Kazhdan–Lusztig Duality in Logarithmic Conformal Field Theory

How to make complicated things simple

AM Semikhatov

Theory Department
Lebedev Physics Institute

4th Sakharov Conference on Physics

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Prejudiced selection

- B Feigin, A Gainutdinov, AMS, I Tipunin, *Commun. Math. Phys.* 265 (2006) 47–93 [hep-th/0504093].
- B Feigin, A Gainutdinov, AMS, I Tipunin, *Theor. Math. Phys.* 148 (2006) 1210–1235 [math.QA/0512621].
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- Y Arike, *Symmetric linear functions of the restricted quantum group* $\bar{U}_q sl_2(\mathbb{C})$, arXiv:0706.1113.
- P Furlan, L Hadjiivanov, I Todorov, *Zero modes' fusion ring and braid group representations for the extended chiral WZNW model*, arXiv:0710.1063.
- H Kondo, Y Saito, *Indecomposable decomposition of tensor products of modules over the restricted quantum universal enveloping algebra associated to* $\mathfrak{sl}_2$, arXiv:0901.4221 [math.QA].
- Y Arike, *Symmetric linear functions on the quantum group* $g_{p,q}$, arXiv:0904.0331 [math.QA].
- D Adamović, A Milas, *The* $N = 1$ *triplet vertex operator superalgebras*, Commun. Math. Phys. 288 (2009) 225–270 arXiv:0712.0379 [math.QA].
- D Adamović, A Milas, *Lattice construction of logarithmic modules for certain vertex algebras*, arXiv:0902.3417 [math.QA].
- K Nagatomo, A Tsuchiya, *The triplet vertex operator algebra* $W(p)$ *and*

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There is a close correspondence (“duality”) between

chiral algebras of 2D logarithmic conformal field theory models

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- (p, p') models, a triplet algebra $W(p, p')$ [FGST].

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Apparently Disparate Things:

$W(p_+ = 2, p_- = 3)$:

$$
\begin{aligned}
W^+ = \Big(&\frac{35}{27}(\partial^4\varphi)^2 + \frac{56}{27}\partial^5\varphi\,\partial^3\varphi + \frac{28}{27}\partial^6\varphi\,\partial^2\varphi + \frac{8}{27}\partial^7\varphi\,\partial\varphi - \frac{280}{9\sqrt{3}}(\partial^3\varphi)^2\,\partial^2\varphi \\
&- \frac{70}{3\sqrt{3}}\partial^4\varphi\,(\partial^2\varphi)^2 - \frac{280}{9\sqrt{3}}\partial^4\varphi\,\partial^3\varphi\,\partial\varphi - \frac{56}{3\sqrt{3}}\partial^5\varphi\,\partial^2\varphi\,\partial\varphi - \frac{28}{9\sqrt{3}}\partial^6\varphi(\partial\varphi)^2 \\
&+ \frac{35}{3}(\partial^2\varphi)^4 + \frac{280}{3}\partial^3\varphi(\partial^2\varphi)^2\,\partial\varphi + \frac{280}{9}(\partial^3\varphi)^2(\partial\varphi)^2 + \frac{140}{3}\partial^4\varphi\,\partial^2\varphi(\partial\varphi)^2 \\
&+ \frac{56}{9}\partial^5\varphi(\partial\varphi)^3 - \frac{140}{\sqrt{3}}(\partial^2\varphi)^3(\partial\varphi)^2 - \frac{560}{3\sqrt{3}}\partial^3\varphi\,\partial^2\varphi\,(\partial\varphi)^2 - \frac{70}{3\sqrt{3}}\partial^4\varphi(\partial\varphi)^4 \\
&+ 70(\partial^2\varphi)^2(\partial\varphi)^4 + \frac{56}{3}\partial^3\varphi(\partial\varphi)^5 - \frac{28}{\sqrt{3}}\partial^2\varphi(\partial\varphi)^6 + (\partial\varphi)^8 - \frac{1}{27\sqrt{3}}\partial^8\varphi\Big)e^{2\sqrt{3}\varphi},
\end{aligned}
$$

