MODIFIED GRAVITY: UNIFICATION OF THE INFLATION, DARK ENERGY AND DARK MATTER

Sergei D. Odintsov

ICREA and ICE(CSIC-IEEC) Barcelona

General introduction

Basic facts:

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2), \tag{1}$$

$a(t)$ - scale factor.
5 percent accuracy of no spatial curvature.
Gravity theory: Einstein.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu} \tag{2}$$

$T_{\mu\nu}$ is stress-tensor of matter.
Typical choice: ideal fluid

$$p = w\rho, \tag{3}$$

p is pressure, ρ is energy density.
Two main cosmological parameters:
evolution of scale factor $a(t)$,
evolution of EoS: $w = w(t)$?

Proposed Universe evolution

Big Bang / String inflationary era / Quantum Gravity - Unknown Era.

Inflationary Universe:

almost de Sitter space:

$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)$

where $a(t) = e^{Ht}$.

Scenarios: most popular

Λ_I - cosmological constant,

or Scalar field,

or ideal fluid $p = -\rho$.

possibility of quintessence and (or) phantom inflation.

Problems

a) enough e-folding to reach observable volume

b) inflationary exit and transition to radiation-dominated stage (preheating, etc.)

Intermediate Universe:
$a(t) = t^{\alpha}$, radiation / matter dominance.
Einstein theory describes it perfectly.

Late Universe: Dark energy era.
Almost de Sitter $a(t) = e^{Ht}$.

Scenarios:
Λ_D - cosmological constant,
scalar fields,
ideal fluid: $p = w\rho$, $w \simeq -1$ (up to 2 percent).
Possibility of phantom $w < -1$ or quintessence: $-1 < w < -\frac{1}{3}$.
Oscillating Universe?

Possible future evolution

ΛCDM most probably continues to be ΛCDM epoch.

If $p = f(\rho)$, where p is negative the following future singularity is possible:

Type I. $t \to t_s$, $a(t) \to \infty$, ρ, $|p| \to \infty$, $a(t) \sim \frac{1}{(t-t_s}$
Type II. $t \to t_s$, $a \to a_s$, $\rho \to \rho_s$, $|p| \to \infty$
Type III. $t \to t_s$, $a(t) \to a_s$, $\rho \to \infty$, $|p| \to \infty$,
Type IV. Only higher derivatives of H diverge.

I. Introduction

The modified gravity approach is extremely attractive in the applications for late accelerating universe and dark energy. Indeed,

1. Modified gravity provides the very natural gravitational alternative for dark energy. The cosmic speed-up is explained simply by the fact of the universe expansion where some sub-dominant terms (like $1/R$) may become essential at small curvature.
2. Modified gravity presents very natural unification of the early-time inflation and late-time acceleration thanks to different role of gravitational terms relevant at small and at large curvature. Moreover, some models of modified gravity are predicted by string/M-theory considerations.
3. It may serve as the basis for unified explanation of dark energy and dark matter. Some cosmological effects (like galaxies rotation curves) may be explained in frames of modified gravity.

4. Assuming that universe is entering the phantom phase, modified gravity may naturally describe the transition from non-phantom phase to phantom one without necessity to introduce the exotic matter (like the scalar with wrong sign kinetic term or ideal fluid with EoS parameter less than -1). In addition, often the phantom phase in modified gravity is transient. Hence, no future Big Rip is usually expected there.
5. Modified gravity quite naturally describes the transition from decceleration to acceleration in the universe evolution.
6. The effective dark energy dominance may be assisted by the modification of gravity. Hence, the coincidence problem is solved there simply by the fact of the universe expansion.
7. Modified gravity is expected to be useful in high energy physics (for instance, for the explanation of hierarchy problem or unification of GUTs with gravity).
8. Despite quite stringent constraints from Solar System tests, there are versions of modified gravity which may be viable theories competing with General Relativity at current epoch.

I0. Class of viable modified $f(R)$ gravities describing inflation and the onset of accelerated expansion, arXiv:0712.4017

Let us recall that, in general, the total action for the modified gravitational models reads

$$S = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} \left[R + f(R) \right] + S_{(m)} \,. \tag{4}$$

Here $f(R)$ is a suitable function, which defines the modified gravitational part of the model. The general equation of motion in $F(R) \equiv R + f(R)$ gravity with matter is given by

$$\frac{1}{2} g_{\mu\nu} F(R) - R_{\mu\nu} F'(R) - g_{\mu\nu} \Box F'(R) + \nabla_\mu \nabla_\nu F'(R) = -\frac{\kappa^2}{2} T_{(m)\mu\nu} \,, \tag{5}$$

where $T_{(m)\mu\nu}$ is the matter energy-momentum tensor.

We investigate two classes of 'viable' modified gravitational models what means, roughly speaking, they have to incorporate the vanishing (or fast decrease) of the cosmological constant in the flat ($R \to 0$) limit, and must exhibit a suitable constant asymptotic behavior for large values of R.
This simple model reads

$$f(R) = -2\Lambda_{\rm eff}\,\theta(R - R_0)\,, \tag{6}$$

where $\theta(R - R_0)$ is Heaviside's step distribution. Models in this class are characterized by the existence of one or more transition scalar curvatures, an example being R_0 in the above toy model.

The other class of modified gravitational models that has been considered contains a sort of 'switching on' of the cosmological constant as a function of the scalar curvature R. A simplest version of this kind reads

$$f(R) = 2\Lambda_{\text{eff}}(e^{-bR} - 1)\,. \tag{7}$$

Here the transition is smooth. The two above models may be combined in a natural way, if one is also interested in the phenomenological description of the inflationary epoch. For example, a two-steps model may be the smooth version of

$$f(R) = -2\Lambda_0\,\theta(R - R_0) - 2\Lambda_I\,\theta(R - R_I)\,, \tag{8}$$

with $R_0 << R_I$, the latter being the inflation scale curvature.

The typical, smooth behavior of $f(R)$ associated with the one- and two-step models is given, in the smooth case, in Figs. 1 and 2, respectively.

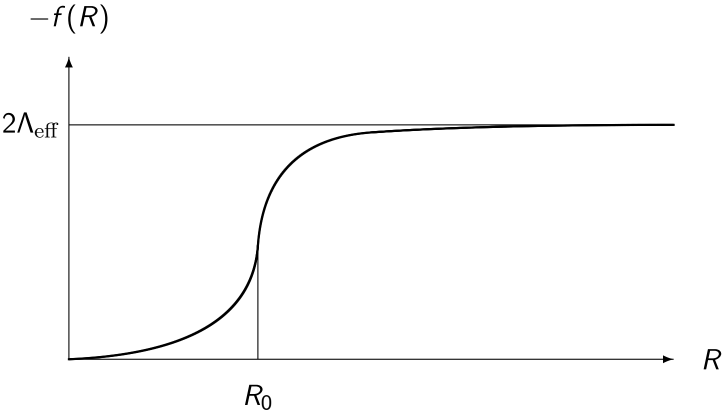

Figure: Typical behavior of $f(R)$ in the one-step model).

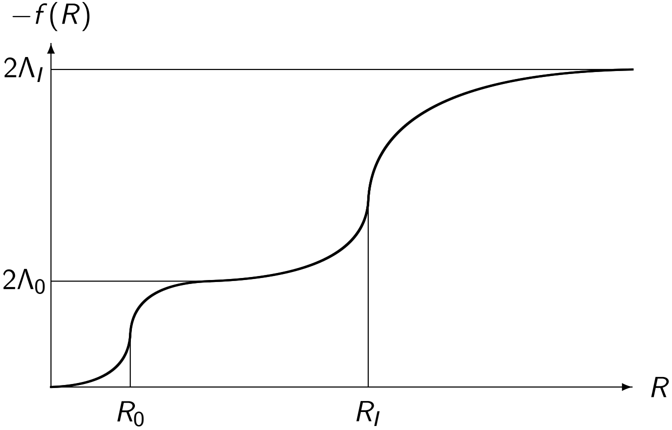

Figure: Typical behavior of $f(R)$ in the two-step model.

Let us recall the two sufficient conditions which often lead to realistic models

$$f(0) = 0\,, \qquad \lim_{R \to R_1} f(R) = -\alpha\,, \tag{9}$$

where α is a suitable curvature scale which represents an effective cosmological constant, being $R_1 >> R_0$, with $R_0 > 0$, the transition point. The condition $f(0) = 0$ ensures the disappearance of the cosmological constant in the limit of flat space-time.
By using these conditions, some models in this class are seen to be able to pass the local tests (with some extra bounds on the theory parameters) and are also capable to explain the observed recent acceleration of the universe expansion, provided that $\alpha = \Lambda_0 = 2H_0^2$, H_0 being the Hubble constant at the epoch of reference. However, they do not incorporate early-time inflation, which comes into play at higher value of R.

