Universal BPS Structure of stationary supergravity solutions

K.S. Stelle

Imperial College London

4^{th} International Sakharov Conference

FIAN, Moscow, 21 May 2009

G. Bossard, H. Nicolai & K.S.S.
0809.5218 and 0902.4438 [hep-th]

Outline

- Introduction: timelike dimensional reductions
- Gravitational and vector charges
- Characteristic equation
- Supersymmetry 'Dirac equation'
- BPS Geology
- Almost Iwasawa decompositions
- Arithmetic subgroups?
- Conclusions

Stationary solutions and timelike dimensional reduction

The search for supergravity solutions with assumed Killing symmetries can be recast as a Kaluza-Klein problem. Consider a $D = 4$ theory with a nonlinear bosonic symmetry $\bar{G}$ (*e.g.* E_7 for maximal $N = 8$ supergravity). Scalar fields take their values in a target space $\bar{\Phi} = \bar{G}/\bar{H}$, where $\bar{H}$ is the corresponding linearly realized subgroup, generally the maximal compact subgroup of $\bar{G}$ (*e.g.* $\mathrm{SU}(8)$ for $N = 8$ SG).

Searching for stationary solutions to such a theory amounts to assuming further that a solution possesses a timelike Killing vector field $\kappa_\mu(x)$.

- We assume that the solution spacetime is asymptotically flat or asymptotically Taub-NUT and that there is a 'radial' function r which is divergent in the asymptotic region, $g^{\mu\nu}\partial_\mu r \partial_\nu r \sim 1 + \mathcal{O}(r^{-1})$.
- The Killing vector κ will be assumed to have $W := -g_{\mu\nu}\kappa^\mu\kappa^\nu \sim 1 + \mathcal{O}(r^{-1})$.

- We assume asymptotic hypersurface orthogonality, $\kappa^{\nu}(\partial_{\mu}\kappa_{\nu} - \partial_{\nu}\kappa_{\mu}) \sim \mathcal{O}(r^{-2})$.
- In any vielbein frame, the curvature will fall off as $R_{abcd} \sim \mathcal{O}(r^{-3})$.
- Lie derivatives with respect to κ are assumed to vanish on all fields.

The $D = 3$ theory dimensionally reduced with respect to the timelike Killing vector κ will have an Abelian principal bundle structure, with a metric

$$ds^2 = -W(dt + \hat{B}_i dx^i)^2 + W^{-1}\gamma_{ij} dx^i dx^j$$

where t is a coordinate adapted to the Killing vector κ and γ is the metric on the 3-dimensional hypersurface Σ_3 at constant t. If the $D = 4$ theory has Abelian vector fields $\mathcal{A}_{\mu}$, they similarly reduce to $D = 3$ as

$$4\sqrt{4\pi G}\mathcal{A}_{\mu} dx^{\mu} = U(dt + \hat{B}_i dx^i) + \hat{A}_i dx^i$$

Comparison to spacelike dimensional reductions

The timelike $D = 3$ reduced theory will have a G/H^* coset space structure similar to the G/H coset space structure of a $D = 3$ theory similarly reduced on a *spacelike* Killing vector. Thus, for a spacelike reduction of maximal supergravity one obtains an $\mathrm{E}_8/\mathrm{SO}(16)$ theory continuing on in the sequence of dimensional reductions originating in $D = 11$. Julia As for the analogous spacelike reduction, the $D = 3$ theory has the possibility of exchanging $D = 3$ Abelian vector fields for scalars by dualization, contributing to the appearance of an enlarged $D = 3$ bosonic 'duality' symmetry. The resulting $D = 3$ theory contains $D = 3$ gravity coupled to a G/H^* nonlinear sigma model.

- However, although the numerator group G is the same for a timelike reduction to $D = 3$ as that obtained for a spacelike reduction, the divisor group H^* is a *noncompact* form of the spacelike divisor group H. Breitenlohner, Gibbons & Maison 1988
- The origin of this $H \to H^*$ change is the appearance of *negative-sign* kinetic terms for scalars descending from $D = 4$ vectors under the timelike reduction.

