BRST structure for quadratically nonlinear superalgebras

Peter Lavrov

Tomsk State Pedagogical University

Based on M. Asorey, PL, O.Radchenko, A. Sugamoto
ArXiv:0809.3322

Moscow, 18 May 2009

Plan

Introduction

– **Conformal field theories** (A.B. Zamolodchikov, 1986; V.G. Knizhnik, 1986; M. Bershadsky, 1986)

– **Higher spin theories in BRST approach** (I.L. Buchbinder, A. Pashnev and M. Tsulaia, 2001)

– **Quantum groups** (A.P. Isaev and O.V. Ogievetsky, 2001)

– **BRST charge for quadratically nonlinear Lie algebras** (K. Schoutens, A. Servin, P. van Nieuwenhuizen, *Commun. Math. Phys.* 124 (1989) 87).

– **BRST structure of polynomial Poisson algebras** (A. Dresse, M. Henneaux, *J. Math. Phys.* 35 (1994) 1334).

– **BRST charge for generic nonlinear algebras** (I. L. Buchbinder, P. M. Lavrov, *J. Math. Phys.* 48 (2007) 082306-1.

BRST charge for constrained systems

Let us consider a phase space M with local coordinates $\{(q^i, p_i), i = 1, 2, .., n; (\epsilon(q^i) = \epsilon(p_i) = \epsilon_i)\}$ and let $\{T_\alpha = T_\alpha(q, p),\ \epsilon(T_\alpha) = \epsilon_\alpha\}$ be a set of independent functions on M. We suppose that T_α satisfy the involution relations in terms of the Poisson superbracket

$$\{T_\alpha, T_\beta\} = T_\gamma U^\gamma_{\alpha\beta},$$

where the Grassmann parities of $U^\gamma_{\alpha\beta} = U^\gamma_{\alpha\beta}(q, p)$ is $\epsilon(U^\gamma_{\alpha\beta}) = \epsilon_\alpha + \epsilon_\beta + \epsilon_\gamma$, and $U^\gamma_{\alpha\beta}$ possess the symmetry properties

$$U^\gamma_{\alpha\beta} = -(-1)^{\epsilon_\alpha\epsilon_\beta} U^\gamma_{\beta\alpha}.$$

The Jacobi identities read

$$U^\mu_{\alpha\sigma} U^\sigma_{\beta\gamma}(-1)^{\epsilon_\alpha\epsilon_\gamma} + cyclic\ perms.(\alpha, \beta, \gamma) = 0$$

The main object of generalized canonical formalism for dynamical systems with the first class constraints is the BRST charge $\mathcal{Q}$. Construction of the BRST charge involves introducing for each constraint T_α an anticommuting ghost c^α and an anticommuting momenta $\mathcal{P}_\alpha$ having the following distribution of the Grassmann parity $\epsilon(c^\alpha) = \epsilon(\mathcal{P}_\alpha) = \epsilon_\alpha + 1$ and the ghost number $gh(c^\alpha) = -gh(\mathcal{P}_\alpha) = 1$ and obeying the relations

$$\{c^\alpha, \mathcal{P}_\beta\} = \delta^\alpha_\beta, \quad \{c^\alpha, c^\beta\} = 0, \quad \{\mathcal{P}_\alpha, \mathcal{P}_\beta\} = 0,$$
$$\{c^\alpha, T_\beta\} = 0, \quad \{\mathcal{P}_\alpha, T_\beta\} = 0.$$

The BRST charge $\mathcal{Q}$ is defined as a solution to the equation

$$\{\mathcal{Q}, \mathcal{Q}\} = 0$$

being odd function of variables $(p, q, c, \mathcal{P})$ with the ghost number $gh(\mathcal{Q}) = 1$ and satisfying the boundary condition

$$\left.\frac{\partial \mathcal{Q}}{\partial c^\alpha}\right|_{c=0} = T_\alpha .$$

Solution to the problem is looked for in the form of power-series expansions in the ghost variables

$$\begin{aligned}
\mathcal{Q} &= T_\alpha c^\alpha + \sum_{k\geq 1} \mathcal{P}_{\beta_k} \cdots \mathcal{P}_{\beta_2} \mathcal{P}_{\beta_1} U^{(k)\beta_1\beta_2\ldots\beta_k}_{\alpha_1\alpha_2\ldots\alpha_{k+1}} c^{\alpha_{k+1}} \cdots c^{\alpha_2} c^{\alpha_1} = \\
&= \mathcal{Q}_1 + \sum_{k\geq 1} \mathcal{Q}_{k+1} .
\end{aligned}$$

(M. Henneaux, *Phys. Repts.* 128 (1985)1).

