Integrability of High Energy Scattering Amplitudes in QCD and in $N = 4$ SUSY

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Content

1 Gluon reggeization

Regge kinematics of scattering amplitudes

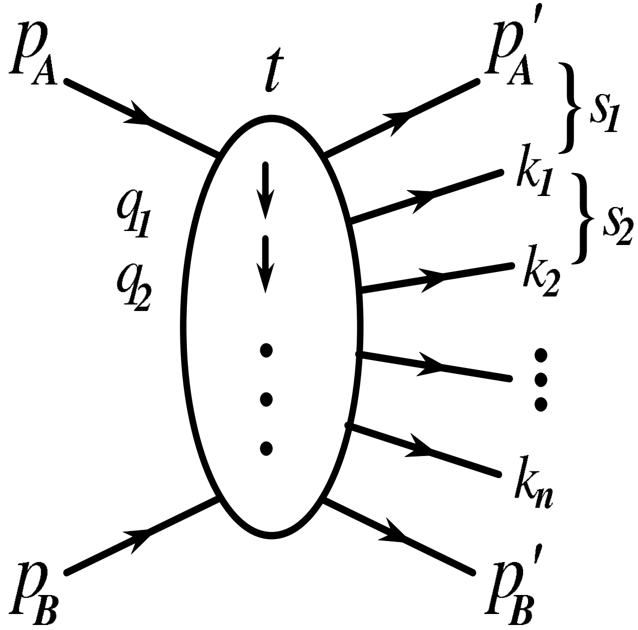
$$s = 4E^2 \gg -t = |q|^2 \approx E^2\,\theta^2$$

Elasic amplitude in LLA of QCD

$$M_{AB}^{A'B'}(s,t)|_{LLA} = 2\,g\,T^c_{A'A}\,\delta_{\lambda_{A'}\lambda_A}\,\frac{s^{1+\omega(t)}}{t}\,g\,T^c_{B'B}\,\delta_{\lambda_{B'}\lambda_B}$$

Gluon Regge trajectory in the leading order

$$\omega(-|q|^2) = -\frac{\alpha_s N_c}{4\pi^2}\int d^2k\,\frac{|q|^2}{|k|^2|q-k|^2} \approx -\frac{\alpha_s N_c}{2\pi}\,\ln\frac{|q^2|}{\lambda^2}$$



Multi-Regge amplitudes (F.,K.,L. (1975))

$$M_{2\to 1+n} \sim \frac{s_1^{\omega_1}}{|q_1|^2}\, gT^{d_1}_{c_2 c_1} C(q_2, q_1)\, \frac{s_2^{\omega_2}}{|q_2|^2} \ldots gT^{d_1}_{c_n c_{n-1}}\, C(q_n, q_{n-1})\, \frac{s_n^{\omega_n}}{|q_n|^2},$$

$$C(q_2, q_1) = \frac{q_2\, q_1^*}{q_2^* - q_1^*}\,,\ \ \sigma_t = \sum_n \int d\Gamma_n\, |M_{2\to 1+n}|^2$$

2 BFKL equation (1975)

Balitsky-Fadin-Kuraev-Lipatov equation

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\;,\;\; \sigma_t \sim s^{\Delta}\;,\;\; \Delta = -\frac{\alpha_s N_c}{2\pi}\,E$$

BFKL Hamiltonian

$$H_{12} = \ln\,|p_1 p_2|^2 + \frac{1}{p_1 p_2^*}(\ln|\rho_{12}|^2)p_1 p_2^*$$

$$+\frac{1}{p_1^* p_2}(\ln|\rho_{12}|^2)p_1^* p_2 - 4\psi(1)\;,\;\; \rho_{12} = \rho_1 - \rho_2$$

Möbius invariance and conformal weights (L. (1986))

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d}\,,$$

$$m = \gamma + n/2\,,\;\; \widetilde{m} = \gamma - n/2\,,\;\; \gamma = 1/2 + i\nu$$