Apparently Disparate Things:

$W(p_+ = 2, p_- = 3)$:

$$
\begin{aligned}
w^- = \Big(&\frac{217}{192}(\partial^5\varphi)^2 - \frac{2653}{3456}\partial^6\varphi\,\partial^4\varphi - \frac{23}{384}\partial^7\varphi\,\partial^3\varphi - \frac{11}{1152}\partial^8\varphi\,\partial^2\varphi - \frac{1}{768}\partial^9\varphi\,\partial\varphi \\
&- \frac{1225}{64\sqrt{3}}\partial^4\varphi(\partial^3\varphi)^2 - \frac{13475}{576\sqrt{3}}(\partial^4\varphi)^2\partial^2\varphi + \frac{2695}{64\sqrt{3}}\partial^5\varphi\,\partial^3\varphi\,\partial^2\varphi + \frac{2555}{192\sqrt{3}}\partial^5\varphi\,\partial^4\varphi\,\partial\varphi \\
&- \frac{2891}{576\sqrt{3}}\partial^6\varphi(\partial^2\varphi)^2 - \frac{1351}{192\sqrt{3}}\partial^6\varphi\,\partial^3\varphi\,\partial\varphi - \frac{103}{192\sqrt{3}}\partial^7\varphi\,\partial^2\varphi\,\partial\varphi - \frac{13}{384\sqrt{3}}\partial^8\varphi(\partial\varphi)^2 \\
&+ \frac{3535}{32}(\partial^3\varphi)^2(\partial^2\varphi)^2 - \frac{735}{16}(\partial^3\varphi)^3\partial\varphi - \frac{3395}{54}\partial^4\varphi(\partial^2\varphi)^3 + \frac{245}{24}\partial^4\varphi\,\partial^3\varphi\,\partial^2\varphi\,\partial\varphi \\
&+ \frac{12635}{576}(\partial^4\varphi)^2(\partial\varphi)^2 + \frac{245}{12}\partial^5\varphi(\partial^2\varphi)^2\partial\varphi + \frac{105}{32}\partial^5\varphi\,\partial^3\varphi(\partial\varphi)^2 \\
&- \frac{2443}{288}\partial^6\varphi\,\partial^2\varphi(\partial\varphi)^2 - \frac{19}{96}\partial^7\varphi(\partial\varphi)^3 - \frac{13405}{144\sqrt{3}}(\partial^2\varphi)^5 + \frac{8225}{24\sqrt{3}}\partial^3\varphi(\partial^2\varphi)^3\partial\varphi \\
&- \frac{105\sqrt{3}}{4}(\partial^3\varphi)^2\partial^2\varphi(\partial\varphi)^2 + \frac{665}{24\sqrt{3}}\partial^4\varphi(\partial^2\varphi)^2(\partial\varphi)^2 + \frac{245}{2\sqrt{3}}\partial^4\varphi\,\partial^3\varphi(\partial\varphi)^3 \\
&- \frac{245}{8\sqrt{3}}\partial^5\varphi\,\partial^2\varphi(\partial\varphi)^3 - \frac{91}{24\sqrt{3}}\partial^6\varphi(\partial\varphi)^4 + \frac{16205}{144}(\partial^2\varphi)^4(\partial\varphi)^2 + \frac{385}{4}\partial^3\varphi(\partial^2\varphi)^2(\partial\varphi)^3 \\
&+ \frac{525}{8}(\partial^3\varphi)^2(\partial\varphi)^4 + \frac{35}{3}\partial^4\varphi\,\partial^2\varphi(\partial\varphi)^4 - 7\,\partial^5\varphi(\partial\varphi)^5 + \frac{665}{3\sqrt{3}}(\partial^2\varphi)^3(\partial\varphi)^4 \\
&+ \frac{105\sqrt{3}}{2}\partial^3\varphi\,\partial^2\varphi(\partial\varphi)^5 - \frac{35}{3\sqrt{3}}\partial^4\varphi(\partial\varphi)^6 + \frac{455}{6}(\partial^2\varphi)^2(\partial\varphi)^6 + 5\,\partial^3\varphi(\partial\varphi)^7 \\
&+ \frac{25}{\sqrt{3}}\partial^2\varphi(\partial\varphi)^8 + (\partial\varphi)^{10} - \frac{1}{13824\sqrt{3}}\partial^{10}\varphi\Big)e^{-2\sqrt{3}\varphi},
\end{aligned}
$$

