eff}}_{b_0^2}(H,b)}{i\Gamma^{\text{eff}}_{H-E}(H)} \sim \left(\frac{b_0}{m}\right)^2 \left(\frac{H_0}{H}\right)^2.$$

In the strong field, $H \gg H_0$:

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# Effective action asymptotics: Magnetic field

Consider the asymptotics of the effective action $i\Gamma^{\text{eff}}_{b_0^2}(H,b)$.

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We have elaborated a new method of calculating the contribution of the CPT- and Lorentz-violating correction $b_\mu$ to the effective action of QED, which accounts for the external field exactly and is based on the proper time technique.

Using this method, we have obtained the following results:

- No effective Chern-Simons term linear in $b^\mu$ in extended QED, in agreement with the publications:
  1. Y.A. Sitenko, K.Y. Rulik `ArXiv/hep-th/0212007`,
  2. B. Altschul `ArXiv/hep-th/0602235`,
- The $b_0^2$-term in the effective action, for the magnetic and the electric fields exactly taken into account.
- The Heisenberg-Euler-to-$b_0^2$ correction ratio $\frac{i\Gamma^{\mathrm{eff}}_{H-E}(H)}{i\Gamma^{\mathrm{eff}}_{b_0^2}(H,b)}$ is evaluated in the weak- and strong-field cases.

References:

[2] V.Ch.Zhukovsky and A.Bubnov, to be published in `arXiv/hep-th`.

Thank you for your attention!