The symmetry properties of $U^{(k)}$ in lower indices coincide with the symmetries of monomials $c^{\alpha_{k+1}} c^{\alpha_k} \cdots c^{\alpha_1}$ while in upper indices they are defined by the symmetries of $\mathcal{P}_{\beta_k} \mathcal{P}_{\beta_{k-1}} \cdots \mathcal{P}_{\beta_1}$. In particular

$$\begin{aligned} U^{(k)\beta_1\beta_2\ldots\beta_k}_{\alpha_1\alpha_2\ldots\alpha_{k+1}} &= (-1)^{(\epsilon_{\alpha_1}+1)(\epsilon_{\alpha_2}+1)} U^{(k)\beta_1\beta_2\ldots\beta_k}_{\alpha_2\alpha_1\ldots\alpha_{k+1}} = \\ &= (-1)^{(\epsilon_{\beta_1}+1)(\epsilon_{\beta_2}+1)} U^{(k)\beta_2\beta_1\ldots\beta_k}_{\alpha_1\alpha_2\ldots\alpha_{k+1}}. \end{aligned}$$

In Yang-Mills theories $U^{\gamma}_{\alpha\beta} = \mathit{const}$ and $\mathcal{Q}$ has the form

$$\mathcal{Q} = T_\alpha c^\alpha + \frac{1}{2} \mathcal{P}_\gamma U^{\gamma}_{\alpha\beta} c^\beta c^\alpha (-1)^{\epsilon_\alpha}.$$

Non-linear superalgebras

Quadratically non-linear superalgebras can be defined by the following specification of structure functions $U^{\gamma}_{\alpha\beta}$ considered above

$$U^{\gamma}_{\alpha\beta} = F^{\gamma}_{\alpha\beta} + T_{\delta} V^{\delta\gamma}_{\alpha\beta}.$$

where the Grassmann parities $\epsilon(F^{\gamma}_{\alpha\beta}) = \epsilon_{\alpha} + \epsilon_{\beta} + \epsilon_{\gamma}$, $\epsilon(V^{\gamma\delta}_{\alpha\beta}) = \epsilon_{\alpha} + \epsilon_{\beta} + \epsilon_{\gamma} + \epsilon_{\delta}$ and structure constants $F^{\gamma}_{\alpha\beta}$ and $V^{\gamma\delta}_{\alpha\beta}$ possess the symmetry properties

$$\begin{aligned} F^{\gamma}_{\alpha\beta} &= -(-1)^{\epsilon_{\alpha}\epsilon_{\beta}} F^{\gamma}_{\beta\alpha}, \\ V^{\gamma\delta}_{\alpha\beta} &= -(-1)^{\epsilon_{\alpha}\epsilon_{\beta}} V^{\gamma\delta}_{\beta\alpha} = (-1)^{\epsilon_{\delta}\epsilon_{\gamma}} V^{\delta\gamma}_{\alpha\beta}. \end{aligned}$$

The Jacobi identities

$$F^{\mu}_{\alpha\sigma} F^{\sigma}_{\beta\gamma} (-1)^{\epsilon_\alpha \epsilon_\gamma} + cycle(\alpha, \beta, \gamma) = 0,$$

$$\Big(V^{\mu\nu}_{\alpha\sigma} F^{\sigma}_{\beta\gamma} + F^{\mu}_{\alpha\sigma} V^{\sigma\nu}_{\beta\gamma} (-1)^{\epsilon_\alpha \epsilon_\nu} + F^{\nu}_{\alpha\sigma} V^{\sigma\mu}_{\beta\gamma} (-1)^{\epsilon_\mu (\epsilon_\alpha + \epsilon_\nu)} \Big) (-1)^{\epsilon_\alpha \epsilon_\gamma} +$$
$$+ cycle(\alpha, \beta, \gamma) = 0,$$

$$\Big(V^{\mu\nu}_{\alpha\sigma} V^{\sigma\lambda}_{\beta\gamma} (-1)^{\epsilon_\lambda (\epsilon_\alpha + \epsilon_\mu)} + cycle(\mu, \nu, \lambda) \Big) (-1)^{\epsilon_\alpha \epsilon_\gamma} +$$
$$+ cycle(\alpha, \beta, \gamma) = 0.$$

Explicit form of BRST charge

Let us now apply the BRST construction to nonlinear superalgebras. In lower order, the following equation is required for $\mathcal{Q}$ to be nilpotent

$$T_{\beta_1}\Big((-1)^{\epsilon_{\alpha_1}}[F^{\beta_1}_{\alpha_1\alpha_2} + T_{\beta_2}V^{\beta_2\beta_1}_{\alpha_1\alpha_2}] - 2U^{(1)\beta_1}_{\alpha_1\alpha_2}\Big)c^{\alpha_2}c^{\alpha_1} = 0.$$

