Spontaneous symmetry breaking in multidimensional gravity: Brane world concept

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Vector order parameter ϕ_I

Symmetry breaking potential $V(\phi^K\phi_K)$ $\phi^K\phi_K = g^{IK}\phi_I\phi_K$

Bilinear combination of derivatives $S_{IKLM} = \phi_{I;K}\phi_{L;M}$

Scalar $S = A\left(\phi^K_{;K}\right)^2 + B\phi^L_{;K}\phi_L^{;K} + C\phi^M_{;K}\phi^K_{;M}$

Covariant derivative $\phi_{P;M} = \dfrac{\partial\phi_P}{\partial x^M} - \dfrac{1}{2}g^{LA}\left(\dfrac{\partial g_{AM}}{\partial x^P} + \dfrac{\partial g_{AP}}{\partial x^M} - \dfrac{\partial g_{MP}}{\partial x^A}\right)\phi_L$

is a sum of symmetric and anti-symmetric tensors

$$\phi_{P;M} = \phi_{s\ P;M} + \phi_{a\ P;M}, \qquad \phi_{s\ P;M} = \phi_{s\ M;P}, \qquad \phi_{a\ P;M} = -\phi_{a\ M;P}$$

Lagrangian $L\left(\phi_I, g^{IK}, \dfrac{\partial g_{IK}}{\partial x^L}\right) = L_g + L_d, \qquad L_g = \dfrac{R}{2\kappa^2},$

$$L_d = A\left(\phi_{s;K}^{\ K}\right)^2 + (B+C)\phi_s^{\ I;K}\phi_{s\,I;K} + (B-C)\phi_a^{\ I;K}\phi_{a\,I;K}$$

The anti-symmetric $\phi_a^{\ I;K}\phi_{a\,I;K}\ (\equiv F^{IK}F_{IK})$ is the ordinary electrodynamics.

The two symmetric terms provide new possibilities.

Vector order parameter specifies a direction.
We chose the coordinate system so that $\phi_I = \delta_{I I_0}\phi.$

In application to the brane world with a topological defect in two extra dimensions we consider the metric in the form

$$ds^2 = g_{IK}dx^I dx^K = e^{2\gamma(l)}\eta_{\mu\nu}dx^\mu dx^\nu - (dl^2 + e^{2\beta(l)}d\varphi^2)$$

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, ..., -1)$$

l is the distance from the brane,

$e^{2\gamma(l)}$ is the warp factor, $\gamma(l)$ is an analoge of gravitation potential,

$r(l) = e^{\beta(l)}$ Is the circular radius.

Three unknowns: $\phi(l),\ \beta(l),\ \gamma(l)$

Covariant derivative $\phi_{I;K} = \delta_I^{d_0}\delta_K^{d_0}\phi' - \frac{1}{2}\delta_{IK}g^{II}(g_{II})'\phi$

is a symmetric tensor: $\phi^I{}_{;K}\phi_I^{;K} = \phi^I{}_{;K}\phi^K_{;I}$

$$L_d = A\left(\phi^K{}_{;K}\right)^2 + B\phi^I{}_{;K}\phi_I^{;K} - V\left(\phi^K\phi_K\right)$$

Equation for field $\phi(l)$

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{-g}L_d}{\partial\phi'}\right)' - \frac{\partial L_d}{\partial\phi} = 0$$

$$L_d = A\left(\phi' + \tfrac{1}{2}\phi\sum_K g^{KK}g'_{KK}\right)^2 + B\left(\phi'^2 + \tfrac{1}{4}\phi^2\sum_L\left(g^{LL}g'_{LL}\right)^2\right) - V\left(-\phi^2\right)$$

$$S_n = \tfrac{1}{2^n}\sum_K\left(g^{KK}g'_{KK}\right)^n = d_0\gamma'^n + \beta'^n, \quad n = 1, 2, \ldots$$

$$g = (-1)^{D-1}\, e^{2(d_0\gamma+\beta)}$$

Vector, type A (A=1/2, B=0)	Vector, type B (A=0, B= 1/2)	Scalar multiplet
$(\phi' + S_1\phi)' + \frac{\partial V}{\partial\phi} = 0$	$\phi'' + S_1\phi' - S_2\phi + \frac{\partial V}{\partial\phi} = 0.$	$\phi'' + S_1\phi' - \phi e^{-2\beta} + \frac{\partial V}{\partial\phi} = 0$

In the flat space-time $\gamma' = 0,\ \beta' = \frac{1}{l},\ \beta'' = -\frac{1}{l^2}, e^{-2\beta} = \frac{1}{l^2}.$

$$\phi'' + \frac{1}{l}\phi' - \frac{1}{l^2}\phi + \frac{\partial V}{\partial\phi} = 0$$

Energy-momentum tensor

$$T_{IK} = \frac{2}{\sqrt{-g}}\left[\frac{\partial\sqrt{-g}\,L_d}{\partial g^{IK}} + g_{QK}g_{PI}\frac{\partial}{\partial x^L}\left(\sqrt{-g}\,\frac{\partial L_d}{\partial\frac{\partial g_{PQ}}{\partial x^L}}\right)\right]$$

Be careful: $V(\phi^K\phi_K) = V(g^{IK}\phi_I\phi_K)$

Differentiation goes first. Only after that one can set

$$g^{d_0 d_0} = -1, \quad \left(g^{d_0 d_0}\right)' = 0$$

Result:

Type A:

$$T_I^K = \frac{1}{2}\delta_I^K\left(\phi' + S_1\phi\right)^2 + \delta_I^K V + \left(\delta_I^{d_0}\delta_{d_0}^K - \delta_I^K\right)\frac{\partial V}{\partial\phi}\phi$$

Type B:

$$T_{I<d_0}^K = \delta_I^K\left[\frac{1}{\sqrt{-g}}\left(\sqrt{-g}\gamma'\right)'\phi^2 + \gamma'\left(\phi^2\right)' - \left(\frac{1}{2}\phi'^2 + \frac{1}{2}S_2\phi^2\right) + V\right]$$

$$T_{d_0}^{d_0} = \frac{1}{2}\left(\phi'^2 + S_2\phi^2\right) + V$$

$$T_{I>d_0}^K = \delta_I^K\left[\frac{1}{\sqrt{-g}}\left(\sqrt{-g}\beta'\right)'\phi^2 + \beta'\left(\phi^2\right)' - \left(\frac{1}{2}\phi'^2 + \frac{1}{2}S_2\phi^2\right) + V\right]$$

Verification: $T_{I\ ;K}^{K} = 0$

Einstein equations

$$R_I^K = \kappa^2 \widetilde{T}_I^K$$

$$R_I^K = \begin{cases} \delta_I^K \left(\gamma'' + \gamma' S_1\right), & I < d_0 \\ \delta_{d_0}^K \left(S_1' + S_2\right), & I = d_0 \\ \delta_\varphi^K \left(\beta'' + S_1 \beta'\right), & I = \varphi \end{cases} \qquad \widetilde{T}_I^K = T_I^K - \tfrac{1}{d_0} \delta_I^K T.$$

