Duality between Wilson Loops and Scattering Amplitudes in QCD

by

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Based on: Y. M., Poul Olesen

- Phys. Rev. Lett. **102**, 071602 (2009) [arXiv:0810.4778 [hep-th]]
- arXiv:0903.4114 [hep-th]
- further developments

Introduction and motivations since 1979

QCD string is not Nambu–Goto but the asymptote at large Wilson loops is universal

$$W(C) \overset{\text{large } C}{\propto} \mathrm{e}^{-K S_{\text{min}}(C)} \qquad \Longrightarrow \quad \text{the area law}$$

What are the consequences for correlators of composite operators?

Regge behavior of scattering amplitudes at high energy and fixed momentum transfer under a few controllable approximations.

Why not 30 years ago?

Three essential constituents:

- Representation of QCD scattering amplitudes through Wilson loops
  Wilson (1975) (lattice), Y.M., Migdal (1979) (continuum)
- Representation of minimal area as quadratic functional of $x_\mu(\cdot)$
  Douglas (1931) (Plateau problem)
- The idea of Wilson-loop/scattering-amplitude duality
  Alday, Maldacena (2007) ($\mathcal{N} = 4$ SYM)

QCD amplitudes through Wilson loops

Y.M., Migdal (1981)

Green's functions of M colorless composite quark operators

$$\bar{q}(x_i)q(x_i) \qquad \bar{q}(x_i)\gamma_5 q(x_i) \qquad \bar{q}(x_i)\gamma_\mu q(x_i) \qquad \bar{q}(x_i)\gamma_\mu\gamma_5 q(x_i)$$

are given by the sum over Wilson loops passing via x_i $(i = 1, \ldots, M)$

$$G \equiv \left\langle \prod_{i=1}^{M} \bar{q}(x_i)q(x_i) \right\rangle_{\text{conn}} = \sum_{\text{paths} \ni \{x_1,\ldots,x_M \equiv x_0\}} J[z(\tau)]\, W[z(\tau)]$$

The weight for the path integration is

$$J[z(\tau)] = \int \mathcal{D}k(\tau)\ \text{sp P}\,\text{e}^{\text{i}\int_0^{\mathcal{T}} \text{d}\tau\,[\dot{z}(\tau)\cdot k(\tau) - \gamma(\tau)\cdot k(\tau)]}$$

for spinor quarks of mass m and scalar operators or

$$J[z(\tau)] = \text{e}^{-\frac{1}{2}\int_0^{\mathcal{T}} \text{d}\tau\,\dot{z}^2(\tau)}$$

for scalar quarks. τ is the proper time.

The Wilson loop $W(C)$ is in pure Yang–Mills at large N (or quenched). For finite N, correlators of several Wilson loops are present.

QCD amplitudes via Wilson loops (cont.)

On-shell scattering amplitudes are given by the LSZ reduction

Momentum-space scattering amplitude (functional Fourier transform)

$$G\,(\Delta p_1, \ldots, \Delta p_M) = \sum_{\text{paths}} \mathrm{e}^{\mathrm{i}\int_0^{\mathcal{T}} \mathrm{d}\tau\, \dot{z}(\tau)\cdot p(\tau)}\, J[z(\tau)]\, W[z(\tau)]$$

for piecewise constant momentum-space loop $p(\tau)$

$$p(\tau) = p_i \qquad \text{for } \tau_i < \tau < \tau_{i+1}$$

$$\dot{p}(\tau) = -\sum_i \Delta p_i \delta(\tau - \tau_i) \qquad \text{with} \quad \Delta p_i \equiv p_{i-1} - p_i$$

representing M momenta of (all incoming) particles.
Then momentum conservation is automatic while an (infinite) volume V is produced, say, by integration over $x_0 = x_M$.

This is because

$$\int \mathrm{d}\tau\, p(\tau) \cdot \dot{z}(\tau) = \sum_i \Delta p_i \cdot x_i$$

reproducing the exponent of the Fourier transformation.

Minimal area and boundary functional

Douglas (1931)

To calculate the scattering amplitudes, we substitute the area-law behavior of asymptotically large Wilson loops:

$$W(C) \overset{\text{large } C}{\propto} \mathrm{e}^{-K S_{\min}(C)},$$

and integrate over the paths.

$S(C)$ is highly nonlinear functional $\Longrightarrow$ hopeless to calculate.

