Lorentz invariant quantization of the Yang-Mills theory free of Gribov ambiguity

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Yang-Mills field Lagrangian

$$L = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} \tag{1}$$

Yang-Mills field stress tensor

$$F^a_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g t^{abc} A^b_\mu A^c_\nu \tag{2}$$

gauge transformation

$$\delta A_\mu = \partial_\mu \eta^a + g t^{abc} A^b_\mu \eta^c \tag{3}$$

The Lagrangian is singular

free propagators do not exist

$$A_{free} = \int d^4k \frac{1}{4} A_\mu(k)(k^2 g^{\mu\nu} - k^\mu k^\nu) A_\nu(k) \tag{4}$$

Canonical quantization:

A_0 is not a canonical variable $\longrightarrow$ Yang-Mills theory is a constrained system.

A gauge condition

$$F(A_\mu) = 0$$

Gribov ambiguity.

The condition $F(A_\mu) = 0$ must separate a unique representative in a gauge equivalent classs.

Abelian case

$$\partial_i A_i = 0$$
$$\tilde{A}_i = A_i + \partial_i \eta$$
$$\partial_i \tilde{A}_i = 0 \longrightarrow \Delta \eta = 0 \tag{5}$$

If $\eta \longrightarrow 0$ at spatial infinity $\eta = 0$.

Non-Abelian case

$$\tilde{A}_i^a = A_i^a + \partial_i \eta^a + g t^{abc} A_i^b \eta^c$$
$$\tilde{A}_i^a = 0 \longrightarrow \Delta \eta^a + g t^{abc} \partial_i (A_i^b \eta^c) = 0 \tag{6}$$

This equation has nontrivial solutions fastly decreasing at spatial infinity.

In perturbation theory

$$\Delta\eta^a + gt^{abc}\partial_i(A_i^b\eta^c) = 0 \rightarrow \eta^a = 0$$

No ambiguity

There is no gauge condition which globally selects a unique representative in a gauge equivalent class.

Higgs model in the unitary gauge $B^a = 0$

In perturbation theory

$$\Delta\eta^a + gt^{abc}\partial_i(A_i^b\eta^c) = 0 \to \eta^a = 0$$

No ambiguity

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Higgs model in the unitary gauge $B^a = 0$

The model ($SU(2)$)

$$L = -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + (D_\mu\varphi)^*(D_\mu\varphi) - (D_\mu\chi)^*(D_\mu\chi)$$
$$+i[(D_\mu b)^*(D_\mu e) - (D_\mu e)^*(D_\mu b)] \quad (7)$$

χ has a negative energy.

$$c = \left(\frac{ic_1 + c_2}{\sqrt{2}}, \frac{c_0 - ic_3}{\sqrt{2}}\right) \quad (8)$$

φ, χ are commuting, e, b are anticommuting

$$S = \int \exp\left\{i\tilde{A}\right\} d\mu d\varphi d\chi db de =$$
$$\int \exp\left\{i\int d^4x(-\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu})\right\}(\det D^{-2})^2(\det D^2)^2 d\mu =$$
$$\int \exp\left\{i\int d^4x(-\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu})\right\} d\mu \quad (9)$$

shift

$$\varphi \to \varphi + g^{-1}\hat{m}; \quad \chi \to \chi - g^{-1}\hat{m}; \quad \hat{m} = (0, m) \qquad (10)$$

The terms quadratic in m compensate $\to$ no mass term for the Yang-Mills field.

$$\tilde{L} = L + g^{-1}[D_\mu\varphi + D_\mu\chi]^* D_\mu\hat{m} + h.c. \qquad (11)$$

The Lagrangian $\tilde{L}$ is invariant with respect to the "shifted" gauge transformations.

$$\begin{aligned} \delta\varphi_0^{\pm} &= \frac{g}{2}\varphi_a^{\pm}\eta^a \\ \delta\varphi_a^{+} &= -\frac{g}{2}\epsilon^{abc}\varphi_b^{+}\eta^c - \frac{g}{2}\varphi_0^{+}\eta^a \\ \delta\varphi_a^{-} &= -m\eta^a - \frac{g}{2}\epsilon^{abc}\varphi_b^{-}\eta^c - \frac{g}{2}\varphi_0^{-}\eta^a \end{aligned} \qquad (12)$$

$$\varphi_\alpha^{\pm} = \frac{\varphi_\alpha \pm \chi_\alpha}{\sqrt{2}}$$

$\tilde{L}$ is also invariant with respect to the supersymmetry transformations:

$$\begin{aligned} \delta\varphi(x) &= i\epsilon b(x) \\ \delta\chi(x) &= -i\epsilon b(x) \\ \delta e(x) &= \epsilon[\varphi(x) + \chi(x)] \\ \delta b(x) &= 0 \end{aligned} \tag{13}$$