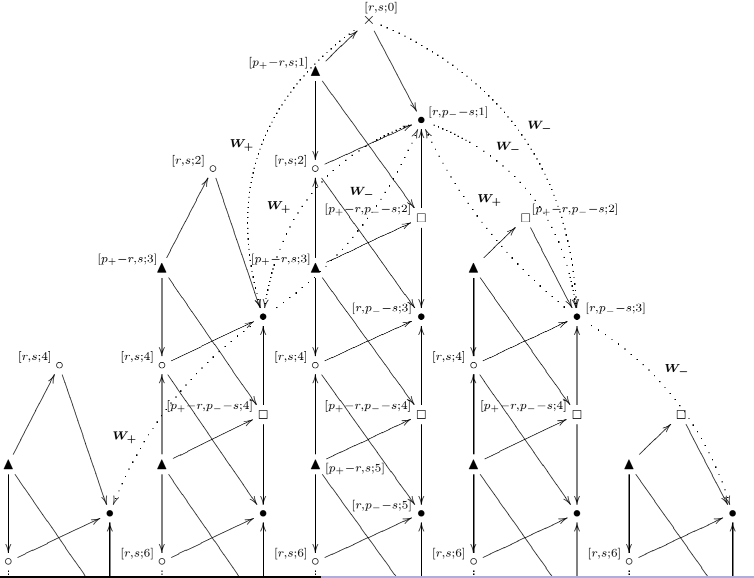
Apparently Disparate Things:

$W(p_+ = 2, p_- = 3)$: with the OPE

$$W^+(z)\,W^-(w) = 2^7 \cdot 3 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 17\,\frac{T(w)}{(z-w)^{28}} + \dots,$$

Apparently Disparate Things:

with a complicated representation theory,



Apparently Disparate Things:

but the "dual" QG is defined by

$$E_\pm^{p_\pm} = F_\pm^{p_\pm} = 0, \quad K^{2p_+p_-} = \mathbf{1},$$
$$KE_\pm K^{-1} = \mathfrak{q}_\pm^2 E_\pm, \quad KF_\pm K^{-1} = \mathfrak{q}_\pm^{-2} F_\pm,$$
$$E_+E_- = E_-E_+, \quad F_+F_- = F_-F_+, \quad E_+F_- = F_-E_+, \quad E_-F_+ = F_+E_-,$$
$$[E_\pm, F_\pm] = \frac{K^{\pm p_\mp} - K^{\mp p_\mp}}{\mathfrak{q}_\pm^{\pm p_\mp} - \mathfrak{q}_\pm^{\mp p_\mp}}$$

and is a finite-dimensional algebra ($\dim = 2p_+^3 p_-^3$).

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$$[E_{\pm}, F_{\pm}] = \frac{K^{\pm p_{\mp}} - K^{\mp p_{\mp}}}{\mathfrak{q}_{\pm}^{\pm p_{\mp}} - \mathfrak{q}_{\pm}^{\mp p_{\mp}}}$$

and is a finite-dimensional algebra ($\dim = 2p_+^3 p_-^3$).