Douglas algorithm for solving the Plateau problem (finding the minimal surface) is to minimize the boundary functional

$$A[x(\theta)] = \frac{1}{8\pi} \int_0^{2\pi} \mathrm{d}\phi \int_0^{2\pi} \mathrm{d}\phi' \, \frac{\left[x(\theta(\phi)) - x(\theta(\phi'))\right]^2}{1 - \cos(\phi - \phi')}$$

with respect to the reparametrizations $\theta(\phi)$ ($\dot{\theta}(\phi) \geq 0$). In general

$$A[x(\theta)] \geq A[x(\theta_*)] = S_{\min}(C)$$

The minimum is reached at $\theta(\phi) = \theta_*(\phi)$ which is contour-dependent.

Minimal area and boundary functional (cont.)

The Douglas functional can be equivalently rewritten as

$$A = -\frac{1}{4\pi}\int_0^{2\pi} \mathrm{d}\theta_1 \int_0^{2\pi} \mathrm{d}\theta_2\, \dot{x}(\theta_1)\cdot\dot{x}(\theta_2)\, \ln\left(1-\cos\left[\phi(\theta_1)-\phi(\theta_2)\right]\right)$$

when only $\phi(\theta)$ is sensitive to reparametrizations.

Simplest example: $\phi_*(\theta)=\theta$ for a circle.

The solution for an ellipse with periods a and b is

$$\theta'_*(\phi)=\frac{\pi}{2K(s)}\frac{1}{\sqrt{(1-s)^2+4s\sin^2\phi}} \qquad \frac{\pi K\left(\sqrt{1-s^2}\right)}{2K\left(s\right)}=\log\frac{a+b}{a-b}$$

where $K(s)$ is the complete elliptic integral of the first kind.

Elliptic integrals also emerge for a rectangle (the Schwarz–Christoffel mapping).

Integration over reparametrizations

Polyakov (1997)

Wilson loop in large-N QCD $\Longleftrightarrow$ the tree-level string disk amplitude integrated over reparametrizations of the boundary contour.

Conformal map of the disk into the upper half-plane:
the disk boundary $\Longrightarrow$ the real axis

$$t(\tau) = -\cot \frac{\pi \tau}{\mathcal{T}} \qquad -\infty < t < +\infty$$

Reparametrization-invariant ansatz

$$W(C) = \int \mathcal{D}s(t) \exp\left(\frac{K}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}t_1 \mathrm{d}t_2 \, \dot{x}(t_1) \cdot \dot{x}(t_2) \ln |s(t_1) - s(t_2)| \right)$$

where the path integral is over reparametrizations $s(t)$ (with $s'(t) \geq 0$).

This classical boundary action is derivable for:

- bosonic string in $d = 26$,
- superstring in $d = 10$.

Area law for asymptotically large C (or very large K) $\Longrightarrow$ a saddle point in the integral over reparametrizations at $s(t) = s_*(t)$.

Large loops and minimal area

Gaussian fluctuations around the saddle-point $\theta_*(\sigma)$ result in a pre-exponential factor

$$W\left[x(\cdot)\right] \stackrel{\text{large loops}}{=} F\left[\sqrt{K}x(\cdot)\right] \, \mathrm{e}^{-KS_{\text{min}}[x(\cdot)]} \left[1 + \mathcal{O}\left(\left(KS_{\text{min}}\right)^{-1}\right)\right],$$

which is contour dependent

$$F\,[\text{circle}] \propto \sqrt{KR^2} \qquad \text{for a circle}$$

Asymptotic area law is recovered modulo the pre-exponential which is not essential for large loops.

More subtle effects (such as the Lüscher term are due to the pre-exponential factor.

Functional Fourier transformation

Reparametrization-invariant functional Fourier transformation

$$W[p(\cdot)] = \int \mathcal{D}x \, \mathrm{e}^{\mathrm{i}\int p\cdot \mathrm{d}x} \, W[x(\cdot)]$$

of the disk amplitude for piecewise constant $p(t)$.

Performing the Gaussian integration:

$$W[p(\cdot)] = \int \mathcal{D}s(t) \exp\left(\alpha' \int_{-\infty}^{+\infty} \mathrm{d}t_1 \fint_{-\infty}^{+\infty} \mathrm{d}t_2 \, \dot{p}(t_1) \cdot \dot{p}(t_2) \ln|s(t_1) - s(t_2)|\right)$$

It looks like the disk amplitude with K replaced by $1/K = 2\pi\alpha'$.
The determinant is a s-independent constant.

The principal-value prescription will be important for stepwise $p(t)$.

$p(t) = p_j$ at the j-th interval for the stepwise discretization $\Longrightarrow$ reparametrization changes t_j's for s_j's keeping their cyclic order — discrete reparametrization transformation.
Stepwise discretization of $x(t)$ itself would violate the continuity of the string end world line.