Thus we receive the following form of the structure function $U^{(1)}$

$$U^{(1)\gamma}_{\alpha\beta} = \frac{1}{2}\Big(F^{\gamma}_{\alpha\beta} + T_{\delta}V^{\delta\gamma}_{\alpha\beta}\Big)(-1)^{\epsilon_\alpha},$$
$$U^{(1)\gamma}_{\alpha\beta} = U^{(1)\gamma}_{\beta\alpha}(-1)^{(\epsilon_\alpha+1)(\epsilon_\beta+1)}$$

and the contribution of the second order in ghosts c^α for $\mathcal{Q}$

$$\mathcal{Q}_2 = \frac{1}{2}\mathcal{P}_\gamma\Big(F^{\gamma}_{\alpha\beta} + T_{\delta}V^{\delta\gamma}_{\alpha\beta}\Big)c^{\beta}c^{\alpha}(-1)^{\epsilon_\alpha}.$$

In the third order the nilpotency equation has the form

$$(-1)^{\epsilon_{\beta_1}\epsilon_{\beta_2}}\mathcal{P}_{\beta_2}T_{\beta_1}\Big(T_{\beta_3}V^{\beta_3\beta_2}_{\alpha_1\sigma}V^{\sigma\beta_1}_{\alpha_2\alpha_3}(-1)^{\epsilon_{\alpha_2}+\epsilon_{\alpha_1}\epsilon_{\beta_1}}+$$
$$+4U^{(2)\beta_2\beta_1}_{\alpha_1\alpha_2\alpha_3}(-1)^{\epsilon_{\beta_2}}\Big)c^{\alpha_3}c^{\alpha_2}c^{\alpha_1}=0.$$

Let us untroduce the following quantities

$$X^{\beta_3\beta_2\beta_1}_{\alpha_1\alpha_2\alpha_3}=V^{\beta_3\beta_2}_{\alpha_1\sigma}V^{\sigma\beta_1}_{\alpha_2\alpha_3}(-1)^{\epsilon_{\alpha_2}+\epsilon_{\beta_1}\epsilon_{\alpha_1}},$$
$$X^{\beta_3\beta_2\beta_1}_{\alpha_1\alpha_2\alpha_3}=X^{\beta_2\beta_3\beta_1}_{\alpha_1\alpha_2\alpha_3}(-1)^{\epsilon_{\beta_2}\epsilon_{\beta_3}}=X^{\beta_3\beta_2\beta_1}_{\alpha_1\alpha_3\alpha_2}(-1)^{(\epsilon_{\alpha_2}+1)(\epsilon_{\alpha_3}+1)}$$

which define the nilpotency equation in the third order. Symmetrization of this quantity with respect to lower indices can be done as

$$X^{\beta_3\beta_2\beta_1}_{[\alpha_1\alpha_2\alpha_3]} = X^{\beta_3\beta_2\beta_1}_{\alpha_1\alpha_2\alpha_3}+X^{\beta_3\beta_2\beta_1}_{\alpha_3\alpha_1\alpha_2}(-1)^{(\epsilon_{\alpha_3}+1)(\epsilon_{\alpha_1}+\epsilon_{\alpha_2})}+$$
$$+X^{\beta_3\beta_2\beta_1}_{\alpha_2\alpha_3\alpha_1}(-1)^{(\epsilon_{\alpha_1}+1)(\epsilon_{\alpha_2}+\epsilon_{\alpha_3})}.$$

Then we rewrite the nilpotency equation in the third order

$$(-1)^{\epsilon_{\beta_1}\epsilon_{\beta_2}}\mathcal{P}_{\beta_2}T_{\beta_1}\left(T_{\beta_3}X^{\beta_3\beta_2\beta_1}_{[\alpha_1\alpha_2\alpha_3]}+12U^{(2)\beta_2\beta_1}_{\alpha_1\alpha_2\alpha_3}(-1)^{\epsilon_{\beta_2}}\right)c^{\alpha_3}c^{\alpha_2}c^{\alpha_1}=0.$$

The third set of the Jacobi identities in terms of $X^{\beta_3\beta_2\beta_1}_{[\alpha_1\alpha_2\alpha_3]}$ can be written as

$$X^{\beta_3\beta_2\beta_1}_{[\alpha_1\alpha_2\alpha_3]}(-1)^{\epsilon_{\beta_1}\epsilon_{\beta_3}}+cyclic\ perms.(\beta_1,\beta_2,\beta_3)=0.$$

