Pure quantum states of particles with rotating spin

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Abstract

The problem of rotation of spin is solved in full agreement with the basic principles of quantum mechanics. In particular, complete system of wave functions for a massive Dirac neutrino possessing anomalous magnetic moment in dense matter and in strong electromagnetic field is obtained. These functions describe neutrino with rotating spin and are eigenfunctions of kinetic momentum operator. Using these wave functions it is possible to calculate probabilities of various processes with neutrino in the framework of the Furry picture. The dispersion law for the neutrino in dense magnetized matter is found. It is shown that group velocity of neutrino is independent of spin orientation.

Sources

- Lobanov A E 2005 *Phys. Lett.* B **619** 136
- Lobanov A E 2006 *J. Phys. A: Math. Theor.* **39**, 7517
- Lobanov A E, Murchikova E M 2008 *Vestn. MGU. Fiz. Astron.* (No 2) 11
- Arbuzova E V, Lobanov A E, Pavlova O S 2009 *Vestn. MGU. Fiz. Astron.* (No 1) 42
- Arbuzova E V, Lobanov A E, Murchikova E M 2009 *Phys. Atom. Nucl.* **72** 149
- Arbuzova E V, Lobanov A E, Murchikova E M arXiv:0903.3358 [hep-ph]

The principal goal of my talk is to answer the question — is it possible to describe rotation of spin of the particles in full agreement with the basic principles of quantum mechanics. Of course, the answer we get in our works is positive. However, the problem is not so simple.
In practice, the quasi-classical Bargmann–Michel–Telegdi (BMT) equation is used for describing spin precession. Note that for description of ultra-relativistic particles quasi-classical approximation is a very good one. However, the consistent quantum approach is needed, for example, for low energy neutrinos playing an important role in several astrophysical problems.

Let us consider simple case — spin dynamics of neutral particle (neutrino) possessing anomalous magnetic moment in a constant homogeneous magnetic field. The Dirac–Pauli equation in this case is

$$\left(i\gamma^{\mu}\partial_{\mu} - \frac{i}{2}\mu_0 F^{\mu\nu}\sigma_{\mu\nu} - m\right)\Psi(x) = 0, \tag{1}$$

where $F^{\mu\nu}$ is electromagnetic field tensor.

In the studies of the influence of a stationary pure magnetic field on the neutrino spin rotation in the pioneer paper

Fujikawa K and Shrock R E 1980 *Phys. Rev. Lett.* **45** 963

as well as in others papers stationary solutions $\Psi_{p\tilde{\zeta}}(x)$ first found in

Ternov I M, Bagrov V G, Khapaev A M 1965 *Zh. Eksp. Teor. Fiz.* **48** 921

were used as wave functions of a particle. These solutions are the eigenfunctions of the canonical momentum operator

$$p^\mu = i\partial^\mu, \tag{2}$$

with eigenvalues $P^\mu = \{P^0_{\tilde{\zeta}}, \mathbf{P}\}$ and of the spin projection operator

$$\tilde{\mathfrak{S}}_{\rm tp} = \frac{\gamma^5 \gamma_\mu H^{\mu\nu} p_\nu}{\sqrt{p^\beta H_{\beta\alpha} H^{\alpha\rho} p_\rho}}, \tag{3}$$

with eigenvalues $\tilde{\zeta} = \pm 1$. Here $H^{\mu\nu} = -\frac{1}{2} e^{\mu\nu\rho\lambda} F_{\rho\lambda}$ is dual electromagnetic field tensor.

