# Geometry of supersymmetric backgrounds

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# Motivation and Outline

Motivation: M-theory, string theory, AdS/CFT, Black Holes, Special Structures in geometry

- Solve the Killing spinor equations of the heterotic supergravity in all cases
- Give all $1/2$ supersymmetric solutions of the heterotic supergravity
- Describe all $1/4$ supersymmetric solutions of the heterotic supergravity
- Report on the progress for IIB and $D = 11$ supergravities

# Method

Originally the supersymmetric solutions 4-D Einstein-Maxwell supergravity have been found using twistor methods, [Tod].

The Killing spinor equations (KSEs) of simple 5-D supergravity have been solved using the Killing spinor form bi-linears, closely related to G-structures, [Gauntlett, Gutowski, Hull, Pakis, Reall].

Throughout, the Spinorial Geometry method to solving KSEs is used, [Gillard, Gran, GP].
This is based on the gauge symmetry of KSEs and a description of spinors in terms of forms.

# Killing spinor equations

The Killing spinor equations of Heterotic supergravities are

$$\mathcal{D}\epsilon = \hat{\nabla}\epsilon = \nabla\epsilon + \frac{1}{2}H\epsilon + \mathcal{O}(\alpha') = 0\,, \quad \mathcal{F}\epsilon = F\epsilon + \mathcal{O}(\alpha') = 0\,,$$
$$\mathcal{A}\epsilon = d\Phi\epsilon - \frac{1}{2}H\epsilon + \mathcal{O}(\alpha') = 0$$

These are valid up to 2-loops in the sigma model calculation.
It is convenient to solve them in the order

$$\text{gravitino} \rightarrow \text{gaugino} \rightarrow \text{dilatino}$$

The gravitino and gaugino have a straightforward Lie algebra interpretation while the solution of the gaugino is more involved. All have been solved [Gran, Lohrmann, GP; hep-th/0510176], [Gran, Roest, Sloane, GP; hep-th/0703143].

# Gravitino and dilatino

The gravitino Killing spinor equation is

$$\mathcal{D}\epsilon = \hat{\nabla}\epsilon = \nabla\epsilon + \frac{1}{2}H\epsilon = 0$$

where $\hat{\nabla}$ is a metric connection with skew-symmetric torsion $H$, and so for generic backgrounds

$$\mathrm{hol}(\hat{\nabla}) = G = Spin(9,1)$$

In addition

$$\hat{\nabla}\epsilon = 0 \Rightarrow \hat{R}\epsilon = 0$$

So either

$$\mathrm{Stab}(\epsilon) = \{1\} \Longrightarrow \hat{R} = 0$$

all spinors are parallel and $M$ is parallelizable (group manifold if $dH = 0$) or

$$\mathrm{Stab}(\epsilon) \neq \{1\} \Longrightarrow \epsilon \text{ singlets}$$

$\mathrm{Stab}(\epsilon) \subset Spin(9,1)$ and $\mathrm{hol}(\hat{\nabla}) \subseteq \mathrm{Stab}(\epsilon)$. The solution to both gravitino and dilatino Killing spinor equations can be summarized as follows:

| $L$ | $\mathrm{Stab}(\epsilon_1, \ldots, \epsilon_L)$ | $N$ |
|---|---|---|
| 1 | $Spin(7) \ltimes \mathbb{R}^8$ | $1(1)$ |
| 2 | $SU(4) \ltimes \mathbb{R}^8$ | $1(1),\ 2(1)$ |
| 3 | $Sp(2) \ltimes \mathbb{R}^8$ | $1(1),\ 2(1),\ 3(1)$ |
| 4 | $(\times^2 SU(2)) \ltimes \mathbb{R}^8$ | $1(1),\ 2(1),\ 3(1),\ 4(1)$ |
| 5 | $SU(2) \ltimes \mathbb{R}^8$ | $1(1),\ 2(1),\ 3(1),\ 4(1),\ 5(1)$ |
| 6 | $U(1) \ltimes \mathbb{R}^8$ | $1(1),\ 2(1),\ 3(1),\ 4(1),\ 5(1),\ 6(1)$ |
| 8 | $\mathbb{R}^8$ | $1(1),\ 2(1),\ 3(1),\ 4(1),\ 5(1),\ 6(1),\ 7(1),\ 8(1)$ |
| 2 | $G_2$ | $1(1),\ 2(1)$ |
| 4 | $SU(3)$ | $1(1),\ 2(2),\ 3(1),\ 4(1)$ |
| 8 | $SU(2)$ | $1(1),\ 2(2),\ 3(3),\ 4(6),\ 5(3),\ 6(2),\ 7(1),\ 8(1)$ |
| 16 | $\{1\}$ | $8(2),\ 10(1),\ 12(1),\ 14(1),\ 16(1)$ |

- $L$ is the number of parallel spinors, ie solutions of the gravitino and $N$ is the number of solutions of both gravitino and dilatino, so $N \leq L$.
- The number in parenthesis denotes the different geometries for a given $N$.