Consider now the quantities $N^{\alpha}_{\alpha_1\alpha_2\alpha_3}$

$$N^{\alpha}_{\alpha_1\alpha_2\alpha_3}=T_{\beta_1}T_{\beta_3}X^{\beta_3\alpha\beta_1}_{[\alpha_1\alpha_2\alpha_3]}(-1)^{\epsilon_\alpha\epsilon_{\beta_1}}.$$

$$T_\alpha N^{\alpha}_{\alpha_1\alpha_2\alpha_3}=0.$$

$$N^{\alpha}_{\alpha_1\alpha_2\alpha_3}=T_\beta N^{\{\alpha\beta\}}_{\alpha_1\alpha_2\alpha_3},\quad N^{\{\alpha\beta\}}_{\alpha_1\alpha_2\alpha_3}=-N^{\{\beta\alpha\}}_{\alpha_1\alpha_2\alpha_3}(-1)^{\epsilon_\alpha\epsilon_\beta}.$$

$$N^{\{\alpha\beta\}}_{\alpha_1\alpha_2\alpha_3}=T_\sigma N^{\{\alpha\beta\}\sigma}_{\alpha_1\alpha_2\alpha_3}.$$

In terms of these quantities the structure functions $U^{(2)}$ read

$$\begin{aligned} U^{(2)\beta_2\beta_1}_{\alpha_1\alpha_2\alpha_3} &= -\frac{1}{12}\,T_\sigma\, N^{\{\beta_2\beta_1\}\sigma}_{\alpha_1\alpha_2\alpha_3}(-1)^{\epsilon_{\beta_2}+\epsilon_{\beta_1}\epsilon_{\beta_2}}, \\ U^{(2)\beta_2\beta_1}_{\alpha_1\alpha_2\alpha_3} &= U^{(2)\beta_1\beta_2}_{\alpha_1\alpha_2\alpha_3}(-1)^{(\epsilon_{\beta_1}+1)(\epsilon_{\beta_2}+1)}. \end{aligned}$$

Equations to define an explicit forms of $N^{\{\alpha\beta\}\sigma}_{\alpha_1\alpha_2\alpha_3}$

$$N^{\{\beta_2\beta_1\}\beta_3}_{\alpha_1\alpha_2\alpha_3} + N^{\{\beta_2\beta_3\}\beta_1}_{\alpha_1\alpha_2\alpha_3}(-1)^{\varepsilon_{\beta_1}\varepsilon_{\beta_3}} = X^{\beta_3\beta_2\beta_1}_{[\alpha_1\alpha_2\alpha_3]} + X^{\beta_1\beta_2\beta_3}_{[\alpha_1\alpha_2\alpha_3]}(-1)^{\epsilon_{\beta_1}\epsilon_{\beta_3}}.$$

$$N^{\{\beta_2\beta_1\}\beta_3}_{\alpha_1\alpha_2\alpha_3} = C\Big(X^{\beta_3\beta_1\beta_2}_{[\alpha_1\alpha_2\alpha_3]} - X^{\beta_3\beta_2\beta_1}_{[\alpha_1\alpha_2\alpha_3]}(-1)^{\epsilon_{\beta_1}\epsilon_{\beta_2}}\Big).$$

$$\Big(3C+1\Big)X^{\beta_3\beta_1\beta_2}_{[\alpha_1\alpha_2\alpha_3]} = 0$$

There are two solutions. The first one

$$C = -\frac{1}{3}.$$

$$N^{\{\beta_2\beta_1\}\beta_3}_{\alpha_1\alpha_2\alpha_3} = -\frac{1}{3}\Big(X^{\beta_3\beta_1\beta_2}_{[\alpha_1\alpha_2\alpha_3]} - X^{\beta_3\beta_2\beta_1}_{[\alpha_1\alpha_2\alpha_3]}(-1)^{\epsilon_{\beta_1}\epsilon_{\beta_2}}\Big).$$

$$U^{(2)\beta_2\beta_1}_{\alpha_1\alpha_2\alpha_3} = -\frac{1}{36}\,T_{\beta_3}\Big(X^{\beta_3\beta_2\beta_1}_{[\alpha_1\alpha_2\alpha_3]} - X^{\beta_3\beta_1\beta_2}_{[\alpha_1\alpha_2\alpha_3]}(-1)^{\epsilon_{\beta_1}\epsilon_{\beta_2}}\Big)(-1)^{\epsilon_{\beta_2}}$$

$$\mathcal{Q}_3 = -\frac{1}{6}\mathcal{P}_1\mathcal{P}_2\,T_{\beta_3}\,V^{\beta_3\beta_2}_{\alpha_1\sigma}\,V^{\sigma\beta_1}_{\alpha_2\alpha_3}(-1)^{\epsilon_{\alpha_2}+\epsilon_{\beta_2}+\epsilon_{\alpha_1}\epsilon_{\beta_1}}\,c^{\alpha_3}\,c^{\alpha_2}\,c^{\alpha_1}.$$