Operator $\tilde{\mathfrak{S}}_{\rm tp}$ has a simple physical meaning. It characterizes — up to the sign — a particle spin projection on the direction of the magnetic field in the rest frame of the particle.
The description of the neutrino spin rotation in the framework of the standard approach to this problem based on solving the Cauchy problem where the initial condition is chosen in a such way that the mean value of neutrino helicity is equal to ± 1. It was taken for granted that the solution of the Cauchy problem can be expressed as a linear combination of the above mentioned wave functions

$$\Psi(x) = \sum_{\tilde{\zeta}=\pm 1} c_{\tilde{\zeta}}(p)\Psi_{p\tilde{\zeta}}(x). \tag{4}$$

However, such an assumption is incorrect. The point is that, once in a pure state the mean value of some spin operator is equal to ± 1, then this state is described by an eigenfunction of this operator. In general case, the construction of the eigenfunction of the spin projection operator as a superposition of only positive-energy solutions of equation (1) is possible only when this spin projection operator commutes with the operator of the sign of the energy. The standard helicity operator $(\boldsymbol{\Sigma}\mathbf{p})/|\mathbf{p}|$ does not feature it.

The given phenomenon is a sort of the famous Klein paradox. To avoid the indicated difficulties, in relativistic quantum mechanics only self-adjoint operators in the subspace of wave functions with a fixed energy sign can be treated as operators of observables. The choice of integrals of motion as operators of observables is the necessary condition to satisfy this requirement.

Landau L D and Peierls R 1931 *Zs. f. Phys.* **69** 56

In the case considered the canonical momentum operator is an integral of motion. However, the conserved operator of the spin projection which should set initial conditions to the Cauchy problem is uniquely — up to the sign — determined by the form of the Dirac–Pauli equation. This operator is $\tilde{\mathfrak{S}}_{\rm tp}$. Therefore, it is impossible to construct a wave function describing a neutrino with rotating spin in the form of an eigenfunction of the canonical momentum operator for its arbitrary eigenvalues.
The solutions

$$\Psi(x) = \sum_{\tilde{\zeta}=\pm 1} c_{\tilde{\zeta}}(p)\Psi_{p\tilde{\zeta}}(x).$$

can exist only when the special values of the canonical momentum are chosen. So, if a particle moves parallel or perpendicular to a constant homogeneous magnetic field, eigenfunctions of the helicity operator are the superpositions of positive-energy solutions alone.

Borisov A V, Ternov A I and Zhukovsky V Ch 1988 *Izv. Vyssh. Uchebn. Zaved. Fiz.* **31**, (No 3) 64

Indeed, the wave functions $\Psi_{p\tilde{\zeta}}(x)$ are the complete system of plane-wave solutions of equation (1). Therefore, any plane-wave solution is a linear combination of this functions.

However, nobody can state that in this linear combination we must choose plane waves with one and the same canonical momentum.

Moreover, group velocities $\mathbf{v}_{\rm gr}^{\tilde{\zeta}=\pm 1}$ of particles with different spin projections are different in general case:

$$\mathbf{v}_{\rm gr}^{\tilde{\zeta}=1} = \frac{\partial P^0_{\tilde{\zeta}=1}}{\partial \mathbf{P}} \neq \frac{\partial P^0_{\tilde{\zeta}=-1}}{\partial \mathbf{P}} = \mathbf{v}_{\rm gr}^{\tilde{\zeta}=-1}. \qquad (5)$$

The particle beam that is described by this linear combination of wave functions is not coherent, if we choose plane waves with the same canonical momentum.

We should emphasize that in the framework of non-relativistic quantum mechanics, where any self-adjoint operator can be treated as operator of observable, we have no problem in describing particle with rotating spin. To solve the problem in relativistic case we should abandon the view that eigenvalues of the canonical momentum operator always impose a direction of the particle propagation. It is necessary to find a self-adjoint operator $\mathfrak{p}^\mu$ with eigenvalues q^μ which obey the condition $q^2 = m^2$. This operator can be interpreted as kinetic momentum operator of the particle. Discuss now how to get this operator.