Type A:

$$\gamma'' + S_1 \gamma' = \kappa^2 \left[-\tfrac{1}{d_0} \left(\phi' + S_1 \phi\right)^2 - \tfrac{2V}{d_0} + \tfrac{1}{d_0} \tfrac{\partial V}{\partial \phi} \phi \right]$$

$$S_1' + S_2 = \kappa^2 \left[-\tfrac{1}{d_0} \left(\phi' + S_1 \phi\right)^2 - \tfrac{2V}{d_0} + \left(1 + \tfrac{1}{d_0}\right) \tfrac{\partial V}{\partial \phi} \phi \right]$$

$$\beta'' + S_1 \beta' = \kappa^2 \left[-\tfrac{1}{d_0} \left(\phi' + S_1 \phi\right)^2 - \tfrac{2V}{d_0} + \tfrac{1}{d_0} \tfrac{\partial V}{\partial \phi} \phi \right]$$

Type B:

$$\gamma'' + S_1 \gamma' = \kappa^2 \left[S_1 \gamma' \phi^2 + \left(\gamma' \phi^2\right)' - \tfrac{1}{d_0} S_1^2 \phi^2 - \tfrac{1}{d_0} \left(S_1 \phi^2\right)' - \tfrac{2}{d_0} V \right]$$

$$S_1' + S_2 = \kappa^2 \left[\phi'^2 + S_2 \phi^2 - \tfrac{1}{d_0} S_1^2 \phi^2 - \tfrac{1}{d_0} \left(S_1 \phi^2\right)' - \tfrac{2}{d_0} V \right]$$

$$\beta'' + S_1 \beta' = \kappa^2 \left[S_1 \beta' \phi^2 + \left(\beta' \phi^2\right)' - \tfrac{1}{d_0} S_1^2 \phi^2 - \tfrac{1}{d_0} \left(S_1 \phi^2\right)' - \tfrac{2}{d_0} V \right]$$

First integral

Type A	Type B
$S_1^2 - S_2 = -\kappa^2\left[(\phi' + S_1\phi)^2 + 2V\right]$	$S_2\left(1 - \kappa^2\phi^2\right) = \kappa^2\phi'^2 + 2\kappa^2 V + S_1^2$

More simplifications: $U = \gamma' - \beta', \quad Z = \phi' + S_1\phi, \quad \psi = \phi'$

$$\gamma' = \frac{U + S_1}{d_0 + 1} \qquad \beta' = \frac{S_1 - d_0 U}{d_0 + 1} \qquad S_2 = \frac{d_0 U^2 + S_1^2}{d_0 + 1}$$

Type A

$$U' = -U S_1$$
$$S_1' = \kappa^2 \frac{d_0+1}{d_0}\frac{\partial V}{\partial\phi}\phi - U^2$$
$$\phi' = Z - S_1\phi$$
$$Z' = -\frac{\partial V}{\partial \phi}$$

Type B

$$\left[U\left(1-\kappa^2\phi^2\right)\right]' + U\left(1-\kappa^2\phi^2\right)S_1 = 0$$
$$\left[S_1\left(1+\frac{\kappa^2\phi^2}{d_0}\right)\right]' + S_1^2\left(1+\frac{\kappa^2\phi^2}{d_0}\right) + \frac{2(1+d_0)}{d_0}\kappa^2 V = 0$$
$$\phi' = \psi$$
$$\psi' + S_1\psi - \frac{d_0U^2+S_1^2}{d_0+1}\phi + \frac{\partial V}{\partial \phi} = 0$$

Type A: Symmetry breaking potential enters the equations only via its derivative.
Type B : The derivative of the potential is eliminated from the Einstein equations

Regularity conditions

Riemann curvature tensor $R^{AB}{}_{CD} = \left\{ \begin{array}{ll} -\gamma'^2 \left(\delta^A_C \delta^B_D - \delta^A_D \delta^B_C\right), & A, B, C, D < d_0, \\ -\beta'\gamma', \quad A = C = \varphi, & B, D < d_0, \\ -\left(\gamma'' + \gamma'^2\right) \delta^B_D, \quad A = C = d_0, & B, D < d_0, \\ -\left(\beta'' + \beta'^2\right), \quad A = C = d_0, & B = D = \varphi. \end{array} \right\}$

γ', $\gamma'' + \gamma'^2$, $\beta'\gamma'$, and $\beta'' + \beta'^2$ must be finite.

$$\beta'' + \beta'^2 = c < \infty \quad \text{at} \quad l = 0 \qquad \beta' = \frac{1}{l} + \frac{1}{3} c l + O\left(l^3\right) \qquad \gamma' = O\left(l\right)$$

Boundary conditioms at $l \to 0$

Type A	**Type B**
$U = \frac{1}{3}\left(d_0 + 1\right)\gamma_0'' l - \frac{1}{l}$ $\quad S_1 = \frac{2}{3}\left(d_0 + 1\right)\gamma_0'' l + \frac{1}{l}$ $\phi = \phi_0' l, \quad Z = 2\phi_0'$ $\quad \gamma_0'' = -\frac{\kappa^2}{d_0}\left(2\phi_0'^2 + V_0\right)$	$U\left(1 - \varkappa^2\phi^2\right) = -\frac{1}{l}$ $\quad S_1\left(1 + \frac{\varkappa^2\phi^2}{d_0}\right) = \frac{1}{l}$ $\phi = \phi_0' l \quad \psi = \phi_0'$ $\quad \gamma_0'' = -\frac{\kappa^2}{d_0}\left(\frac{1}{2}\phi_0'^2 + V_0\right)$

One constant γ_0" or ϕ_0' remains arbitrary.

Analytical solution. Type A, case $\frac{\partial V}{\partial \phi} \equiv 0$

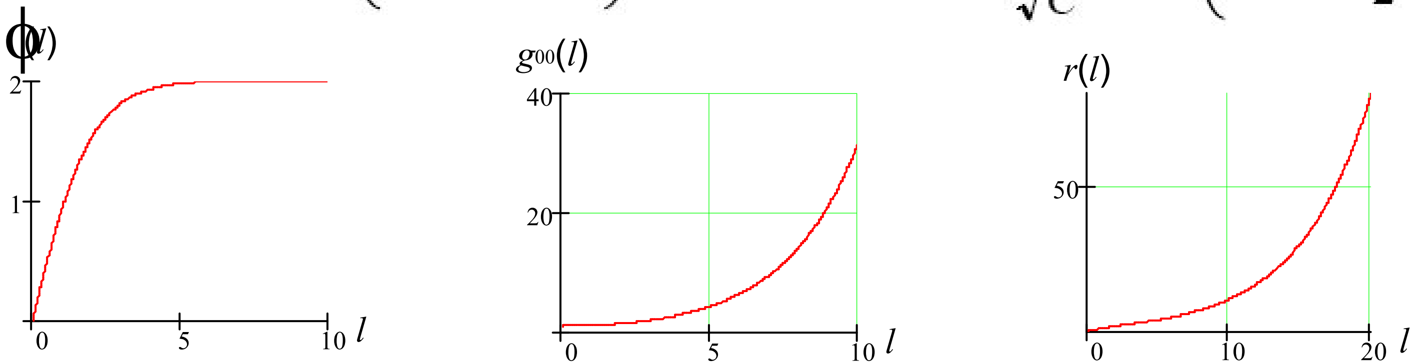
$\Lambda = \chi^2 V_0$ *- cosmological constant*

$$U = -\frac{\sqrt{C}}{\sinh\left(\sqrt{C}l\right)}, \quad S_1 = \sqrt{C}\coth\left(\sqrt{C}l\right), \quad \phi(l) = \frac{2\phi_0'}{\sqrt{C}}\tanh\frac{\sqrt{C}l}{2}$$

$$C = 2(d_0+1)\gamma_0'' = -\frac{2(d_0+1)}{d_0}\left(2\chi^2\phi_0'^2 + \Lambda\right)$$

The solution is regular if $C \geq 0$, i.e. $\Lambda \leq -2\chi^2\phi_0'^2$.

$$g_{00}(l) = e^{2\gamma} = \left(\cosh\frac{\sqrt{C}\,l}{2}\right)^{\frac{4}{d_0+1}} \qquad r(l) = \frac{2\sinh\left(\frac{\sqrt{C}\,l}{2}\right)}{\sqrt{C}}\left(\cosh\frac{\sqrt{C}\,l}{2}\right)^{-\frac{d_0-1}{d_0+1}}$$



$C = 1$

The existence of a (negative) cosmological constant is sufficient for the symmetry breaking of the initially flat bulk.