The second possibility corresponds to restriction on structure constants of nonlinear superalgebras

$$X^{\beta_3\beta_2\beta_1}_{[\alpha_1\alpha_2\alpha_3]} = 0$$

or

$$V^{\beta_3\beta_2}_{\alpha_1\sigma} V^{\sigma\beta_1}_{\alpha_2\alpha_3} (-1)^{\epsilon_{\alpha_1}(\epsilon_{\alpha_3}+\epsilon_{\beta_1})} + cycle(\alpha_1, \alpha_2, \alpha_3) = 0.$$

It means that $N^{\{\beta_2\beta_1\}\beta_3}_{\alpha_1\alpha_2\alpha_3} = 0$ and

$$U^{(2)\beta_1\beta_2}_{\alpha_1\alpha_2\alpha_3} = 0, \quad \mathcal{Q}_3 = 0.$$

The condition of nilpotency in the forth order of ghost fields c^{α} has the form

$$(-1)^{\epsilon_{\beta_3}}\mathcal{P}_{\beta_3}\mathcal{P}_{\beta_2}T_{\beta_1}\Big(Y^{\beta_1\beta_2\beta_3}_{[\alpha_1\alpha_2\alpha_3\alpha_4]}+T_{\beta_4}X^{\beta_4\beta_1\beta_2\beta_3}_{[\alpha_1\alpha_2\alpha_3\alpha_4]}+$$
$$+144U^{(3)\beta_1\beta_2\beta_3}_{\alpha_1\alpha_2\alpha_3\alpha_4}\Big)c^{\alpha_4}c^{\alpha_3}c^{\alpha_2}c^{\alpha_1}=0,$$

$$\begin{aligned}
Y^{\beta_1\beta_2\beta_3}_{\alpha_1\alpha_2\alpha_3\alpha_4} &= F^{\beta_1}_{\gamma\sigma}V^{\sigma\beta_2}_{\alpha_1\alpha_2}V^{\gamma\beta_3}_{\alpha_3\alpha_4}(-1)^{p_{\alpha_1\alpha_2\alpha_3\gamma\beta_2\beta_3}},\\
X^{\beta_4\beta_1\beta_2\beta_3}_{\alpha_1\alpha_2\alpha_3\alpha_4} &= V^{\beta_4\beta_1}_{\gamma\sigma}V^{\sigma\beta_2}_{\alpha_1\alpha_2}V^{\gamma\beta_3}_{\alpha_3\alpha_4}(-1)^{p_{\alpha_1\alpha_2\alpha_3\gamma\beta_2\beta_3}},\\
p_{\alpha_1\alpha_2\alpha_3\gamma\beta_2\beta_3} &= \epsilon_{\alpha_1}+\epsilon_{\alpha_3}+\epsilon_{\beta_2}+(\epsilon_{\alpha_1}+\epsilon_{\alpha_2})\epsilon_{\beta_3}+\epsilon_{\gamma}(\epsilon_{\alpha_1}+\epsilon_{\alpha_2}+\epsilon_{\beta_2})
\end{aligned}$$

Symmetrization in four below indices means

$$\begin{aligned}
Y_{[\alpha_1\alpha_2\alpha_3\alpha_4]} &= Y_{\alpha_1[\alpha_2\alpha_3\alpha_4]}+Y_{\alpha_4[\alpha_1\alpha_2\alpha_3]}(-1)^{(\epsilon_{\alpha_4}+1)(\epsilon_{\alpha_1}+\epsilon_{\alpha_2}+\epsilon_{\alpha_3}+1)}+\\
&+ Y_{\alpha_3[\alpha_4\alpha_1\alpha_2]}(-1)^{(\epsilon_{\alpha_1}+\epsilon_{\alpha_2})(\epsilon_{\alpha_3}+\epsilon_{\alpha_4})}+\\
&+ Y_{\alpha_2[\alpha_3\alpha_4\alpha_1]}(-1)^{(\epsilon_{\alpha_1}+1)(\epsilon_{\alpha_2}+\epsilon_{\alpha_3}+\epsilon_{\alpha_4}+1)}.
\end{aligned}$$

One can prove that

$$X^{\beta_4\beta_1\beta_2\beta_3}_{[\alpha_1\alpha_2\alpha_3\alpha_4]} = 0$$

and

$$Y^{\beta_1\beta_2\beta_3}_{[\alpha_1\alpha_2\alpha_3\alpha_4]} = Y^{\beta_2\beta_1\beta_3}_{[\alpha_1\alpha_2\alpha_3\alpha_4]}(-1)^{(\epsilon_{\beta_1}+1)(\epsilon_{\beta_2}+1)}.$$

$$U^{(3)\beta_1\beta_2\beta_3}_{\alpha_1\alpha_2\alpha_3\alpha_4} = -\frac{1}{144}Y^{\beta_1\beta_2\beta_3}_{[\alpha_1\alpha_2\alpha_3\alpha_4]}$$

and for contribution to BRST charge in the forth order

$$\mathcal{Q}_4 = -\frac{1}{24}\mathcal{P}_{\beta_3}\mathcal{P}_{\beta_2}\mathcal{P}_{\beta_1}F^{\beta_1}_{\gamma\sigma}V^{\sigma\beta_2}_{\alpha_1\alpha_2}V^{\gamma\beta_3}_{\alpha_3\alpha_4}(-1)^{p_{\alpha_1\alpha_2\alpha_3\gamma\beta_2\beta_3}}c^{\alpha_4}c^{\alpha_3}c^{\alpha_2}c^{\alpha_1}.$$