3 BKP equation (1980)

Bartels-Kwiecinski-Praszalowicz equation

$$E\,\Psi(\vec{\rho}_1, ..., \vec{\rho}_n) = H\,\Psi(\vec{\rho}_1, ..., \vec{\rho}_n)\ ,\ \ H = \sum_{k<l} \frac{\vec{T}_k \vec{T}_l}{-N_c}\, H_{kl}$$

Holomorphic separability at large N_c (L. (1988))

$$H = h + h^*\ ,\ \ h = \ln p_1 + \ln p_2 + \frac{1}{p_1}(\ln \rho_{12})\, p_1 + \frac{1}{p_2}(\ln \rho_{12})\, p_2 - 2\psi(1)$$

Holomorphic factorization of wave functions

$$\Psi(\vec{\rho}_1, \vec{\rho}_2, ..., \vec{\rho}_n) = \sum_{r,s} a_{r,s}\, \Psi_r(\rho_1, ..., \rho_n)\, \Psi_s(\rho_1^*, ..., \rho_n^*)$$

4 Integrability at $N_c \to \infty$

Monodromy and transfer matrices (L. (1993))

$$t(u) = L_1 L_2 ... L_n = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \; T(u) = A(u) + D(u)\,,$$

$$[T(u), T(v)] = [T(u), h] = 0\,, \; L_k = \begin{pmatrix} u + \rho_k\, p_k & p_k \\ -\rho_k^2\, p_k & u - \rho_k\, p_k \end{pmatrix}$$

Yang-Baxter equation (L. (1993))

$$t_{r_1'}^{s_1}(u)\, t_{r_2'}^{s_2}(v)\, l_{r_1 r_2}^{r_1' r_2'}(v-u) = l_{s_1' s_2'}^{s_1 s_2}(v-u)\, t_{r_2}^{s_2'}(v)\, t_{r_1}^{s_1'}(u)\,, \; \hat{l} = u\,\hat{1} + i\,\hat{P}$$

Duality symmetry (L. (1999))

$$p_r \longrightarrow \rho_{r+1,r} \longrightarrow p_{r+1}$$

Heisenberg spin model (L. (1994); F., K.(1995))

5 Effective action approach

Gluon and Reggeized gluon fields

$$v_\mu(x) = -iT^a v^a_\mu(x)\,,\;\; A_\pm(x) = -iT^a A^a_\pm(x)$$

Local gauge transformations

$$\delta v_\mu(x) = \frac{1}{g}[D_\mu, \chi(x)],\; \delta\psi(x) = -\chi(x)\,\psi(x),\; \delta A_\pm(x) = 0$$

Effective action for reggeized gluons (L., 1995)

$$S = \int d^4x \left(L_0 + L^{GR}_{ind}\right)\,,\;\; L_0 = i\bar{\psi}\hat{D}\psi + \frac{1}{2}\,Tr\,G^2_{\mu\nu}$$

$$L^{GR}_{ind} = -\frac{1}{g}\partial_+\,P\exp\left(-g\frac{1}{2}\int_{-\infty}^{x^+} v_+(x')d(x')^+\right)\,\partial^2_\sigma A_- + (+\rightarrow -)$$

6 Pomeron in $N = 4$ SUSY

BFKL kernel in two loops (F., L. (1998))

$$\omega = 4\,\hat{a}\;\chi(n,\gamma) + 4\;\hat{a}^2\,\Delta(n,\gamma)\,,\;\; \hat{a} = g^2 N_c/(16\pi^2)\,,$$

Hermitian separability in $N = 4$ SUSY (K.,L. (2000))

$$\Delta(n,\gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M) + \rho(M^*)}{2\hat{a}/\omega}\,,\; M = \gamma + \frac{|n|}{2}\,,$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2)\,,\; \beta'(z) = \frac{1}{4}\left[\Psi'\Big(\frac{z+1}{2}\Big) - \Psi'\Big(\frac{z}{2}\Big)\right]$$

Maximal transcendentality (2002) and integrability (1997)

$$\phi(M) = 3\zeta(3) + \Psi^{''}(M) - 2\Phi(M) + 2\beta^{'}(M)\Big(\Psi(1) - \Psi(M)\Big),$$

$$\Phi(M) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+M}\left(\Psi'(k+1)\; - \;\frac{\Psi(k+1) - \Psi(1)}{k+M}\right)$$

