MONOPOLES AND INTEGRABLE SYSTEMS

4th international Sakharov Conference, 2009

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Integrable system

d=4	$\boxed{\mathcal{N}=4\,,\ \text{SUSY YM}}$		$\boxed{\text{self}-\text{duality YM eq.}}$
	$\downarrow$	$\swarrow$	$\downarrow$
d=3	$\boxed{\text{Bogomolny eq.}}$		$\vert$
		$\searrow$	$\downarrow$
d=2	$\boxed{\text{Integrable systems}}$	$\sim$	$\boxed{\text{Hitchin equations}}$

In our case

$$R^4 \to R^2 \times \Sigma_\tau\,, \quad \Sigma_\tau = \mathbb{C}/(\tau\mathbb{Z}+\mathbb{Z})$$

Integrable system

Integrable systems related to elliptic curve.

Hitchin type systems:

Elliptic Calogero-Moser systems, Elliptic Gauden systems, Elliptic tops, Painleve VI, XYZ-model, Landau-Ginzburg equation, Calogero-Moser field theory;
Their trigonometric, Whittaker-Inozemtsev and rational degenerations - XXZ, XXX-models, sin-Gordon equation, Toda models,...

PLAN

1. Characteristic classes
2. New Integrable systems
3. Relations between old and new systems
4. Bogomolny equation

Integrable system

LAX EQUATION $\boxed{\dot{L} = [L, M]}$

L - Lax operator $L = L(\mathbf{v}, \mathbf{u}, \mathbf{S}; z)$

$(\mathbf{v}, \mathbf{u}, \mathbf{S})$ - dynamical variables (with Poisson brackets),

z - spectral parameter $z \in \Sigma_\tau$ - elliptic curve.

L takes values in $\mathfrak{g}$=Lie(G), where G is a gauge group.

$M = M(L)$,

Commuting integrals:

$$H_{k,j} \sim \int_{\Sigma_\tau} \operatorname{tr} L^k(z) \mu_{k,j}(z, \bar{z}) \, , \;\; \{H_{k_1,j_1}, H_{k_2,j_2}\} = 0$$

$L(z)$ is a section of the Higgs bundle over $\Sigma_t au$.

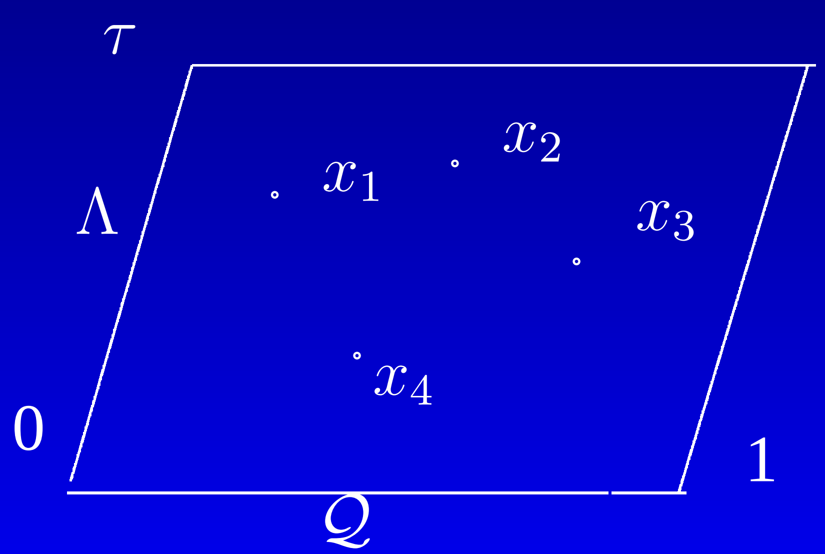
Lax operator

1. $L(z)$ is a meromorphic map

$$L(z)\ :\ (\Sigma_\tau = \mathbb{C}/(\tau\mathbb{Z}+\mathbb{Z})) \to \mathfrak{g} -\text{ a simple complex Lie algebra}$$

2. $Res\, L(z)_{z=x_a} = \mathbf{S}_a \in \mathcal{O}_a -$ a coadjoint orbit of G
3. $L(z+1) = \mathcal{Q}L(z)\mathcal{Q}^{-1},\ L(z+\tau) = \Lambda L(z)\Lambda^{-1}.$

$$\boxed{\Lambda\mathcal{Q}\Lambda^{-1}\mathcal{Q}^{-1} = Id}$$



Monodromies

$\bar{G}$ - a simply connected simple complex Lie group.

$\mathcal{Z}(\bar{G})$ - its center. $G_{ad} = \bar{G}/\mathcal{Z}(\bar{G})$.

