Simplicity of $\mathcal{N} = 8$ Supergravity Amplitudes

Pierre Vanhove

Sakharov Conference, Moscow, May 19, 2009

based on

- 0805.3682, 0811.3405 + work in progress with N.E.J. Bjerrum-Bohr and S. Badger
- hep-th/0610299, hep-th/0611273 + work in progress with M.B. Green, J.G. Russo

Motivations

Recently we have experience fantastic progress in the evaluation of on-shell gravity amplitude in field theory.

Motivations

Recently we have experience fantastic progress in the evaluation of on-shell gravity amplitude in field theory.

Although gravity is intrinsically non-linear, with a dimensional coupling constant and on-shell S-matrix computations in $\mathcal{N} = 8$ supergravity showed that *many* simplifications take place leading to *surprisingly simple* results compared to the Feynman graph approach in particular the theory is **much well behaved in perturbation than expected**

Motivations

Recently we have experience fantastic progress in the evaluation of on-shell gravity amplitude in field theory.

Although gravity is intrinsically non-linear, with a dimensional coupling constant and on-shell S-matrix computations in $\mathcal{N} = 8$ supergravity showed that *many* simplifications take place leading to *surprisingly simple* results compared to the Feynman graph approach in particular the theory is **much well behaved in perturbation than expected**

In this talk we will explain what could be the role of supersymmetry and diffeomorphism invariance in this supergravity amplitudes

Outline

1. Multiloop amplitudes in string theory & $\mathcal{N} = 8$ supergravity
2. The no triangle property of $\mathcal{N} = 8$ supergravity amplitudes
3. Conclusion & Outlook

Part I

Multiloop amplitudes in $\mathcal{N} = 8$ supergravity

Constraints on $\mathcal{N}=8$ supergravity amplitudes

Gravity has a dimensional coupling constant

$$[1/\kappa^2_{(D)}] = \mathrm{mass}^{D-2}$$

An L-loop n-point gravity amplitude in D-dimensions has the dimension

$$[\mathfrak{M}_L^{(D)}] = \mathrm{mass}^{(D-2)L+2}$$

Constraints on $\mathcal{N}=8$ supergravity amplitudes

Gravity has a dimensional coupling constant

$$[1/\kappa_{(D)}^2] = \mathrm{mass}^{D-2}$$

An L-loop n-point gravity amplitude in D-dimensions has the dimension

$$[\mathfrak{M}_L^{(D)}] = \mathrm{mass}^{(D-2)L+2}$$

$\mathcal{N}=8$ 4-graviton amplitudes factorize an R^4 term and possibly higher derivatives

$$[\mathfrak{M}_L^{(D)}] = \mathrm{mass}^{(D-2)L-6-2\beta_L}\, D^{2\beta_L} R^4$$

Constraints on $\mathcal{N}=8$ supergravity amplitudes

Gravity has a dimensional coupling constant

$$[1/\kappa^2_{(D)}] = \mathrm{mass}^{D-2}$$

An L-loop n-point gravity amplitude in D-dimensions has the dimension

$$[\mathfrak{M}_L^{(D)}] = \mathrm{mass}^{(D-2)L+2}$$

$\mathcal{N}=8$ 4-graviton amplitudes factorize an R^4 term and possibly higher derivatives

$$[\mathfrak{M}_L^{(D)}] = \mathrm{mass}^{(D-2)L-6-2\beta_L}\, D^{2\beta_L} R^4$$

- Critical dimension for UV divergences is

$$D \geq D_c = 2 + \frac{6+2\beta_L}{L}$$

Superspace counting and UV behaviour

- Critical dimension for UV divergence is

$$D \geq D_c = 2 + \frac{6 + 2\beta_L}{L}$$

- Depending on the various implementations of supersymmetry

$$6 \leq 6 + 2\beta_L \leq 18$$

- With a first possible divergence in $D = 4$ at
  - $L \geq 3$ [Howe, Lindstrom, Stelle '81]
  - $L \geq 5$ [Howe, Stelle '06; Bossard, Howe, Stelle '09]
  - $L \geq 8$ [Kallosh '81]
  - $L \geq 9$: $\beta_L \leq 6$ [Green,Russo,Vanhove '06]
  - $L = \infty$: $\beta_L = L$ [Green,Russo,Vanhove '06]

