Gauge symmetry of local dynamics

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The problem

- Given: General local dynamical system.
  It is defined by not necessarily variational or Hamiltonian system of (differential) equations of motion
- Find: The gauge symmetry of the system.

The motivation:

The methods have recently become available for the BRST embedding and quantising of non-Lagrangian/non-Hamiltonian systems.

Refs: `P.O.Kazinski, S.L.Lyakhovich and A.A.Sharapov, JHEP (2005); S.L.Lyakhovich and A.A.Sharapov, JHEP (2005-2006), PLB (2007)`

Classical BRST differential in non-variational dynamics.

In the general irreducible dynamics the *proper* BRST differential Q

$$[Q,Q]=0 \quad gh(Q)=1, \tag{1}$$

is subject to the boundary conditions

$$Q=E_a(x)\frac{\partial}{\partial\eta_a}+C^\alpha R^i_\alpha(x)\frac{\partial}{\partial x^i}+\eta_a\Gamma^a_A(x)\frac{\partial}{\partial\zeta_A}+\cdots \tag{2}$$

defined by (non-)variational eqs $E_a(x)=0$, generators of gauge identities $\Gamma^a_A(x)$, and generators of gauge symmetry transformations $R^i_\alpha(x)$:

$$\Gamma^a_A E_a(x)\equiv 0, \qquad R^i_\alpha(x)\frac{\partial E_a(x)}{\partial x^i}=U(x)^b_{\alpha a}E_b(x) \tag{3}$$

Γ's can be derived in many ways, but they don't define R's in non-variational case. Hence, the procedure of deriving all the gauge symmetries is a pre-requisite for constructing the classical BRST embedding of the non-Lagrangian dynamics.

Quantization pre-requisites for non-variational dynamics. 1. Deformation quantization:

- The equations of motion are to be of the form:

$$\dot{x}^i = v^i(x) + Z^i_\alpha(x)\lambda^\alpha, T_a(x) = 0; \quad ZT \sim T, [Z,Z] \sim Z+T, [v,Z] \sim Z+T$$

- Weak Poisson bi-vector P should exist such that:

$$[Z,P] \sim Z+T, \quad [v,P] \sim Z+T, \quad [P,T] \sim Z+T, \quad [P,P] \sim Z+T$$

Bi-victor P turns the variety of on-shell gauge invariants into Poisson algebra. The gauge generators and time drift differentiate this algebra, not being Hamiltonian vector fields. Upon BRST embedding this is turned into the Poisson algebra of the cohomology classes. Quantisation results in associative $*$ in the cohomology. Quantum BRST operator and drift are the differentiations of $*$, although not interior.

Hamiltonian constrained system:

$$P^{ij} = \{x^i, x^j\}, \quad [P,P] = 0, \quad v^i = \{x^i, H(x)\}, \quad Z^i_\alpha = \{x^i, T_\alpha\}, \quad \alpha \equiv a$$

Quantization pre-requisites for non-variational dynamics.
2. Covariant quantization:

- The equations $E_a(x)=0$ with known gauge symmetry;
- The*Lagrange anchor* $V_a^i(x)$ should exist such that

$$V_a^i(x)\partial_i E_b(x) - V_b^i(x)\partial_i E_a(x) = C_{ab}^c(x) E_c(x)$$

The BRST embedding converts the on-shell gauge invariants converts into the BRST cohomology classes. The anchor is promoted to the weak anti-bracket, satisfying Jacoby identity only among the BRST invariants and modulo BRST exact terms. The anti-bracket is differentiated by Q, which is not necessarily $(\cdot,S)$. Upon quantisation, the classical BRST differential Q and the anchor together give rise to the quantum BRST operator $\hat{Q}=Q+i\hbar(\cdots), \hat{Q}^2=0$ defining the probability amplitude $\hat{Q}\Psi(x)=0$. Lagrangian system:

$a\equiv i, \quad E_i=\partial_i S(x), \quad V_a^j=\delta_a^j, \quad \hat{Q}=(\cdot,S)+i\hbar\Delta, \quad \Psi=e^{\frac{i}{\hbar}S}$

Primary normal form of local dynamics.