Some examples of G/H^* and G/H theories in $D = 3$

G/H	G/H^*	$\bar{G}/\bar{H}$	3+1 dimensional theory
$\frac{SL(n+2)}{SO(n+2)}$	$\frac{SL(n+2)}{SO(n,2)}$	$GL(n)/SO(n)$	$n+4$ dimensional Einstein gravity with n Killing vectors
$\frac{SU(2,1)}{S(U(2)\times U(1))}$	$\frac{SU(2,1)}{S(U(1,1)\times U(1))}$	$U(1)/U(1)$	Einstein-Maxwell ($N=2$ supergravity)
$\frac{SO(8,2)}{SO(8)\times SO(2)}$	$\frac{SO(8,2)}{SO(6,2)\times SO(2)}$	$\frac{SO(6)\times SO(2,1)}{SO(6)\times SO(2)}$	$N=4$ supergravity
$\frac{SO(8,8)}{SO(8)\times SO(8)}$	$\frac{SO(8,8)}{SO(6,2)\times SO(2,6)}$	$\frac{SO(6,6)\times SO(2,1)}{SO(6)\times SO(6)\times SO(2)}$	$N=4$ supergravity +supersym. Maxwell (10 dim. supergravity)
$E_{8(+8)}/SO(16)$	$E_{8(+8)}/SO^*(16)$	$E_{7(+7)}/SU(8)$	$N=8$ supergravity (11 dim. supergravity)

The $D = 3$ classification of extended supergravity stationary solutions *via* timelike reduction generalizes the $D = 3$ supergravity systems obtained from spacelike reduction. de Wit, Tollsten & Nicolai

Charges

Define the Komar two-form $K \equiv \partial_\mu \kappa_\nu dx^\mu \wedge dx^\nu$. This is invariant under the action of the timelike isometry and, by the asymptotic hypersurface orthogonality assumption, is asymptotically horizontal. This condition is equivalent to a requirement that the scalar field B dual to the Kaluza-Klein vector arising by dimensional reduction out of the metric vanish like $\mathcal{O}(r^{-1})$ as $r \to \infty$. In this case, one can define the Komar mass and NUT charge by (where s^* indicates a pull-back to a section) Bossard, Nikolai & K.S.S.

$$m \equiv \frac{1}{8\pi} \int_{\partial\Sigma} s^* \star K \qquad n \equiv \frac{1}{8\pi} \int_{\partial\Sigma} s^* K$$

The Maxwell field also defines charges. Using the Maxwell field equation $d \star \mathcal{F} = 0$, where $\mathcal{F} \equiv \delta\mathcal{L}/\delta F$ is a linear combination of the two-form field strengths F depending on the four-dimensional scalar fields, and using the Bianchi identity $dF = 0$ one obtains conserved electric and magnetic charges

$$q \equiv \frac{1}{2\pi} \int_{\partial\Sigma} s^* \star \mathcal{F} \qquad p \equiv \frac{1}{2\pi} \int_{\partial\Sigma} s^* F$$

Now consider these charges from the three-dimensional point of view in order to clarify their transformation properties under the three dimensional duality group G (in a simple Maxwell-Einstein example, $G = \mathrm{SU}(2,1)$).

The three-dimensional theory is described in terms of a coset representative $\mathcal{V} \in G/H^*$. The Maurer–Cartan form $\mathcal{V}^{-1}d\mathcal{V}$ decomposes as

$$\mathcal{V}^{-1}d\mathcal{V} = Q + P \quad , \qquad Q \equiv Q_\mu dx^\mu \in \mathfrak{h}^* \quad , \quad P \equiv P_\mu dx^\mu \in \mathfrak{g} \ominus \mathfrak{h}^*$$

Then the three-dimensional equations of motion can be rewritten as $d \star \mathcal{V}P\mathcal{V}^{-1} = 0$, so the $\mathfrak{g}$-valued Noether current is $\star\mathcal{V}P\mathcal{V}^{-1}$.

Since the three-dimensional theory is Euclidean, one cannot properly speak of a conserved charge. Nevertheless, since $\star\mathcal{V}P\mathcal{V}^{-1}$ is d-closed, the integral of this 2-form on a given homology cycle does not depend on the representative of the cycle.