The case of $\mathcal{N} = 4$ SYM

The coupling constant of $\mathcal{N} = 4$ SYM has dimension $[g_{YM}] = (length)^{D-4}$

The four point amplitude at L-loop has the superficial power counting

$$[\mathcal{A}^{(D)}_{4;L}] = \Lambda^{(D-4)L}$$

UV finite in $D < 4$

- could be logarithmically diverging in four dimensions by power counting
- Supersymmetry improves this power counting according

The case of $\mathcal{N}=4$ SYM

The coupling constant of $\mathcal{N}=4$ SYM has dimension $[g_{YM}]=(length)^{D-4}$

The four point amplitude at L-loop has the superficial power counting

$$[\mathcal{A}^{(D)}_{4;L}]=\Lambda^{(D-4)L-4}\,F^4$$

$$\text{UV finite in } D<4+\frac{4}{L}$$

- Off-shell $\mathcal{N}=2$ superspace is enough to assure finiteness in four dimensions by factorizing a F^4 term

  [Mandelstam; Howe, Stelle; Marcus, Sagnotti]

  - BUT: contradicts the non-renormalisation theorems for F^4 for $L\geq 2$
  - BUT: does not lead to the correct divergences structure at $L=2$ in $D=8$ and $L=3$ in $D=6$

The case of $\mathcal{N} = 4$ SYM

The coupling constant of $\mathcal{N} = 4$ SYM has dimension $[g_{YM}] = (length)^{D-4}$

The four point amplitude at L-loop has the superficial power counting

$$[\mathcal{A}^{(D)}_{4;L}] = \Lambda^{(D-4)L-6}\, s\, F^4 \quad L \geq 2$$

$$\text{UV finite in } D < 4 + \frac{6}{L}$$

- Perturbative computations at $L = 1, 2, 3, 4$ loops order indicates a better power counting rule with the factorization of $s\, F^4$

  [Bern, Dixon, Perelstein, Rozowski]

- confirmed by superspace arguments

  [Howe, Stelle; Howe, Stelle, Bossard]

Supersymmetry in $\mathcal{N} = 8$ amplitudes

[Berkovits] pure spinor formalism leads to particular series superspace integrals expressed in terms of the dim 1 superfield $W_{\alpha\beta,ij}$

- Zero-mode saturation in 4-graviton amplitudes

$$\mathfrak{M}_L \sim \int d^{16}\theta d^{16}\bar{\theta}\, \theta^{12-2L}\bar{\theta}^{12-2L}\, W^4\, I_L + \cdots \sim D^{2L} R^4\, I_L + \cdots$$

$$W_{\alpha\beta,a_1a_2} = F_{\alpha\beta,a_1a_2} + \cdots + \theta^{\gamma}_{a_1}\bar{\theta}^{\delta}_{a_2}\, R_{\alpha\gamma\beta\delta} + \cdots$$

- The first full superspace integral is met at six loops

$$\mathfrak{M}_6 \sim \int d^{16}\theta d^{16}\bar{\theta}\, W^4 \sim D^{12} R^4 + \text{susy completion}$$

Supersymmetry in $\mathcal{N} = 8$ amplitudes

[Berkovits] pure spinor formalism leads to particular series superspace integrals expressed in terms of the dim 1 superfield $W_{\alpha\beta,ij}$