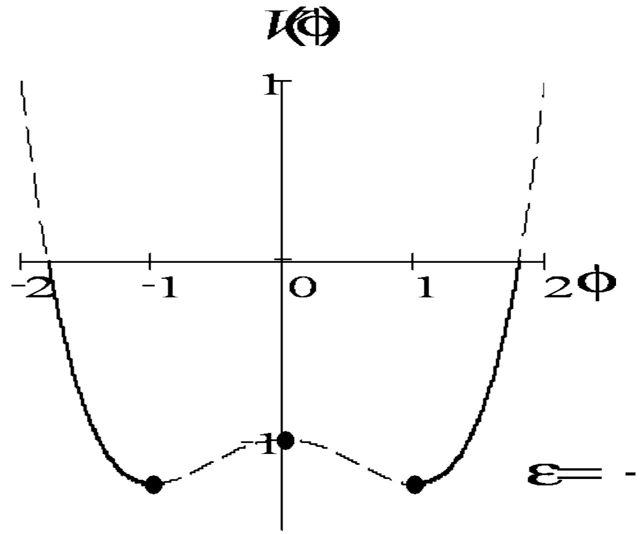
Numerical analysis

"Mexican hat" potential

$$V = \frac{\lambda\eta^4}{4}\left[\varepsilon + \left(1 - \frac{\phi^2}{\eta^2}\right)^2\right]$$

Three extreme points

$$V'_\infty = 0, \quad V''_\infty = 2\eta^2, \qquad \phi_\infty = \pm\eta$$
$$V'_\infty = 0, \quad V''_\infty = -\eta^2, \qquad \phi_\infty = 0.$$



Dimensionless parameters

$$d_0, \varepsilon, \Gamma, \text{ and } \phi'_0 \qquad \Gamma = \chi^2\eta^2$$

Solutions with fixed d_0 , ε , *and* Γ *exist within the interval*

$$0 < \phi'_0 \le \phi'_{0\max}(d_0, \varepsilon, \Gamma)$$

We set $d_0 = 4$, $a = (\lambda\eta^2)^{-1/2} = 1$, $\eta = 1$ *in computations.*

Analysis of equations

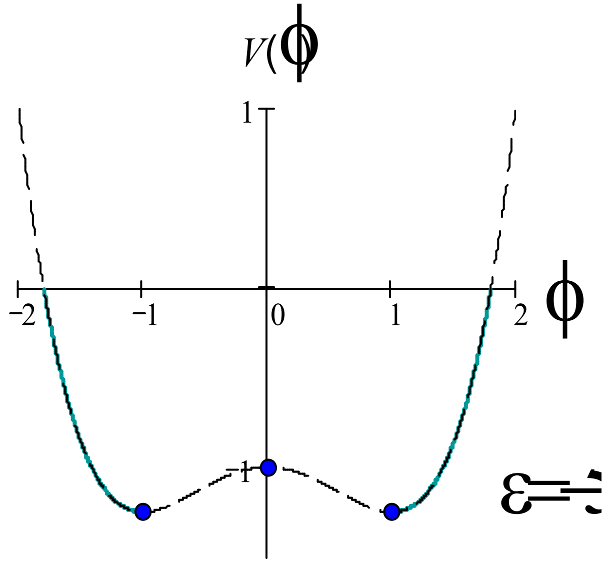
Type A	Type B
$\gamma' - \beta' = -e^{-(d_0\gamma+\beta)}$	$\gamma' - \beta' = -\dfrac{e^{-(d_0\gamma+\beta)}}{1-\kappa^2\phi^2}$

$\beta' > \gamma'$ $\beta' - \gamma' \to 0, \quad l \to \infty$ **Type B: singularity at** $\kappa^2\phi^2 = 1$

Regular solutions exist within the whole interval $0 < l < \Pi$; $r \to \Pi$ **at** $l \to \Pi$

Limiting values

Type A	$\gamma'_\infty = \sqrt{-\dfrac{2\kappa^2 V_\infty}{(d_0+1)\left[d_0+(d_0+1)\,\kappa^2\phi^2_\infty\right]}}$ $V'(\phi_\infty) = 0 \qquad V_\infty < 0$
Type B	$\gamma'_\infty = \sqrt{\dfrac{1}{(d_0+1)\,\phi_\infty}\dfrac{\partial V_\infty}{\partial\phi}} \qquad \dfrac{\partial V_\infty}{\partial\phi^2} > 0$ $\kappa^2 V_\infty = -\left(d_0+\kappa^2\phi^2_\infty\right)\dfrac{\partial V_\infty}{\partial\phi^2} \qquad V_\infty < 0$



Analysis of equations

Asymptotic behavior $\phi(l)$ $\phi = \phi_\infty + \delta\phi$

Type A

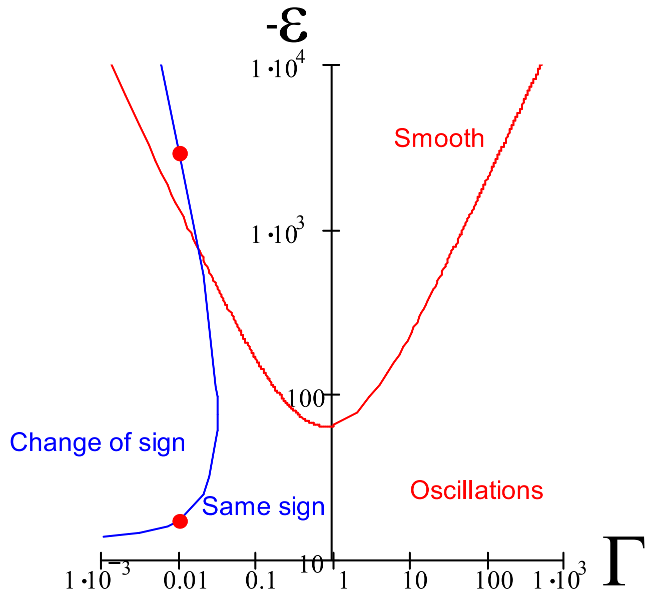
$$\delta\phi'' + (d_0+1)\gamma'_\infty\delta\phi' + \frac{2\chi^2|V_\infty|V''_\infty}{d_0(d_0+1)\gamma'^2_\infty}\delta\phi = 0$$

$$\lambda_\pm = -\frac{(d_0+1)\gamma'_\infty}{2}\left(1 \mp \sqrt{1 - \frac{8\chi^2|V_\infty|V''_\infty}{d_0(d_0+1)^3\gamma'^4_\infty}}\right)$$

$$\delta\phi = Ae^{\lambda_+ l} + Be^{\lambda_- l},$$

If $V''_\infty > 0$, ***then both eigenvalues are either negative, or have negative real parts***

$$\frac{8\chi^2|V_\infty|V''_\infty}{d_0(d_0+1)^3\gamma'^4_\infty} > 1$$ *- oscillations*