7 Elastic BDS amplitude

Regge asymptotics at $s/t \to \infty$

$$M_{2\to 2} = \Gamma(t) \left(\frac{-s}{\mu^2}\right)^{\omega(t)} \Gamma(t)$$

Reggeized gluon trajectory

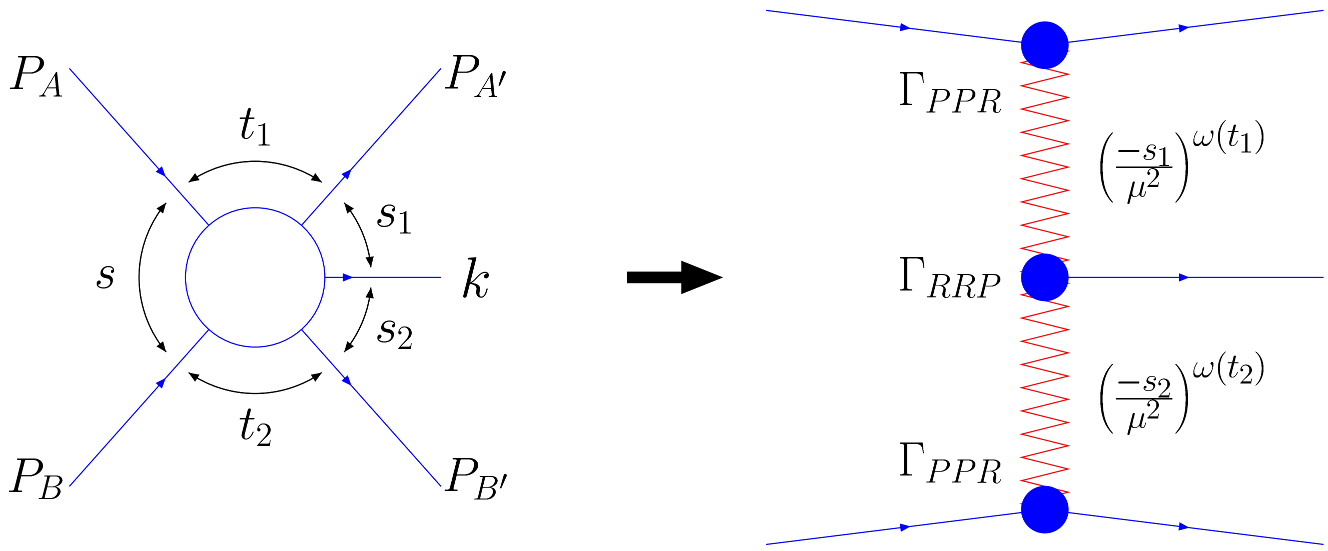
$$\omega(t) = -\frac{\gamma_K(a)}{4} \ln \frac{-t}{\mu^2} + \int_0^a \frac{da'}{a'} \left(\frac{\gamma_K(a')}{4\epsilon} + \beta(a')\right)$$

Reggeon residues

$$\ln \Gamma(t) = \ln \frac{-t}{\mu^2} \int_0^a \frac{da'}{a'} \left(\frac{\gamma_K(a')}{8\epsilon} + \frac{\beta(a')}{2}\right) + \frac{C(a)}{2} + \frac{\gamma_K(a)}{2} \zeta_2$$

$$- \int_0^a \frac{da'}{a'} \ln \frac{a}{a'} \left(\frac{\gamma_K(a')}{4\epsilon^2} + \frac{\beta(a')}{\epsilon} + \delta(a')\right)$$

8 One particle production



$$\ln \Gamma_{\kappa=s_1 s_2/s} = -\frac{1}{2}\left(\omega(t_1)+\omega(t_2)-\int_0^a \frac{da'}{a'}\left(\frac{\gamma_K(a')}{4\epsilon}+\beta(a')\right)\right)\ln\frac{-\kappa}{\mu^2}-$$
$$\frac{\gamma_K(a)}{16}\left(\ln^2\frac{-\kappa}{\mu^2}-\ln^2\frac{-t_1}{-t_2}-\zeta_2\right)-\frac{1}{2}\int_0^a \frac{da'}{a'}\ln\frac{a}{a'}\left(\frac{\gamma_K(a')}{4\epsilon^2}+\frac{\beta(a')}{\epsilon}+\delta(a')\right)$$