- Zero-mode saturation in 4-graviton amplitudes

$$\mathfrak{M}_L \sim \int d^{16}\theta d^{16}\bar\theta\, \theta^{12-2L}\bar\theta^{12-2L}\, W^4\, I_L + \cdots \sim D^{2L} R^4\, I_L + \cdots$$

$$W_{\alpha\beta,a_1a_2} = F_{\alpha\beta,a_1a_2} + \cdots + \theta^\gamma_{a_1}\bar\theta^\delta_{a_2}\, R_{\alpha\gamma\beta\delta} + \cdots$$

- The first full superspace integral is met at six loops

$$\mathfrak{M}_6 \sim \int d^{16}\theta d^{16}\bar\theta\, W^4 \sim D^{12} R^4 + \text{susy completion}$$

Supersymmetry in $\mathcal{N} = 8$ amplitudes

[Berkovits] pure spinor formalism leads to particular series superspace integrals expressed in terms of the dim 1 superfield $W_{\alpha\beta,ij}$

- Zero-mode saturation in 4-graviton amplitudes

$$\mathfrak{M}_L \sim \int d^{16}\theta d^{16}\bar\theta\, \theta^{12-2L}\bar\theta^{12-2L}\, W^4\, I_L + \cdots \sim D^{2L}R^4\, I_L + \cdots$$

$$W_{\alpha\beta,a_1a_2} = F_{\alpha\beta,a_1a_2} + \cdots + \theta^\gamma_{a_1}\bar\theta^\delta_{a_2}\, R_{\alpha\gamma\beta\delta} + \cdots$$

- The first full superspace integral is met at six loops

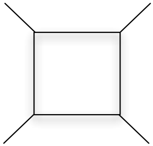
$$\mathfrak{M}_6 \sim \int d^{16}\theta d^{16}\bar\theta\, W^4 \sim D^{12}R^4 + \text{susy completion}$$

Gives a rational for the $\beta_L = L$ rule until $L = 6$ loops

The $L = 1, 2, 3, 4$ $\mathcal{N} = 8$ supergravity amplitudes follow this saturation rule of the zero modes [Bern et al.]

Loop amplitudes in $\mathcal{N} = 8$ supergravity

One loop amplitude is given by the scalar box integral with $\beta_1 = 0$

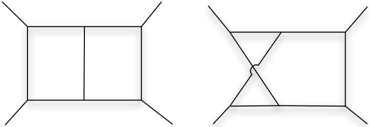


[Green,Schwarz,Brink]

$$\mathfrak{M}_1^{(D)} = R^4 \int_0^\infty \frac{dT}{T}\, T^{\frac{(8-D)}{2}} \int \prod_{i=1}^{3} du_i \, 1 \prod_{1\leq i<j\leq 4} e^{-(k^i\cdot k^j)\, G^{(1)}_{ij}}$$

Loop amplitudes in $\mathcal{N} = 8$ supergravity

Two-loop amplitude is given by the sum of the planar and non-planar double-box scalar integral $\beta_2 = 2$

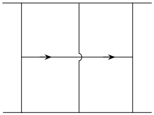


[Bern, Dunbar, Dixon, Perelstein, Rozowsky]

$$\mathfrak{M}_2^{(D)} = R^4 \int \prod_{i=1}^{3} dL_i \oint \prod_{i=1}^{4} du_i \, ((k^i \cdot k^j)\Delta_{ij})^2 \prod_{1\leq i<j\leq 4} e^{-(k^i \cdot k^j)\, G_{ij}^{(2)}}$$

Loop amplitudes in $\mathcal{N}=8$ supergravity

Three-loop amplitude have an integrand that satisfies the rule $\beta_3 = 3$



[Bern, Dixon,Roiban; Bern, Carrasco, Dixon, Johansson, Kosower, Roiban]