Analysis of equations

Asymptotic behavior $\phi(l)$ $\phi = \phi_\infty + \delta\phi$

Type B

$$\delta\phi'' + S_{1\infty}\delta\phi' + \left(\frac{\partial^2 V_\infty}{\partial\phi^2} - \frac{d_0 - 3\varkappa^2\phi_\infty^2}{(d_0+1)(d_0+\varkappa^2\phi_\infty^2)}S_{1\infty}^2\right)\delta\phi = 0$$

$$\delta\phi = Ae^{\lambda_+ l} + Be^{\lambda_- l},$$

$$\lambda_\pm = -\sqrt{\frac{d_0+1}{4\phi_\infty}\frac{\partial V_\infty}{\partial\phi}} \pm \sqrt{\left(\frac{d_0+1}{4} + \frac{d_0 - 3\varkappa^2\phi_\infty^2}{d_0+\varkappa^2\phi_\infty^2}\right)\frac{\partial V_\infty}{\phi_\infty\partial\phi} - \frac{\partial^2 V_\infty}{\partial\phi^2}}.$$

In practice both eigenvalues are complex with negative real parts.

Type A

Left:

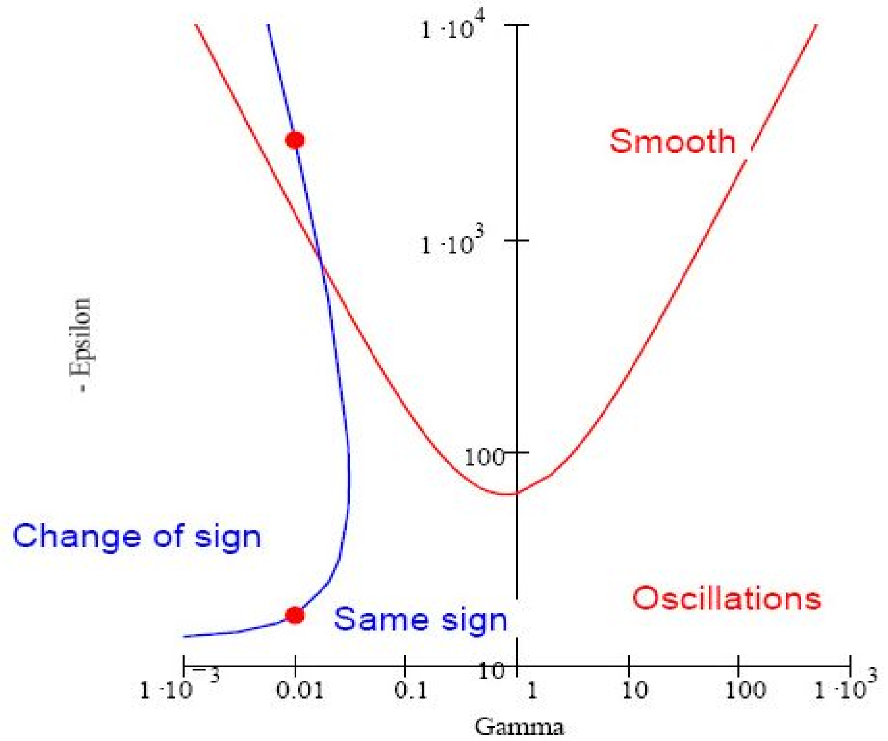
Red line is a border between oscillating and smooth solutions $\phi(l)$. To the left of the blue line $\phi(l)$ does not change sign.

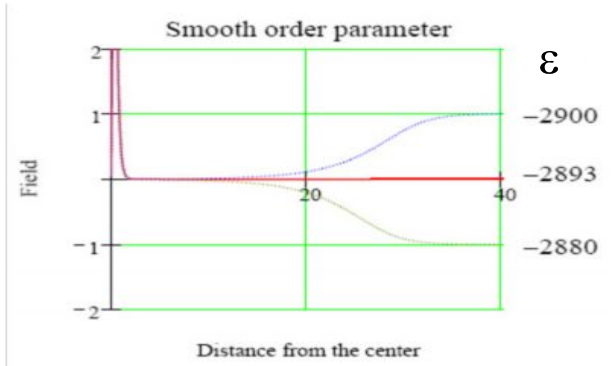
$$\varepsilon_b = -16\frac{(1+G)^2}{G}, \quad G = \frac{d_0+1}{d_0\Gamma}$$

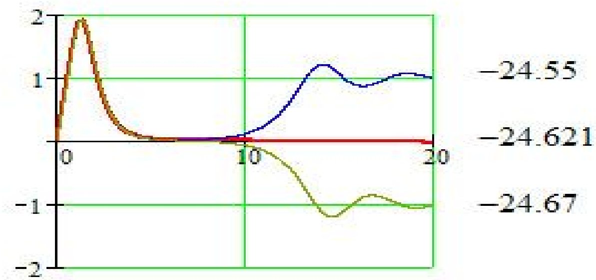
Right:

Order parameter $\phi(l)$ in the vicinity of red points on the blue line in left figure.