As a result, for stationary solutions, the integral of this three-dimensional current, over any space-like closed surface containing in its interior all the singularities and topologically non-trivial subspaces of a solution, defines a $\mathfrak{g} \ominus \mathfrak{h}^*$-valued charge matrix $\mathscr{C}$

$$\mathscr{C} \equiv \frac{1}{4\pi} \int_{\partial\Sigma} \star \mathcal{V} P \mathcal{V}^{-1}$$

This transforms in the adjoint representation of G according to the standard non-linear action. For asymptotically flat solutions, $\mathcal{V}$ goes to the identity matrix asymptotically and the charge matrix $\mathscr{C}$ in that case is given by the asymptotic value of the one-form P:

$$P = \mathscr{C} \, \frac{dr}{r^2} + \mathcal{O}(r^{-2})$$

Now set up some general notation for the relevant group structure.

Let $\mathfrak{g}_4$ be the algebra of the $D = 4$ symmetry group $\bar{G}$ and let $\mathfrak{h}_4$ be the algebra of its $D = 4$ divisor group $\bar{H}$. $\mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{so}(2, 1)$ is the algebra of the Ehlers group (*i.e.* the $D = 3$ duality group of pure $D = 4$ gravity); $\mathfrak{so}(2)$ is the algebra of its divisor group. Let $\mathfrak{l}_4$ be the $\mathfrak{h}_4$ representation carried by the electric and magnetic charges q and p. Then $\mathscr{C}$ can be decomposed into three irreducible representations with respect to $\mathfrak{so}(2) \oplus \mathfrak{h}_4$ according to

$$\mathfrak{g} \ominus \mathfrak{h}^* \cong \big(\mathfrak{sl}(2, \mathbb{R}) \ominus \mathfrak{so}(2)\big) \oplus \mathfrak{l}_4 \oplus \big(\mathfrak{g}_4 \ominus \mathfrak{h}_4\big)$$

The metric induced by the Cartan-Killing metric of $\mathfrak{g}$ on this coset space is positive definite for the first and last terms, and negative definite for $\mathfrak{l}_4$.

One associates the $\mathfrak{sl}(2, \mathbb{R}) \ominus \mathfrak{so}(2)$ component with the Komar mass and the Komar NUT charge, and one associates the $\mathfrak{l}_4$ component with the electromagnetic charges. The remaining $\mathfrak{g}_4 \ominus \mathfrak{h}_4$ charges come from the Noether current of the four-dimensional theory.

Characteristic equation

Breitenlohner, Gibbons and Maison proved that if G is simple, all the non-extremal single-black-hole solutions of a given theory lie on the H^* orbit of a Kerr solution. Moreover, all *static* solutions regular outside the horizon with a charge matrix satisfying $\mathrm{Tr}\,\mathscr{C}^2 > 0$ lie on the H^*-orbit of a Schwarzschild solution. (Turning on and off angular momentum requires consideration of the $D = 2$ duality group generalizing the Geroch A_1^1 group, and will be considered in future work.)

Using Weyl coordinates, the coset representative $\mathcal{V}$ associated to the Schwarzschild solution with mass m can be written in terms of the non-compact generator $\mathbf{h}$ of the Ehlers $\mathfrak{sl}(2, \mathbb{R})$ only, *i.e.*

$$\mathcal{V} = \exp\left(\frac{1}{2}\ln\frac{r-m}{r+m}\,\mathbf{h}\right) \quad \rightarrow \qquad \mathscr{C} = m\mathbf{h}$$

For the maximal $N = 8$ theory with symmetry $\mathrm{E}_{8(8)}$ (and also for the exceptional 'magic' $N = 2$ supergravity Gunaydin, Sierra & Townsend with symmetry $\mathrm{E}_{8(-24)}$), one finds

$$\mathbf{h}^5 = 5\mathbf{h}^3 - 4\mathbf{h}$$

- Consequently, the charge matrix $\mathscr{C}$ satisfies in all cases

$$\boxed{\mathscr{C}^5 = 5c^2\mathscr{C}^3 - 4c^4\mathscr{C}}$$

  where $c^2 \equiv \frac{1}{k} \operatorname{Tr} \mathscr{C}^2$ is the extremality parameter (vanishing for extremal static solutions) and $k \equiv \operatorname{Tr} \mathbf{h}^2 > 0$.
- Moreover, for all but the two exceptional E_8 cases, a stronger constraint is actually satisfied by the charge matrix $\mathscr{C}$:

$$\boxed{\mathscr{C}^3 = c^2\mathscr{C}}$$

The characteristic equations select acceptable orbits of solutions, *i.e.* orbits not exclusively containing solutions with naked singularities. They determine $\mathscr{C}$ in terms of the mass and NUT charge and the $D = 4$ electromagnetic charges.