- There are differences with the holonomy groups that appear in the Berger classification
- There are compact and non-compact isotropy groups which lead to geometries with different properties
- There is a restriction on the number of parallel spinors.
- The isotropy group of more than 8 spinors is $\{1\}$
- The conditions on the geometry of the spacetime which arise from the KSEs are known in all cases

| $L$ | $\mathrm{Stab}(\epsilon_1, \dots, \epsilon_L)$ | $N$ |
|---|---|---|
| 1 | $Spin(7) \ltimes \mathbb{R}^8$ | $1(1)$ |
| 2 | $SU(4) \ltimes \mathbb{R}^8$ | $1(1),\ 2(1)$ |
| 3 | $Sp(2) \ltimes \mathbb{R}^8$ | $1(1),\ 2(1),\ 3(1)$ |
| 4 | $(\times^2 SU(2)) \ltimes \mathbb{R}^8$ | $1(1),\ 2(1),\ 3(1),\ 4(1)$ |
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| 6 | $U(1) \ltimes \mathbb{R}^8$ | $1(1),\ 2(1),\ 3(1),\ 4(1),\ 5(1),\ 6(1)$ |
| 8 | $\mathbb{R}^8$ | $1(1),\ 2(1),\ 3(1),\ 4(1),\ 5(1),\ 6(1),\ 7(1),\ 8(1)$ |
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- If the isotropy group is the identity, the $1/2$ susy solutions are WZNW models.

# $N = 8, SU(2)$

The solution of the conditions that arise from the Killing spinor equations [GP] give

$$\begin{aligned} ds^2 &= \eta_{ab}\lambda^a\lambda^b + h\, ds^2_{\text{hk}}\,, \quad e^{2\Phi} = h \\ H &= \frac{1}{3}\eta_{ab}\lambda^a \wedge d\lambda^b + \frac{2}{3}\eta_{ab}\lambda^a \wedge \mathcal{F}^b - \star_{\text{hk}}\tilde{d}h \end{aligned}$$

- The spacetime is a Principal bundle $M = P(G, B; \pi)$ equipped with local frame $(\lambda^a, e^i)$, where $\lambda^a$ is an anti-self dual instanton connection and

$$\mathcal{F}^a \equiv d\lambda^a - \frac{1}{2}H^a{}_{bc}\lambda^b\lambda^c = \frac{1}{2}H^a{}_{ij}e^i \wedge e^j$$

- The base space $B$ is a hyper-Kähler 4-manifold
- $\mathfrak{Lie}G = \mathbb{R}^{6,1}\,,\quad \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)\,,\quad \mathfrak{cw}_6$, self-dual, [Chamseddine,Figueroa, Sabra]
- $\Phi$ depends only on the coordinates of $B$
- Field equations require

$$\star_{\text{hk}}dH \equiv -\tilde{\nabla}^2_{\text{hk}}h - \frac{1}{2}\eta_{ab}\,\mathcal{F}^a_{ij}\,\mathcal{F}^{bij} = \frac{\alpha'}{8}\left(\text{tr}\check{R}_{ij}\check{R}^{ij} - \text{tr}F_{ij}F^{ij}\right) + \mathcal{O}(\alpha'^2)$$

where $\check{\nabla} = \nabla - \frac{1}{2}H$.

# Solutions

Suppose that $\lambda$ is flat, $\mathcal{F}=0$, and $F=\check{R}$, so $dH=0$. There are two classes of solutions.

$$G\times B_{\rm hk}~,$$

where

$$G=\mathbb{R}^{5,1}~,\quad AdS_3\times S^3~,\quad CW_6$$

These include vacua for compactifications to 6 and 3 dimensions.