[Bern, Carrasco, Dixon, Johansson, Roiban]

$$\mathfrak{M}_3^{(D)} = R^4 \int \prod_{i=1}^{6} L_i \prod_{i=1}^{4} \oint du_i \qquad (k^m \cdot k^n)(k^p \cdot k^q)(k^r \cdot k^s)\Delta_{mnp}\bar{\Delta}_{rsq} \times$$

$$\times \prod_{1\leq i<j\leq 4} e^{-(k^i \cdot k^j)\, G_{ij}^{(3)}}$$

Non-renormalisation theorems

The $\beta_L = L$ rule implies that $[\mathfrak{M}_L^{(D)}] = \text{mass}^{(D-4)L-6}\, D^{2L} R^4$

- 1-loop non-renormalisation of R^4: $\beta_L \geq 2$ for $L \geq 2$

$$\begin{aligned} \text{UV divergence for } L=1: \quad & D \geq 8 \\ \text{First UV divergence in 4D:} \quad & L \geq 3 + \beta_L \geq 5 \,\text{loops} \end{aligned}$$

- 2-loop non-renormalisation of $D^4 R^4$: $\beta_L \geq 3$ for $L \geq 3$
- 3-loop non-renormalisation of $D^6 R^4$: $\beta_L \geq 4$ for $L \geq 4$
- After 6-loop supersymmetric protection runs out: $\beta_L = 6$ for $L \geq 6$

Non-renormalisation theorems

The $\beta_L = L$ rule implies that $[\mathfrak{M}_L^{(D)}] = \text{mass}^{(D-4)L-6}\, D^{2L} R^4$

- 1-loop non-renormalisation of R^4: $\beta_L \geq 2$ for $L \geq 2$
- 2-loop non-renormalisation of $D^4 R^4$: $\beta_L \geq 3$ for $L \geq 3$

$$\begin{aligned} \text{UV divergence for } L = 2: &\qquad D \geq 7 \\ \text{First UV divergence in 4D:} &\qquad L \geq 3 + \beta_L \geq 6 \text{ loops} \end{aligned}$$

- 3-loop non-renormalisation of $D^6 R^4$: $\beta_L \geq 4$ for $L \geq 4$
- After 6-loop supersymmetric protection runs out: $\beta_L = 6$ for $L \geq 6$

Non-renormalisation theorems

The $\beta_L = L$ rule implies that $[\mathfrak{M}_L^{(D)}] = \text{mass}^{(D-4)L-6}\, D^{2L} R^4$

- 1-loop non-renormalisation of R^4: $\beta_L \geq 2$ for $L \geq 2$
- 2-loop non-renormalisation of $D^4 R^4$: $\beta_L \geq 3$ for $L \geq 3$
- 3-loop non-renormalisation of $D^6 R^4$: $\beta_L \geq 4$ for $L \geq 4$

$$\begin{aligned} \text{UV divergence for } L = 3: &\qquad D \geq 6 \\ \text{First UV divergence in 4D:} &\qquad L \geq 3 + \beta_L \geq 7 \text{ loops} \end{aligned}$$

- After 6-loop supersymmetric protection runs out: $\beta_L = 6$ for $L \geq 6$

Non-renormalisation theorems

The $\beta_L = L$ rule implies that $[\mathfrak{M}_L^{(D)}] = \text{mass}^{(D-4)L-6}\, D^{2L} R^4$

- 1-loop non-renormalisation of R^4: $\beta_L \geq 2$ for $L \geq 2$
- 2-loop non-renormalisation of $D^4 R^4$: $\beta_L \geq 3$ for $L \geq 3$
- 3-loop non-renormalisation of $D^6 R^4$: $\beta_L \geq 4$ for $L \geq 4$
- After 6-loop supersymmetric protection runs out: $\beta_L = 6$ for $L \geq 6$

$$[\mathfrak{M}_L^{(D)}] = \text{mass}^{(D-2)L-18}\, D^{12} R^4$$

leading to a 9-loop divergence in $D = 4$ with for counter-term

$$\int d^{32}\theta\, (W^{ij}_{\alpha\beta})^4 = D^{12} R^4 + \textit{susy completion}$$