For $(p, 1)$ models, the QG is even simpler,

$$KE\,K^{-1} = \mathfrak{q}^{2}E, \qquad KF\,K^{-1} = \mathfrak{q}^{-2}F, \qquad [E, F] = \frac{K - K^{-1}}{\mathfrak{q} - \mathfrak{q}^{-1}},$$
$$E^{p} = 0, \quad F^{p} = 0, \quad K^{2p} = 1,$$

$\dim = 2p^3$

The correspondence

- Irreducible representations:
  $(p,1)$ models: $W(p)$ irreps $\stackrel{1:1}{\longleftrightarrow}$ QG irreps
  (p,p') models: $W(p,p')$ irreps $\longleftarrow$ Quantum Group irreps
- Modular transformation properties:
  $SL(2,\mathbb{Z})$ acting on (generalized) characters $\stackrel{=}{\longleftrightarrow}$ $SL(2,\mathbb{Z})$ acting on QG center

  $(3p-1)$ generalized characters $\dim Z = (3p-1)$

  $\frac{1}{2}(3p-1)(3p'-1)$ gen'd characters $\dim Z = \frac{1}{2}(3p'-1)(3p'-1)$
- Indecomposable representations:
  FGST conjecture, recently refined/proved by Nagatomo–Tsuchiya
- Fusion:
  BFGT, supported by statistical mechanics models
  (Pierce–Rasmussen–Zuber, Rasmussen et al.)

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Drinfeld double

- Start with the algebra B of screenings and zero modes.
- Take the dual space B^*,
  then $\langle\beta\gamma, b\rangle = \langle\beta, b'\rangle\langle\gamma, b''\rangle$ and $\langle\Delta(\beta), a\otimes b\rangle = \langle\beta, ab\rangle$.

How?

Drinfeld double

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$(p,1)$ example:

$kE = \mathfrak{q}Ek,\ \ E^p = 0,\ \ k^{4p} = 1; \quad \mathfrak{q} = e^{i\pi/p}$
and *comultiplication*: $\Delta(E) = 1 \otimes E + E \otimes k^2,\ \ \Delta(k) = k \otimes k$.

write $\Delta b = \sum b' \otimes b'' = b' \otimes b''$

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$(p, 1)$ example:

$\langle F, E^m k^n\rangle = \delta_{m,1}\dfrac{\mathfrak{q}^{-n}}{\mathfrak{q} - \mathfrak{q}^{-1}},\ \langle \varkappa, E^m k^n\rangle = \delta_{m,0}\mathfrak{q}^{-n/2},$

then $\varkappa F = \mathfrak{q}F\varkappa,\ \ F^p = 0,\ \ \varkappa^{4p} = 1; \quad \Delta(F) = \varkappa^2 \otimes F + F \otimes 1,\ \ \Delta(\varkappa) = \varkappa \otimes \varkappa$.

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  then $\langle\beta\gamma, b\rangle = \langle\beta, b'\rangle\langle\gamma, b''\rangle$ and $\langle\Delta(\beta), a\otimes b\rangle = \langle\beta, ab\rangle$.
- Some abstract nonsense:

  B acts on B^*: $h\rightharpoonup\beta = \langle\beta'', h\rangle\beta'$ (left regular action)

  $\beta\leftharpoonup h = \langle\beta', h\rangle\beta''$ (right regular action)

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$a\leftharpoonup\beta = \langle\beta, a'\rangle a''$

- Then the Drinfeld double: $\mathcal{D}(B) = B^*\otimes B$ with the composition

$$(\alpha\otimes a)(\beta\otimes b) = ().$$

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$$B^* \text{ acts on } B:\quad \beta\rightharpoonup a = \langle\beta, a''\rangle a'$$