This symmetry is crucial for the unitarity of the model.

possible gauges:

$$\partial_i A_i^a = 0$$

$$\varphi_a^- = 0 \quad (\text{no ambiguity})$$

The equation $(\varphi^{\Omega})^-_a = 0$ at the surface $\varphi^-_a = 0$ has no nontrivial solutions fastly decreasing at spatial infinity

Effective action in the gauge $\varphi^-_a = 0$

$$\tilde{A} = \int d^4x\{-\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \partial_\mu\varphi^+_0\partial_\mu\varphi^-_0 + m\varphi^+_a\partial_\mu A^a_\mu$$
$$+i[(D_\mu b)^*(D_\mu e) - (D_\mu e)^*(D_\mu b)]$$
$$+\frac{mg}{4}A^2_\mu\varphi^+_0 + \frac{g^2}{8}A^2_\mu\varphi^+_0\varphi^-_0 + g\partial_\mu\varphi^-_0 A^a_\mu\varphi^+_a + \frac{g}{2}\varphi^-_0\varphi^+_a\partial_\mu A^a_\mu\} \quad (14)$$

The construction of the canonical formalism is straightforward.

The canoncal momentum for A_0^a is nonzero:

$$p_0^a = m\varphi_a^+ (1 + \frac{g}{2m}\varphi_0^-) \tag{15}$$

The Hamiltonian:

$$\tilde{A}_H = \int d^4x \{ p_i^a \dot{A}_i^a + p_0^a \dot{A}_0^a + p_\varphi \dot{\varphi}_0 + p_\chi \dot{\chi}_0$$
$$- \frac{(p_i^a)^2}{2} + \frac{(p_0^a)^2}{2(1 + g/(2m)\varphi_0^-)^2} - \frac{p_\varphi^2}{2} + \frac{p_\chi^2}{2} +$$
$$+ A_0^a \partial_i p_i^a - \frac{p_0^a \partial_i A_i^a}{1 + g/(2m)\varphi_0^-} - \frac{1}{4} F_{ik}^a F_{ik}^a + \ldots \} \tag{16}$$

Integration over canonical momenta produces a nontrivial Jacobian

$$\prod_x (1 + \frac{g}{2m}\varphi_0^-)^3$$

The scattering matrix in terms of the integral over trajectories in the coordinate space is:

$$S = \int \exp\left\{ i\tilde{A} \right\} \prod_x (1 + \frac{g}{2m}\varphi_0^-)^3 dA_\mu d\varphi_\alpha^+ d\varphi_0^- db^\alpha de^\alpha \qquad (17)$$

PHYSICAL UNITARITY.

The scattering matrix (17) acts in the space which contains many unphysical exitations. Nevertheless the supersymmetry of the effective action provides the unitarity of this S-matrix in the subspace which includes only physical vectors.

Invariance with respect to the supersymmetry transformations via Noether theorem leads to the existence of the conserved charge Q, which may be used to select the physical subspace:

$$Q|\psi >= 0 \rightarrow Q_0|\psi >_{as} = 0 \tag{18}$$

The vectors $|\psi >_{as}$ may be presented in the form:

$$|\psi >_{as} = |\psi >_{tr} + |N>, \quad <N|N>=0$$
$$<\psi|O|\psi >_{as} = <\psi|O|\psi >_{tr} \tag{19}$$

for any observable O.

The gauge condition $\varphi_a^- = 0$ breaks "old" supersummetry.

However there exists $\Omega(\eta)$ such that

$$\int d^4x \lambda^a(x) \partial_i A_i^a(x) = \int d^4x \lambda^a(x) (\varphi^\Omega)_a^-(x) \qquad (20)$$

For the free theory

$$\eta^a(x) = \frac{-\varphi_a^-(x) + \partial_i A_i^a(x)}{m}$$

Therefore the function η^a changes under the supersymmetry transformations

$$\eta^a(x) \to \eta^a(x) - i\frac{\sqrt{2}\epsilon b^a(x)}{m}$$

The supersymmetry transformations in the gauge $\varphi_a^- = 0$ to zero order in g

$$\tilde{A}^a_\mu(x) = A^a_\mu(x) - \partial_\mu \eta^a(x) \longrightarrow \tilde{A}^a_\mu(x) + i\frac{\sqrt{2}\epsilon\partial_\mu \tilde{b}^a(x)}{m}$$
$$\tilde{e}_\alpha(x) \longrightarrow \tilde{e}_\alpha(x) + \epsilon\sqrt{2}\tilde{\varphi}^+_\alpha(x)$$
$$\tilde{\varphi}^-_0(x) \longrightarrow \tilde{\varphi}^-_0(x) + i\sqrt{2}\epsilon\tilde{b}_0(x) \qquad (21)$$