If we additionally propose the fulfilment of restrictions on structure constants of superalgebra

$$Y^{\beta_1\beta_2\beta_3}_{[\alpha_1\alpha_2\alpha_3\alpha_4]} = 0$$

then we can state that there exists the unique form of nilpotent BRST charge

$$\mathcal{Q} = T_\alpha c^\alpha + \frac{1}{2}\mathcal{P}_\gamma\Big(F^\gamma_{\alpha\beta} + T_\delta V^{\delta\gamma}_{\alpha\beta}\Big)c^\beta c^\alpha(-1)^{\epsilon_\alpha}.$$

Examples

The simple case of superalgebras really involving fermionic functions is a superalgebra with three generators T, G_1, G_2 where T is a bosonic ($\epsilon(T) = 0$) and G_1, G_2 are fermionic ($\epsilon(G_1) = \epsilon(G_2) = 1$) ones. In particular, it means that $G_1^2 = G_2^2 = 0$. The most general relations for the Poisson superbracket of generators preserving the Grassmann parities have the form

$$\begin{aligned}
\{T, G_1\} &= a_1(T)G_1 + a_2(T)G_2, \\
\{T, G_2\} &= b_1(T)G_1 + b_2(T)G_2, \\
\{G_1, G_1\} &= \alpha_1(T) + \alpha_2(T)G_1G_2, \\
\{G_2, G_2\} &= \beta_1(T) + \beta_2(T)G_1G_2, \\
\{G_1, G_2\} &= \gamma_1(T) + \gamma_2(T)G_1G_2.
\end{aligned}$$

Here $a_i, b_i, \alpha_i, \beta_i, \gamma_i, i = 1, 2$ are polynomial functions of T.

The Jacobi identities for this algebra require the fulfilment of equations

$$\begin{aligned}
&\alpha_1^{'} a_1 + \alpha_2\gamma_1 = 0, \quad \beta_1^{'} b_1 + \beta_1\beta_2 = 0,\\
&\alpha_1^{'} a_2 - \alpha_1\alpha_2 = 0, \quad \beta_1^{'} b_2 + \beta_2\gamma_1 = 0,\\
&a_1\gamma_1 + a_2\beta_1 + b_1\alpha_1 + b_2\gamma_1 = 0,\\
&b_2^{'} a_1 - a_1^{'} + a_2^{'} b_1 + a_2 b_1^{'} - b_1\alpha_2 + a_2\beta_2 = 0,\\
&2\gamma_1^{'} a_1 + \alpha_1^{'} b_1 + 2\gamma_1\gamma_2 + \alpha_2\beta_1 = 0,\\
&2\gamma_1^{'} a_2 + \alpha_1^{'} b_2 - 2\gamma_2\alpha_1 - \alpha_2\gamma_1 = 0,\\
&\beta_1^{'} a_1 + 2\gamma_1^{'} b_1 + \beta_2\gamma_1 + 2\gamma_2\beta_1 = 0,\\
&\beta_1^{'} a_2 - 2\gamma_1^{'} b_2 - \alpha_1\beta_2 - 2\gamma_1\gamma_2 = 0,
\end{aligned}$$

where $f^{'}$ denotes the derivative of $f = f(T)$ with respest to T. We have the nine first order differential and one algebraic nonlinear equations in ten unknowns $a_i, b_i, \alpha_i, \beta_i, \gamma_i, i = 1, 2$.

We will not study the general solution to this system and will just list below some special cases. We have the following examples:

$$1. \quad \{T, G_1\} = 0, \quad \{T, G_2\} = 0, \quad \{G_1, G_1\} = \alpha(T),$$
$$\{G_2, G_2\} = \beta(T), \quad \{G_1, G_2\} = \gamma(T).$$

For quadratically nonlinear superalgebras $((T, G_1, G_2) = (T_1, T_2, T_3), \epsilon_1 = 0, \epsilon_2 = \epsilon_3 = 1)$

$$\begin{aligned}
&\alpha = A_1 T + A_2 T^2, \ \beta = B_1 T + B_2 T^2, \\
&\gamma = D_1 T + D_2 T^2, \\
&F^1_{22} = A_1, \quad F^1_{33} = B_1, \quad F^1_{23} = D_1, \\
&V^{11}_{22} = A_2, \quad V^{11}_{33} = B_2, \quad V^{11}_{23} = \frac{1}{2} D_2.
\end{aligned}$$

In what follows we will identify the ghost variables $(c^1, c^2, c^3) = (c, \eta_1, \eta_2), (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3) = (\mathcal{P}, P_1, P_2)$.