$$a\leftharpoonup\beta = \langle\beta, a'\rangle a''$$

- Then the Drinfeld double: $\mathcal{D}(B) = B^*\otimes B$ with the composition

$$(\alpha\otimes a)(\beta\otimes b) = \alpha(\,)\otimes b.$$

Drinfeld double

- Start with the algebra B of screenings and zero modes.
- Take the dual space B^*,
  then $\langle\beta\gamma, b\rangle = \langle\beta, b'\rangle\langle\gamma, b''\rangle$ and $\langle\Delta(\beta), a\otimes b\rangle = \langle\beta, ab\rangle$.
- Some abstract nonsense:

$$B \text{ acts on } B^*: \quad h\rightharpoonup\beta = \langle\beta'', h\rangle\beta' \text{ (left regular action)}$$
$$\beta\leftharpoonup h = \langle\beta', h\rangle\beta'' \text{ (right regular action)}$$
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$$(\alpha\otimes a)(\beta\otimes b) = \alpha(a'\rightharpoonup\beta\leftharpoonup(a'''))\otimes a''b.$$

How?

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  then $\langle\beta\gamma, b\rangle = \langle\beta, b'\rangle\langle\gamma, b''\rangle$ and $\langle\Delta(\beta), a\otimes b\rangle = \langle\beta, ab\rangle$.
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$$(\alpha\otimes a)(\beta\otimes b) = \alpha(a' \rightharpoonup \beta \leftharpoonup S^{-1}(a''')) \otimes a''b.$$

How?

Drinfeld double

$$kE = \mathfrak{q}Ek, \quad E^p = 0, \quad k^{4p} = 1,$$
$$\varkappa F = \mathfrak{q}F\varkappa, \quad F^p = 0, \quad \varkappa^{4p} = 1,$$
$$k\varkappa = \varkappa k, \quad kFk^{-1} = \mathfrak{q}^{-1}F, \quad \varkappa E\varkappa^{-1} = \mathfrak{q}^{-1}E, \quad [E,F] = \frac{k^2 - \varkappa^2}{\mathfrak{q} - \mathfrak{q}^{-1}}.$$

$$\Downarrow$$

$\overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2)$ at $2p$th root of unity

$$k^2E = \mathfrak{q}Ek^2, \quad E^p = 0, \quad k^{4p} = 1,$$
$$k^2F = \mathfrak{q}^{-2}Fk^2, \quad F^p = 0,$$
$$[E,F] = \frac{k^2 - k^{-2}}{\mathfrak{q} - \mathfrak{q}^{-1}}.$$

has all the remarkable properties, such as an $SL(2,\mathbb{Z})$ representation on the $(3p-1)$-dimensional center.

Anything else?

Can we have a **manifestly** QG-invariant description of LCFT?

What would be the QG analogue of the algebra of fields in such description?

Assuming that the QG acts on fields, it has to act on *products* of fields:

$$a \triangleright (\phi\psi) = \,?$$

So we need a **module algebra**
— to begin with, a module algebra over the Drinfeld double $\mathcal{D}(B)$

Theorem

For a Hopf algebra B with invertible antipode, the Heisenberg double $\mathcal{H}(B^*)$ is a $\mathcal{D}(B)$-module algebra.

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For a Hopf algebra B with invertible antipode, the Heisenberg double $\mathcal{H}(B^*)$ is a $\mathcal{D}(B)$-module algebra.

Heisenberg double

$$\mathcal{H}(B^*) = B^* \otimes B \quad \text{as a vector space}$$

(the same vector space as $\mathcal{D}(B) = B^* \otimes B$(!))

with the composition law

$$(\alpha \# a)(\beta \# b) = \alpha(a' \rightharpoonup \beta) \# a'' b, \qquad \alpha, \beta \in B^*, \quad a, b \in B.$$

Theorem (continued):

The $\mathcal{D}(B)$ action on $\mathcal{H}(B^*)$ is given by

$$(\mu \otimes m) \triangleright (\alpha \# a) = ((\mu \otimes m)' \rightharpoonup \alpha) \# ((\mu \otimes m)'' \triangleright a),$$

$$\mu \otimes m \in \mathcal{D}(B), \quad \alpha \# a \in \mathcal{H}(B^*),$$

where

$$(\mu \otimes m) \rightharpoonup \alpha = \mu''(m \rightharpoonup \alpha) S^{*-1}(\mu'),$$

$$(\mu \otimes m) \triangleright a = (m' a S(m'')) \leftharpoonup S^{*-1}(\mu).$$