The $\beta_L = L$ rule

When the $\beta_L = L$ rule is satisfied [Green,Russo,Vanhove]

$$[\mathfrak{M}_L^{(D)}] = \mathrm{mass}^{(D-4)L-6}\, D^{2L} R^4$$

The $\beta_L = L$ rule

When the $\beta_L = L$ rule is satisfied [Green,Russo,Vanhove]

$$[\mathfrak{M}_L^{(D)}] = \mathrm{mass}^{(D-4)L-6}\, D^{2L} R^4$$

- $D^{2L} R^4$ not renormalized after L-loop order both in string theory and supergravity (if no IR divergences are met)

The $\beta_L = L$ rule

When the $\beta_L = L$ rule is satisfied [Green,Russo,Vanhove]

$$[\mathfrak{M}_L^{(D)}] = \mathrm{mass}^{(D-4)L-6}\, D^{2L} R^4$$

- $D^{2L} R^4$ not renormalized after L-loop order both in string theory and supergravity (if no IR divergences are met)
- *Same critical dimension for UV divergences* as $\mathcal{N} = 4$ SYM

$$D \geq D_c = 4 + \frac{6}{L}$$

- If true to all order the theory would be perturbatively UV finite in 4D

Part II

The no triangle property in $\mathcal{N} = 8$ supergravity

Basis of one-loop scalar integral functions

In $D = 4 - 2\epsilon$ one expands the amplitudes on a basis of scalar integral functions with massive external legs

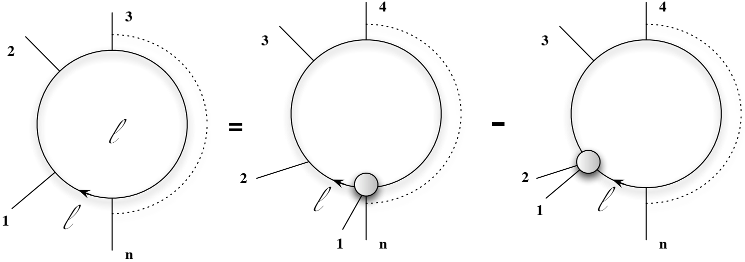
$$\mathfrak{M}_{n;1}^{(4-2\epsilon)} = \sum_i bo_i \, I_{\Box}^{(i)} + \sum_i t_i \, I_{\triangleright}^{(i)} + \sum_i bu_i \, I_{\circ}^{(i)} + c_{\text{rational pieces}}$$

This basis of scalar integral functions captures the IR and UV divergences of the one-loop amplitudes

The Reduction formulas

On-shell integrals are reduced to the scalar integral functions by cancelling one power of loop momentum with one propagator

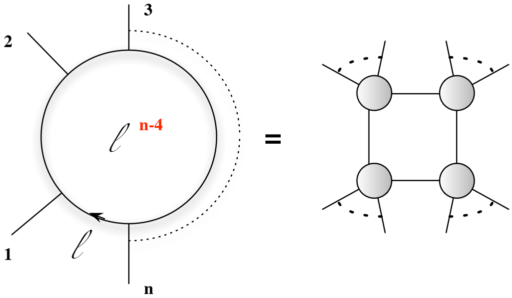
$$\int d^D\ell \, \frac{2(\ell \cdot k_1)}{\ell^2 \, (\ell - k_1)^2} \times (\cdots) = \int d^D\ell \, \left(\frac{1}{(\ell - k_1)^2} - \frac{1}{\ell^2}\right) \times (\cdots)$$



These reduction formula are well adapted to the soft and collinear singularities of QCD/ $\mathcal{N} = 4$ SYM amplitudes

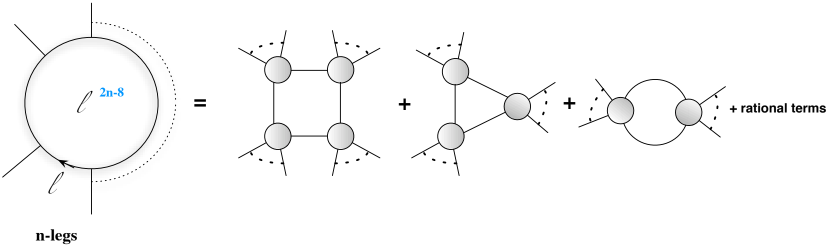
The no triangle property in $\mathcal{N} = 4$ SYM