9 Steinman relations

Overlapping channels

$$(s_1, s_2)\ (2 \to 3)\,;\ \ (s_1, s_2),\ (s_2, s_3),\ (s_{012}, s_2),\ (s_{123}, s)\ (2 \to 4)$$

Dispersion representation for $M_{2\to3}$ in the Regge anzatz

$$M_{2\to3} = c_1(-s)^{j(t_2)}(-s_1)^{j(t_1)-j(t_2)} + c_2(-s)^{j(t_1)}(-s_2)^{j(t_2)-j(t_1)}$$

Violation of the dispersion representation for $M_{2\to4}$

$$M_{2\to4} \neq d_1(-s)^{j_3}(-s_{012})^{j_2-j_3}(-s_1)^{j_1-j_2} + d_2(-s)^{j_1}(-s_{123})^{j_2-j_1}(-s_3)^{j_3-j_2}$$

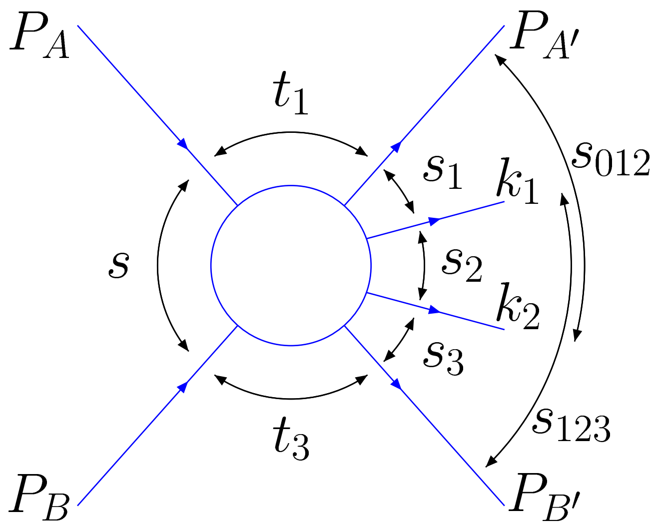
$$+d_3(-s)^{j_3}(-s_{012})^{j_1-j_3}(-s_2)^{j_2-j_1} + d_4(-s)^{j_1}(-s_{123})^{j_3-j_1}(-s_2)^{j_2-j_3}$$

$$+d_5(-s)^{j_2}(-s_1)^{j_1-j_2}(-s_3)^{j_3-j_2}\,,\ \ j_r = j(t_r)$$

Important relations

$$\Phi \equiv \frac{(-s)(-s_2)}{(-s_{012})(-s_{123})}\,,\ \ Li_2(1-\Phi)_{\Phi\to\exp(2\pi i)} = \ln(1-\Phi) \approx \ln\frac{(\vec{k}_1+\vec{k}_2)^2}{s_2}$$

10 Regge factorization violation



$$M_{2\to 4}|_{s_2>0;\, s_1,s_3<0} = \exp\left[\frac{\gamma_K(a)}{4}\, i\pi\, \left(\ln\frac{t_1 t_2}{(\vec{k}_1+\vec{k}_2)^2\mu^2} - \frac{1}{\epsilon}\right)\right]$$

$$\times\Gamma(t_1)\left(\frac{-s_1}{\mu^2}\right)^{\omega(t_1)}\Gamma(t_2,t_1)\left(\frac{-s_2}{\mu^2}\right)^{\omega(t_2)}\Gamma(t_3,t_2)\left(\frac{-s_3}{\mu^2}\right)^{\omega(t_3)}\Gamma(t_3)$$