Examples:

1. $\bar{G} = \mathrm{SL}(N, \mathbb{C})$, $\mathcal{Z}(\mathrm{SL}(N, \mathbb{C})) = \mathrm{diag}(\omega, \ldots, \omega)$,
$G_{ad} = \mathrm{PSL}(N, \mathbb{C})$.

2 $\bar{G} = Spin$, $G_{ad} = SO$.

$$\zeta - \text{generator of } \mathcal{Z}(\bar{G}) \qquad \boxed{\Lambda \mathcal{Q} \Lambda^{-1} \mathcal{Q}^{-1} = \zeta}$$

Λ and $\mathcal{Q}$ are not monodromies of $\bar{G}$-bundle, but transition operators for G_{ad}-bundle. ζ is an obstruction to lift G_{ad}-bundle to $\bar{G}$-bundle.

$$\zeta \in H^2(\Sigma_\tau, \mathcal{Z}(\bar{G})) \sim \mathcal{Z}(\bar{G})$$

ζ is the characteristic class of the bundle

Example

$$\bar{G} = \mathrm{SL}(N, \mathbb{C})\,,\ \mathcal{Z}(\mathrm{SL}(N, \mathbb{C})) = \mathrm{diag}(\omega, \dots, \omega)\ \ \omega^N = 1$$

$$\mathbf{1}.\ \mathcal{Q} = Id\,,\ \ \Lambda = \exp 2\pi i \,\mathrm{diag}(u_1, \dots, u_N)\,,\ \ \mathcal{Q}\Lambda\mathcal{Q}^{-1}\Lambda^{-1} = Id$$

$L^{ad}(z)$ - Lax operator of the Elliptic Calogero-Moser System.

$$\mathbf{2}.\ \mathcal{Q} = \mathrm{diag}(1, \omega, \dots, \omega^{N-1})\,,\ \ \mathcal{Q}\Lambda\mathcal{Q}^{-1}\Lambda^{-1} = \omega Id$$

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

$L^{sc}(z)$ - Lax operator of Elliptic Top.

Centers of universal covering groups

$\bar{G}$	Lie $(\bar{G})$	$\mathcal{Z}(\bar{G})$	G_{ad}
$\mathrm{SL}(n, \mathbb{C})$	A_{n-1}	$\mathbb{Z}_n$	$\mathrm{SL}(n, \mathbb{C})/\mathbb{Z}_n$
$\mathrm{Spin}_{2n+1}(\mathbb{C})$	B_n	$\mathbb{Z}_2$	$\mathrm{SO}(2n+1)$
$\mathrm{Sp}_n(\mathbb{C})$	C_n	$\mathbb{Z}_2$	$\mathrm{Sp}_n(\mathbb{C})/\mathbb{Z}_2$
$\mathrm{Spin}_{4n}(\mathbb{C})$	D_{2n}	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathrm{SO}(2n)/\mathbb{Z}_2$
$\mathrm{Spin}_{4n+2}(\mathbb{C})$	D_{2n+1}	$\mathbb{Z}_4$	$\mathrm{SO}(2n)/\mathbb{Z}_2$
$E_6(\mathbb{C})$	E_6	$\mathbb{Z}_3$	$E_6(\mathbb{C})/\mathbb{Z}_3$
$E_7(\mathbb{C})$	E_7	$\mathbb{Z}_2$	$E_7(\mathbb{C})/\mathbb{Z}_2$

Monodromies

$$\boxed{\Lambda \mathcal{Q} \Lambda^{-1} \mathcal{Q}^{-1} = \zeta\,.} \quad (\maltese)$$

$$\boxed{\Lambda \longrightarrow g\Lambda g^{-1}\,, \;\; \mathcal{Q} \longrightarrow g\mathcal{Q}g^{-1}}\,.$$

• $\zeta \leftrightarrow \Lambda$ - Weyl group transformation - a symmetry of extended Dynkin graph.

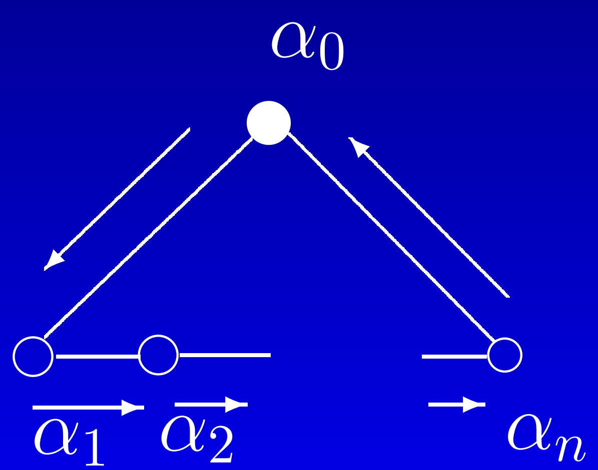


Fig.1: A_n and action of Λ

Monodromies

- •• $\mathcal{Q} \in \mathcal{H}$ -Cartan subgroup of G.