Thus, one might also reasonably require that

$$f(0) = 0\,, \quad \lim_{R \to R_2} f(R) = -(\alpha + \alpha_I)\,, \tag{10}$$

where $\alpha_I >> \alpha$ is associated with the inflation cosmological constant, Λ_I, and where $R_2 >> R_I >> R_0$, R_I being the corresponding transition large scalar curvature.
Further restrictions, like small corrections to Newton's law and the stability of planet-like gravitational solutions need to be fulfilled too.

The starting point is the trace of the equations of motion, which is trivial in the Einstein theory but gives precious dynamical information in the modified gravitational models. It reads

$$3\nabla^2 f'(R) = R + 2f(R) - Rf'(R) - \kappa^2 T\,. \qquad (11)$$

The above trace equation can be interpreted as an equation of motion for the non trivial 'scalaron' $f'(R)$ (since it is indeed associated with the corresponding scalar field in the other frame). For solutions with constant scalar curvature R_*, the scalaron field is constant and one obtains the following vacuum solution:

$$R_* + 2f(R_*) - R_* f'(R_*) = 0\,. \qquad (12)$$

Furthermore, we can describe the degree of freedom associated with the scalaron by means of a scalar field χ, defined by $F'(R) = 1 + f'(R) = e^{-\chi}$. If we consider a perturbation around the vacuum solution of constant curvature R_*, given by $R = R_* + \delta R$, where

$$\delta R = -\frac{1 + f'(R_*)}{f''(R_*)}\delta\chi \,, \tag{13}$$

then the equation of motion for the scalaron field is

$$\Box\delta\chi - \frac{1}{3}\left(\frac{1 + f'(R_*)}{f''(R_*)} - R_*\right)\delta\chi = -\frac{\kappa^2}{6(1 + f'(R_*)}T \,. \tag{14}$$

As a result, in connection with the local and with the planetary tests, the following effective mass plays a very crucial role:

$$M^2 \equiv \frac{1}{3}\left(\frac{1 + f'(R_*)}{f''(R_*)} - R_*\right) \,. \tag{15}$$

If $M^2 < 0$, a tachyon appears and this leads to an instability. Even if $M^2 > 0$, when M^2 is small, it is $\delta R \neq 0$ at long ranges, which generates a large correction to Newton's law. As a result, M^2 has to be positive and very large in order to pass both the local and the astronomical tests.
Concerning the matter instability this might occur when the curvature is rather large, as on a planet, as compared with the average curvature of the universe $R \sim \left(10^{-33}\,\mathrm{eV}\right)^2$. In order to arrive to a stability condition, we can start by noting that the scalaron equation can be rewritten in the form

$$\Box R + \frac{f'''(R)}{f''(R)} \nabla_\rho R \nabla^\rho R + \frac{(1 + f'(R)R}{3f''(R)} - \frac{2(R + f(R))}{3f''(R)} = \frac{\kappa^2}{6f''(R)} T \,. \quad (16)$$

If we now consider a perturbation, δR, of the Einstein gravity solution $R = R_e = -\frac{k^2 T}{2} > 0$, we obtain

$$0 \simeq (-\partial_t^2 + U(R_e))\delta R + C\,, \tag{17}$$

with the effective potential

$$\begin{aligned} U(R_e) \quad \equiv \quad & \left(\frac{F''''(R_e)}{F''(R_e)} - \frac{F'''(R_e)^2}{F''(R_e)^2}\right)\nabla_\rho R_e \nabla^\rho R_e + \frac{R_e}{3} - \\ & -\frac{F'(R_e)F'''(R_e)R_e}{3F''(R_e)^2} - \frac{F'(R_e)}{3F''(R_e)} + \\ & +\frac{2F(R_e)F'''(R_e)}{3F''(R_e)^2} - \frac{F'''(R_e)R_e}{3F''(R_e)^2}\,. \end{aligned} \tag{18}$$

If $U(R_e)$ is positive, then the perturbation δR becomes exponentially large and the whole system becomes unstable. Thus, the matter stability condition is, in this case,

$$U(R_e) < 0\,. \tag{19}$$

We will here present some new viable $f(R)$ models. We start with a most simple one

$$f(R) = \alpha(e^{-bR} - 1)\,. \tag{20}$$

Since $f(0) = 0$ and $f(R) \to -\alpha$ for large R, conditions (9) are satisfied. Moreover,

$$f'(R) = -b\alpha e^{-bR}\,, \quad f''(R) = b^2\alpha e^{-bR}\,. \tag{21}$$

We have seen that in the discussion of the viability of modified gravitational models, the existence of vacuum constant curvature solutions plays a very crucial role, namely the existence of solutions of Eq. (12). With regard to the trivial fixed point $R_* = 0$, this model has the properties

$$1 + f'(0) = 1 - \alpha b\,, \qquad f''(0) = \alpha b^2\,. \tag{22}$$

Thus, the effective mass for $R_* = 0$ is

$$M^2(0) = \frac{1 - \alpha b}{3\alpha b^2}\,, \tag{23}$$

and then Minkowski space time is stable as soon as $\alpha b < 1$. Such condition is equivalent to $1 + f'(0) > 0$.

A simple modification of the above model which incorporates the inflationary era, namely the requirement (10), is

$$f(R) = \alpha(e^{-bR} - 1) - \alpha_I \frac{e^{bR} - 1}{e^{bR} + e^{bR_I}}\,, \tag{24}$$

or, as a two-step model,

$$f(R) = -\alpha \frac{e^{bR} - 1}{e^{bR} + e^{bR_0}} - \alpha_I \frac{e^{bR} - 1}{e^{bR} + e^{bR_I}}\,. \tag{25}$$

Again, $f(0) = 0$ and, at the value $R = R_I$, there is a transition to a higher constant value $-(\alpha + \alpha_I)$ which can be related to inflation. A possible modification of the previous model is the following:

$$f(R) = -\alpha(e^{-bR} - 1) + cR^N \frac{e^{bR} - 1}{e^{bR} + e^{bR_I}}\,, \tag{26}$$

with $N > 2$ and $c > 0$. In this variant, during the inflationary era at $R > R_I$, $f(R)$, the model acquires also a power dependence on the scalar curvature, which may help to exit from the inflationary stage.

For the sharp, theta models, besides the problem of antigravity, for $R_0 << \alpha$ and $R_I << \alpha_I$, they posses, generically, two De Sitter critical points, one around the transition point $R_* \simeq \frac{5R_0}{4}$ and the other being

$$R_{*,2} \simeq 2\alpha \,. \tag{27}$$

We can also investigate the matter instability. For the two-step model (25), we now assume

$$R_0 \ll R \sim R_e \ll R_I \ . \tag{28}$$

Then $f(R)$ in (25) can be approximated as

$$f(R) \sim -\alpha \left\{ -1 + \left(1 + \mathrm{e}^{-bR_0} \right) \mathrm{e}^{-b(R-R_0)} \right\} - \frac{\alpha_I bR}{1 + \mathrm{e}^{bR_I}} \ . \tag{29}$$

We may assume

$$\frac{\alpha_I b}{1+\mathrm{e}^{bR_I}} \ll 1 \ , \tag{30}$$

since bR_I could be very large. Then we find

$$U(R_e) \simeq -\frac{\mathrm{e}^{b(R_e-R_0)}}{3\alpha b^2 \left(1+\mathrm{e}^{-bR_0}\right)} \ , \tag{31}$$

which is negative and there is no instability.
As a model which is able to describe both the inflation and the late acceleration epochs, we can consider the following two-step model:

$$\begin{aligned} f(R) &= -\alpha_0 \left(\tanh\left(\frac{b_0\left(R-R_0\right)}{2}\right) + \tanh\left(\frac{b_0 R_0}{2}\right)\right) - \\ & -\alpha_I \left(\tanh\left(\frac{b_I\left(R-R_I\right)}{2}\right) + \tanh\left(\frac{b_I R_I}{2}\right)\right) \ . \end{aligned} \tag{32}$$