Derivation of Koba–Nielsen amplitudes

First note that

$$\int_{-\infty}^{+\infty} \mathrm{d}t_1 \int_{-\infty}^{+\infty} \mathrm{d}t_2\, \dot{p}(t_1)\cdot\dot{p}(t_2) \ln|s(t_1)-s(t_2)|$$
$$= -\frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\mathrm{d}s_1 \mathrm{d}s_2}{(s_1-s_2)^2}\left[p\left(t(s_1)\right)-p\left(t(s_2)\right)\right]^2$$

Integration over s_1 or s_2 has divergences for adjacent sides $k=l\pm1$.

Principal-value regularization ⇒ omitting sides with $k=l\pm1$ ⟹ the integrals over s_1 and s_2 are finite:

$$-\frac{1}{2}\sum_{k\neq l\pm1}\int_{s_{k-1}}^{s_k}\mathrm{d}s_1\int_{s_{l-1}}^{s_l}\mathrm{d}s_2\,\frac{(p_k-p_l)^2}{(s_1-s_2)^2}$$
$$= \sum_{k\neq l}\Delta p_k\cdot\Delta p_l \log|s_k-s_l|$$
$$+\sum_j \Delta p_j^2 \log\frac{(s_j-s_{j-1})(s_{j+1}-s_j)}{(s_{j+1}-s_{j-1})}$$

which is invariant under projective transformations.

Projective transformation

Stepwise discretization of $p(t)$ naturally results in M-particle (off-shell) Koba–Nielsen amplitudes invariant under the $PSL(2;\mathbb{R})$ projective (or Möbius) transformation

$$s \Rightarrow \frac{as+b}{cs+d} \qquad \text{with} \quad ad-bc=1$$

because the projective group is a subgroup of reparametrizations.

The main formulas:

$$\begin{aligned}(s_i - s_j) \;&\Rightarrow\; \frac{(s_i - s_j)}{(cs_i+d)(cs_j+d)} \\ \mathrm{d}s_i \;&\Rightarrow\; \frac{\mathrm{d}s_i}{(cs_i+d)^2}\end{aligned}$$

under the projective transformation.

Derivation of Koba–Nielsen amplitudes (cont.)

Integrating over reparametrizations at intermediate points (s_{i-1}, s_i) results in the following measure

$$D^{(M)}s = \prod_{i=1}^{M} \frac{\mathrm{d}s_i}{|s_i - s_{i-1}|}$$

for the integration over s_i's.
It is invariant under the projective transformation and gives

$$W(\Delta p_1, \ldots, \Delta p_M)$$
$$= \int_{s_{i-1}<s_i} \prod_i \frac{\mathrm{d}s_i}{|s_i - s_{i-1}|} \prod_{k \neq l} |s_k - s_l|^{\alpha' \Delta\vec{p}_k \Delta\vec{p}_l} \prod_j \left(\frac{|s_j - s_{j-1}|\,|s_{j+1} - s_j|}{|s_{j+1} - s_{j-1}|} \right)^{\alpha' \Delta p_j^2}$$

where the integration over s_i emerges from the path integral over reparametrizations.

Fixing the $PSL(2;\mathbb{R})$ invariance in the standard way

$$s_1 = 0, \quad s_{M-1} = 1, \quad s_M = \infty$$

$\Longrightarrow$ scalar amplitudes in the Koba–Nielsen variables.

Path integrals over reparametrization

The measure on $Diff(\mathbb{R})$

$$\int_{\substack{t(s_0)=t_0\\ t(s_f)=t_f}} \mathcal{D}_{\text{diff}} t(s) \cdots = \lim_{L\to\infty} \int_{t_0}^{t_f} \frac{1}{(t_f - t_L)} \prod_{j=1}^{L} \int_{t_0}^{t_{j+1}} \mathrm{d}t_j \frac{1}{(t_j - t_{j-1})} \cdots$$

is invariant under reparametrizations

$$s \to t(s)\,, \quad t(s_0) = s_0\,, \quad t(s_f) = s_f\,, \quad \frac{\mathrm{d}t}{\mathrm{d}s} \geq 0$$

The main integral for the integration at the intermediate point t_i

$$\int_{t_{i-1}}^{t_{i+1}} \mathrm{d}t_i \frac{\delta}{(t_{i+1} - t_i)^{1-\delta}(t_i - t_{i-1})^{1-\delta}} = \frac{2}{(t_{i+1} - t_{i-1})^{1-2\delta}}$$

where small δ is introduced to control a logarithmic divergence.
This is an analogue of the well-known formula