A space of unitary representation is defined by the condition called "the wave equation for a particle with mass m and spin s". The wave equation for particles with spin $s = 1/2$ is the Dirac equation

$$(i\gamma^{\mu}\partial_{\mu} - m)\,\Psi_0(x) = 0. \tag{6}$$

In this case the realization of generators of the Poincaré group and the Pauli–Lubanski–Bargmann vector in the coordinate representation is

$$\begin{gathered} p^{\mu} = i\partial^{\mu}, \quad m^{\mu\nu} = i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}) + \frac{i}{2}\sigma^{\mu\nu}, \\ w^{\mu} = \frac{i}{2}\,\gamma^{5}(\gamma^{\mu}\gamma^{\nu}\partial_{\nu} - \partial^{\mu}). \end{gathered} \tag{7}$$

These operators commute with the operator of the Dirac equation and can be identified with observables. They have a self-adjoint extension on the subsets of solutions of the Dirac equation with fixed sign of the energy with regard to the standard scalar product

$$(\Phi, \Psi) = \int d\mathbf{x}\, \Phi^{\dagger}(\mathbf{x}, t)\, \Psi(\mathbf{x}, t). \tag{8}$$

The above description of the particle characteristics cannot be directly used in the presence of external fields, where the Dirac equation has the form

$$(i\gamma^{\mu} D_{\mu} - m)\, \Psi(x) = 0, \tag{9}$$

and operators $p^{\mu}, m^{\mu\nu}, w^{\mu}$ are not necessarily integrals of motion.

Since an irreducible representation of group is defined accurately up to an equivalence transformation, it is reasonable to state the problem of finding such realization of the Lie algebra of the Poincaré group for which the condition of irreducibility of the representation leads to wave equation describing a particle in a given external background. To solve this problem it is necessary to find a unitary operator $U(x, x_0)$ which converts solutions of the wave equation for a free particle $\Psi_0(x)$ to solutions $\Psi(x)$ of equation for the particle in external background:

$$U(x, x_0)\Psi_0(x) = \Psi(x). \tag{10}$$

Thus, $U(x, x_0)$ is an intertwining operator in the sense of Darboux. This operator should satisfy the equation

$$(i\gamma^\mu D_\mu)\, U(x, x_0) - U(x, x_0)\, (i\gamma^\mu \partial_\mu) = 0. \tag{11}$$

Therefore, operators

$$\mathfrak{p}^\mu = U(x, x_0) p^\mu U^{-1}(x, x_0), \quad \mathfrak{m}^{\mu\nu} = U(x, x_0) m^{\mu\nu} U^{-1}(x, x_0) \qquad (12)$$

commute with the operator of the wave equation.
As a consequence, the Pauli–Lubanski–Bargmann vector $\mathfrak{W}^\mu$ and the components of the three-dimensional spin projection operator $\mathfrak{S}_i$ can be constructed in the same way as in the case of a free particle:

$$\mathfrak{W}^\mu = -\frac{1}{2} e^{\mu\nu\rho\lambda} \mathfrak{m}_{\nu\rho} \mathfrak{p}_\lambda, \quad \mathfrak{S}_i = -\frac{1}{m} \mathfrak{W}_\mu S_i^\mu(\mathfrak{p}), \qquad (13)$$

The choice of unit vectors $S_i^\mu(\mathfrak{p})$ is not unique, and it is possible to construct operators that determine the spin projection on any direction in an arbitrary Lorentz frame.

The above statement may be reduced to the following: the wave function of particle in an external background can be derived with the help of a solution of the Dirac equation for a free particle and of some unitary evolution operator. A complete set of integrals of motion may be constructed with the help of operators $\mathfrak{p}^\mu$ and $\mathfrak{m}^\mu$. The physical meaning of eigenvalues of observables, i.e. quantum numbers, is clear enough then. However, $U(x, x_0)$ is an integral operator and, so it is very difficult to find explicit form of the evolution operator in general case.

Discuss the model based on the Dirac–Pauli equation

$$\left(i\gamma^{\mu}\partial_{\mu} - \frac{1}{2}\gamma^{\mu}f_{\mu}(1+\gamma^{5}) - \frac{i}{2}\mu_{0}F^{\mu\nu}\sigma_{\mu\nu} - m\right)\Psi(x) = 0, \qquad (14)$$

where f^{μ} and $F^{\mu\nu}$ are constant functions with the restriction

$$F^{\mu\nu}f_{\nu} = 0. \qquad (15)$$

This equation can be used for describing a Dirac neutrino dynamics in dense magnetized matter. In this case f^{μ} is effective four-potential

Wolfenstein L 1978 *Phys. Rev.* D **17**, 2369

which is a linear combination of the currents and of the polarizations of background fermions with the proper choice of coupling constants.