We now assume

$$R_I \gg R_0 \ , \quad \alpha_I \gg \alpha_0 \ , \quad b_I \ll b_0 \ , \tag{33}$$

and

$$b_I R_I \gg 1 \ . \tag{34}$$

When $R \to 0$ or $R \ll R_0$, R_I, $f(R)$ behaves as

$$f(R) \to -\left(\frac{\alpha_0 b_0}{2\cosh^2\left(\frac{b_0 R_0}{2}\right)} + \frac{\alpha_I b_I}{2\cosh^2\left(\frac{b_I R_I}{2}\right)}\right) R \ . \tag{35}$$

and find $f(0) = 0$ again. When $R \gg R_I$, we find

$$\begin{aligned} f(R) \quad & \to -2\Lambda_I \equiv \\ & \equiv -\alpha_0 \left(1 + \tanh\left(\frac{b_0 R_0}{2}\right)\right) - \alpha_I \left(1 + \tanh\left(\frac{b_I R_I}{2}\right)\right) \sim \\ & \sim -\alpha_I \left(1 + \tanh\left(\frac{b_I R_I}{2}\right)\right) \ . \end{aligned} \tag{36}$$

On the other hand, when $R_0 \ll R \ll R_I$, we find

$$f(R) \;\to\; -\alpha_0\left[1+\tanh\left(\frac{b_0 R_0}{2}\right)\right] - \frac{\alpha_I b_I R}{2\cosh^2\left(\frac{b_I R_I}{2}\right)} \sim -2\Lambda_0 \equiv$$
$$\equiv -\alpha_0\left[1+\tanh\left(\frac{b_0 R_0}{2}\right)\right] \ . \qquad (37)$$

Here we have assumed (34). We also find

$$f'(R) = -\frac{\alpha_0 b_0}{2\cosh^2\left(\frac{b_0(R-R_0)}{2}\right)} - \frac{\alpha_I b_I}{2\cosh^2\left(\frac{b_I(R-R_I)}{2}\right)} \ , \qquad (38)$$

which has two valleys when $R \sim R_0$ or $R \sim R_I$. When $R = R_0$, we obtain

$$f'(R_0) = -\alpha_0 b_0 - \frac{\alpha_I b_I}{2\cosh^2\left(\frac{b_I(R_0-R_I)}{2}\right)} > -\alpha_I b_I - \alpha_0 b_0 \ . \qquad (39)$$

On the other hand, when $R = R_I$, we get

$$f'(R_I) = -\alpha_I b_I - \frac{\alpha_0 b_0}{2\cosh^2\left(\frac{b_0(R_0 - R_I)}{2}\right)} > -\alpha_I b_I - \alpha_0 b_0 \ . \quad (40)$$

Then, in order to avoid the antigravity period, we find

$$\alpha_I b_I + \alpha_0 b_0 < 2 \ . \quad (41)$$

The existence of the de Sitter critical points in this two-step model is much more difficult to investigate. However, in order to get the acceleration of the Universe expansion it is sufficient that $\omega_{eff} < -\frac{1}{3}$.

We now investigate the correction to the Newton's law and the matter instability issue. In the solar system domain, on or inside the earth, where $R \gg R_0$, $f(R)$ can be approximated by

$$f(R) \sim -2\Lambda_{\rm eff} + 2\alpha \mathrm{e}^{-b(R-R_0)} \,. \tag{42}$$

On the other hand, since $R_0 \ll R \ll R_I$, by assuming Eq. (34), $f(R)$ in (32) could be also approximated by

$$f(R) \sim -2\Lambda_0 + 2\alpha \mathrm{e}^{-b_0(R-R_0)} \,, \tag{43}$$

which has the same expression, after having identified $\Lambda_0 = \Lambda_{\rm eff}$ and $b_0 = b$. Then, we may check the case of (42) only.

We find that the effective mass has the following form

$$M^2 \sim \frac{\mathrm{e}^{b(R-R_0)}}{4\alpha b^2}\,, \tag{44}$$

which could be very large again, as in the last section, and the correction to Newton's law can be made negligible. We also find that $U(R_b)$ in (18) has the form

$$U(R_e) = -\frac{1}{2\alpha b}\left(2\Lambda + \frac{1}{b}\right)\mathrm{e}^{-b(R_e-R_0)}\ , \tag{45}$$

which could be negative, what would suppress any instability.
The perturbations story?

Ia. Modified non-local-F(R) gravity as the key for the inflation and dark energy

The starting action of the non-local gravity is given by

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R\left(1+f(\Box^{-1}R)\right)+\mathcal{L}_{\rm matter}\right\} . \tag{46}$$

Here f is some function and $\Box$ is the d'Almbertian for scalar field. The above action can be rewritten by introducing two scalar fields ϕ and ξ in the following form:

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}\left\{R\left(1+f(\phi)\right)+\xi\left(\Box\phi-R\right)\right\}+\mathcal{L}_{\rm matter}\right] \\ &= \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}\left\{R\left(1+f(\phi)\right)-\partial_\mu\xi\partial^\mu\phi-\xi R\right\}+\mathcal{L}_{\rm matter}\right] \end{aligned} \tag{47}$$

Varying (47) with respect to the metric tensor $g_{\mu\nu}$ gives

$$\begin{aligned} 0 &= \frac{1}{2} g_{\mu\nu} \left\{ R \left(1 + f(\phi) - \xi\right) - \partial_\rho \xi \partial^\rho \phi \right\} - R_{\mu\nu} \left(1 + f(\phi) - \xi\right) \\ &\quad + \frac{1}{2} \left(\partial_\mu \xi \partial_\nu \phi + \partial_\mu \phi \partial_\nu \xi\right) - \left(g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu\right) \left(f(\phi) - \xi\right) + \kappa^2 T_{\mu\nu} \end{aligned} \quad (48)$$

On the other hand, the variation with respect to ϕ gives

$$0 = \Box \xi + f'(\phi) R \ . \quad (49)$$

Now we assume the FRW metric

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} \left(dx^i\right)^2 \ , \quad (50)$$

and the scalar fields ϕ and ξ only depend on time. Then Eq.(48) has the following form:

$$\begin{aligned} 0 &= -3H^2 \left(1 + f(\phi) - \xi\right) + \frac{1}{2} \dot{\xi} \dot{\phi} - 3H \left(f'(\phi) \dot{\phi} - \dot{\xi} \right) + \kappa^2 \rho \ , \\ 0 &= \left(2\dot{H} + 3H^2\right) \left(1 + f(\phi) - \xi\right) + \frac{1}{2} \dot{\xi} \dot{\phi} + \left(\frac{d^2}{dt^2} + 2H \frac{d}{dt} \right) \left(f(\phi) - \xi\right) \end{aligned}$$

On the other hand, scalar equations are:

$$0 = \ddot{\phi} + 3H\dot{\phi} + 6\dot{H} + 12H^2 , \tag{53}$$

$$0 = \ddot{\xi} + 3H\dot{\xi} - \left(6\dot{H} + 12H^2\right) f'(\phi) . \tag{54}$$

We now assume deSitter solution $H = H_0$, then Eq.(53) can be solved as

$$\phi = -4H_0 - \phi_0 \mathrm{e}^{-3H_0 t} + \phi_1 , \tag{55}$$

with constants of integration, ϕ_0 and ϕ_1. For simplicity, we only consider the case that $\phi_0 = \phi_1 = 0$. We also assume $f(\phi)$ is given by

$$f(\phi) = f_0 \mathrm{e}^{b\phi} = f_0 \mathrm{e}^{-4bH_0\phi} . \tag{56}$$

Then Eq.(54) can be solved as follows,

$$\xi = -\frac{3f_0}{3-4b}\mathrm{e}^{-4bH_0 t} + \frac{\xi_0}{3H_0}\mathrm{e}^{-3H_0 t} - \xi_1 . \tag{57}$$

Here ξ_0 and ξ_1 are constants. For the deSitter space a behaves as $a = a_0 \mathrm{e}^{H_0 t}$. Then for the matter with constant equation of state w, we find

$$\rho = \rho_0 \mathrm{e}^{-3(w+1)H_0 t} . \tag{58}$$

Then by substiruting (55), (57), and (58) into (51), we obtain

$$0 = -3H_0^2\left(1+\xi_1\right) + 6H_0^2 f_0\left(2b-1\right)\mathrm{e}^{-4H_0 bt} + \kappa^2\rho_0\mathrm{e}^{-3(w+1)H_0 t} \ . \tag{59}$$

When $\rho_0 = 0$, if we choose

$$b = \frac{1}{2}\ , \quad \xi_1 = -1\ , \tag{60}$$

deSitter space can be a solution. Even if $\rho \neq 0$, if we choose

$$b = \frac{3}{4}(1+w)\ , \quad f_0 = \frac{\kappa^2\rho_0}{3H_0^2\left(1+3w\right)}\ , \quad \xi_1 = -1\ , \tag{61}$$

there is a deSitter solution.