Supersymmetry 'Dirac equation'

Extremal solutions have $c^2 = 0$, implying that the charge matrix $\mathscr{C}$ becomes *nilpotent*: $\mathscr{C}^5 = 0$ in the E_8 cases and $\mathscr{C}^3 = 0$ otherwise.

For $\mathcal{N}$ extended supergravity theories, one finds $H^* \cong \mathrm{Spin}^*(2\mathcal{N}) \times H_0$ and the charge matrix $\mathscr{C}$ transforms as a Weyl spinor of $\mathrm{Spin}^*(2\mathcal{N})$ valued in a representation of $\mathfrak{h}_0$. Define the $\mathrm{Spin}^*(2\mathcal{N})$ fermionic oscillators

$$a_i := \frac{1}{2}\Big(\Gamma_{2i-1} + i\Gamma_{2i}\Big) \qquad a^i \equiv (a_i)^\dagger = \frac{1}{2}\Big(\Gamma_{2i-1} - i\Gamma_{2i}\Big)$$

for $i, j, \dots = 1, \dots, \mathcal{N}$. These obey standard anticommutation relations

$$\{a_i, a_j\} = \{a^i, a^j\} = 0 \quad , \qquad \{a_i, a^j\} = \delta_i^j$$

Using this creation/annihilation oscillator basis, the charge matrix $\mathscr{C}$ can be represented as a state

$$|\mathscr{C}\rangle \equiv \Big(W + Z_{ij} a^i a^j + \Sigma_{ijkl} a^i a^j a^k a^l + \cdots\Big)\,|0\rangle$$

From the requirement that dilatino fields be left invariant under an unbroken supersymmetry of a BPS solution, one derives a 'Dirac equation' for the charge state vector,

$$\left(\epsilon^i_\alpha a_i + \Omega_{\alpha\beta}\epsilon^\beta_i a^i\right)|\mathscr{C}\rangle = 0$$

where $(\epsilon^i_\alpha, \epsilon^\alpha_i)$ is the asymptotic (for $r \to \infty$) value of the Killing spinor and $\Omega_{\alpha\beta}$ is a symplectic form on $\mathbb{C}^{2n}$ in cases with n/N preserved supersymmetry.

This condition turns out to be equivalent to the algebraic requirement that $\mathscr{C}$ be a *pure spinor* of $\mathrm{Spin}^*(2\mathcal{N})$ For BPS solutions, it has the consequence that the characteristic equations can be explictly solved in terms of rational functions.

Note that $c^2 = 0$ is a *weaker* condition than the supersymmetry Dirac equation. Extremal and BPS are not always synonymous conditions, although they coincide for $\mathcal{N} \leq 5$ pure supergravities. They are not synomymous for $\mathcal{N} = 6\,\&\,8$ or for theories with vector matter coupling.

BPS Geology

Analysis of the 'Dirac equation' or nilpotency degree of the charge matrix $\mathscr{C}$ leads to a decomposition of the moduli space $\mathcal{M}$ of supergravity solutions into *strata* of various BPS degrees. Letting $\mathcal{M}_0$ be the non-BPS stratum, $\mathcal{M}_1$ being the $\frac{1}{2}$ BPS stratum, etc., the dimensions of the strata for pure supergravity theories turn out to be

	$\mathcal{N}=2$	$\mathcal{N}=3$	$\mathcal{N}=4$	$\mathcal{N}=5$	$\mathcal{N}=6$	$\mathcal{N}=8$
$\dim(\mathcal{M}_0)$	4	8	14	22	34	58
$\dim(\mathcal{M}_1)$	3	7	13	21	33	57
$\dim(\mathcal{M}_2)$			8	16	26	46
$\dim(\mathcal{M}_4)$					17	29