Explicit form of structure constants allows us to conclude that indices $\beta_1, \beta_2, \beta_3, \sigma$ for non-trivial relations in restrictions should be $\beta_1 = \beta_2 = \beta_3 = \sigma = 1$.

$$V^{11}_{\alpha_1 1} V^{11}_{\alpha_2 \alpha_3} (-1)^{\epsilon_{\alpha_1} \epsilon_{\alpha_3}} + cycle(\alpha_1, \alpha_2, \alpha_3) = 0(?)$$

where (?) denotes the statement which should be checked. These relations are satisfied because of $V^{11}_{\alpha 1} = 0$. Non-trivial relations in the second class of restrictions may occur when $\gamma = \sigma = 1$ and all terms in these relations contain $F^{\beta}_{11} = 0$. Therefore the nilpotent BRST charge for this example has the form

$$\begin{aligned} \mathcal{Q} \;\; = \;\; & Tc + G_1 \eta_1 + G_2 \eta_2 + \frac{1}{2} A_1 \mathcal{P} \eta_1^2 + \frac{1}{2} B_1 \mathcal{P} \eta_2^2 + D_1 \mathcal{P} \eta_1 \eta_2 + \\ & + \frac{1}{2} A_2 \mathcal{P} T \eta_1^2 + \frac{1}{2} B_2 \mathcal{P} T \eta_2^2 + \frac{1}{2} D_2 \mathcal{P} T \eta_1 \eta_2 . \end{aligned}$$

In this example there are no restrictions on parameters $(A_1, B_1, D_1, A_2, B_2, D_2)$ which define superalgebras.

2. $\{T, G_1\} = a(T)G_1, \quad \{T, G_2\} = a(T)G_2, \quad \{G_1, G_1\} = 0,$
$\{G_2, G_2\} = \beta(T)G_1 G_2, \quad \{G_1, G_2\} = \gamma(T)G_1 G_2.$

$$
\begin{aligned}
&a = A_0 + A_1 T, \quad \beta = B_0, \quad \gamma = D_0, \\
&F^2_{12} = A_0, \quad F^3_{13} = A_0, \\
&V^{12}_{12} = \frac{1}{2}A_1, \quad V^{13}_{13} = \frac{1}{2}A_1, \quad V^{23}_{33} = \frac{1}{2}B_0, \quad V^{23}_{23} = \frac{1}{2}D_0.
\end{aligned}
$$

When $A_0 = B_0 = 0$ the algebra belongs to *self − reproducing algebras* (Dresse, Henneaux, *J. Math. Phys.* 35 (1994) 1334).

Analysis of the relations gives us the following restrictions on structure constants of superalgebras in this case

$$V^{12}_{12} = V^{13}_{13} = V^{23}_{23} = 0, \quad (A_1 = D_0 = 0).$$

Due to the facts $V^{1\alpha}_{\beta\gamma} = 0$ and $F^{1}_{\alpha\beta} = 0$ for all values of α, β, γ, the relations are satisfied. The nilpotent BRST charge can be written in the form

$$\mathcal{Q} = Tc + G_1\eta_1 + G_2\eta_2 + A_0(P_1\eta_1 + P_2\eta_2)c + \frac{1}{2}B_0(P_2 G_1 - P_1 G_2)\eta_2^2.$$

3. $\{T, G_1\} = a(T)G_2, \quad \{T, G_2\} = b(T)G_1, \quad \{G_2, G_2\} = 0,$
$\{G_1, G_1\} = \alpha(T)G_1G_2, \quad \{G_1, G_2\} = \gamma(T)G_1G_2.$

$$a = A_0 + A_1T, \quad b = B_0 + B_1T, \quad \alpha = C_0, \quad \gamma = D_0,$$
$$F^3_{12} = A_0, \quad F^2_{13} = B_0,$$
$$V^{13}_{12} = \frac{1}{2}A_1, \quad V^{12}_{13} = \frac{1}{2}B_1, \quad V^{23}_{22} = \frac{1}{2}C_0, \quad V^{23}_{23} = \frac{1}{2}D_0.$$