In the presence of matter with $w \neq 0$, we may have a deSitter solution $H = H_0$ even if $f(\phi)$ given by

$$f(\phi) = f_0\mathrm{e}^{\phi/2} + f_1\mathrm{e}^{3(w+1)\phi/4}\ . \tag{62}$$

Then the following solution exists:

$$\phi = -4H_0 t\,, \quad \xi = 1+3f_0 \mathrm{e}^{-2H_0 t}+\frac{f_1}{w}\mathrm{e}^{-3(w+1)H_0 t}\,, \quad \rho = -\frac{3(3w+1)H_0^2 f_1}{\kappa^2} \tag{63}$$

Note that H_0 in (55) can be arbitrary and can be determined by an initial condition. Since H_0 can be small or large, the theory with function NLdS2 with $b = 1/2$ could describe the early-time inflation or current cosmic acceleration. Motivated by this, we may propose the following model:

$$f(\phi) = \begin{cases} f_0 \mathrm{e}^{\phi/2} & 0 > \phi > \phi_1 \\ f_0 \mathrm{e}^{\phi_1/2} & \phi_1 > \phi > \phi_2 \\ f_0 \mathrm{e}^{(\phi-\phi_2+\phi_1)/2} & \phi < \phi_2 \end{cases} . \tag{64}$$

Here ϕ_1 and ϕ_2 are constants. We also assume that matter could be neglected when $0 > \phi > \phi_1$ or $\phi < \phi_2$. Since the above function $f(\phi)$ is not smooth around $\phi = \phi_1$ and ϕ_2, one may replace the above $f(\phi)$ with a more smooth function. When $0 > \phi > \phi_1$ or $\phi < \phi_2$, the universe is described by the deSitter solution although corresponding H_0 might be different.

When $\phi_1 > \phi > \phi_2$, since $f(\phi)$ is a constant, the universe is described by the Einstein gravity, where effective gravitational constant $\kappa_{\rm eff}$ is given by

$$\frac{1}{\kappa_{\rm eff}^2} = \frac{1}{\kappa^2}\left(1 + f_0 {\rm e}^{\phi_1/2}\right) \ . \tag{65}$$

Then due to the matter contribution there could occur matter dominated phase. In this phase, the Hubble rate H behaves as $H = \frac{2}{3(t_0+t)}$ with a constant t_0 and the scalar curvature is given by $R = \frac{4}{3(t_0+t)^2}$. Now we assume that the universe started at $t=0$ with a rather big but constant curvature $R = R_I = 12H_I^2$ with a constant H_I, that is, the universe is in deSitter phase. Then in the model (64), by following (55), ϕ behaves as $\phi = -4H_I t$. Subsequently, at $t = t_1 \equiv -\phi_1/4H_I$, we have $\phi = \phi_1$ and the universe enters into the matter dominated phase. If the curvature is continuous at $t = t_1$, t_0 can be found by solving

$$R = \frac{4}{3\left(t_0 + t_1\right)^2} = 12H_I^2 \ . \tag{66}$$

If ϕ and $\dot{\phi}$ are also continuous, when $\phi_1 > \phi > \phi_2$, ϕ is given by solving (53) as

$$\phi = -\frac{4}{3}\ln\left(\frac{t}{t_1}\right) - \tilde{\phi}\left(t - t_1\right) + \phi_1\ , \quad \tilde{\phi} \equiv -4H_I\left(t_0 + t_1\right)^2 + \frac{4}{3}\left(t_0 + t_1\right)\ . \tag{67}$$

When $\phi = \phi_2$, the deSitter phase, which corresponds to the accelerating expansion of the present universe, could have started. The solution corresponds to deSitter space (with some shifts of parameters) and $H_0 = H_L$ could be given by solving

$$12H_L^2 = \frac{4}{3\left(t_0 + t_2\right)^2}\ . \tag{68}$$

if the curvature is continuous at $\phi = \phi_2$. In (68), t_2 is defined by $\phi(t_2) = \phi_2$. Thus, we got the cosmological FRW model with inflation, radiation/matter dominated phase, and current accelerating expansion.

Unification of the inflation with cosmic acceleration in the non-local-F(R) gravity

The starting action is:

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R\left(1+f(\Box^{-1}R)\right)+F(R)+\mathcal{L}_{\rm matter}\right\}\ . \tag{69}$$

Here $F(R)$ is some function of R. FRW equations look like

$$\begin{aligned} 0 &= -3H^2\left(1+f(\phi)-\xi\right)+\frac{1}{2}\dot\xi\dot\phi-3H\left(f'(\phi)\dot\phi-\dot\xi\right) \\ &\quad -F(R)+6\left(H^2+\dot H\right)F'(R)-36\left(4H^2\dot H+H\ddot H\right)F''(R)+\kappa^2\rho \end{aligned} \tag{70}$$

$$\begin{aligned} 0 &= \left(2\dot H+3H^2\right)\left(1+f(\phi)-\xi\right)+\frac{1}{2}\dot\xi\dot\phi+\left(\frac{d^2}{dt^2}+2H\frac{d}{dt}\right)\left(f(\phi)-\xi\right) \\ &\quad F(R)-2\left(\dot H+3H^2\right)F'(R)+\kappa^2 p\ . \end{aligned} \tag{71}$$

Here $R=12H^2+6\dot H$.

We may propose several scenarios. One is that the inflation at the early universe is generated mainly by $F(R)$ part but the current acceleration is defined mainly by $f\left(\Box^{-1}R\right)$ part. One may consider the inverse, that is, the inflation is generated by $f\left(\Box^{-1}R\right)$ part but the late-time acceleration by $F(R)$.

For instance, for the first scenario one can take: $F(R) = \beta R^2$. Here β is a constant. We choose $f(\Box^{-1}R)$ part as in (56) with $b = 1/2$ but f_0 is taken to be very small and ϕ starts with $\phi = 0$. Hence, at the early universe $f\left(\Box^{-1}R\right)$ is very small and could be neglected. Then due to the $F(R)$-term (71), there occurs (slightly modified) R^2-inflation. After the end of the inflation, there occurs the radiation/matter dominance era. In this phase, ϕ behaves as in (67): $\phi = -\frac{4}{3}\ln\left(\frac{t}{\hat{t}_0}\right) - \hat{\phi}_1\left(t - \hat{t}_0\right) + \hat{\phi}_2$. However, the constants $\hat{t}_0$, $\hat{\phi}_1$, and $\hat{\phi}_2$ should be determined by the proper initial conditions, which may differ from that in (67). We now assume $\hat{\phi}_1$ is very small but negative. ¿From the expression of (56) it follows $f(\phi)$ becomes large as time goes by and finally this term dominates. As a result, deSitter expansion occurs at the present universe.

II. The modified $f(R)$ gravity

Let us start from the rather general 4-dimensional action:

$$S = \int d^4x\sqrt{-g}\left\{f(R) + L_m\right\} \ . \tag{72}$$

Here R is the scalar curvature, $f(R)$ is an arbitrary function and L_m is a matter Lagrangian density. The equation of the motion is given by

$$0 = \frac{1}{2}g_{\mu\nu}f(R) - R_{\mu\nu}f'(R) - \nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\nabla^2 f'(R) + \frac{1}{2}T_{\mu\nu} \ . \tag{73}$$

With no matter and for the Ricci tensor $R_{\mu\nu}$ being covariantly constant, the equation of motion corresponding to the action (72) is:

$$0 = 2f(R) - Rf'(R) \ , \tag{74}$$

which is the algebraic equation with respect to R. If the solution of Eq.(74) is positive, it expresses deSitter universe and if negative, anti-deSitter universe.