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}t_i}{\sqrt{2\pi}} \frac{\mathrm{e}^{-(t_f - t_i)^2/2\nu_1}}{\sqrt{\nu_1}} \frac{\mathrm{e}^{-(t_i - t_0)^2/2\nu_2}}{\sqrt{\nu_2}} = \frac{\mathrm{e}^{-(t_f - t_0)^2/2(\nu_1+\nu_2)}}{\sqrt{(\nu_1 + \nu_2)}}$$

which is used for calculations with the usual Wiener measure

Projective-invariant off-shell amplitudes

For 4 scalars this reproduces the Veneziano amplitude

$$A(\Delta p_1, \Delta p_2, \Delta p_3, \Delta p_4) = \int_0^1 \mathrm{d}x\, x^{-\alpha(s)-1}(1-x)^{-\alpha(t)-1},$$

where $\alpha(t) = \alpha' t$ – linear Regge trajectory – and

$$s = -(\Delta p_1 + \Delta p_2)^2, \qquad t = -(\Delta p_2 + \Delta p_3)^2$$

are usual Mandelstam's variables (for Euclidean metric).

The tachyonic condition $\alpha' \Delta p_j^2 = 1$ has not to be imposed.

This is of the type of Lovelace choice that reproduces some projective-invariant off-shell string amplitudes known since late 1960's.

The more familiar on-shell tachyon amplitudes can be obtained by setting $\alpha' \Delta p_j^2 = 1$.

Regge–Veneziano behavior from the area law

To substitute the area-law behavior of $W(C)$ into the path integral and to find out for what momenta the asymptotically large loops dominate. Typical momenta will be large for large loops.

Interchanging the order of integration over $z(\tau)$ and $\sigma(\tau)$, we obtain for the QCD scattering amplitude (fixed s_M)

$$G(\Delta p_1, ..., \Delta p_M) \propto \prod_{i=1}^{M-1} \int_{-\infty}^{s_{i+1}} \frac{\mathrm{d}s_i}{1+s_i^2} \left[\frac{|s_{i+1}-s_i||s_i-s_{i-1}|}{|s_{i+1}-s_{i-1}|}\right]^{\Delta p_i^2/2\pi K}$$
$$\times |s_i - s_j|^{\Delta p_i \cdot \Delta p_j/2\pi K} \, \mathcal{K}(s_1, \ldots, s_{M-1}; \Delta p_1, \ldots, \Delta p_M)$$

which is a convolution of the Koba–Nielsen integrand and a kernel

$$\mathcal{K} = \int \mathcal{D}s(t) \int \mathcal{D}k(t) \int_0^{\infty} \mathrm{d}\mathcal{T} \, \mathcal{T}^{M-1} \, \mathrm{e}^{-m\mathcal{T}} \, \mathrm{sp} \; \mathrm{P} \exp\left(-\frac{\mathrm{i}\mathcal{T}}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}t \, \gamma(t) k(t)\right)$$
$$\times \exp\left(\frac{1}{4\pi K} \int_{-\infty}^{+\infty} \mathrm{d}t_1 \int_{-\infty}^{+\infty} \mathrm{d}t_2 \, \dot{k}(t_1) \dot{k}(t_2) \ln|s(t_1) - s(t_2)|\right)$$
$$\times \exp\left(\frac{1}{2\pi K} \sum_i \Delta p_i \int_{-\infty}^{+\infty} \mathrm{d}t \, \dot{k}(t) \ln|s_i - s(t)|\right)$$

Regge–Veneziano behavior from the area law (cont.1)

The spectrum is still of the Regge–Veneziano type (= linear).

This is rather close to the disk amplitude, except for the additional integration over k.

For small m (and/or very large M), the integral over $\mathcal{T}$ is dominated by large $\mathcal{T} \sim (M-1)/m$ because of $\mathcal{T}^{M-1}$. Typical values of $k \sim 1/\mathcal{T}$ are essential in the path integral over k for large $\mathcal{T}$
$\Longrightarrow$ $\mathcal{K}$ becomes momentum independent:

$$\mathcal{K}(s_1,\ldots,s_{M-1};\Delta p_1,\ldots,\Delta p_M) = \prod_{i=1}^{M} \frac{1}{|s_{i+1}-s_i|}$$

so the (off-shell) Koba–Nielsen amplitudes are reproduced.

Regge–Veneziano behavior from the area law (cont.2)