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$$\mathcal{H}(B^*) = B^* \# B \quad \text{as a vector space}$$

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where

$$(\mu \otimes m) \rightharpoonup \alpha = \mu''(m \rightharpoonup \alpha) S^{*\,-1}(\mu'),$$

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where

$$(\mu \otimes m) \rightharpoonup \alpha = \mu''(m \rightharpoonup \alpha) S^{*-1}(\mu'),$$
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with the composition law

$$(\alpha \# a)(\beta \# b) = \alpha(a' \rightharpoonup \beta) \# a''b, \qquad \alpha, \beta \in B^*, \quad a, b \in B.$$

whereas in $\mathcal{D}(B)$,

$$(\alpha \otimes a)(\beta \otimes b) = \alpha(a' \rightharpoonup \beta \leftharpoonup S^{-1}(a''')) \otimes a''b$$

Theorem (continued):

The $\mathcal{D}(B)$ action on $\mathcal{H}(B^*)$ is given by

$$(\mu \otimes m) \triangleright (\alpha \# a) = ((\mu \otimes m)' \rightharpoonup \alpha) \# ((\mu \otimes m)'' \triangleright a),$$
$$\mu \otimes m \in \mathcal{D}(B), \quad \alpha \# a \in \mathcal{H}(B^*),$$

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$$(\mu \otimes m) \rightharpoonup \alpha = \mu''(m \rightharpoonup \alpha) S^{*\,-1}(\mu'),$$
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invented in: A Alekseev, L Faddeev, *Commun. Math. Phys.* 141 (1991) 413–422;
N Reshetikhin, M Semenov-Tian-Shansky, *Lett. Math. Phys.* 19 (1990) 133–142;
M Semenov-Tyan-Shanskii, *Theor. Math. Phys.* 93 (1992) 1292–1307.

Theorem (continued):

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Interpretation:

View $a, b \in B$ as operators and $\alpha, \beta \in B^*$ as functions: “Leibnitz rule”

Theorem (continued):

The $\mathcal{D}(B)$ action on $\mathcal{H}(B^*)$ is given by

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Theorem (continued):

The $\mathcal{D}(B)$ action on $\mathcal{H}(B^*)$ is given by

$$(\mu \otimes m) \triangleright (\alpha \# a) = \big((\mu \otimes m)' \rightharpoonup \alpha\big) \# \big((\mu \otimes m)'' \triangleright a\big),$$
$$\mu \otimes m \in \mathcal{D}(B), \quad \alpha \# a \in \mathcal{H}(B^*),$$

where

$$(\mu \otimes m) \rightharpoonup \alpha = \mu''(m \rightharpoonup \alpha) S^{*\,-1}(\mu'),$$
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Back to $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$

Recall that we had

$$\mathcal{D}(B)$$
$$\Downarrow$$
$$\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$$

$$\dim \overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2) = 2p^3,$$

Basis: $E,\ F,\ k^2,$

$$k^2 E = \mathfrak{q} E k^2, \quad E^p = 0, \quad k^{4p} = 1,$$
$$k^2 F = \mathfrak{q}^{-2} F k^2, \quad F^p = 0,$$
$$[E, F] = \frac{k^2 - k^{-2}}{\mathfrak{q} - \mathfrak{q}^{-1}}$$

Dually, we have

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$$\dim \overline{\mathcal{H}}_{\mathfrak{q}} s\ell(2) = 2p^3$$