In the following, the metric is assumed to be in the FRW form:

$$ds^2 = -dt^2 + \hat{a}(t)^2 \sum_{i=1}^{3} \left(dx^i\right)^2 \ . \tag{75}$$

Here we assume that the spatial part is flat as suggested by the observation of the Cosmic Microwave Background (CMB) radiation. Without the matter and in FRW background, Eq.(73) gives

$$0 = -\frac{1}{2}f(R) + 3\left(H^2 + \dot{H}\right) f'(R) - 6\frac{\dot{H}}{H} f''(R) - 18H^2 \frac{d}{dt}\left(\frac{\dot{H}}{H^2}\right) f''(R) \, . \tag{76}$$

Here R is given by $R = 12H^2 + 6\dot{H}$. Our main purpose is to look for accelerating cosmological solutions of the following form: de Sitter (dS) space, where H is constant and $a(t) \propto \mathrm{e}^{Ht}$, quintessence and phantom like cosmologies:

$$a = \left\{ \begin{array}{ll} a_0 t^{h_0}, & \text{when } h_0 > 0 \text{ (quintessence)} \\ a_0 \left(t_s - t\right)^{h_0}, & \text{when } h_0 < 0 \text{ (phantom)} \end{array} \right. \ . \tag{77}$$

Introducing the auxiliary fields, A and B, one can rewrite the action (72) as follows:

$$S = \int d^4 x \sqrt{-g} \left[\frac{1}{\kappa^2} \left\{ B \left(R - A \right) + f(A) \right\} + \mathcal{L}_{\mathrm{matter}} \right] . \quad (78)$$

One is able to eliminate B, and to obtain

$$S = \int d^4 x \sqrt{-g} \Big[\frac{1}{\kappa^2} \left\{ f'(A) \left(R - A \right) + f(A) \right\} + \mathcal{L}_{\mathrm{matter}} \Big] , \quad (79)$$

and by using the conformal transformation $g_{\mu\nu} \rightarrow \mathrm{e}^{\sigma} g_{\mu\nu}$ ($\sigma = - \ln f'(A)$), the action (79) is rewritten as the Einstein-frame action:

$$S_E = \int d^4 x \sqrt{-g} \Big[\frac{1}{\kappa^2} \left(R - \frac{3}{2} g^{\rho\sigma} \partial_\rho \sigma \partial_\sigma \sigma - V(\sigma) \right) + \mathcal{L}^{\sigma}_{\mathrm{matter}} \Big] . \quad (80)$$

Here,

$$V(\sigma) = \mathrm{e}^{\sigma} G \left(\mathrm{e}^{-\sigma} \right) - \mathrm{e}^{2\sigma} f \left(G \left(\mathrm{e}^{-\sigma} \right) \right) = \frac{A}{f'(A)} - \frac{f(A)}{f'(A)^2} . \quad (81)$$

The action (79) is called the Jordan-frame action. In the Einstein-frame action, the matter couples with the scalar field σ. One may define the effective EoS parameter w_{eff} in Jordan frame as

$$w_{\mathrm{eff}} = \frac{p}{\rho} = -1 - \frac{2\dot{H}}{3H^2} \ , \tag{82}$$

The scale factor in Einstein frame (when the two frames appear) is denoted as $a(t)$.

A. Modified gravity with negative and positive powers of the curvature

As the first gravitational alternative for dark energy we consider the following action

$$f(R) = R - \frac{c}{(R - \Lambda_1)^n} + b(R - \Lambda_2)^m \ . \tag{83}$$

Here we assume the coefficients n, m, c, $b > 0$ but n, m may be fractional.

For the action (83), Eq.(74) has the following form:

$$0 = -R + \frac{(n+2)c}{(R - \Lambda_1)^n} + (m-2)b(R - \Lambda_2)^m \ . \tag{84}$$

Especially when $n = 1$ and $m = 2$, one gets

$$R = R_{\pm} = \frac{\Lambda_1 \pm \sqrt{\Lambda_1^2 + 12c}}{2} \ . \tag{85}$$

If $c > 0$, one solution corresponds to deSitter space and another to anti-deSitter. If $-\frac{\Lambda_1^2}{12} < c < 0$ and $\Lambda_1 > 0$, both of solutions express the deSitter space. Hence, the natural possibility for the unification of early-time inflation with late-time acceleration appears.
By assuming the FRW universe metric (75), one may define the Hubble rate by $H = \dot{\hat{a}}/\hat{a}$. The contribution from matter may be neglected. Especially when $n = 1$, $m = 2$, and $\Lambda_1 = \Lambda_2 = 0$ in (83) and the curvature is small, we obtain
$\hat{a} \propto t^2$. We now consider the more general case that $f(R)$ is given by (83) when the curvature is small. Neglecting the contribution from the matter again, solving (73), we obtain $\hat{a} \propto t^{\frac{(n+1)(2n+1)}{n+2}}$.

B. ln R gravity

Other gravitational alternatives for dark energy may be suggested along the same line. As an extension of the theory of the previous section, one may consider the model containing the logarithm of the scalar curvature R:

$$f(R) = R + \alpha' \ln \frac{R}{\mu^2} + \beta R^m \ . \tag{86}$$

We should note that $m = 2$ choice simplifies the model.
We can consider late FRW cosmology when the scalar curvature R is small. Solving (73), it follows that the power law inflation could occur: $\hat{a} \propto t^{\frac{1}{2}}$. Since $\dot{\hat{a}} > 0$ but $\ddot{\hat{a}} < 0$, the deccelerated expansion occurs.
One may discuss further generalizations

$$f(R) = R + \gamma R^{-n} \left(\ln \frac{R}{\mu^2} \right)^m \ . \tag{87}$$

Here n is rectricted by $n > -1$ (m is an arbitrary) in order that the second term could be more dominant than the Einstein term when R is small.

For this model, we find

$$\hat{a} \sim t^{\frac{(n+1)(2n+1)}{n+2}} \ . \tag{88}$$

This does not depend on m. The effective w_{eff} is given by

$$w_{\mathrm{eff}} = -\frac{6n^2 + 7n - 1}{3(n+1)(2n+1)} \ . \tag{89}$$

Then w_{eff} can be negative if

$$-1 < n < -\frac{1}{2} \text{ or } n > \frac{-7 + \sqrt{73}}{12} = 0.1287\cdots \ . \tag{90}$$

From (88), the condition that the universe could accelerate is $\frac{(n+1)(2n+1)}{n+2} > 1$, that is:

$$n > \frac{-1 + \sqrt{3}}{2} = 0.366\cdots \ . \tag{91}$$

Clearly, the effective EoS parameter w may be within the existing bounds.

C. Modified gravity coupled with matter

The ideal fluid is taken as the matter with the constant w: $p = w\rho$. Then from the energy conservation law it follows $\rho = \rho_0 a^{-3(1+w)}$. In a some limit, strong cuvature or weak one, $f(R)$ may behave as $f(R) \sim f_0 R^\alpha$, with constant f_0 and α. An exact solution of the equation of motion is found to be

$$a = a_0 t^{h_0} \ , \quad h_0 \equiv \frac{2\alpha}{3(1+w)} \ ,$$

$$a_0 \equiv \left[-\frac{6 f_0 h_0}{\rho_0} \left(-6h_0 + 12h_0^2 \right)^{\alpha-1} \left\{ (1-2\alpha)(1-\alpha) - (2-\alpha) h_0 \right\} \right]^{-\frac{}{3}}$$

When $\alpha = 1$, the result $h_0 = \frac{2}{3(1+w)}$ in the Einstein gravity is reproduced. The effective $w_{\rm eff}$ may be defined by $h_0 = \frac{2}{3(1+w_{\rm eff})}$. By using (92), one finds the effective $w_{\rm eff}$ (82) is given by

$$w_{\rm eff} = -1 + \frac{1+w}{\alpha} \ . \tag{93}$$

Hence, if w is greater than -1 (effective quintessence or even usual ideal fluid with positive w), when α is negative, we obtain the effective phantom phase where $w_{\rm eff}$ is less than -1.
One may now take $f(R)$ as

$$f(R) = \frac{1}{\kappa^2}\left(R - \gamma R^{-n} + \eta R^2\right) \ . \tag{94}$$

When the cuvature is small, the second term becomes dominant and one may identify $f_0 = -\frac{\gamma}{\kappa^2}$ and $\alpha = -n$. Then from (93), it follows $w_{\rm eff} = -1 - \frac{1+w}{n}$. Hence, if $n > 0$, an effective phantom era occurs even if $w > -1$.

D. The equivalence with scalar-tensor theory

It is very interesting that $f(R)$ gravity is in some sense equivalent to the scalar-tensor theory with the action:

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R - \frac{1}{2}\omega(\phi)\partial_\mu\phi\partial^\mu\phi - V(\phi)\right\} ,$$

$$\omega(\phi) = -\frac{2}{\kappa^2}h'(\phi) , \quad V(\phi) = \frac{1}{\kappa^2}\left(3h(\phi)^2 + h'(\phi)\right) . \qquad (95)$$

Here $h(\phi)$ is a proper function of the scalar field ϕ. Imagine the following FRW cosmology is constructed:

$$\phi = t , \quad H = h(t) . \qquad (96)$$

Then *any* cosmology defined by $H = h(t)$ in (96) can be realized by (95).