$M = 4$ QCD scattering amplitude (with no $PSL(2;\mathbb{R})$)

$$G_4 \;=\; \int_{-\infty}^{+\infty} \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3\, \theta_{\mathrm{C}}(s_1, s_2, s_3, s_4) \frac{1}{(1+s_1^2)(1+s_2^2)(1+s_3^2)}$$

$$\times \frac{1}{|s_{43}||s_{32}||s_{21}||s_{41}|} \left(\frac{s_{21}s_{43}}{s_{31}s_{42}}\right)^{-\alpha' s} \left(\frac{s_{41}s_{32}}{s_{31}s_{42}}\right)^{-\alpha' t}$$

$$s_{ij} = s_i - s_j.$$

Introducing the variable

$$x = \frac{s_{21}s_{43}}{s_{31}s_{42}} \qquad 0 \le x \le 1 \qquad x = 0 \text{ for } s_2 = s_1 \qquad x = 1 \text{ for } s_2 = s_3$$

we get

$$G_4 = \int_0^1 \mathrm{d}x\, x^{-\alpha' s - 1} (1-x)^{-\alpha' t - 1} \int_{-\infty}^{+\infty} \mathrm{d}s_1 \mathrm{d}s_3\, \theta_{\mathrm{C}}(s_1, s_3, s_4) \frac{1}{|s_{43}||s_{31}||s_{41}|}$$

because of the linear divergence of the integral over s_1 and s_3

Dominating C_* in the sum over paths

Loop C_* dominating in the sum over paths:

$$\begin{aligned} x_* &= -\frac{\mathrm{i}}{K} G * \dot{p}\,, \\ x^\mu(\tau_*(\sigma)) &= \frac{\mathrm{i}}{K} \sum_j \Delta p_j^\mu\, G\left(\sigma - \sigma_j\right) \end{aligned}$$

with arbitrary $\tau_*(\sigma)$: $\tau_*(\sigma_j) = \sigma_j$.

It bounds the minimal surface of the area

$$K S_{\min}(C) = \alpha' \left[\ln \frac{s}{t} + 1\right]$$

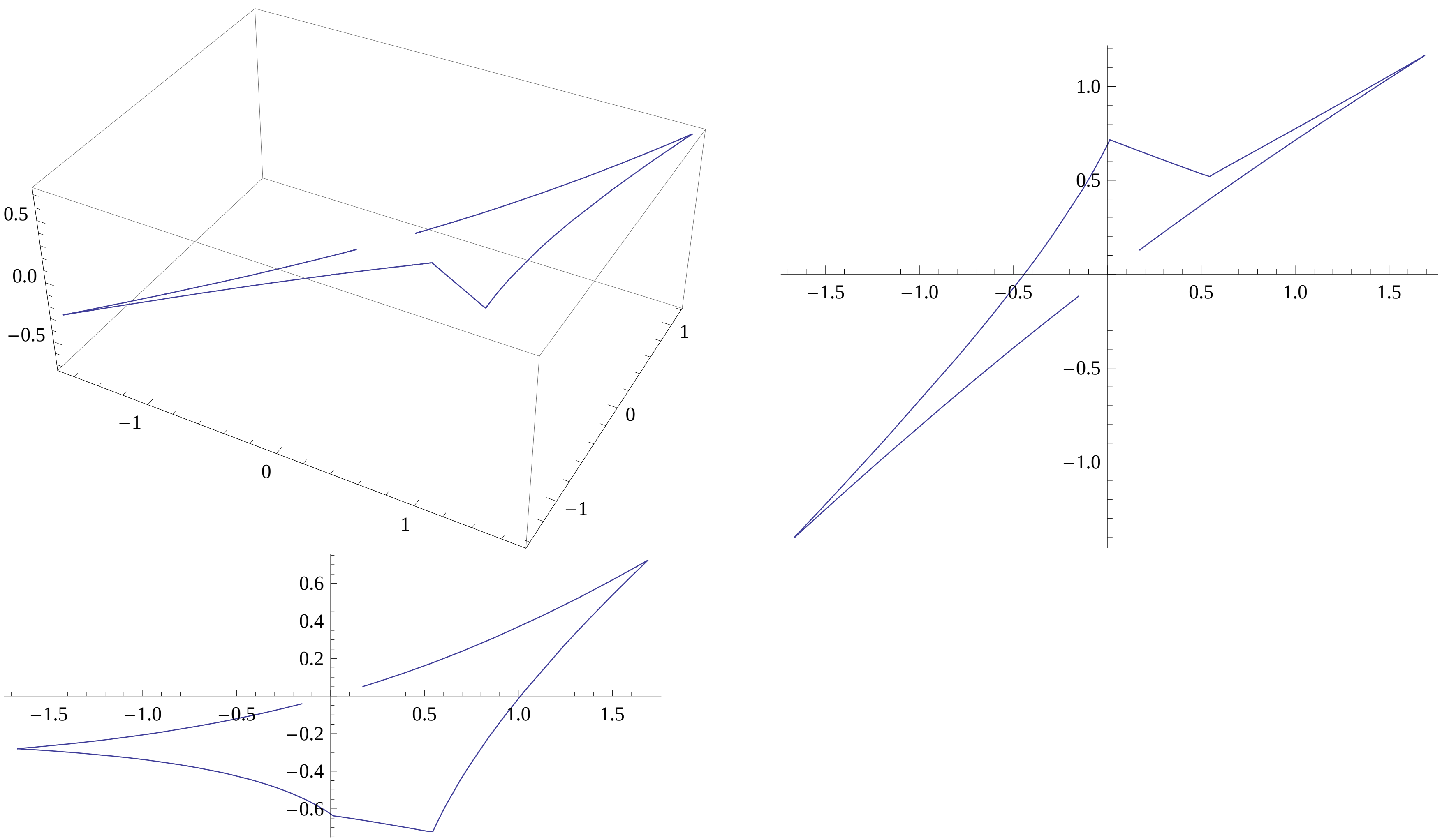
for $s \gg t \gtrsim K$. It is large for very large s.