Though the explicit form of a kinetic momentum operator for a particle with spin interacting with dense matter and electromagnetic field is not known beforehand, the correspondence principle allows us to construct solutions characterized by its eigenvalues:

$$\Psi_{q\zeta_0}(x) = \frac{1}{2}\sum_{\zeta=\pm 1} e^{-i(P_\zeta x)}(1-\zeta\gamma^5\gamma_\mu S^\mu_{\rm tp}(q))(1-\zeta_0\gamma^5\gamma_\mu S^\mu_0(q))(\gamma^\mu q_\mu + m)\psi_0, \tag{16}$$

$$P^\mu_\zeta == q^\mu\left(1+\zeta\frac{(f\varphi)}{2\sqrt{(\varphi q)^2 - m^2\varphi^2}}\right)$$
$$+\frac{1}{2}f^\mu\left(1-\frac{\zeta\sqrt{(\varphi q)^2 - m^2\varphi^2}}{(\varphi q)}\right) - \varphi^\mu\frac{\zeta(f\varphi)m^2}{2(\varphi q)\sqrt{(\varphi q)^2 - m^2\varphi^2}}, \tag{17}$$

$$S^\mu_{\rm tp}(q) = \frac{q^\mu(\varphi q)/m - \varphi^\mu m}{\sqrt{(\varphi q)^2 - \varphi^2 m^2}}, \tag{18}$$

where

$$\varphi^\mu = f^\mu/2 + \mu_0 H^{\mu\nu} q_\nu/m.$$

This system represents the complete system of solutions of equation (14) characterized by kinetic momentum of the particle q^μ and the quantum number $\zeta_0 = \pm 1$ which can be interpreted as the neutrino spin projection on the initial direction of polarization $S_0^\mu(q)$.

This system is non-stationary in the general case. The solutions are stationary only when the initial polarization vector $S_0^\mu(q)$ is equal to the vector of the total polarization $S_{\rm tp}^\mu(q)$. In this case the wave functions are eigenfunctions of the spin projection operator $\mathfrak{S}_{\rm tp} = -\gamma^5 \gamma_\mu S_{\rm tp}^\mu(q)$ with eigenvalues $\zeta = \pm 1$, and of the canonical momentum operator $p^\mu = i\partial^\mu$ with eigenvalues P_ζ^μ.

The orthonormal system of the stationary solutions, the basis of solutions of equation (14), can be written in the form

$$\Psi_{q\zeta}(x) = e^{-i(P_\zeta x)}\sqrt{|J_\zeta(q)|}(1-\zeta\gamma^5\gamma_\mu S^\mu_{\mathrm{tp}}(q))(\gamma^\mu q_\mu + m)\psi_0, \qquad (19)$$

where $J_\zeta(q)$ is the transition Jacobian between the variables q^μ and P^μ_ζ:

$$J_\zeta(q) = \left(1+\zeta\frac{(f\varphi)}{2\sqrt{(\varphi q)^2 - m^2\varphi^2}}\right)^2\left(1+\zeta\frac{f_\mu\mu_0 H^{\mu\nu}q_\nu/(2m) - 2I_1}{\sqrt{(\varphi q)^2 - m^2\varphi^2}}\right). \qquad (20)$$

Here $I_1 = \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ is the first invariant of the tensor $F^{\mu\nu}$.

Note that to obtain complete system of solutions for antineutrino it is necessary to change the sign of the kinetic momentum q^{μ}.