$$\phi'(0) = \sqrt{-\frac{\varepsilon+1}{8}}$$

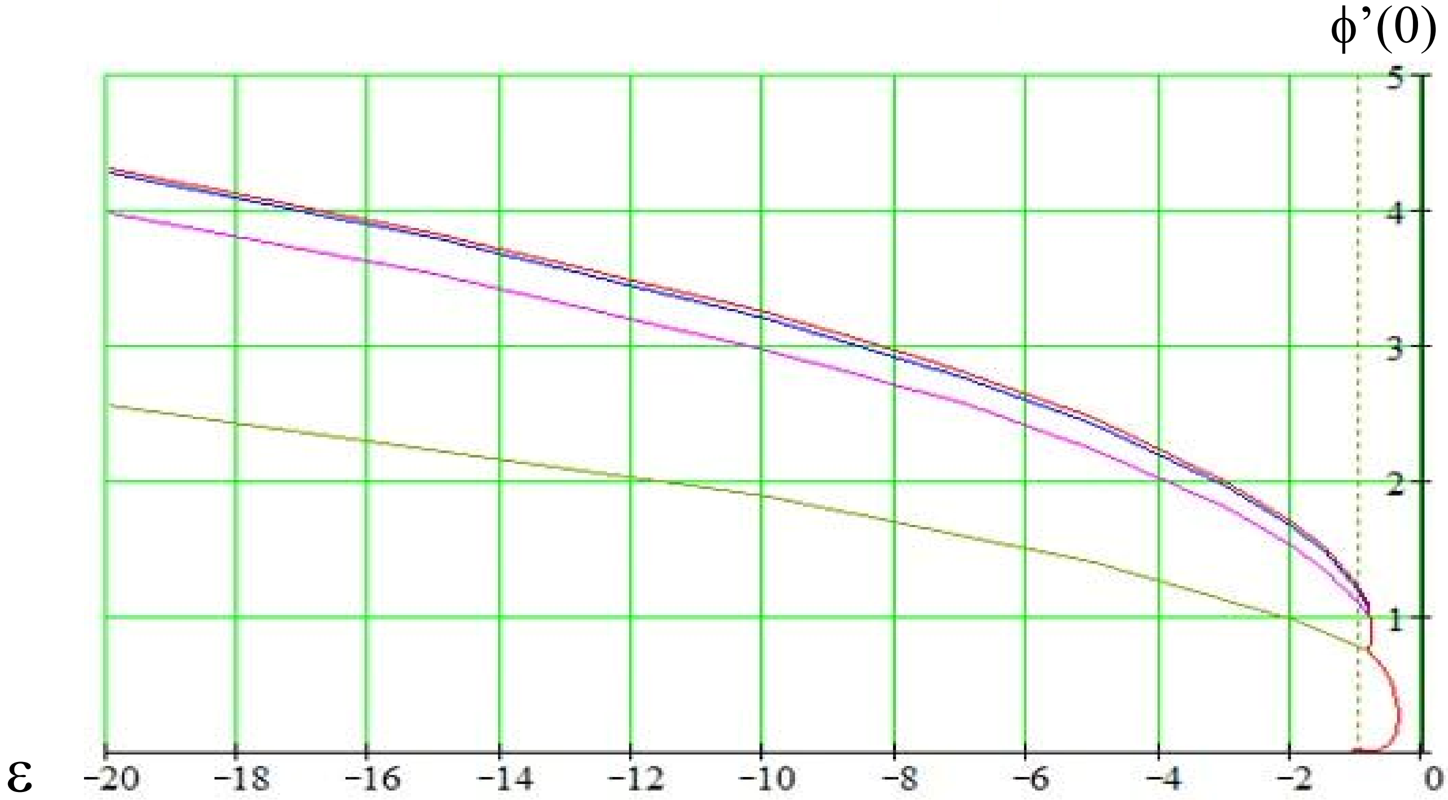






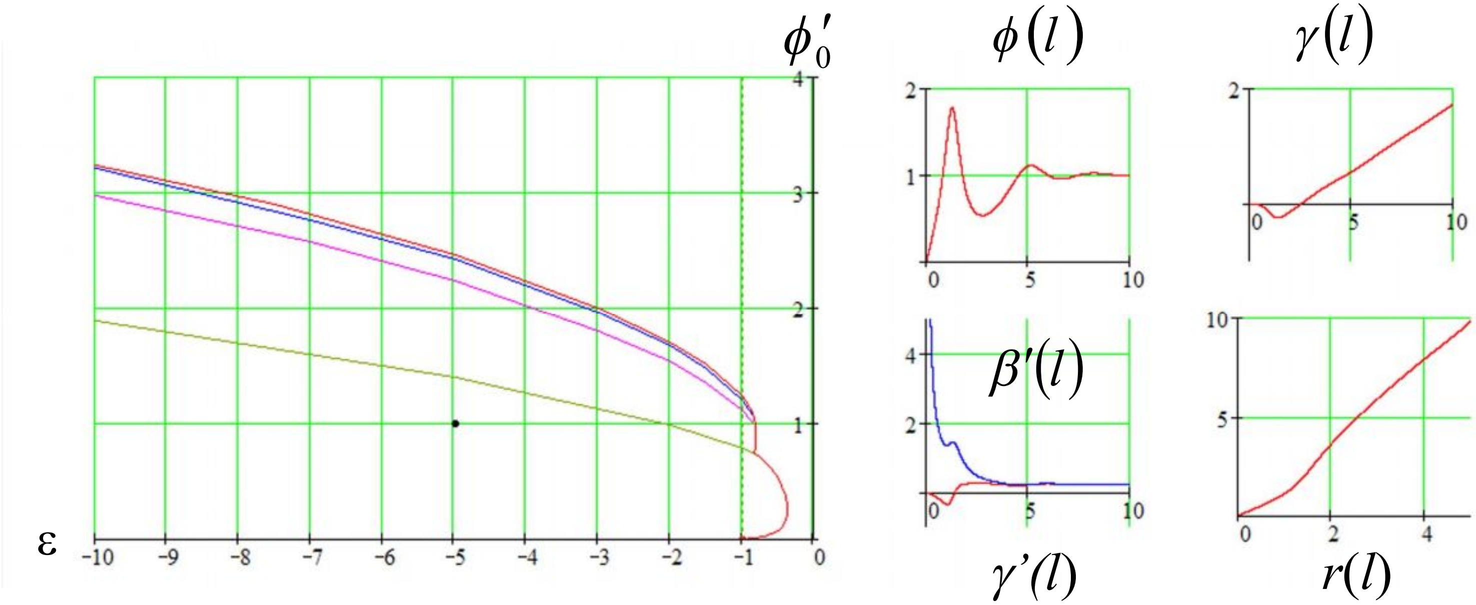
Type A

Domain of regular solutions in the plane of parameters (ϕ_0', ε) for $d_0 = 4$, $\Gamma = 1$.



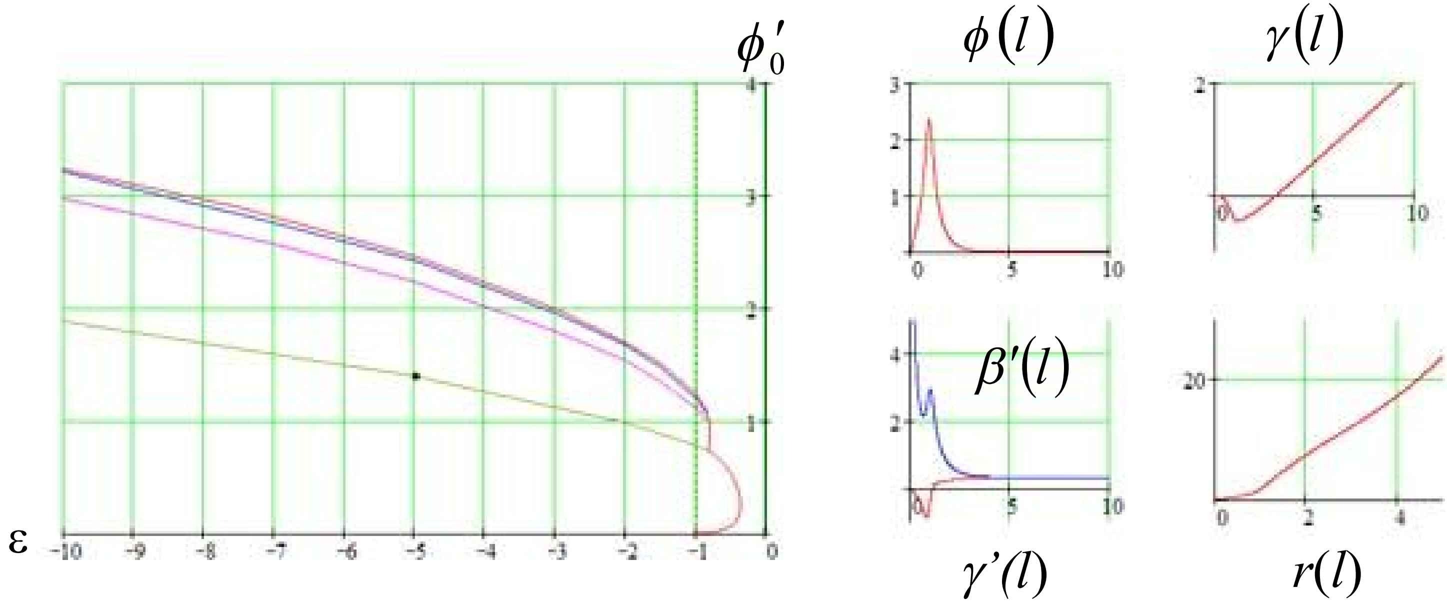
Type A solutions with the order parameter not changing sign