Actually, the integrand oscillates because of $\mathrm{e}^{\mathrm{i} \int p \cdot \mathrm{d}x}$ that results in the factor of i. But an estimate of the order of magnitude is correct.

Probably t has to be large but $\ll s$ for the width of C_* to be $\gg$ 1fm. Then the value of $\alpha(0)$ (coming from the measure) is not essential.

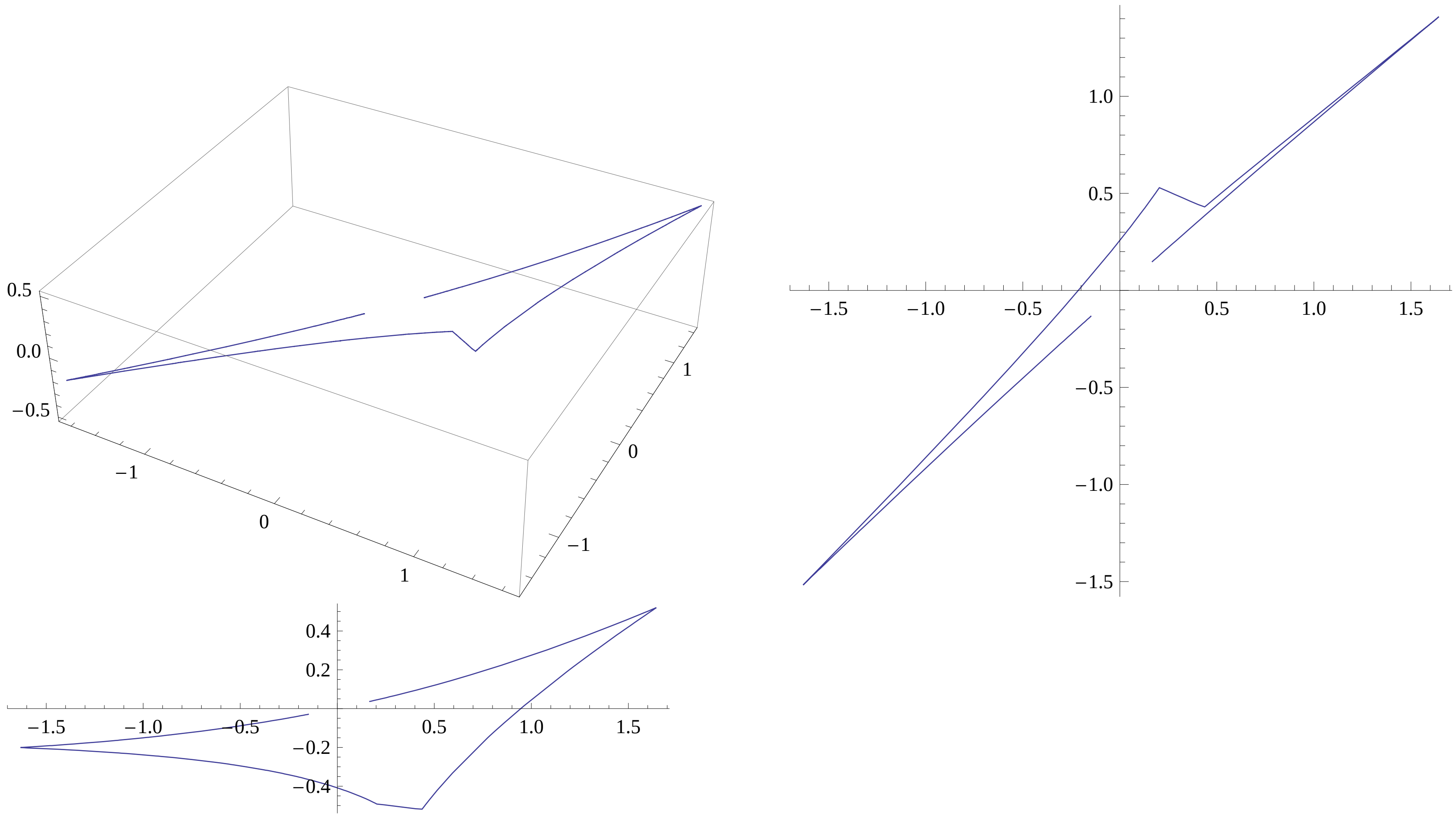
Dominating C_* in the sum over paths (cont.1)