Indeed, if one defines a new field φ as

$$\varphi = \int d\phi \sqrt{|\omega(\phi)|} \, , \tag{97}$$

the action (95) can be rewritten as

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R \mp \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \tilde{V}(\varphi) \right\} \, . \tag{98}$$

In case the sign in front of the kinetic term of φ in (98) is $-$, we can use the conformal transformation $g_{\mu\nu} \to \mathrm{e}^{\pm \kappa \varphi \sqrt{\frac{2}{3}}} g_{\mu\nu}$, and make the kinetic term of φ vanish. Hence, one obtains

$$S = \int d^4x \sqrt{-g} \left\{ \frac{\mathrm{e}^{\pm \kappa \varphi \sqrt{\frac{2}{3}}}}{2\kappa^2} R - \mathrm{e}^{\pm 2\kappa \varphi \sqrt{\frac{2}{3}}} \tilde{V}(\varphi) \right\} \, . \tag{99}$$

The action (99) may be called as Jordan frame action and the action (98) as the Einstein frame action.

Since φ becomes the auxiliary field, one may delete φ by using an equation of motion:

$$R = \mathrm{e}^{\pm\kappa\varphi\sqrt{\frac{2}{3}}} \left(4\kappa^2 \tilde{V}(\varphi) \pm 2\kappa\sqrt{\frac{3}{2}}\tilde{V}'(\varphi) \right) \ , \qquad (100)$$

which may be solved with respect to R as $\varphi = \varphi(R)$. One can rewrite the action (99) in the form of $f(R)$ gravity :

$$\begin{aligned} S &= \int d^4x\sqrt{-g}f(R) \ , \\ f(R) &\equiv \frac{\mathrm{e}^{\pm\kappa\varphi(R)\sqrt{\frac{2}{3}}}}{2\kappa^2} R - \mathrm{e}^{\pm 2\kappa\varphi(R)\sqrt{\frac{2}{3}}} \tilde{V}\left(\varphi(R)\right) \ . \end{aligned} \qquad (101)$$

III. String-inspired Gauss-Bonnet gravity as dark energy

We consider a model of the scalar field ϕ coupled with gravity. As a stringy correction, the term proportional to the GB invariant $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is added. The starting action is given by

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R - \frac{\gamma}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) + f(\phi)G\right\} ,$$

$$V = V_0 \mathrm{e}^{-\frac{2\phi}{\phi_0}} , \quad f(\phi) = f_0 \mathrm{e}^{\frac{2\phi}{\phi_0}} . \tag{102}$$

Here $\gamma = \pm 1$.

For the canonical scalar, $\gamma = 1$ but at least when GB term is not included, the scalar behaves as phantom only when $\gamma = -1$. Starting with FRW universe metric (82) in sect.37 and assuming (77) in sect.37, the following solutions may be obtained

$$
\begin{aligned}
V_0 t_1^2 &= -\frac{1}{\kappa^2 (1 + h_0)} \left\{ 3h_0^2 (1 - h_0) + \frac{\gamma \phi_0^2 \kappa^2 (1 - 5h_0)}{2} \right\} , \\
\frac{48 f_0 h_0^2}{t_1^2} &= -\frac{6}{\kappa^2 (1 + h_0)} \left(h_0 - \frac{\gamma \phi_0^2 \kappa^2}{2} \right) .
\end{aligned}
\tag{103}
$$

Even if $\gamma = -1$, there appear the solutions describing non-phantom cosmology coresponding the quintessence or matter.

As an example, we consider the case that $h_0 = -\frac{80}{3} < -1$, which gives $w_{\rm eff} = -1.025$. Simple tuning gives other acceptable values of the effective w in the range close to -1. This is consistent with the observational bounds for effective w Then from (103), one obtains

$$\begin{aligned} V_0 t_1^2 &= \frac{1}{\kappa^2}\left(\frac{531200}{231} + \frac{403}{154}\gamma\phi_0\kappa^2\right) , \\ \frac{f_0}{t_1^2} &= -\frac{1}{\kappa^2}\left(\frac{9}{49280} + \frac{27}{7884800}\gamma\phi_0\kappa^2\right) . \end{aligned} \tag{104}$$

Therefore even starting from the canonical scalar theory with positive potential, we may obtain a solution which reproduces the observed value of w.

If ϕ and H are constants: $\phi = \varphi_0$, $H = H_0$, this corresponds to deSitter space. Then the solution of equations of motion gives:

$$H_0^2 = -\frac{\mathrm{e}^{-\frac{2\varphi_0}{\phi_0}}}{8 f_0 \kappa^2} . \tag{105}$$

Therefore in order for the solution to exist, the condition is $f_0 < 0$. In (105), φ_0 can be arbitrary.

IV. Modified gravity: non-linear coupling, cosmic acceleration
A. Gravitational solution of coincidence problem

As an example of such theory, the following action is considered:

$$S = \int d^4 x \sqrt{-g} \left\{ \frac{1}{\kappa^2} R + \left(\frac{R}{\mu^2} \right)^{\alpha} L_d \right\} \ . \tag{106}$$

Here L_d is matter-like action (dark energy). The choice of parameter μ may keep away the unwanted instabilities which often occur in higher derivative theories.
By the variation over $g_{\mu\nu}$, the equation of motion follows:

$$0 = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} = \frac{1}{\kappa^2} \left\{ \frac{1}{2} g^{\mu\nu} R - R^{\mu\nu} \right\} + \tilde{T}^{\mu\nu} \ . \tag{107}$$

Here the effective EMT tensor $\tilde{T}_{\mu\nu}$ is defined by

$$\tilde{T}^{\mu\nu} \equiv \frac{1}{\mu^{2\alpha}} \left\{ -\alpha R^{\alpha-1} R^{\mu\nu} L_d + \alpha \left(\nabla^\mu \nabla^\nu - g^{\mu\nu} \nabla^2 \right) \left(R^{\alpha-1} L_d \right) + R^\alpha T^{\mu\nu} \right.$$

$$T^{\mu\nu} \equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \left(\int d^4x \sqrt{-g} L_d \right)$$

Let free massless scalar be a matter

$$L_d = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \ . \tag{109}$$

Then the equation given by the variation over ϕ has the following form:

$$0 = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta \phi} = \frac{1}{\sqrt{-g}} \partial_\mu \left(R^\alpha \sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right) \ . \tag{110}$$

The metric again corresponds to FRW universe with flat 3-space. If we assume ϕ only depends on t ($\phi = \phi(t)$), the solution of scalar field equation (110) is given by

$$\dot{\phi} = qa^{-3}R^{-\alpha} \ . \tag{111}$$

Here q is a constant of the integration. Hence $R^{\alpha}L_d = \frac{q^2}{2a^6 R^{\alpha}}$, which becomes dominant when R is small (large) compared with the Einstein term $\frac{1}{\kappa^2}R$ if $\alpha > -1$ ($\alpha < -1$). Thus, one arrives at the remarkable possibility that dark energy grows to asymptotic dominance over the usual matter with decrease of the curvature. At current universe, this solves the coincidence problem (the equality of the energy density for dark energy and for matter) simply by the fact of the universe expansion.