$$\boxed{\mathcal{Q} = \exp\, 2\pi i(\kappa)}\,, \quad \boxed{\kappa = \frac{\rho}{h} + \mathbf{u} \in \mathfrak{H}\,,}$$

$\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ -half-sum of positive roots,
h is the Coxeter number.

$$\Lambda \mathbf{u} \Lambda^{-1} = \mathbf{u}\,, \quad (\Lambda^l = Id)$$

$$\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 + \ldots + \mathfrak{g}_{l-1}$$

$\mathfrak{g}_0$ - invariant subalgebra.

$\mathbf{u} \in \mathfrak{H}_0$- the moduli space

Invariant subalgebras

$\mathfrak{g}$	Π_1	l =ord Λ	$\mathfrak{g}_0$
A_{N-1} , N=pl	$\cup_1^l A_{p-1}$	N/p	A_{p-1}
B_n	B_{n-1}	2	B_{n-1}
C_{2l+1}	A_{2l}	2	B_l
C_{2l}	A_{2l-1}	2	C_l
D_{2l+1}	A_{2l-2}	4	B_{l-1}
D_{2l}	A_{2l-3}	2	C_{l-1}
E_6	D_4	3	G_2
E_7	E_6	2	F_4

Relations between systems

$$G_{ad} \rightarrow L^{ad}(z)\,, \quad (\Lambda \mathcal{Q} \Lambda^{-1} \mathcal{Q}^{-1} = Id)$$

$$\bar{G} \rightarrow L^{sc}(z)\,, \quad (\Lambda \mathcal{Q} \Lambda^{-1} \mathcal{Q}^{-1} = \zeta)$$

Modification

Symplectic Hecke Correspondence

$$L^{sc}(z) = \Xi^{-1} L^{ad}(z) \Xi$$

Ξ is a singular gauge transformation.

Example

$\mathfrak{g} = A_1$

$$\bar{G} = \mathrm{SL}(2,\mathbb{C})\,, \quad G_{ad} = \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2 = \mathrm{PSL}(2,\mathbb{C})\,.$$

G_{ad} - Calogero-Moser system for two particles:

$$H_{G_{ad}} = \frac{1}{2}v^2 + \nu^2\wp(2u)\,, \quad \{v,u\} = 1$$

$\bar{G}$ - Euler top:

$$H_{\bar{G}} = \frac{1}{2}(S_1^2\wp(\tau/2) + S_2^2\wp(\frac{1+\tau}{2}) + S_3^2\wp(\frac{1}{2}))\,,$$

$\{S_j, S_k\} = \epsilon_{ijk}S_i$

Example

$$S_1 = -v\frac{\theta_{10}(0)}{\theta'_{11}(0)}\frac{\theta_{10}(2u)}{\theta_{11}(2u)} - \nu\frac{\theta^2_{10}(0)}{\theta_{00}(0)\theta_{01}(0)}\frac{\theta_{00}(2u)\theta_{01}(2u)}{\theta^2_{11}(2u)}$$