Typical loop C_* dominating in the sum over paths for $t/s = .2$



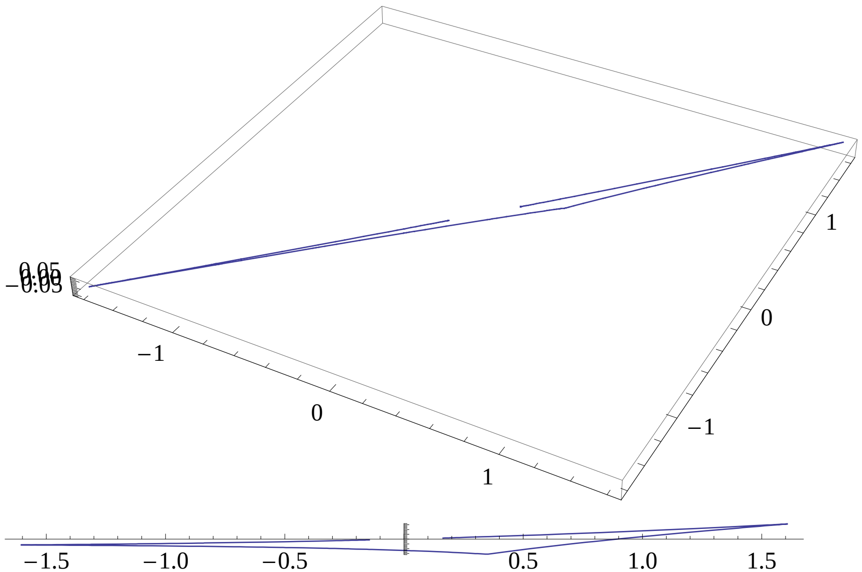
Dominating C_* in the sum over paths (cont.2)

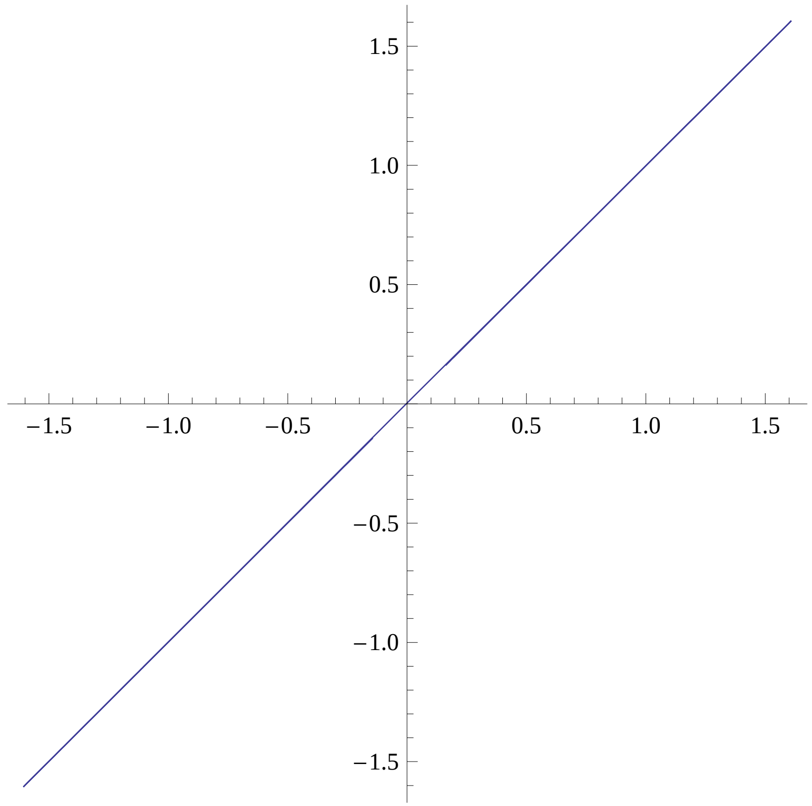
Typical loop C_* dominating in the sum over paths for $t/s = .1$



Dominating C_* in the sum over paths (cont.3)

Typical loop C_* dominating in the sum over paths for $t/s = .01$





Conclusions

- Regge–Veneziano behavior of QCD scattering amplitudes follows from the area law. The only approximation is large N. Great simplification occurs for small m and/or large M (Veneziano-type).

- It was crucial for the success of calculations that all integrals are Gaussian except for the one over reparametrizations which reduces to integration over the Koba–Nielsen variables.