Dispersion law for the neutrino in dense magnetized matter is different from the one for the free particle and can be written as

$$\tilde{P}^2 = m^2 - f^2/4 - 2I_1 - 2\zeta\Delta\sqrt{(\tilde{P}\tilde{\Phi})^2 - \tilde{\Phi}^2 m^2}, \qquad (21)$$

where

$$\tilde{P}^\mu = P^\mu_\zeta - f^\mu/2, \quad \tilde{\Phi}^\mu = f^\mu/2 + \mu_0 H^{\mu\nu}\tilde{P}_\nu/m,$$
$$\Delta = \mathrm{sign}\left(1 + \frac{f_\mu \mu_0 H^{\mu\nu}\tilde{P}_\nu/m - 4I_1}{\tilde{P}^2 - m^2 + f^2/4 + 2I_1 - (\tilde{\Phi}f)}\right). \qquad (22)$$

The appearance of the factor Δ in equation (21) is a consequence of the fact that ζ is projection of the particle spin on the direction defined by the kinetic momentum instead of the canonical one.

In spite of the modifications in the dispersion law described above we see that the neutrino moving through dense matter and electromagnetic fields may still behave as a free particle, i.e. its group velocity

$$\mathbf{v}_{\mathrm{gr}} = \frac{\partial P^0_\zeta}{\partial \mathbf{P}_\zeta} = \frac{\mathbf{q}}{q^0} \qquad (23)$$

is the same for both polarization states of the particle. However, in interactions with other particles some channels of reactions which are closed for a free neutrino can be opened due to the modification of the dispersion law.

Let us discuss now properties of non-stationary solutions in more detail. Solution $\Psi_{q\zeta_0}(x)$ is a plane-wave solution of the Dirac–Pauli equation (14), describing a pure quantum-mechanical state of a neutral particle with a non-conserved spin projection on the fixed space axis. Solutions $\Psi_{q\zeta_0}(x)$ do not form an orthogonal basis. However, the considered system is not overcomplete, since the spectrum of the spin projection operator is finite. So the system can be easily orthogonalized. Generalization of the stationary basis (19) is

$$\tilde{\Psi}_{q\zeta_0}(x) = \frac{1}{2}\sum_{\zeta=\pm1} e^{-i(P_\zeta x)}\sqrt{|J_\zeta(q)|} \\ \times(1-\zeta\gamma^5\gamma_\mu S^\mu_{\rm tp}(q))(1-\zeta_0\gamma^5\gamma_\mu S^\mu_0(q))(\gamma^\mu q_\mu + m)\psi_0. \tag{24}$$

Thus we have just established that unitary intertwining operator $U(x, x_0)$ in this case is the Fourier integral operator and it acts on elements of the plain-wave basis of solutions of the free particle Dirac equation in the following way

$$\tilde{\Psi}_{q\zeta_0}(x) = U(x, x_0)\Psi_0(x) = \frac{1}{2}\sum_{\zeta=\pm 1} e^{-i((P_\zeta - q)x)}\sqrt{|J_\zeta(q)|} \times (1 - \zeta\gamma^5\gamma_\mu S^\mu_{\mathrm{tp}}(q))\Psi_0(x). \quad (25)$$

Action of the inverse operator is defined by the formula

$$\Psi_0(x) = U^{-1}(x, x_0)\tilde{\Psi}_{q\zeta_0}(x) = \frac{1}{2}\sum_{\zeta=\pm 1} e^{i((P_\zeta - q)x)}\frac{1}{\sqrt{|J_\zeta(q)|}} \times (1 - \zeta\gamma^5\gamma_\mu S^\mu_{\mathrm{tp}}(q))\tilde{\Psi}_{q\zeta_0}(x). \quad (26)$$

Since the intertwining operator is defined on the elements of the basis, its action on an arbitrary solution is defined as well. Hence, the explicit form of this operator as a function of coordinates and differential operators can be easily obtained.