$d_0 = 4, \quad \Gamma = 1$



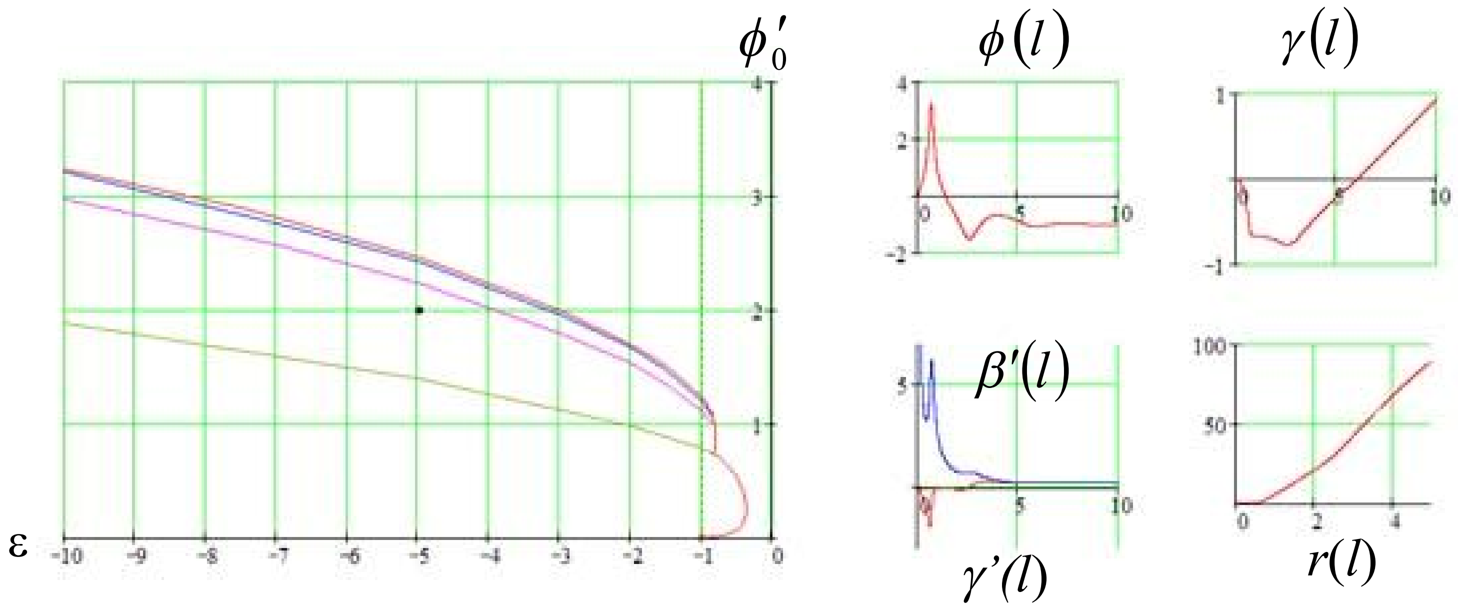
Type A solutions with the order parameter not changing sign, and terminating with $\phi = 0$ at $l \to \infty$

$d_0 = 4, \quad \Gamma = 1$



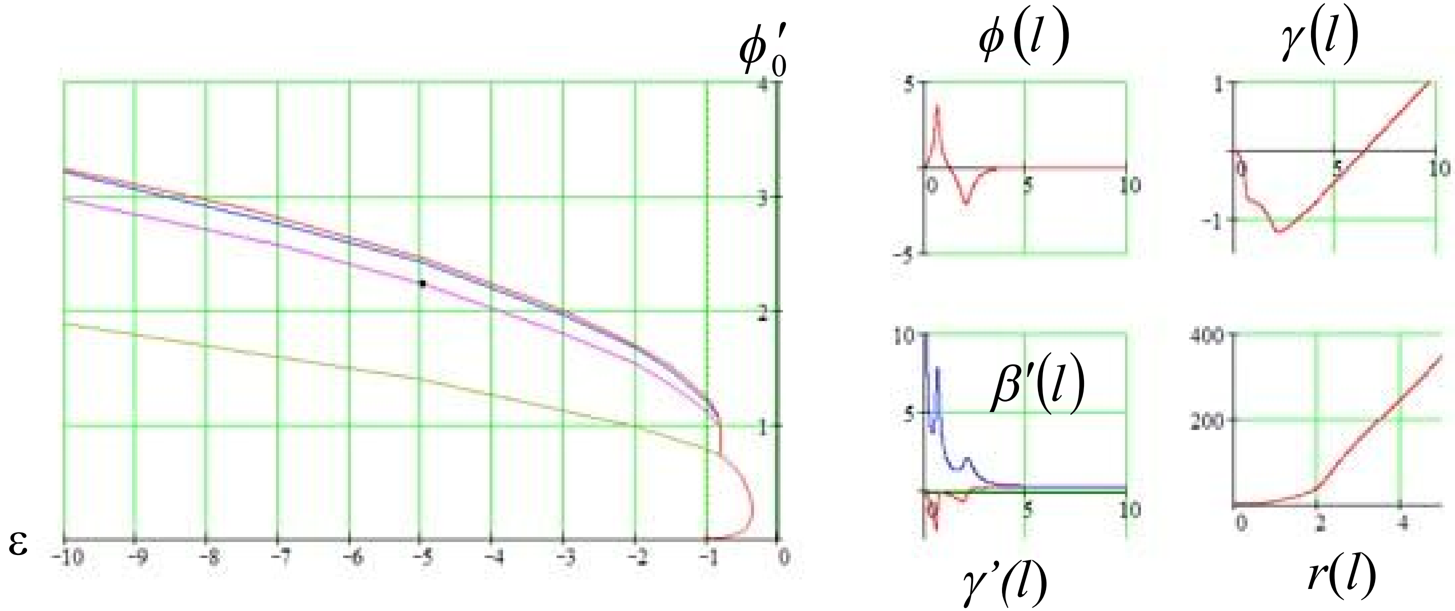
Type A solutions with the order parameter changing sign once