Substituting (111) into (107), the $(\mu, \nu) = (t, t)$ component of equation of motion has the following form:

$$
\begin{aligned}
0 &= -\frac{3}{\kappa^2}H^2 + \frac{36q^2}{\mu^{2\alpha}a^6\left(6\dot{H}+12H^2\right)^{\alpha+2}}\left\{\frac{\alpha(\alpha+1)}{4}\ddot{H}H + \frac{\alpha+1}{4}\dot{H}^2\right. \\
&\quad \left. + \left(1+\frac{13}{4}\alpha+\alpha^2\right)\dot{H}H^2 + \left(1+\frac{7}{2}\alpha\right)H^4\right\} \ . \qquad (112)
\end{aligned}
$$

The accelerated FRW solution of (112) exists

$$
a = a_0 t^{\frac{\alpha+1}{3}} \quad \left(H = \frac{\alpha+1}{3t}\right) , \quad a_0^6 \equiv \frac{\kappa^2 q^2 (2\alpha-1)(\alpha-1)}{\mu^{2\alpha}3(\alpha+1)^{\alpha+1}\left(\frac{2}{3}(2\alpha-1)\right)^{\alpha+2}} \ . \qquad (113)
$$

Eq.(113) tells that the universe accelerates, that is, $\ddot{a} > 0$ if $\alpha > 2$. If $\alpha < -1$, the solution (113) describes shrinking universe if $t > 0$. If the time is shifted as $t \to t - t_s$ (t_s is a constant), the accelerating and expanding universe occurs when $t < t_s$. In the solution with $\alpha < -1$ there appears a Big Rip singularity at $t = t_s$. For the matter with the relation $p = w\rho$, where p is the pressure and ρ is the energy density, from the usual FRW equation, one has $a \propto t^{\frac{2}{3(w+1)}}$. For $a \propto t^{h_0}$ it follows $w = -1 + \frac{2}{3h_0}$, and the accelerating expansion ($h_0 > 1$) of the universe occurs if $-1 < w < -\frac{1}{3}$. For the case of (113), one finds

$$w = \frac{1-\alpha}{1+\alpha} \ . \tag{114}$$

Then if $\alpha < -1$, we have $w < -1$, which is an effective phantom. For the general matter with the relation $p = w\rho$ with constant w, the energy E and the energy density ρ behave as $E \sim a^{-3w}$ and $\rho \sim a^{-3(w+1)}$. Thus, for the standard phantom with $w < -1$, the density becomes large with time and might generate the Big Rip.

B. Dynamical cosmological constant theory: an exact example

the following action similar to the one under consideration has been proposed:

$$I = \int d^4x \sqrt{-g} \left[\frac{R}{2\kappa^2} + \alpha_0 R^2 + \frac{\left(\kappa^4 \partial_\mu \varphi \partial^\mu \varphi\right)^q}{2q\kappa^4 f(R)^{2q-1}} - V(\varphi) \right] , \tag{115}$$

where $f(R)$ is a proper function. When the curvature is small, it is assumed $f(R)$ behaves as

$$f(R) \sim \left(\kappa^2 R^2\right)^m . \tag{116}$$

Here m is positive. When the curvature is small, the vacuum energy, and therefore the value of the potential becomes small. Then one may assume, for the small curvature, $V(\varphi)$ behaves as

$$V(\varphi) \sim V_0 \left(\varphi - \varphi_c\right) . \tag{117}$$

Here V_0 and φ_c are constants. If $q > 1/2$, the factor in front of the kinetic term of φ in (115) becomes large.

There is an exactly solvable model which realizes the above scenario. Let us choose

$$f(R) = \beta R^2 \ , \quad V(\varphi) = V_0 \left(\varphi - \varphi_c\right) \ . \tag{118}$$

Here β is a constant. R^2 term is neglected by putting $\alpha_0 = 0$ in (115) since the curvature is small. Searching for the solution (77) in sect.37 and choosing $\varphi = \varphi_c + \varphi_0/t^2$ or $\varphi = \varphi_c + \varphi_0/\left(t_s - t\right)^2$, the following restrictions are obtained

$$\varphi_0^2 = \frac{54\beta\left(-1+2h_0\right)^3 h_0^4}{\kappa^2\left(12h_0^2 - 2h_0 - 1\right)} \ , \quad V_0 = \pm \frac{3h_0 + 1}{\sqrt{6\kappa^2\left(12h_0^2 - 2h_0 - 1\right)\left(-1+2h_0\right.}} \tag{119}$$

Since φ_0^2 should be positive, one finds

$$\begin{aligned} &\text{when } \beta > 0 \ , \qquad \frac{1-\sqrt{13}}{12} < h_0 < \frac{1+\sqrt{13}}{12} \text{ or } h_0 \geq \frac{1}{2} \ , \\ &\text{when } \beta < 0 \ , \qquad h_0 < \frac{1-\sqrt{13}}{12} \text{ or } \frac{1+\sqrt{13}}{12} < h_0 \leq \frac{1}{2} \end{aligned} \tag{120}$$

For example, if $h_0 = -1/60$, which gives $w_{\rm eff} = -1.025$, we find

$$\kappa V_0 = \pm \frac{19}{34}\sqrt{\frac{15}{31}} = \pm 0.388722... \ . \tag{121}$$

For $h_0 > 0$ case, since $R = 6\dot{H} + 12H^2$, the curvature R decreases as t^{-2} with time t and φ approaches to φ_c but does not arrive at φ_c in a finite time, as expected .
As H behaves as h_0/t or $h_0/(t_s - t)$ for (77) in sect.37, if we substitute the value of the age of the present universe 10^{10}years$\sim (10^{-33}\mathrm{eV})^{-1}$ into t or $t_s - t$, the observed value of H could be reproduced, which could explain the smallness of the effective cosmological constant $\Lambda \sim H^2$. Note that even if there is no potential term, that is, $V_0 = 0$, when $\beta < 0$, there is a solution

$$h_0 = -\frac{1}{3} < \frac{1-\sqrt{13}}{12} = -0.2171... \ , \tag{122}$$

which gives the EoS parameter : $w = -3$, although w is not realistic. Playing with different choices of the potential and non-linear coupling more realistic predictions may be obtained.

V. Late-time cosmology in modified Gauss-Bonnet gravity

A. $f(G)$ gravity

Our next example is modified Gauss-Bonnet gravity. Let us start from the action :

$$S = \int d^4 x \sqrt{-g} \left(\frac{1}{2\kappa^2} R + f(G) + \mathcal{L}_m \right) \ . \tag{123}$$

Here $\mathcal{L}_m$ is the matter Lagrangian density and G is the GB invariant: $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma}$. By variation over $g_{\mu\nu}$ one gets:

$$\begin{aligned} 0 = & \frac{1}{2\kappa^2}\left(-R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}R\right) + T^{\mu\nu} + \frac{1}{2}g^{\mu\nu}f(G) - 2f'(G)RR^{\mu\nu} \\ & +4f'(G)R^{\mu}{}_{\rho}R^{\nu\rho} - 2f'(G)R^{\mu\rho\sigma\tau}R^{\nu}{}_{\rho\sigma\tau} - 4f'(G)R^{\mu\rho\sigma\nu}R_{\rho\sigma} + 2\left(\nabla^\mu\nabla^\nu \right. \\ & -2g^{\mu\nu}\left(\nabla^2 f'(G)\right)R - 4\left(\nabla_\rho\nabla^\mu f'(G)\right)R^{\nu\rho} - 4\left(\nabla_\rho\nabla^\nu f'(G)\right)R^{\mu\rho} \\ & +4\left(\nabla^2 f'(G)\right)R^{\mu\nu} + 4g^{\mu\nu}\left(\nabla_\rho\nabla_\sigma f'(G)\right)R^{\rho\sigma} - 4\left(\nabla_\rho\nabla_\sigma f'(G)\right)R^{\mu\rho\nu} \end{aligned}$$

where $T^{\mu\nu}$ is the matter EM tensor.

By choosing the spatially-flat FRW universe metric (75) in sect.37, the equation corresponding to the first FRW equation has the following form:

$$0 = -\frac{3}{\kappa^2} H^2 + G f'(G) - f(G) - 24 \dot{G} f''(G) H^3 + \rho_m \ , \qquad (125)$$

where ρ_m is the matter energy density. When $\rho_m = 0$, Eq. (125) has a deSitter universe solution where H, and therefore G, are constant. For $H = H_0$, with constant H_0, Eq. (125) turns into

$$0 = -\frac{3}{\kappa^2} H_0^2 + 24 H_0^4 f'\left(24 H_0^4\right) - f\left(24 H_0^4\right) \ . \qquad (126)$$

For a large number of choices of the function $f(G)$, Eq. (126) has a non-trivial ($H_0 \neq 0$) real solution for H_0 (deSitter universe).