Since $\mathcal{N} = 4$ super-Yang-Mills amplitudes have $n - 4$ powers of loop momenta they are reducible to boxes only



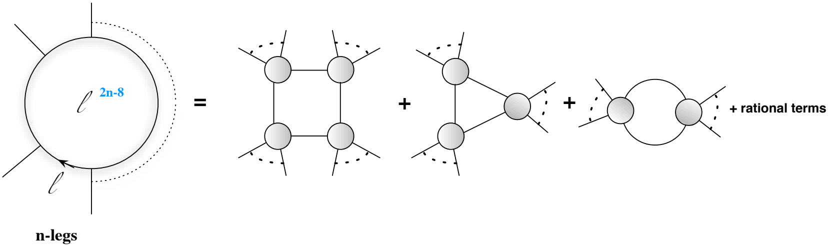
The no triangle property in $\mathcal{N} = 8$

$\mathcal{N} = 8$ amplitudes $2n - 8$ powers of loop momenta should contains boxes, triangles, bubbles and rational terms



The no triangle property in $\mathcal{N}=8$

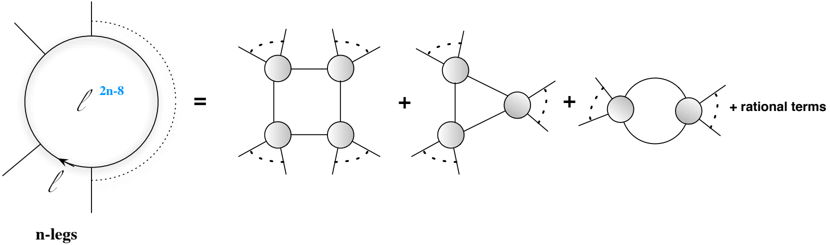
$\mathcal{N}=8$ amplitudes $2n-8$ powers of loop momenta should contains boxes, triangles, bubbles and rational terms



Explicit computations by [Bjerrum-Bohr et al., Bern et al.] showed that the amplitudes reduce to scalar box integral functions like for $\mathcal{N}=4$ SYM

The no triangle property in $\mathcal{N} = 8$

$\mathcal{N} = 8$ amplitudes $2n - 8$ powers of loop momenta should contains boxes, triangles, bubbles and rational terms



This result was unexpected because the counting was based on reduction formula that did not take into account *all* the cancellations occurring in Gravity and did not reflect the softer IR behaviour of gravity amplitudes [Bjerrum-Bohr, Vanhove]

The no triangle property of $\mathcal{N} = 8$ amplitudes

- Gravity does not have color factor
  - summation over all the permutations at one-loop
  - Sum over all the planar and non-planar diagrams at higher loop order
- Gauge invariance $\varepsilon_{\mu\nu} \to \varepsilon_{\mu\nu} + \partial_\mu v_\nu + \partial_\nu v_\mu$

Unordered amplitudes are more than just the sum over all orderings of color ordered amplitudes.
All the various orderings have the *same* tensorial structure

$$\mathfrak{M} = \sum_r t_r \int_0^\infty \frac{dT}{T}\, T^{\frac{2n-D}{2}} \int_0^1 \prod_{i=1}^{n-1} d\nu_i\, \mathcal{P}(\partial Q_n)\, e^{-T \sum_{r,s}(k_r \cdot k_s)\, G^{(1)}_{r,s}}$$