$d_0 = 4, \quad \Gamma = 1$



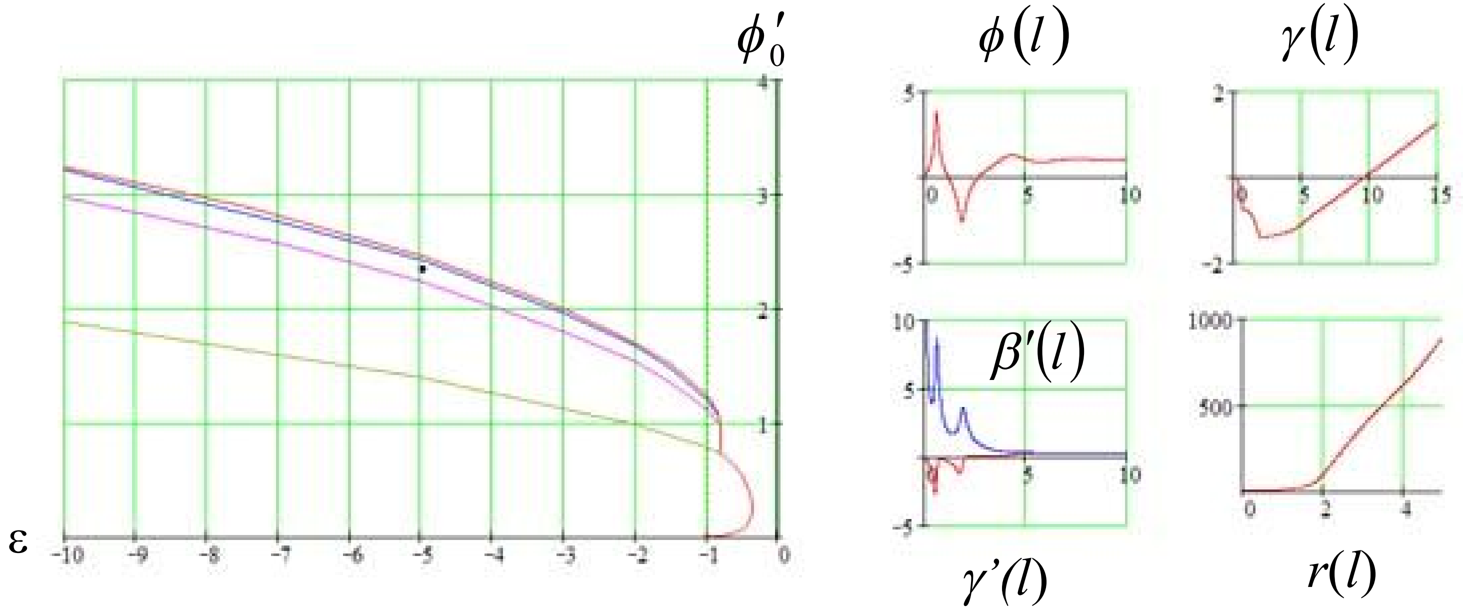
Type A solutions with the order parameter changing sign once, and terminating with $\phi = 0$ при $l \to \infty$

$d_0 = 4, \quad \Gamma = 1$



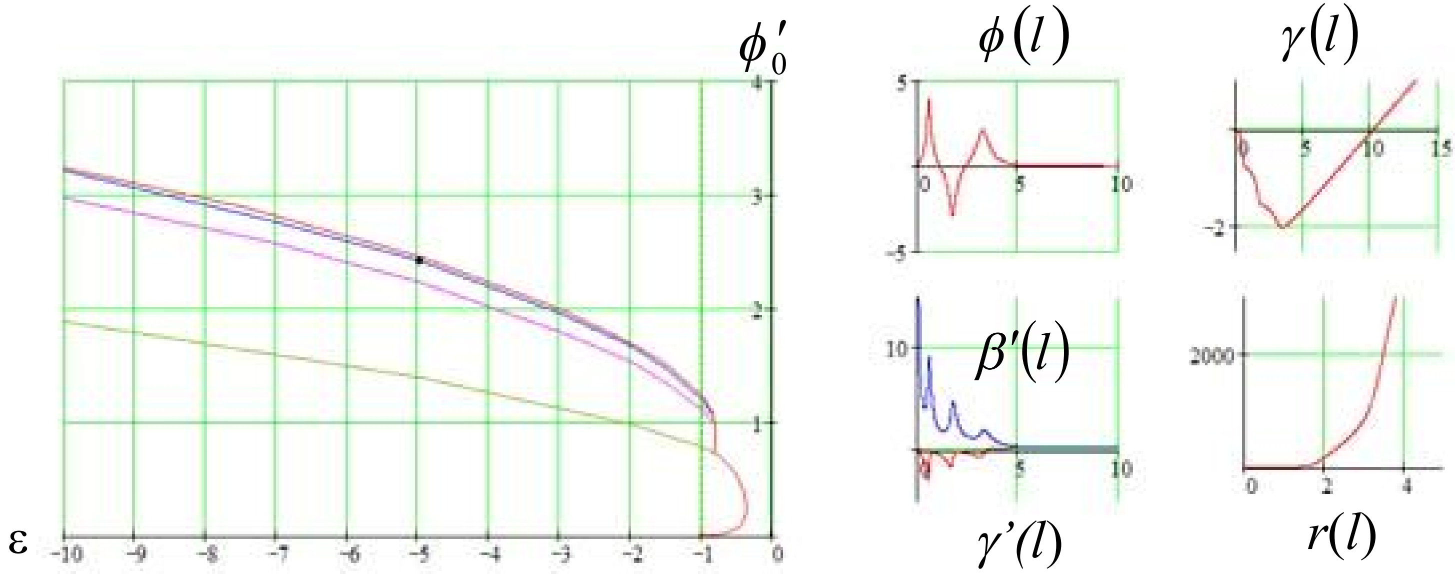
Type A solutions with the order parameter changing sign twice

$d_0 = 4, \quad \Gamma = 1$



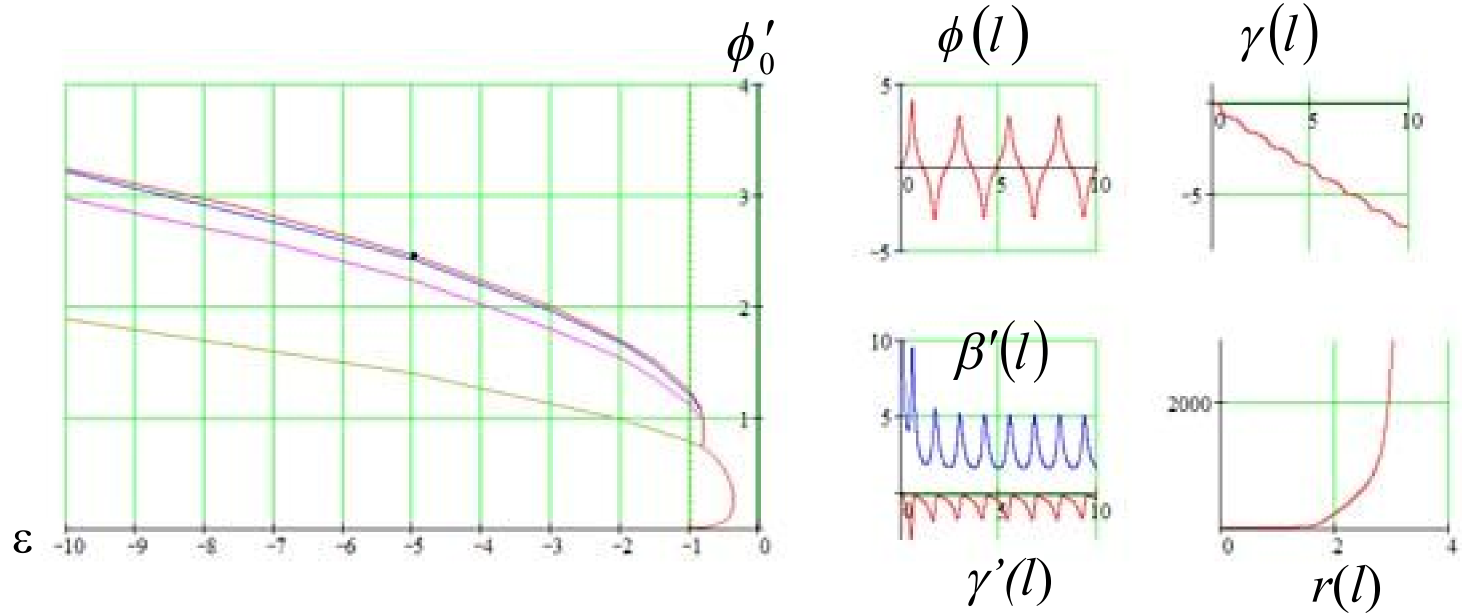
Type A solutions with the order parameter changing sign twice, and terminating with $\phi = 0$ at $l \to \infty$