11 Mandelstam cuts in j_2-plane

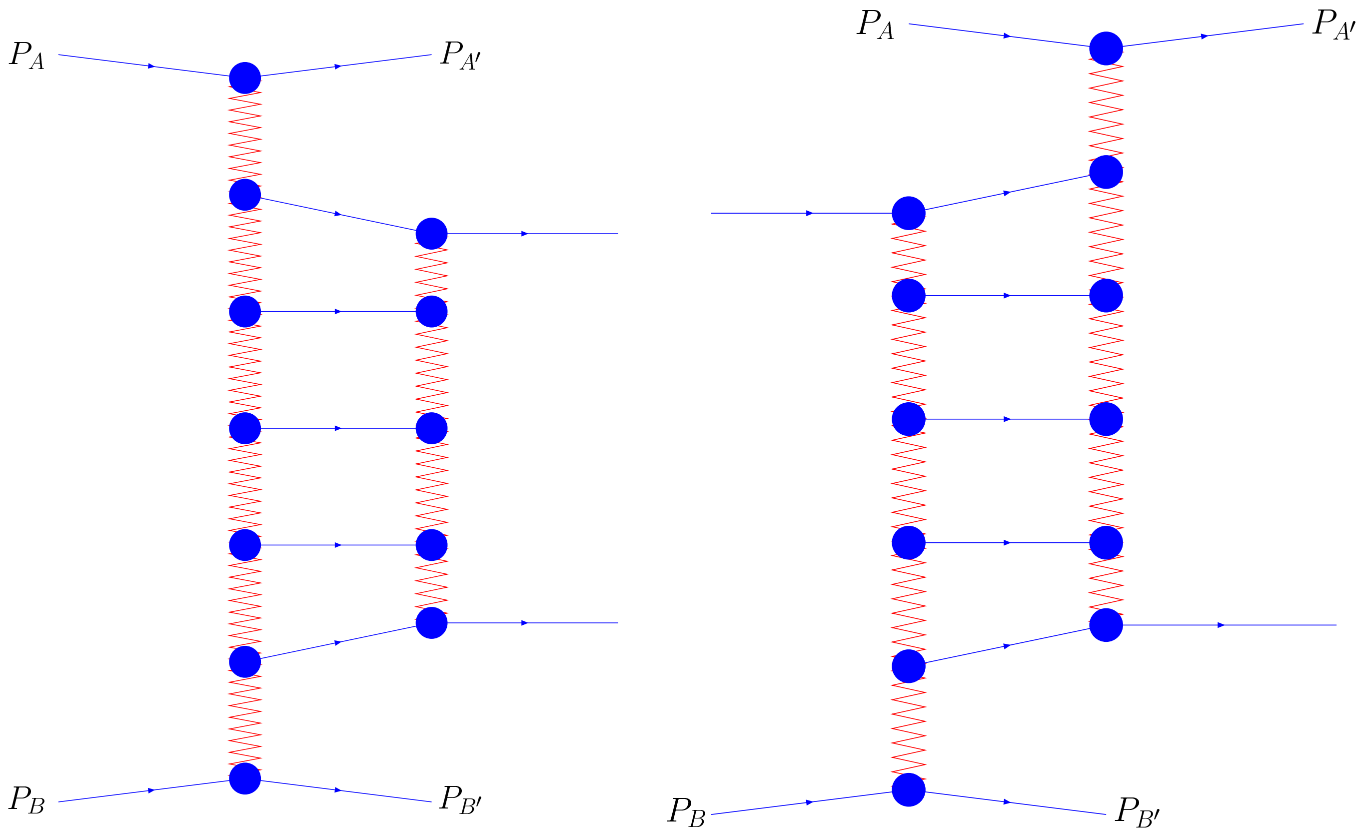


Figure 1: BFKL ladders in $M_{2\to4}$ and $M_{3\to3}$

12 BFKL equation for octet states

Renormalization of the intercept in the s_2-channel

$$\Delta_2 = -a\left(E + \ln\frac{t_2}{\mu^2} - \frac{1}{\epsilon}\right)$$

BFKL hamiltonian for the partial wave f_{j_2}

$$H = \ln\frac{|p_1p_2|^2}{|p_1+p_2|^2} + \frac{1}{2}\,p_1p_2^*\ln|\rho_{12}|^2\frac{1}{p_1p_2^*} + \frac{1}{2}\,p_1^*p_2\ln|\rho_{12}|^2\frac{1}{p_1^*p_2} + 2\gamma$$