Let us find the explicit forms of the kinetic momentum operator $\mathfrak{p}^{\mu}$ and the spin projection operator $\mathfrak{S}_{\rm tp}$ in the coordinate representation.
The stationary solutions $\Psi_{q\zeta}(x)$ are classified by eigenvalues of the operators $\mathfrak{p}^{\mu}$ and $\mathfrak{S}_{\rm tp}$, so

$$\mathfrak{p}^{\mu}\Psi_{q\zeta}(x) = q^{\mu}\Psi_{q\zeta}(x), \quad \mathfrak{S}_{\rm tp}\Psi_{q\zeta}(x) = \zeta\Psi_{q\zeta}(x). \tag{27}$$

Since this solutions are also eigenfunctions of the canonical momentum operator $p^{\mu} = i\partial^{\mu}$ with eigenvalues P^{μ}_{ζ}, we have

$$p^{\mu}\Psi_{q\zeta}(x) = P^{\mu}_{\zeta}\Psi_{q\zeta}(x). \tag{28}$$

Now we should express eigenvalues of the kinetic momentum operator q^{μ} in terms of eigenvalues of the canonical momentum operator P^{μ}_{ζ}:

$$q^{\mu} = \tilde{P}^{\mu} + \frac{\tilde{P}^{\mu}(\tilde{\Phi}f) - f^{\mu}(f\tilde{P})/2 - 2m\mu_0 H^{\mu\nu}\tilde{\Phi}_{\nu}}{\tilde{P}^2 - m^2 + f^2/4 + 2I_1 - (\tilde{\Phi}f)}. \tag{29}$$

The vector of total polarization in terms of the new variable is

$$S^{\mu}_{\rm tp}(q) = \Delta \frac{q^{\mu}(\tilde{\Phi}\tilde{P})/m - \tilde{\Phi}^{\mu} m}{\sqrt{(\tilde{\Phi}\tilde{P})^2 - m^2\tilde{\Phi}^2}}. \qquad (30)$$

We can interpret P^{μ}_{ζ} as a result of action of operator $p^{\mu} = i\partial^{\mu}$ on the wave function. So by changing $\tilde{P}^{\mu} \Rightarrow p^{\mu} - f^{\mu}/2$ and $\tilde{\Phi}^{\mu} \Rightarrow f^{\mu}/2 + \mu_0 H^{\mu\nu}(p_{\nu} - f_{\nu}/2)/m$ in formulas (29), (30), we obtain kinetic momentum operator $\mathfrak{p}^{\mu}$ and spin projection operator $\mathfrak{S}_{\rm tp} = -\gamma^5\gamma_{\mu}S^{\mu}_{\rm tp}(q)$ in the explicit form. These operators are pseudodifferential ones and are determined on the solutions of equation (14) with fixed mass m.

To extend the domain of definition of constructed operators, we need to replace mass m in (29) and (30) by the matrix operator from the Dirac–Pauli equation. Unfortunately, the result of this substitution cannot be written as a compact formula, so we do not present it here. However, even if we do not know a covariant form of the operator $\mathfrak{p}^\mu$, we may conclude that on the solutions of equation (14) the relations

$$\mathfrak{p}^2 = m^2, \quad \gamma^\mu \mathfrak{p}_\mu = m, \tag{31}$$

should hold.

Consider now special cases where the presented technique looks quite clear. Discuss the influence on the neutrino dynamics of the electromagnetic field alone, i.e. assume that $f^\mu = 0$. In this case the covariant form of the kinetic momentum operator is

$$\mathfrak{p}^\mu = p^\mu + \gamma^5 \frac{\mu_0 H^{\mu\alpha} H_{\alpha\nu} p^\nu H_{\beta\alpha} p^\alpha \gamma^\beta}{p^\beta H_{\beta\alpha} H^{\alpha\rho} p_\rho}, \tag{32}$$

and the spin projection operator $\mathfrak{S}_{\mathrm{tp}}$ is defined by the formula

$$\mathfrak{S}_{\mathrm{tp}} = \mathrm{sign}\left(1 + \frac{2\mu_0 I_1 \gamma^5 \gamma_\mu H^{\mu\nu} p_\nu}{p^\beta H_{\beta\alpha} H^{\alpha\rho} p_\rho}\right) \tilde{\mathfrak{S}}_{\mathrm{tp}}. \tag{33}$$