Invariance with respect to these supersymmetry transformations generates a conserved charge which for the asymptotic states may be presented as $Q_0 = \tilde{Q}^0_0 + Q^0_0$

$$\tilde{Q}^0_0 = \sqrt{2}\int d^3x\{m^{-1}(\partial_i A_0 - \partial_0 A_i)^a(\partial_i b^a) - \varphi^+_a \partial_0 b^a\} \qquad (22)$$

$$Q^0_0 = \sqrt{2}\int d^3x\{\partial_0 \varphi^+_0 b_0 + \partial_0 b_0 \varphi^+_0\} \qquad (23)$$

Q_0^0 and $\tilde{Q}_0^0$ are nilpotent, anticommuting and independent. Hence

$$Q_0|\psi>_{as}= 0 \rightarrow \tilde{Q}_0^0|\psi>_{as}= 0, \quad Q_0^0|\psi>_{as}= 0$$

$\tilde{Q}_0^0$ coincides with the BRST charge for the Lorentz gauge condition, introduced via the Lagrange multiplier $\int d^4x\varphi_a^+\partial_\mu A_\mu^a$, if one identifies $b^a = c^a, e^a = \bar{c}^a$

Any vector annihilated by $\tilde{Q}_0^0$ has a form

$$|\psi>= |\tilde{\psi}> +|\tilde{N}>$$

the vector $|\tilde{\psi}>$ contains only three dimensionally transversal components of the Yang-Mills field and the exitations coresponding to $\varphi_0^{\pm}, b^0, e^0$ and $|\tilde{N}>$ has a zero norm.

Any vector annihilated by Q_0^0 has a form

$$|\psi> = |\hat{\psi}> + |\hat{N}>$$

where the vector $|\hat{\psi}>$ does not contain the exitations corresponding to $\varphi_0^{\pm}, b^0, e^0$ and $|\hat{N}>$ is a zero norm vector

Hence

$$|\psi>_{as} = |\psi>_{tr} + |N>, \quad <\psi|O|\psi>_{as} = <\psi|O|\psi>_{tr}$$

After factorization with respect to zero norm vectors, the space of $|\psi>_{as}$ coincides with $|\psi>_{tr}$.

For perturbative calculations the Lorentz gauge $\partial_\mu A_\mu$ is more convenient.

A generating functional for gauge invariant Green functions:

$$Z(J_\mu) = \int \exp\{i(A + \int d^4x J_\mu F_\mu)\}\delta(\varphi_a^-)\Delta_- dA_\mu d\varphi_\alpha^- \varphi_\alpha^+ db_\alpha de_\alpha \tag{24}$$

where F_μ denotes a gauge invariant functional, A is the gauge invariant action and the factor Δ_- is the gauge invariant functional defined by the equation

$$\Delta_- \int d\Omega \delta(\partial_\mu A_\mu^\Omega) = 1$$

at the surface $\varphi_a^- = 0$

$$\prod_x (a + \frac{g}{2}\varphi_0^-)^{-3} = \int d\Omega \delta(\varphi_a^{-\Omega})|_{\varphi_a^- = 0}$$

Multiplying the integral (24) by "one"

$$\Delta_L \int \delta(\partial_\mu A_\mu^\Omega) d\Omega = 1$$

and making change of variables $A_\mu \to A_\mu^\Omega, \varphi \to \varphi^\Omega, b \to b^\Omega, e \to e^\Omega$, we get the generating functional in the Lorentz gauge:

$$Z(J_\mu = \int \exp\left\{ i(A + \int d^4x J_\mu F_\mu) \right\} \Delta_L \delta(\partial_\mu A_\mu) dA_\mu d\varphi_\alpha db_\alpha de_\alpha \tag{25}$$

The transition is legitimate only in perturbation theory as beyond the perturbation theory Δ_L may be singular.

Problems

Perturbative renormalization directly in the gauge $\varphi_a^- = 0$? Mixed propagator A_μ^a, φ_b^+ at large k decreases as k^{-1}. It leads to appearance of the infinite number of divergent diagrams differing by the number of φ_0^- lines. But the degree of divergency of all these diagrams is limited by 2. A similar situation exists in the supersymmetric gauge theories in a manifestly supersymmetric gauge. The number of divergent diagrams is infinite, but the number of counterterms is finite due to supersymmetry. Does the supersymmetry in this case restricts the number of counterterms?

Conclusion

1.The Yang-Mills theory allows unambigous Lorentz invariant quantization.

2.The ghost field Lagrangian in a covariant gauge may be gauge invariant.

3. It simplifies regularization and renormalization.

4.This method may be useful for supersymmetric gauge theories.