Basis: $\lambda,\ z,\ \partial$

$$\lambda^{2p} = 1, \quad \lambda z = z\lambda, \quad \lambda\partial = \partial\lambda,$$
$$z^p = 0, \quad \partial^p = 0,$$
$$\partial z = (\mathfrak{q} - \mathfrak{q}^{-1})1 + \mathfrak{q}^{-2} z\partial$$

$\overline{\mathcal{H}}_{\mathfrak{q}} s\ell(2) =$
$(\mathbb{C}[\lambda]/(\lambda^{2p} - 1)) \otimes \mathbb{C}_{\mathfrak{q}}[z, \partial]$

Back to $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$

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$$\partial z = (\mathfrak{q} - \mathfrak{q}^{-1})1 + \mathfrak{q}^{-2} z\partial$$

$\overline{\mathcal{H}}_{\mathfrak{q}} s\ell(2) =$
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Back to $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$

Recall that we had

$$\mathcal{D}(B)$$

$$\Downarrow$$

$$\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$$

$$\dim \overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2) = 2p^3,$$

Basis: $E,\ F,\ k^2,$

$$k^2 E = \mathfrak{q} E k^2, \quad E^p = 0, \quad k^{4p} = 1,$$

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$$\overline{\mathcal{H}}_{\mathfrak{q}} s\ell(2) = (\mathbb{C}[\lambda]/(\lambda^{2p} - 1)) \otimes \mathbb{C}_{\mathfrak{q}}[z, \partial]$$

The reduced Heisenberg double:

$$\overline{\mathcal{H}}_{\mathfrak{q}} s\ell(2) = (\mathbb{C}[\lambda]/(\lambda^{2p} - 1)) \otimes \mathbb{C}_{\mathfrak{q}}[z, \partial],$$

$$\lambda^{2p} = 1, \quad \lambda z = z\lambda, \qquad \lambda\partial = \partial\lambda,$$

$$z^p = 0, \quad \partial^p = 0,$$

$$\partial z = (\mathfrak{q} - \mathfrak{q}^{-1})1 + \mathfrak{q}^{-2} z\partial$$

with the $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ action

$$E \rhd \lambda = \frac{1}{\mathfrak{q}+1} \lambda\, z, \qquad k^2 \rhd \lambda = \mathfrak{q}^{-1}\lambda, \qquad F \rhd \lambda = -\frac{\mathfrak{q}}{\mathfrak{q}+1} \partial\,\lambda,$$

$$E \rhd z^m = -\mathfrak{q}^m [m] z^{m+1}, \; k^2 \rhd z^m = \mathfrak{q}^{2m} z^m, \quad F \rhd z^m = [m]\mathfrak{q}^{1-m} z^{m-1},$$

$$E \rhd \partial^n = \mathfrak{q}^{1-n}[n]\partial^{n-1}, \quad k^2 \rhd \partial^n = \mathfrak{q}^{-2n}\partial^n, \quad F \rhd \partial^n = -\mathfrak{q}^n [n]\partial^{n+1}$$

The reduced Heisenberg double:

$$\overline{\mathcal{H}}_{\mathfrak{q}} s\ell(2) = (\mathbb{C}[\lambda]/(\lambda^{2p}-1)) \otimes \mathbb{C}_{\mathfrak{q}}[z,\partial],$$
$$\lambda^{2p} = 1, \quad \lambda z = z\lambda, \qquad \lambda\partial = \partial\lambda,$$
$$z^p = 0, \quad \partial^p = 0,$$
$$\partial z = (\mathfrak{q}-\mathfrak{q}^{-1})1 + \mathfrak{q}^{-2} z\partial$$

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$$E \triangleright \partial^n = \mathfrak{q}^{1-n}[n]\partial^{n-1}, \quad k^2 \triangleright \partial^n = \mathfrak{q}^{-2n}\partial^n, \quad F \triangleright \partial^n = -\mathfrak{q}^n[n]\partial^{n+1}$$