Any system of differential equations can be depressed to the first order in the form of inhomogeneous constrained Pfaffian system:

$$\boldsymbol{\theta}_{Ji}(x)\dot{x}^i = V_J(x), \quad T_a(x) = 0$$

Let the vectors $Z^i_{\boldsymbol{\alpha}}(x)$ span the on-shell kernel of the Pfaff one-forms $Z^i_{\boldsymbol{\alpha}}(x)\boldsymbol{\theta}_{Ji}(x) \sim T(x)$, then the equations can be rewritten in the *primary normal form*:

$$\dot{x}^i = v^i(x) + Z^i_{\boldsymbol{\alpha}}(x)\boldsymbol{\lambda}^{\boldsymbol{\alpha}}, \quad T_a(x) = 0.$$

The vector field v is called a *primary drift*, and the vector distribution $\mathcal{Z} = \mathrm{span}\{Z_{\boldsymbol{\alpha}}\}$ is called a *primary characteristic distribution*. $\mathcal{Z}$ is not necessarily integrable, and it is not necessarily tangential to the primary constraint surface.

Extension of Dirac-Bergmann algorithm: key steps

1.Derivation of compatibility conditions for primary eqs:

- Checking the conservation of primary constraints.

  $$\dot{T}_a(x) \approx v^i(x)\partial_i T_a + \boldsymbol{\lambda}^\alpha Z^i_\alpha(x)\partial_i T_a = 0$$

  Results can be three-fold:
  (i) some primary constraints can conserve identically;
  (ii) determining some of the multipliers $\boldsymbol{\lambda}_\perp$ as functions of x;
  (iii) appearance of the secondary constraints $T^{(2)}$.

- $\dot{T}^{(2)} \approx 0 \Rightarrow$ further secondary ('tetriary') constraints $T^{(3)}$, more fixed multipliers, identical conservation. $\dot{T}^{(3)} \approx 0 \Rightarrow \ldots$

- The iterative procedure ends when the new constraints stop appearing and/or all the multipliers are determined.

2. Complete normal form of the local dynamics.

After excluding determined multipliers and finding all the secondary constraints, the equations take the *complete normal form*:

$$\dot{x}^i = \tilde{v}^i(x) + \boldsymbol{\lambda}_{||}^{\alpha} Z^i_{\alpha||}, \qquad \tilde{T}(x) = 0, \quad \tilde{T} = (T, T^{(2)}, T^{(3)}, \ldots)$$

The primary distribution is decomposed into tangential and transverse sub-distributions w.r.t. the complete constraint surface:

$$\mathcal{Z} = \mathcal{Z}_{\perp} \oplus \mathcal{Z}_{||}, \quad \mathcal{Z}_{||} \tilde{T}(x) \approx 0, \quad \dim \mathcal{Z}_{\perp} = rank Z_{\alpha} \tilde{T}_a$$

The complete constraint set is also decomposed into transverse and tangential subsets w.r.t. to $\mathcal{Z}$:

$$\tilde{T} = (T_{\perp}, T_{||}) \quad Z T_{||} \approx 0, \quad Z_{\perp} T_{\perp} = D, \quad \det D \neq 0$$

Complete drift $\tilde{v}^i = v^i - v^j \partial_j T_{\perp a} (D^{-1})^{ab} Z^i_{b\perp}$ is tangential to the complete constraint surface, $\tilde{v}\tilde{T} \approx 0$. Conservation of the transverse constraints determines all the multipliers corresponding to the transverse sub-distribution: $\boldsymbol{\lambda}^a_{\perp} = -(D^{-1})^{ab} v T_{\perp b}$.