Analyzing the relations we obtain the following restrictions for the superalgebra to a linear one

$$V^{12}_{13} = V^{13}_{12} = V^{23}_{22} = V^{23}_{23} = 0, \quad (A_1 = B_1 = C_0 = D_0 = 0)$$

with the usual nilpotent BRST charge for linear superalgebras

$$\mathcal{Q} = Tc + G_1\eta_1 + G_2\eta_2 + A_0P_2\eta_1c + B_0P_1\eta_2c.$$

4. $\{T, G_1\} = a(T)G_1, \quad \{T, G_2\} = 0, \quad \{G_1, G_1\} = \alpha(T)G_1G_2,$
$\{G_2, G_2\} = \beta(T)G_1G_2, \quad \{G_1, G_2\} = \gamma(T)G_1G_2.$

$$a = A_0 + A_1T, \quad \alpha = C_0, \quad \beta = B_0, \quad \gamma = D_0,$$
$$F_{12}^2 = A_0,$$
$$V_{12}^{12} = \frac{1}{2}A_1, \quad V_{22}^{23} = \frac{1}{2}C_0, \quad V_{33}^{23} = \frac{1}{2}B_0, \quad V_{23}^{23} = \frac{1}{2}D_0.$$

As in previous case, analysis of the relations restricts the superalgebra to a linear one

$$V_{12}^{12} = V_{22}^{23} = V_{33}^{23} = V_{23}^{23} = 0, \quad (A_1 = B_0 = C_0 = D_0 = 0).$$

The nilpotent BRST charge has the form

$$\mathcal{Q} = Tc + G_1\eta_1 + G_2\eta_2 + A_0P_1\eta_1c.$$

5. $\{T, G_1\} = 0, \quad \{T, G_2\} = b_1(T)G_1 + b_2(T)G_2, \quad \{G_1, G_1\} = 0,$
$\{G_2, G_2\} = \beta(T)G_1G_2, \quad \{G_1, G_2\} = \gamma(T)G_1G_2.$

$$b_1 = B_0 + B_1T, \quad b_2 = B_2 + B_3T, \quad \beta = B_4, \quad \gamma(T) = D_0,$$
$$F^2_{13} = B_0, \quad F^3_{13} = B_2,$$
$$V^{23}_{33} = \frac{1}{2}B_4, \quad V^{12}_{13} = \frac{1}{2}B_1, \quad V^{13}_{13} = \frac{1}{2}B_3, \quad V^{23}_{23} = \frac{1}{2}D_0.$$

Analysis of the relations gives us the following restrictions on structure constants of superalgebras

$$V^{13}_{13} = V^{23}_{23} = V^{23}_{33} = 0, \quad (B_3 = D_0 = B_4 = 0).$$

Due to the facts $V^{3\alpha}_{\beta\gamma} = 0$ for all values of α, β, γ and $F^2_{13} \neq 0, F^3_{13} \neq 0$, the relations are satisfied. The nilpotent BRST charge can be written in the form

$$\mathcal{Q} = Tc + G_1\eta_1 + G_2\eta_2 + \Big(B_0P_1 + B_2P_2 + \frac{1}{2}B_1(\mathcal{P}G_1 + P_1T)\Big)\eta_2 c.$$

Discussion

We have analyzed possible restrictions on structure constants of nonlinear superalgebras which can be dictated by the nilpotency equation for the BRST charge. These restrictions may occur beginning from the third order in Faddeev-Popov ghost variables. In this order the restrictions

$$X^{\beta_1\beta_2\beta_3}_{[\alpha_1\alpha_2\alpha_3]} = 0$$

lead to $\mathcal{Q}_3 = 0$.

In bosonic case these restrictions have the form

$$V^{\beta_1\beta_2}_{\sigma[\alpha_1} V^{\sigma\beta_3}_{\alpha_2\alpha_3]} = 0.$$

In the paper written by Schoutens, Servin, van Nieuwenhuizen, (*Commun.Math.Phys.*124(1989)87) the restrictions

$$V^{\beta_1\beta_2}_{\sigma\alpha_1} V^{\sigma\beta_3}_{\alpha_2\alpha_3} = 0.$$

were used.

In the fourth order the restrictions

$$Y^{\beta_1\beta_2\beta_3}_{[\alpha_1\alpha_2\alpha_3\alpha_4]} = 0$$

which in bosonic case have the form

$$F^{\beta_1}_{\gamma\sigma} V^{\sigma\beta_2}_{[\alpha_1\alpha_2} V^{\gamma\beta_3}_{\alpha_3\alpha_4]} = 0$$

allow to state the existence of BRST charge in the unique form

$$\mathcal{Q} = T_\alpha c^\alpha + \frac{1}{2}\mathcal{P}_\gamma \Big(F^\gamma_{\alpha\beta} + T_\delta V^{\delta\gamma}_{\alpha\beta} \Big) c^\beta c^\alpha (-1)^{\epsilon_\alpha}.$$

Thank you!