'Almost Iwasawa' decomposition

Earlier analysis of the orbits of the $D = 4$ symmetry groups $\bar{G}$ Cremmer, Lü, Pope & K.S.S. heavily used the Iwasawa decomposition

$$g = u_{(g,Z)} \exp\Big(\ln \lambda_{(g,Z)} \mathbf{z}\Big) b_{(g,Z)}$$

with $u_{(g,Z)} \in \bar{H}$ and $b_{(g,Z)} \in \mathfrak{B}_Z$ where $\mathfrak{B}_Z \subset \bar{G}$ is the 'parabolic' (Borel) subgroup that leaves the charges Z invariant up to a multiplicative factor $\lambda_{(g,Z)}$. This multiplicative factor can be compensated for by 'trombone' transformations combining Weyl scalings with compensating dilational coordinate transformations, leading to a formulation of active symmetry transformations that map solutions into other solutions with *unchanged asymptotic values* of the spacetime metric and scalar fields.

- The $D = 3$ structure is characterized by the fact that the Iwasawa decomposition *breaks down* for noncompact divisor groups H^*.
- The Iwasawa decomposition does, however work "almost everywhere" in the $D = 3$ solution space. The places where it fails are precisely the extremal suborbits of the duality group.

Arithmetic subgroups?

Since the work of Hull & Townsend, there has been a ‘folk’ expectation that all Cremmer-Julia type duality symmetries should be reduced to arithmetic subgroups like $E_8(\mathbb{Z})$ as a result of Dirac charge quantization. However, consider the explicit transformations of the pure gravity charge matrix

$$\mathscr{C} \equiv \begin{pmatrix} m & n \\ n & -m \end{pmatrix} \quad \in \quad \mathfrak{sl}(2,\mathbb{R}) \ominus \mathfrak{so}(2)$$

yielding

$$m' = \frac{(\alpha^2-\gamma^2+\beta^2-\delta^2)c + (\alpha^2-\gamma^2-\beta^2+\delta^2)m + 2(\alpha\beta-\gamma\delta)n}{\sqrt{2(\alpha^2+\gamma^2+\beta^2+\delta^2)+2(\alpha^2+\gamma^2-\beta^2-\delta^2)\frac{m}{c}+4(\alpha\beta+\gamma\delta)}}$$

$$n' = \frac{2(\alpha\gamma+\beta\delta)c + 2(\alpha\gamma-\beta\delta)m + 2(\alpha\delta+\beta\gamma)n}{\sqrt{2(\alpha^2+\gamma^2+\beta^2+\delta^2)+2(\alpha^2+\gamma^2-\beta^2-\delta^2)\frac{m}{c}+4(\alpha\beta+\gamma\delta)}}$$

It is very hard to see how such transformations can be discretized in such a way as to preserve a Dirac type quantization rule.

Conclusions

The understanding of duality group orbits for stationary supergravity solutions has been deepened in the following ways.

- The Noether charge matrix $\mathscr{C}$ satisfies a characteristic equation $\mathscr{C}^5 = 5c^2\mathscr{C}^3 - 4c^4\mathscr{C}$ in the maximal E_8 cases and $\mathscr{C}^3 = c^2\mathscr{C}$ in the non-maximal cases, where $c^2 \equiv \frac{1}{k}\,\mathrm{Tr}\,\mathscr{C}^2$ is the extremality parameter.
- Extremal solutions are characterized by $c^2 = 0$, and $\mathscr{C}$ becomes nilpotent ($\mathscr{C}^5 = 0$ *viz.* $\mathscr{C}^3 = 0$) on the corresponding suborbits.
- BPS solutions have a charge matrix $\mathscr{C}$ satisfying an algebraic 'supersymmetry Dirac equation' which encodes the general properties of such solutions. This is a stronger condition than the $c^2 = 0$ extremality condition.
- The orbits of the $D = 3$ duality group G are not always acted upon transitively by G. This is related to the failure of the Iwasawa decomposition for noncompact divisor groups H^*. The Iwasawa failure set corresponds to the extremal suborbits.