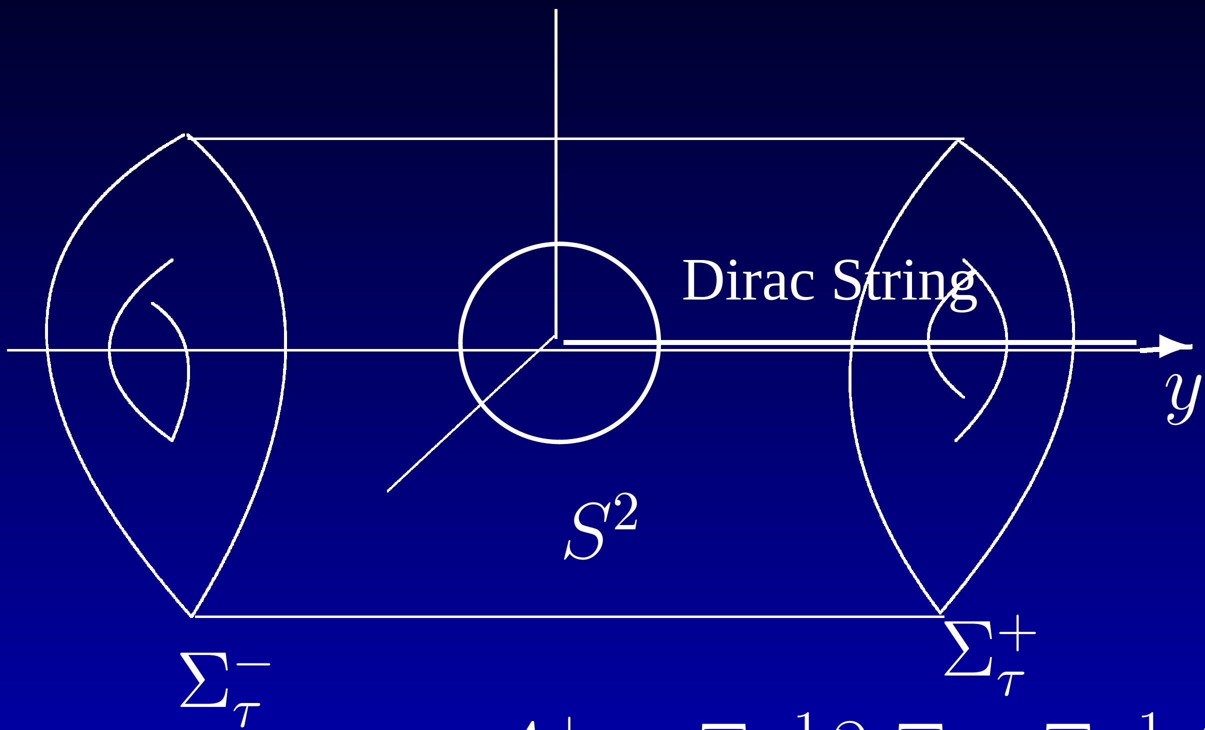
$$S_2 = -v\frac{\theta_{00}(0)}{i\theta'_{11}(0)}\frac{\theta_{00}(2u)}{\theta_{11}(2u)} - \nu\frac{\theta^2_{00}(0)}{i\theta_{10}(0)\theta_{01}(0)}\frac{\theta_{10}(2u)\theta_{01}(2u)}{\theta^2_{11}(2u)}$$

$$S_3 = -v\frac{\theta_{01}(0)}{\theta'_{11}(0)}\frac{\theta_{01}(2u)}{\theta_{11}(2u)} - \nu\frac{\theta^2_{01}(0)}{\theta_{00}(0)\theta_{10}(0)}\frac{\theta_{00}(2u)\theta_{10}(2u)}{\theta^2_{11}(2u)}$$

θ_{ab}- theta functions with characteristics.

$(S_1, S_2, S_3) = \Xi(v, u, \nu)$

Bogomolny equation



$$A_z^+ = \Xi^{-1}\partial_z \Xi + \Xi^{-1} A_z^- \Xi$$

$$A_z^\pm \to L^\pm(z)$$

Bogomolny equation

Σ_τ – elliptic curve, $(z, \bar{z})$ - complex coordinates.

$W(y, z, \bar{z}) = \mathbb{R} \times \Sigma_g$

Fields:

1)$\mathbf{A} = (A_z, A_{\bar{z}}, A_y)$ - connections in the adjoint representation of simple Lie algebra $\mathfrak{g}$.

$\mathbf{F} = (F_{z,\bar{z}}, F_{z,y}, F_{\bar{z},y})$.

2)ϕ- scalar field in the adjoint representation of simple Lie algebra $\mathfrak{g}$ - (the Higgs field).

The Bogomolny equation on W

$$\boxed{F = *D\phi} \quad (* \; - \; \text{ the Hodge operator on } W)$$

Bogomolny equation

$\vec{x} = (z, \bar{z}, y)\,,\ \vec{x}^0 = (0,0,0)$

$$\star D \star D\phi = \varpi^\vee \delta(\vec{x} - \vec{x}^0)$$

$\varpi^\vee \in \mathfrak{H}\,,\, \varpi^\vee$ belongs to the coweight lattice $P^\vee$.
$\zeta = \exp 2\pi i\, \varpi^\vee$ - characteristic class.

$$\mathcal{Z}(\bar{G}) \sim P^\vee / Q^\vee$$

Bogomolny equation

Abelian case - Dirac monopole

$$\phi \sim \frac{im}{2}\frac{1}{\sqrt{y^2+z\bar{z}}}$$

$$A_z^+(z,\bar{z},y) \sim -\frac{im}{2}\left(\frac{1}{z}\frac{y}{\sqrt{y^2+z\bar{z}}} - \frac{1}{z}\right) + const\,,$$
$$A_z^-(z,\bar{z},y) \sim -\frac{im}{2}\left(\frac{1}{z}\frac{y}{\sqrt{y^2+z\bar{z}}} + \frac{1}{z}\right) + const\,,$$

$$A_z^+ = A_z^- + i\partial_z \log z^m\,, \quad \Xi = z^m$$

$$\int_{\Sigma_\tau^+} F = \int_{\Sigma_\tau^-} F + m\,,$$