Eigenfunctions and eigenvalues

$$\Psi_{n,\nu} = \left(\frac{p_1}{p_2}\right)^{i\nu+n/2}\left(\frac{p_1^*}{p_2^*}\right)^{i\nu-n/2}\,,\;\; E_{n,\nu} = 2Re\,\psi(i\nu+\frac{|n|}{2}) - 2\psi(1)$$

Factorization of infrared divergencies in LLA

$$M_{2\to4}^{LLA} = \left(1 - 2i\pi\,a^2\ln s_2\ln\frac{|k_1+k_2||q_2|}{|k_2||q_1|}\ln\frac{|k_1+k_2||q_2|}{|k_1||q_3|} + ...\right)M_{2\to4}^{BDS}\,,$$

13 Multi-gluon states in octet channels

Holomorphic hamiltonian for n-gluon composite states

$$h = \ln(Z_1^2 \partial_1) - 2\psi(1) + \ln \partial_{n-1} + \sum_{k=1}^{n-2} h_{k,k+1}\,,\; p_k = Z_{k-1,k}\,,\; Z_0 = 0\,,\; Z_n = \infty$$

Pair hamiltonian of the spin chain

$$h_{1,2} = \ln(Z_{12}^2 \partial_1) + \ln(Z_{12}^2 \partial_2) - 2\ln Z_{12} - 2\psi(1)$$

Sklyanin ansatz

$$\Omega = \prod_k Q(\hat{u}_k)\,\Omega_0\,,\;\; \Omega_0 = \prod_{l=1}^{n-1} \frac{1}{|Z_l|^4}$$

Integrals of motion and Baxter equation for the open spin chain

$$[D(u), h] = 0\,,\; D(u)Q(u) = (u+i)^{n-1} Q(u+i)$$

14 Three-gluon composite state

Wave function in the coordinate representation

$$\Psi = Z_2^{a_1+a_2} (Z_2^*)^{\widetilde{a}_1+\widetilde{a}_2} \int \frac{d^2y}{|y|^2} \, y^{-a_2} \, (y^*)^{\widetilde{a}_2} \left(\frac{y-1}{y - Z_2/Z_1} \right)^{a_1} \left(\frac{y^*-1}{y^* - Z_2^*/Z_1^*} \right)^{\widetilde{a}_1}$$

Fourier transformation

$$\Psi(\vec{Z}_1, \vec{Z}_2) = \int d^2p_1 \, d^2p_2 \, \exp(i\vec{p_1}\vec{Z}_1) \, \exp(i\vec{p_2}\vec{Z}_2) \, \Psi(\vec{p_1}, \vec{p_2}) \,, \;\; E = E(a_1) + E(a_2)$$

Baxter-Sklyanin representation

$$\Psi^t(\vec{p_1}, \vec{p_2}) = P^{-a_1-a_2} \, (P^*)^{-\widetilde{a}_1-\widetilde{a}_2} \int d^2u \, u \, \widetilde{u} \, Q(u, \widetilde{u}) \left(\frac{p_1}{p_2} \right)^{u} \left(\frac{p_1^*}{p_2^*} \right)^{u^*}$$

Baxter function

$$Q(u, \widetilde{u}) = \frac{\Gamma(-u) \, \Gamma(-\widetilde{u})}{\Gamma(1+u) \, \Gamma(1+\widetilde{u})} \, \frac{\Gamma(u-a_1) \, \Gamma(u-a_2)}{\Gamma(1-\widetilde{u}+\widetilde{a}_1) \, \Gamma(1-\widetilde{u}+\widetilde{a}_2)} \,, \;\; \int d^2u = \int d\nu \sum_n$$

15 Discussion

1. Multi-Regge amplitudes.
2. Integrability of BFKL dynamics in LLA.
3. Remarkable properties of NLLA in $N = 4$ SUSY.
4. BDS amplitudes in the multi-Regge kinematics.
5. Breakdown of the Regge factorization.
6. Mandelstam cuts in the planar amplitudes M_n for $n > 5$.
7. Integrable open spin chain for scattering amplitudes.