Here

$$\tilde{\mathfrak{S}}_{\mathrm{tp}} = \frac{\gamma^5 \gamma_\mu H^{\mu\nu} p_\nu}{\sqrt{p^\beta H_{\beta\alpha} H^{\alpha\rho} p_\rho}}, \tag{34}$$

just the spin projection operator that was used in

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When electromagnetic field is absent, but f^μ is non-trivial, the covariant forms of the kinetic momentum and spin projection operators are

$$\mathfrak{p}^\mu = p^\mu - \frac{f^\mu}{2} - \gamma^5 \frac{p^\mu f^2 - f^\mu (fp)}{2((pf)^2 - p^2 f^2)} \sigma^{\mu\nu} f_\mu p_\nu, \qquad (35)$$

$$\mathfrak{S}_{\rm tp} = \frac{\gamma^5 \sigma^{\mu\nu} f_\mu p_\nu}{\sqrt{(pf)^2 - p^2 f^2}}. \qquad (36)$$

Note that if the matter is at rest and non-polarized ($\mathbf{f} = 0$), then

$$\mathfrak{S}_{\rm tp} = {\rm sign}(f^0) \frac{(\boldsymbol{\Sigma}\mathbf{p})}{|\mathbf{p}|}, \qquad (37)$$

in other words $\mathfrak{S}_{\rm tp}$ is equal to the standard helicity operator up to the sign.

We can find now spin projection operators for non-stationary wave functions $\tilde{\Psi}_{q\zeta_0}(x)$ and $\Psi_{q\zeta_0}(x)$. For this purpose introduce operators $\mathfrak{S}_\pm$ that act on the elements of stationary system of solution as follows

$$\mathfrak{S}_+\Psi_{q\zeta}(x)=\frac{(1-\zeta)}{2}\Psi_{q(-\zeta)}(x),\quad \mathfrak{S}_-\Psi_{q\zeta}(x)=\frac{(1+\zeta)}{2}\Psi_{q(-\zeta)}(x). \quad (38)$$

Then operators $\mathfrak{S}_1=\frac{1}{2}(\mathfrak{S}_++\mathfrak{S}_-)$, $\mathfrak{S}_2=\frac{1}{2i}(\mathfrak{S}_+-\mathfrak{S}_-)$ and $\mathfrak{S}_3=\frac{1}{2}\mathfrak{S}_{\rm tp}$ correspond to elements of the Lie algebra of $SU(2)$ group. Commutation relations for these operators are

$$[\mathfrak{S}_i,\mathfrak{S}_j]=ie_{ijk}\mathfrak{S}_k. \quad (39)$$

To determine the explicit realization of operators $\mathfrak{S}_\pm$ on eigenfunctions of operator $\mathfrak{p}^\mu$ let us choose the basis $S_i^\mu(q)$ in the form $S_{\rm tp}^\mu(q), S_1^\mu(q), S_2^\mu(q)$. Here space-like unit vectors $S_1^\mu(q), S_2^\mu(q)$ are

$$S_1^\mu(q)=\frac{S_0^\mu(q)+S_{\rm tp}^\mu(q)(S_0(q)S_{\rm tp}(q))}{\sqrt{1-(S_0(q)S_{\rm tp}(q))^2}},\quad S_2^\mu(q)=\frac{e^{\mu\nu\rho\lambda}q_\nu S_{0\rho}(q)S_{{\rm tp}\lambda}(q)}{m\sqrt{1-(S_0(q)S_{\rm tp}(q))^2}}. \quad (40)$$

As a result we have

$$\mathfrak{S}_{\pm} = -\frac{1}{2}\frac{\sqrt{|J_{\zeta=\pm 1}(q)|}}{\sqrt{|J_{\zeta=\mp 1}(q)|}} e^{\pm 2i\theta}\gamma^5\gamma_\mu \left(S_1^\mu(q) \pm iS_2^\mu(q)\right), \qquad (41)$$

where

$$\theta = \zeta((q^\mu + f^\mu/2 - P_\zeta^\mu)x)\sqrt{(\varphi q)^2 - \varphi^2 m^2}/m. \qquad (42)$$