We now consider the case $\rho_m \neq 0$. Assuming that the EoS parameter $w \equiv p_m/\rho_m$ for matter (p_m is the pressure of matter) is a constant then, by using the conservation of energy: $\dot{\rho}_m + 3H\left(\rho_m + p_m\right) = 0$, we find $\rho = \rho_0 a^{-3(1+w)}$. The function $f(G)$ is chosen as

$$f(G) = f_0 \left|G\right|^{\beta} \ , \tag{127}$$

with constant f_0 and β. If $\beta < 1/2$, $f(G)$ term becomes dominant compared with the Einstein term when the curvature is small. If we neglect the contribution from the Einstein term in (125), the following solution may be found

$$h_0 = \frac{4\beta}{3(1+w)}\,, \quad a_0 = \Big[-\frac{f_0(\beta-1)}{\left(h_0-1\right)\rho_0}\left\{24\left|h_0^3\left(-1+h_0\right)\right|\right\}^{\beta}\left(h_0-1+4\beta \tag{128}$$

Then the effective EoS parameter $w_{\rm eff}$ (82) in sect.37 is less than -1 if $\beta < 0$, and for $w > -1$ is

$$w_{\rm eff} = -1 + \frac{2}{3h_0} = -1 + \frac{1+w}{2\beta} \ , \tag{129}$$

which is again less than -1 for $\beta < 0$. Thus, if $\beta < 0$, we obtain an effective phantom with negative h_0 even in the case when $w > -1$. In the phantom phase, there might seem to occur the Big Rip at $t = t_s$ [**?**]. Near this Big Rip , however, the curvature becomes dominant and then the Einstein term dominates, so that the $f(G)$ term can be neglected. Therefore, the universe behaves as $a = a_0 t^{2/3(w+1)}$ and as a consequence the Big Rip does not eventually occur. The phantom era is transient.

B. $f(G,R)$ gravity

It is interesting to study late-time cosmology in generalized theories, which include both the functional dependence from curvature as well as from the Gauss-Bonnet term :

$$S = \int d^4x\sqrt{-g}\left(f(G,R) + \mathcal{L}_m\right) \ . \tag{130}$$

The following solvable model is considered:

$$f(G,R) = R\tilde{f}\left(\frac{G}{R^2}\right) \ , \quad \tilde{f}\left(\frac{G}{R^2}\right) = \frac{1}{2\kappa^2} + f_0\left(\frac{G}{R^2}\right) \ . \tag{131}$$

The FRW solution may be found again:

$$H = \frac{h_0}{t} \ , \quad h_0 = \frac{\frac{3}{\kappa^2} - 2f_0 \pm \sqrt{8f_0\left(f_0 - \frac{3}{8\kappa^2}\right)}}{\frac{6}{\kappa^2} + 2f_0} \ . \tag{132}$$

Then, for example, if $\kappa^2 f_0 < -3$, there is a solution describing a phantom with $h_0 < -1 - \sqrt{2}$ and a solution describing the effective matter with $h_0 > -1 + \sqrt{2}$. Late-time cosmology in other versions of such theory may be constructed.

Inhomogeneous equation of state of the universe dark fluid

Let us remind several simple facts about the universe filled with ideal fluid. By using the energy conservation law $0 = \dot{\rho} + 3H(p + \rho)$, when ρ and p satisfy the following simple EOS $p = w\rho$ with constant w, we find $\rho = \rho_0 a^{-3(1+w)}$. Then by using the first FRW equation $(3/\kappa^2)H^2 = \rho$, the well-known solution follows $a = a_0 (t - t_1)^{\frac{2}{3(w+1)}}$ $(w > -1)$ or $a_0 (t_2 - t)^{\frac{2}{3(w+1)}}$ $w \neq -1$ $(w < -1)$ and $a = a_0 \mathrm{e}^{\kappa t \sqrt{\frac{\rho_0}{3}}}$ when $w = -1$, which is the deSitter universe. Here t_1 and t_2 are constants of the integration. When $w < -1$, there appears a Big Rip singularity in a finite time at $t = t_2$.

In general, the singularities in dark energy universe may behave in a different way. Type I ("Big Rip") : For $t \to t_s$, $a \to \infty$, $\rho \to \infty$ and $|p| \to \infty$ Type II ("sudden") : For $t \to t_s$, $a \to a_s$, $\rho \to \rho_s$ or 0 and $|p| \to \infty$ Type III : For $t \to t_s$, $a \to a_s$, $\rho \to \infty$ and $|p| \to \infty$ Type IV : For $t \to t_s$, $a \to a_s$, $\rho \to 0$, $|p| \to 0$ and higher derivatives of H diverge. This also includes the case when ρ (p) or both of them tend to some finite values while higher derivatives of H diverge. Here t_s, a_s and ρ_s are constants with $a_s \neq 0$.

The singularities in the inhomogeneous EoS dark fluid universe

One may start from the dark fluid with the following EOS:

$$p = -\rho - f(\rho) \ , \tag{133}$$

where $f(\rho)$ can be an arbitrary function in general. The choice $f(\rho) \propto \rho^{\alpha}$ with a constant α was proposed. Then the scale factor is given by

$$a = a_0 \exp\left(\frac{1}{3}\int \frac{d\rho}{f(\rho)}\right) \ , \tag{134}$$

and the cosmological time may be found

$$t = \int \frac{d\rho}{\kappa\sqrt{3\rho}f(\rho)} \ , \tag{135}$$

As an example we may consider the case that

$$f(\rho) = A\rho^{\alpha} \ . \tag{136}$$

Then we find :

- In case $\alpha = 1/2$ or $\alpha = 0$, there does not appear any singularity.
- In case $\alpha > 1$, when $t \to t_0$, the energy density behaves as $\rho \to \infty$ and therefore $|p| \to \infty$. Then the scale factor a is finite even if $\rho \to \infty$. Therefore $\alpha > 1$ case corresponds to type III singularity.
- In $\alpha = 1$ case, if $A > 0$, there occurs the Big Rip or type I singularity but if $A \leq 0$, there does not appear future singularity.
- In case $1/2 < \alpha < 1$, when $t \to t_0$, all of ρ, $|p|$, and a diverge if $A > 0$ then this corresponds to type I singularity.

- In case $0 < \alpha < 1/2$, when $t \to t_0$, we find ρ, $|p| \to 0$ and $a \to a_0$ but

$$\ln a \sim |t - t_0|^{\frac{\alpha-1}{\alpha-1/2}} \quad . \tag{137}$$

  Since the exponent $(\alpha - 1)/(\alpha - 1/2)$ is not always an integer, even if a is finite, the higher derivatives of H diverge in general. Therefore this case corresponds to type IV singularity.
- In case $\alpha < 0$, when $t \to t_0$, we find $\rho \to 0$, $a \to a_0$ but $|p| \to \infty$. Therefore this case corresponds to type II singularity.

At the next step, we consider the inhomogeneous EOS for dark fluid, so that the dependence from Hubble parameter is included in EOS. This new terms may origin from string/M-theory, braneworld or modified gravity

$$p = -\rho + f(\rho) + G(H) \ . \tag{138}$$

where $G(H)$ is some function.
In general, EOS needs to be double-valued in order for the transition (crossing of phantom divide) to occur between the region $w < -1$ and the region $w > -1$. Then there could not be one-to-one correspondence between p and ρ. In such a case, we may suggest the implicit, inhomogeneous equation of the state

$$F(p, \rho, H) = 0 \ . \tag{139}$$

The following example may be of interest:

$$(p+\rho)^2 - C_0\rho^2\left(1-\frac{H_0}{H}\right) = 0 \ . \tag{140}$$

Here C_0 and H_0 are positive constants. Hence, the Hubble rate looks as

$$H = \frac{16}{9C_0^2 H_0\left(t-t_-\right)\left(t_+-t\right)} \ . \tag{141}$$

and

$$p = -\rho\left\{1+\frac{3C_0^2}{4H_0}\left(t-t_0\right)\right\} \ , \quad \rho = \frac{2^8}{3^3 C_0^4 H_0^2 \kappa^2 \left(t-t_-\right)^2\left(t_+-t\right)^2} \ . \tag{142}$$

In (141), since $t_- < t_0 < t_+$, as long as $t_- < t < t_+$, the Hubble rate H is positive. The Hubble rate H has a minimum $H = H_0$ when $t = t_0 = (t_- + t_+)/2$ and diverges when $t \to t_\pm$. Then one may regard $t \to t_-$ as a Big Bang singularity and $t \to t_+$ as a Big Rip one. As clear from (142), the parameter $w = p/\rho$ is larger than -1 when $t_- < t < t_0$ and smaller than -1 when $t_0 < t < t_+$. Therefore there occurs the crossing of phantom divide $w = -1$ when $t = t_0$ thanks to the effect of inhomogeneous term in EOS. In principle, the more general EOS may contain the derivatives of H, like $\dot{H}$, $\ddot{H}$, ... More general EOS than (139) may have the following form:

$$F\left(p, \rho, H, \dot{H}, \ddot{H}, \cdots\right) = 0 \ . \tag{143}$$