- Derivation is legible for those momenta Δp_i for which asymptotically large loops are essential in the sum over C:

$K S_{\text{min}}(C_*) = \alpha' t \ln \frac{s}{\max\{t, K\}}$ i.e. asymptotically large s and $K \lesssim t \ll s$.

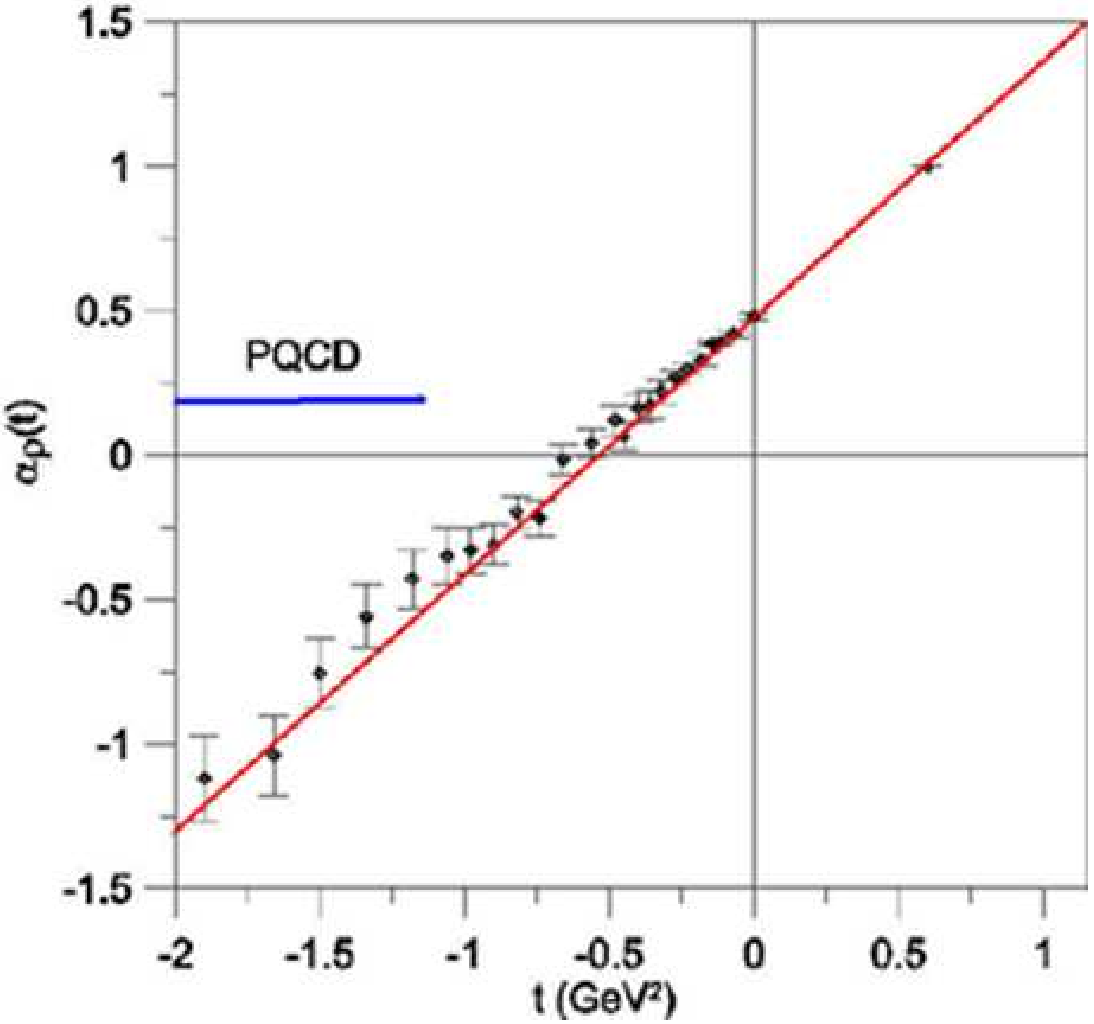
- This region is broader than classical string when $t \gg 1/\alpha'$ but the intercept $\alpha(0)$ of the $q\bar{q}$ Regge trajectory is not yet fixed.

- 4-point scattering amplitude is valid only for asymptotically large s and fixed t associated with small angle or fixed momentum transfer.

- When $-t \ll s$ becomes large, there are no longer reasons to expect the contribution of large loops to dominate over perturbation theory, which comes from integration over small loops.

Effective ρ-trajectory and pQCD prediction

The figure taken from A. B. Kaidalov, hep-ph/0612358



It is hard to believe that pQCD Kirschner, Lipatov (1983) is relevant