Operators $\mathfrak{S}_{\mathrm{tp}}$ and $\mathfrak{S}_{\pm}$ are integrals of motion. So the spin projection operator $\tilde{\mathfrak{S}}_0$ that has eigenfunctions $\tilde{\Psi}_{q\zeta_0}(x)$ and eigenvalues $\zeta_0 = \pm 1$ is a linear combination of these operators:

$$\tilde{\mathfrak{S}}_0 = -(S_0(q)S_{\mathrm{tp}}(q))\mathfrak{S}_{\mathrm{tp}} + \sqrt{1-(S_0(q)S_{\mathrm{tp}}(q))^2}\,[\,\mathfrak{S}_+ + \mathfrak{S}_-]\,. \tag{43}$$

Similarly one can construct the integral of motion $\mathfrak{S}_0$ with eigenfunctions $\Psi_{q\zeta_0}(x)$ and eigenvalues $\zeta_0 = \pm 1$:

$$\mathfrak{S}_0 = -(S_0(q)S_{\rm tp}(q))\mathfrak{S}_{\rm tp} + \sqrt{1-(S_0(q)S_{\rm tp}(q))^2}\left[\frac{\sqrt{|J_{\zeta=-1}(q)|}}{\sqrt{|J_{\zeta=+1}(q)|}}\,\mathfrak{S}_+ + \frac{\sqrt{|J_{\zeta=+1}(q)|}}{\sqrt{|J_{\zeta=-1}(q)|}}\,\mathfrak{S}_-\right]. \quad (44)$$

Note that operator (44) is not self-adjoint operator with respect to the standard scalar product. It seems quite natural, since wave functions $\Psi_{q\zeta_0}(x)$ do not form an orthogonal system. However, the system of wave functions is orthonormalized to the condition "one particle in the unit volume". In this sense wave functions $\Psi_{q\zeta_0}(x)$ minimize the uncertainty relation for spin projection operators:

$$\langle(\mathfrak{S}_1 - \langle\mathfrak{S}_1\rangle)^2\rangle\langle(\mathfrak{S}_2 - \langle\mathfrak{S}_2\rangle)^2\rangle = \frac{1}{4}\langle\mathfrak{S}_3\rangle^2. \tag{45}$$

Therefore, these wave functions describe spin-coherent states of neutrino. The given system of spin-coherent states is parameterized by four-vector S_0^μ.

Spin-coherent states have very simple quasi-classical interpretation. We can introduce effective electromagnetic fields

$$\mathbf{H} \Rightarrow \mathbf{B} = \mathbf{H} + \mathbf{M}, \quad \mathbf{D} \Rightarrow \mathbf{E} = \mathbf{D} - \mathbf{P}, \tag{46}$$

where

$$\mathbf{M} = (f^0\mathbf{q} - q^0\mathbf{f})/(2m), \quad \mathbf{P} = -\big[\mathbf{q} \times \mathbf{f}\big]/(2m). \tag{47}$$

In the rest frame of the particle its spin vector precesses around the direction of effective magnetic field $\mathbf{B}_0$ with the frequency $\omega = 2m\mu_0|\mathbf{B}_0|/q^0$, the angle between $\mathbf{B}_0$ and the vector of spin being $\vartheta = \arccos((\mathbf{B}_0\zeta_0)/|\mathbf{B}_0|)$, where

$$\zeta_0 = \mathbf{S}_0 - \frac{\mathbf{q}S_0^0}{q^0 + m}. \tag{48}$$

Conclusions

Consequently, the problem of neutrino spin rotation in dense matter and in strong electromagnetic field is solved in full agreement with the basic principles of quantum mechanics. Using the wave functions of stationary basis or the wave functions of non-stationary basis it is possible to calculate probabilities of various processes with neutrino in the framework of the Furry picture. When choosing one or another type of the basis, it is necessary to take into account, that due to the time-energy uncertainty, stationary states of the neutrino can be generated only when the linear size of the area occupied by the electromagnetic field and the matter is comparable in the order of magnitude with the formation length of the process.