The 5-brane solution with flat [Callan, Harvey, Strominger] or curved worldvolume and transverse space $B_{\rm hk}$. For $B_{\rm hk}=\mathbb{R}^4$

$$\begin{aligned} ds^2 &= ds^2(G)+h\,ds^2(\mathbb{R}^4)~,\quad H=\frac{1}{3}\eta_{ab}\lambda^a\wedge d\lambda^b-\star_{\rm hk}\tilde{d}h~,\\ e^{2\Phi} &= h~,\quad h=1+\sum_\ell\frac{N_\ell}{|x-x_\ell|^2} \end{aligned}$$

At infinity is $G\times\mathbb{R}^4$, and $G\times S^3\times\mathbb{R}$ with linear dilaton near the position of the branes. The 5-brane charge at infinity is $p=\sum_\ell N_\ell$

# New solutions

Again suppose that $F = \check{R}$, ie $dH = 0$. Take $G = AdS_3 \times S^3$ and $\lambda$ a $SU(2) = S^3$ instanton on $B_{\rm hk} = \mathbb{R}^4$. Using t'Hooft's ansatz to set

$$C_i^r = (I_r)^j{}_i \partial_j \log(1 + \frac{\rho^2}{|x|^2}) \ , \quad r = 1, 2, 3$$

the smooth 1-instanton solution, $\lambda = dgg^{-1} - gCg^{-1}$, $g \in SU(2)$,

$$\begin{aligned} ds^2 &= ds^2(AdS_3) + \delta_{rs}\lambda^r\lambda^s + h ds^2(\mathbb{R}^4) \ , \\ H &= d\text{vol}(AdS_3) + \frac{1}{3}\delta_{rs}\lambda^r \wedge d\lambda^s + \frac{2}{3}\delta_{rs}\lambda^r \wedge \mathcal{F}^s - \star_{\rm hk}\tilde{d}h \ , \\ e^{2\Phi} &= h \ , \quad h = 1 + 4\frac{|x|^2 + 2\rho^2}{(|x|^2 + \rho^2)^2} \end{aligned}$$

where $\rho$ is the size of the instanton.

This solution $M = AdS_3 \times X_7$ is smooth with one $AdS_3 \times S^3 \times \mathbb{R}^4$ asymptotic region. There is no throat at $|x| = 0$, the location of the instanton. The dilaton is bounded everywhere on spacetime. Using the ADHM construction, solutions can be constructed that depend on $8\nu - 3$ continuous parameters.

# $N = 8$, $\mathbb{R}^8$

There is a choice of coordinates [GP] such that

$$\begin{aligned} ds^2 &= 2e^- e^+ + ds^2(\mathbb{R}^8)\,, \quad H = d(e^- \wedge e^+)\,, \\ e^- &= h^{-1} dv\,, \quad e^+ = du + V dv + n_i dx^i \end{aligned}$$

All components depend on $v$ and $x$, and $e_+ = \partial_u$ is Killing.
If there is no $v$ dependence, the field equations imply that

$$\partial_i^2 h = \partial_i^2 V = 0\,, \quad \partial^i dn_{ij} = 0$$

The solutions is a superposition of fundamental strings [Dabholkar, Gibbons, Harvey, Ruis-Ruis], pp-waves and null rotations (also known as chiral null models).

# $N = 4, SU(2)$

The solution of the conditions that arise from the Killing spinor equations give

$$\begin{aligned} ds^2 &= \eta_{ab}\lambda^a\lambda^b + h\, ds_4^2 \,, \\ H &= \frac{1}{3}\eta_{ab}\lambda^a \wedge d\lambda^b + \frac{2}{3}\eta_{ab}\lambda^a \wedge \mathcal{F}^b - \star_4 \tilde{d}h \end{aligned}$$

- The spacetime is a Principal bundle $M = P(G, B; \pi)$ equipped with an anti-self dual connection $\lambda^a \equiv e^a$ with curvature, $\mathcal{F}$ is either an anti-self-dual instanton or $\mathcal{F} \in \mathfrak{su}(2) \oplus \mathfrak{u}(1)$.
- The base space $B$ is either hyper-Kähler or Kähler 4-manifold
- $\mathfrak{Lie}G =$
  $\mathbb{R}^{6,1}\,, \quad \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)\,, \quad \mathfrak{cw}_6\,, \quad \mathbb{R}^{2,1} \oplus \mathfrak{su}(2)\,, \quad \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}^3\,, \quad \mathfrak{cw}_4 \oplus \mathbb{R}^2$
- $\Phi$ depends not only on the coordinates of $B$ but also on the coordinates of the fibre $G$
- These solutions can be thought of as WZNW models with linear dilaton twisted over $B$.