$Q_n = \sum_{i<j}(k_i \cdot k_j)\,[(\nu_i - \nu_j)^2 - |\nu_i - \nu_j|]$

The no triangle property of $\mathcal{N}=8$ amplitudes

Loop momentum is a total derivative $k_i \cdot \ell \sim \partial_{\nu_i} Q_n$ which can be freely integrated

$$\int_0^\infty \frac{dT}{T} \int_0^1 d^{n-1}\nu \, \partial_{\nu_i} Q_n(\cdots) = -\int_0^\infty \frac{dT}{T} \int_0^1 d^{n-1}\nu \, Q_n \partial_{\nu_i}(\cdots)$$

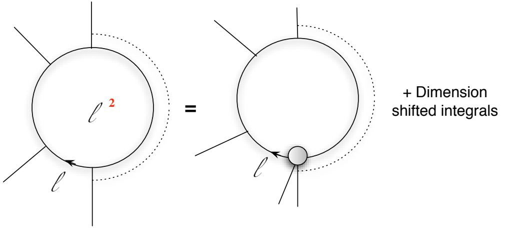
$\partial_{\nu_i}\partial_{\nu_j} Q_n \sim (k_i \cdot k_j)[\delta(\nu_i - \nu_j) - 1]$ does not contain any loop momenta

two powers of loop momentum are cancelled at each steps

The no triangle property of $\mathcal{N} = 8$ amplitudes

This implies new reduction formulas for unordered integrals

[Bjerrum-Bohr, Vanhove]

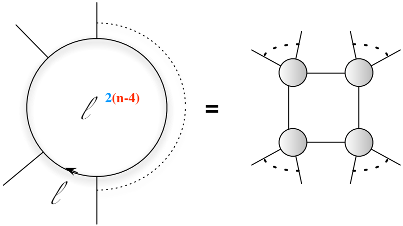


- These reduction formulas reflect that the graviton amplitudes have softer IR singularities than for QCD [Weinberg]
- The dimension shifted contributions cancel in the total amplitude

Gauge invariance implies that one can push all the 'triangles' into total derivative which cancel in the total amplitude (no boundary contributions)

The no triangle property of $\mathcal{N} = 8$ amplitudes

For $\mathcal{N} = 8$ sugra amplitude the no triangle property arises because the amplitude has $n - 4$ powers of ℓ^2



Summary & Outlook

We have explained that colorless gauge theory like gravity exhibit important cancellations in on-shell amplitudes.

- Cancellations are already seen at tree level which have a better high-momentum limit as naively expected
  - Need a very good control of the tree amplitudes
- Need a reorganisation of the expansion on a basis of integral functions
- Could these extra on-shell cancellations help extending the rule $\beta_L = L$ beyond 6 loops?

Summary & Outlook

We have explained that colorless gauge theory like gravity exhibit important cancellations in on-shell amplitudes.

- Cancellations are already seen at tree level which have a better high-momentum limit as naively expected
  - Need a very good control of the tree amplitudes
- Need a reorganisation of the expansion on a basis of integral functions
- Could these extra on-shell cancellations help extending the rule $\beta_L = L$ beyond 6 loops?

Summary & Outlook

We have explained that colorless gauge theory like gravity exhibit important cancellations in on-shell amplitudes.

- Cancellations are already seen at tree level which have a better high-momentum limit as naively expected
  - Need a very good control of the tree amplitudes
- Need a reorganisation of the expansion on a basis of integral functions
- Could these extra on-shell cancellations help extending the rule $\beta_L = L$ beyond 6 loops?

Summary & Outlook

We have explained that colorless gauge theory like gravity exhibit important cancellations in on-shell amplitudes.

- Cancellations are already seen at tree level which have a better high-momentum limit as naively expected
    - Need a very good control of the tree amplitudes
- Need a reorganisation of the expansion on a basis of integral functions
- Could these extra on-shell cancellations help extending the rule $\beta_L = L$ beyond 6 loops?

14th Itzykson Meeting on String theory
CEA - Saclay
June 17 - 19, 2009

http://ipht.cea.fr/Meetings/Itzykson2009/