$\overline{\mathcal{H}}_{\mathfrak{q}} s\ell(2)$

The reduced Heisenberg double:

$$\overline{\mathcal{H}}_{\mathfrak{q}} s\ell(2) = (\mathbb{C}[\lambda]/(\lambda^{2p}-1)) \otimes \mathbb{C}_{\mathfrak{q}}[z,\partial],$$
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$$E \triangleright z^m = -\mathfrak{q}^m[m]z^{m+1}, \; k^2 \triangleright z^m = \mathfrak{q}^{2m} z^m, \quad F \triangleright z^m = [m]\mathfrak{q}^{1-m} z^{m-1},$$
$$E \triangleright \partial^n = \mathfrak{q}^{1-n}[n]\partial^{n-1}, \quad k^2 \triangleright \partial^n = \mathfrak{q}^{-2n}\partial^n, \quad F \triangleright \partial^n = -\mathfrak{q}^n[n]\partial^{n+1}$$

The reduced Heisenberg double:

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$$\lambda^{2p} = 1, \quad \lambda z = z\lambda, \qquad \lambda\partial = \partial\lambda,$$

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$$E \triangleright \partial^n = \mathfrak{q}^{1-n}[n]\partial^{n-1}, \quad k^2 \triangleright \partial^n = \mathfrak{q}^{-2n}\partial^n, \quad F \triangleright \partial^n = -\mathfrak{q}^n[n]\partial^{n+1}$$

The reduced Heisenberg double:

$$\overline{\mathcal{H}}_{\mathfrak{q}} s\ell(2) = (\mathbb{C}[\lambda]/(\lambda^{2p} - 1)) \otimes \mathbb{C}_{\mathfrak{q}}[z, \partial],$$

$$\lambda^{2p} = 1, \quad \lambda z = z\lambda, \qquad \lambda\partial = \partial\lambda,$$

$$z^p = 0, \quad \partial^p = 0,$$

$$\partial z = (\mathfrak{q} - \mathfrak{q}^{-1})1 + \mathfrak{q}^{-2} z\partial$$

with the $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ action

$$E \triangleright \lambda = \frac{1}{\mathfrak{q}+1} \lambda\, z, \qquad k^2 \triangleright \lambda = \mathfrak{q}^{-1} \lambda, \qquad F \triangleright \lambda = -\frac{\mathfrak{q}}{\mathfrak{q}+1} \partial\, \lambda,$$

$$E \triangleright z^m = -\mathfrak{q}^m [m] z^{m+1}, \; k^2 \triangleright z^m = \mathfrak{q}^{2m} z^m, \quad F \triangleright z^m = [m] \mathfrak{q}^{1-m} z^{m-1},$$

$$E \triangleright \partial^n = \mathfrak{q}^{1-n} [n] \partial^{n-1}, \quad k^2 \triangleright \partial^n = \mathfrak{q}^{-2n} \partial^n, \quad F \triangleright \partial^n = -\mathfrak{q}^n [n] \partial^{n+1}$$

— algebra of *q*-differential operators on a line.

$p = 2$, the simplest case:

$$\{z, z\} = 0, \quad \{\partial, \partial\} = 0,$$
$$\{\partial, z\} = 2i$$

$$E \triangleright z = 0, \quad k^2 \triangleright z = -z, \quad F \triangleright z = 1,$$
$$E \triangleright \partial = 1, \quad k^2 \triangleright \partial = -\partial, \quad F \triangleright \partial = 0$$

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$$\{z, z\} = 0, \quad \{\partial, \partial\} = 0,$$
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finite-dimensional counterpart of free fermions, *which are known to describe the* $(p = 2, 1)$ *logarithmic conformal field model*.

Conclusion

- Kazhdan–Lusztig duality with LCFT is based on the pair

$$(\mathcal{D}(B),\ \mathcal{H}(B^*))$$

(quantum group, its module algebra)

- $\mathcal{D}(B)$: well known (a serendipitous finding of FGST (2005))
- $\mathcal{H}(B^*)$: new

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