# $N = 4, SU(3)$

The solution of the conditions that arise from the Killing spinor equations give

$$\begin{aligned} ds^2 &= \eta_{ab}\lambda^a\lambda^b + ds^2(B)\,, \\ H &= \frac{1}{3}\eta_{ab}\lambda^a \wedge d\lambda^b + \frac{2}{3}\eta_{ab}\lambda^a \wedge \mathcal{F}^b + \pi^*\tilde{H} \end{aligned}$$

- The spacetime is a Principal bundle $M = P(G, B; \pi)$ equipped with an connection $\lambda^a \equiv e^a$ with curvature $\mathcal{F}$.
- $\mathfrak{Lie}G = \mathbb{R}^{3,1}\,,\quad \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}\,,\quad \mathfrak{su}(2) \oplus \mathbb{R}\,,\quad \mathfrak{cw}_4$
- $\Phi$ depends only on the coordinates of $B$
- If $G$ is abelian, $B$ is a complex, conformally balanced, with an $SU(3)$-structure compatible with a skew-symmetric connection with torsion, and $\mathcal{F} \in \mathfrak{su}(3)$. (CYs with torsion)
- If $G$ non-abelian, $B$ is a complex , conformally balanced, with a $U(3)$-structure compatible with a skew-symmetric connection with torsion, and $\mathcal{F} \in \mathfrak{su}(3) \oplus u(1)$ (Hermitian-Einstein).

# $N = 4, \times^2 SU(2) \ltimes \mathbb{R}^8$

The solution of the conditions that arise from the Killing spinor equations give

$$
\begin{aligned}
ds^2 &= 2e^- e^+ + h_1\, d\sigma^2_{\rm hk} + h_2\, d\delta^2_{\rm hk}\ , \\
H &= e^+ \wedge de^- + \frac{1}{2} h_{ij}\, e^- \wedge e^i \wedge e^j - \star_\sigma dh_1 - \star_\delta dh_2 \\
e^- &= (dv + m_i x^i)\ , \quad e^+ = du + V dv + n_i dx^i
\end{aligned}
$$

where $e_+ = \partial_u$ is a null Killing vector field and $e^i$ is a transverse frame to the lightcone.

- These solutions are in the same universality class as those of rotating intersecting 5-branes with transverse space a hyper-Kähler manifold and superposed with a pp-wave and a fundamental string .

# IIB

[Gutowski, Gran, Roest, GP]

- $N = 1$, the KSEs have been solved in all cases
- $N = 2$, the KSEs have been solved provided $P = G = 0$
- $N = 28$, there is a unique plane wave solution constructed by Bena and Roidan
- $N > 28$, all solutions are maximally supersymmetric
- $N = 32$, the maximally supersymmetric solutions, classified by [Figueroa, GP], are $\mathbb{R}^{9,1}$, $AdS_5 \times S^5$ [Freund, Rubin; Schwarz], and the maximally supersymmetric plane wave [Blau, Hull, Figueroa, GP]

# $D = 11$

- $N = 1$, the KSEs have been solved in all case [Gauntlett, Pakis, Gutowski]
- $N > 29$, all solutions are maximally supersymmetric [Gutowski, Gran, Roest, GP]
- $N = 32$, the maximally supersymmetric solutions, classified by [Figueroa, GP], are $\mathbb{R}^{10,1}$, $AdS_4 \times S^7$, $AdS_7 \times S^4$, [Freund, Rubin], and the maximally supersymmetric plane wave [Kowalski-Glikman]
- IIA, $N = 31$, all solutions are maximally supersymmetric [Bandos, Azcarraga, Varela]

# Conclusions

- The Killing spinor equations of heterotic supergravity have been solved in ALL cases.
- The $1/2$-supersymmetric heterotic solutions are either fundamental rotating strings superposed with pp-waves, or can be constructed from hyper-Kähler 4-manifolds and their anti-self dual instantons.
- The $1/4$-supersymmetric heterotic solutions are associated with either hyper-Kähler or Kähler 4-manifolds, or suitable 6-manifolds with a either $U(3)$ or $SU(3)$ structure and their anti-self dual instantons or Hermitian-Einstein connections.