$d_0 = 4, \quad \Gamma = 1$

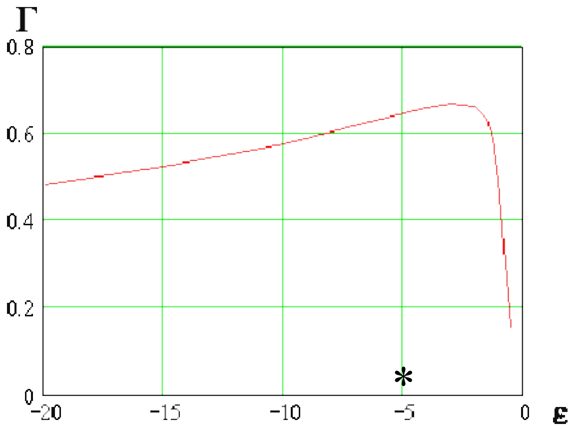


Type A solutions on the border of regularity

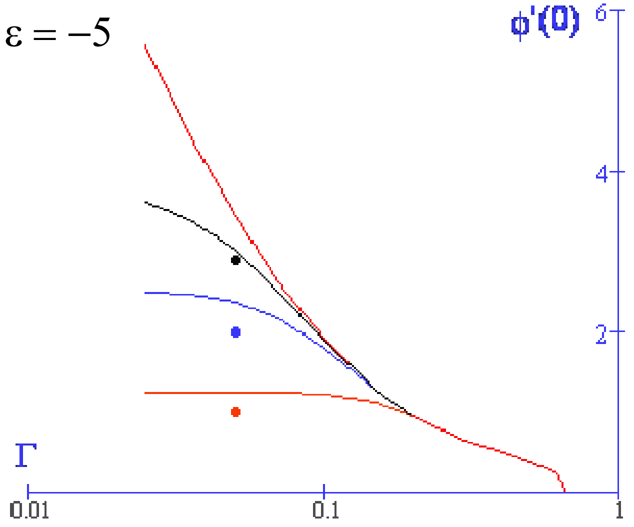
$d_0 = 4, \quad \Gamma = 1$



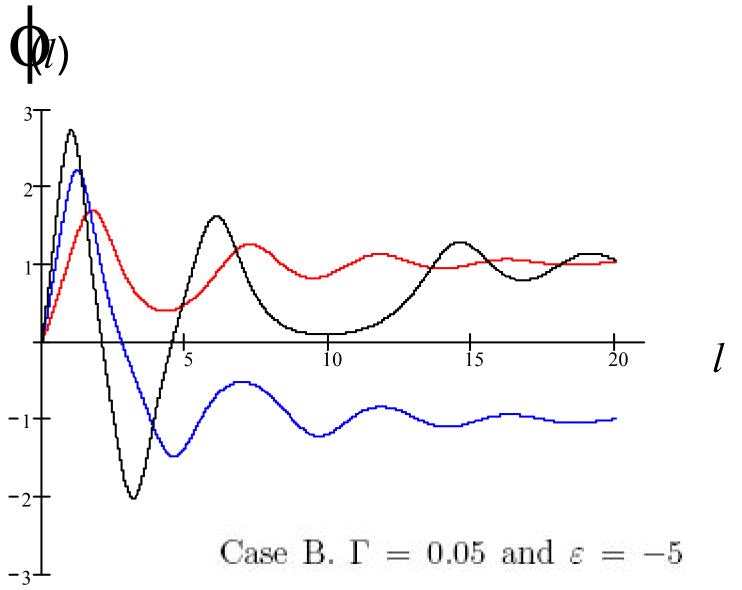
Type B



Upper border of regularity in the plane of parameters (ε, Γ)



Map of regular solutions in the plane (ϕ'(0), Γ)



Neutral quantum particle

Lagrangian of a spin-less particle $L_\chi = \frac{1}{2} g^{AB} \chi^*_{,B} \chi_{,A} - \frac{1}{2} m_0^2 \chi^* \chi$

Wave equation $\frac{1}{\sqrt{-g}} \left(\sqrt{-g} g^{AB} \chi_{,A} \right)_{,B} + m_0^2 \chi = 0$

Wave function $\chi\left(x^A\right) = X(l) \exp\left(-ip_\mu x^\mu + in\varphi\right)$

$$X'' + S_1 X' + \left(p^2 e^{-2\gamma} - n^2 e^{-2\beta} - m_0^2\right) X = 0 \qquad p^2 = E^2 - \mathbf{p}^2$$

Schrödinger equation $dl = e^\gamma dx, \qquad X(l) = y(x) / \sqrt{f(x)},$

$$y_{xx} + \left[p^2 - V_g(x)\right] y = 0 \qquad f(x) = \exp\left\{-\frac{1}{2}\left[(d_0 - 1)\gamma + \beta\right]\right\}.$$

Gravitational potential $\mathrm{V}_g(x) = e^{2\gamma}\left(e^{-2\beta} n^2 + m_0^2\right) + \frac{1}{2} \frac{1}{\sqrt{f}} \frac{d}{dx}\left(\frac{1}{f^{1/2}} \frac{df}{dx}\right)$

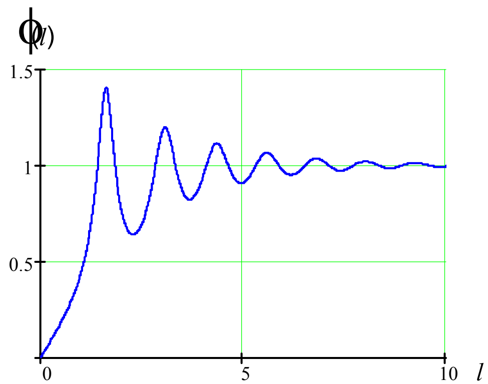
Type A

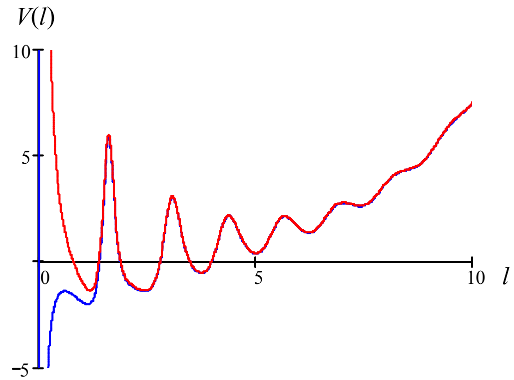
$d_0 = 4$

$\varepsilon = -2$

$\Gamma = 10$

$\phi'_0 = \sqrt{-\frac{\varepsilon+1}{8}}$





Spin-less particles, identical in the plain bulk, acquire integer spins and different masses being trapped within different points of minimum of the gravitational potential

Comparison of vector and multi-scalar order parameters

Property	Multi-scalar	Vector, **Type A**	Vector, **Type B**
Order of Einstein equations	4	3	3
Number of parameters	n_V	$n_V + 1$	$n_V + 1$
Fine tuning	sometimes	no need	no need
Trapping of matter on brane	yes	yes	yes
Behavior $r(l)$	$\to \infty, \to r_m, \to 0$	$\to \infty$	$\to \infty$
In equations:	V	$dV/d\phi$	V
Behavior $V(\phi)$ at $l \to \infty$	$dV/d\phi = 0$	$dV/d\phi = 0$	$dV / d\phi^2 > 0$
Derivation of T_{IK}	easier	more difficult	more difficult
Strength of grav. field Γ	unlimited	unlimited	limited from above