3. Gauge symmetry of the complete normal form.

Given the equations in the complete normal form

$$\dot{x}^i - v^i(x) - \boldsymbol{\lambda}^{\boldsymbol{\alpha}} Z^i_{\boldsymbol{\alpha}}(x) = 0, \quad T_a(x) = 0; \qquad ZT \approx 0, \quad vT \approx 0 \qquad (4)$$

they are fully consistent, having no further consequences.
Let us find all the infinitesimal local gauge transformations for (4):

$$\delta_\epsilon x^i = \sum_{n=0}^{p} R^i_{(p-n)}(x, \boldsymbol{\lambda}, \dot{\boldsymbol{\lambda}}, \ddot{\boldsymbol{\lambda}}, \dots) \overset{(n)}{\varepsilon}, \quad \delta_\varepsilon \boldsymbol{\lambda}^{\boldsymbol{\alpha}} = \sum_{n=0}^{p+1} U^{\boldsymbol{\alpha}}_{(p+1-n)}(x, \boldsymbol{\lambda}, \dot{\boldsymbol{\lambda}}, \ddot{\boldsymbol{\lambda}}, \dots) \overset{(n)}{\varepsilon},$$

such that the equations are left invariant in the sense that their variations vanish on shell with ε being arbitrary function of time.
The first fact we find about the transformations is that the number of the independent parameters coincides to the dimension of $\mathcal{Z}$,
and the choice is always possible $\delta_\epsilon x^i = Z^i_{\boldsymbol{\alpha}} \overset{(p)}{\varepsilon}{}^{\boldsymbol{\alpha}} + \cdots, \quad \delta_\varepsilon \boldsymbol{\lambda}^{\boldsymbol{\alpha}} = \overset{(p+1)}{\varepsilon}{}^{\boldsymbol{\alpha}}$
where $\cdots$ stand for the lower order derivatives of the parameter.

4. Gauge distribution.

The lower order terms can be iteratively found for the gauge transformation, and their general structure is as follows.

- The derivatives of all the orders from the parameters are involved in the transformation without gaps, and with linear independent coefficients;
- The coefficients at the derivatives $\overset{(n)}{\varepsilon}{}^{\alpha}$ in the transformation span and are spanned by the *gauge distribution* $span\{R_{(0)}\} \cup \cdots \cup span\{R_{(p)}\} = \mathcal{Z}_V$
- The gauge distribution is a closure of the primary characteristic distribution

$$\mathcal{Z}_V = \mathcal{Z} \cup [\mathcal{Z}, \mathcal{Z}] \cup [\mathcal{Z}, v] \cup \cdots,$$

where $\cdots$ mean higher iterated commutators $\mathcal{Z}$ and v

5. Local gauge invariants.

The physical observables are the on-shell gauge invariants:

$$\delta_\varepsilon O(x,\lambda,\dot\lambda,\ddot\lambda,\ldots)\approx 0$$

As $\delta_\varepsilon\lambda = \overset{(p+1)}{\varepsilon} + \ldots$, the local physical observables are defined as the phase space on-shell invariants of the gauge distribution:

$$ZO(x)_{|T(x)=0}=0, \quad \forall Z\in\mathcal{Z}_V \quad \Leftrightarrow \quad \delta_\varepsilon O(x)_{|T(x)=0}=0$$

The observables are considered equivalent if their difference vanishes on shell,

$$O_1\sim O_2 \quad \Leftrightarrow \quad (O_1-O_2)_{T(x)=0}=0.$$

The time evolution of the equivalence classes is consistent with the invariance, and only the invariants evolve causally:

$$\dot O = vO+\lambda^\alpha Z_\alpha O, \quad T(x)=0; \qquad \delta_\varepsilon \dot O\approx 0 \quad \Leftrightarrow \quad Z_V O\approx 0$$

6. The involutive normal form of the local dynamics.

The complete normal form is sufficient for classical BRST embedding and covariant quantisation. But it is insufficient for the deformation quantization. Introduce the *involutive normal form*

$$\dot{x}^i = v^i(x) + Z^i_{V\boldsymbol{\alpha}(x)}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}, \qquad T_a(x) = 0$$

where independent $\boldsymbol{\lambda}$'s are included entire gauge distribution $\mathcal{Z}_V$. These equations involve more variables than the complete normal equations, and even for the original variables, they have different gauge symmetry transformations:

$$\delta_\varepsilon x^i = Z_{V\boldsymbol{\alpha}}\varepsilon^{\boldsymbol{\alpha}}$$

These transformations involve more parameters, but without time derivatives. The involutive normal form is equivalent to the complete normal form in the sense that the gauge invariants remain the same, and have the same time evolution.