Local BRST cohomology and the inverse problem of variational calculus for sigma models of AKSZ type

Maxim Grigoriev

Based on:

M.G. and Glenn Barnich, arXiv:0905.0547

(Local) BRST cohomology:

Batalin-Vilkovisky formalism:

Given equations T_a, gauge symmetries R^i_α, reducibility relations,.... the BRST differential:

$$s = \delta + \gamma + \dots\,, \qquad s^2 = 0\,, \quad \mathrm{gh}(s) = 1$$
$$\delta = T_a \frac{\partial}{\partial \mathcal{P}_a} + \dots\,, \qquad \gamma = c^\alpha R^i_\alpha \frac{\partial}{\partial \phi_i} + \dots\,.$$

ϕ^i – fields, c^α – ghosts, $\mathcal{P}_a$ – ghost momenta/antifields, $\dots$

δ – (Koszule-Tate) restriction to the stationary surface

γ – implements gauge invariance condition

H^0 – observables (gauge invariant functions on the stationary surface)

Other cohomology groups (including those in the the space of tensor fields) encode physically interesting quantities – anomalies, consistent deformations, etc.

In the context of local gauge field theory:

Physically interesting – cohomology groups in local functionals

Local functions – functions in x, z^α, $\partial_\mu z^\alpha$, $\partial_\mu \partial_\nu z^\alpha \ldots$

Total derivative:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} + z^\alpha_{;\mu} \frac{\partial}{\partial z^\alpha} + z^\alpha_{;\mu\nu} \frac{\partial}{\partial z^\alpha_{;\nu}} + \ldots$$

BRST differential is an evolutionary vector field:

$$[\partial_\mu, s] = 0\,, \qquad s z^\alpha = s^A[z]$$

Local functionals:

Quotient space: $\qquad f[z] \sim f[z] + \partial_\mu j^\mu[z]$

More invariant way: $H^n(d = dx^\mu \partial_\mu,$ local forms$)$

$$s\omega^n + d\omega^{n-1} = 0\,, \qquad \omega^n_k \sim \omega^n_k + d\chi^{n-1}_k + s\chi^n_{k-1}$$

In the local field theory – local BRST cohomology encode physically interesting quantities.

Lagrangian Batalin–Vilkovisky formalism

In addition to s there is a natural odd symplectic structure:

$$s = \{S,\,\cdot\,\}\,, \qquad \frac{1}{2}\,\{S,S\} = 0\,, \qquad S = S_0[\phi] + c^\alpha R^i_\alpha \mathcal{P}_i + \ldots$$

BV master action and the BV master equation.

$\{\,\cdot\,,\,\cdot\,\}$ – Lie superalgebra structure on local functionals.

In the Lagrangian case:

$$\begin{aligned}
H^0(s) &- \text{1st order consistent deformations}\,,\\
H^1(s) &- \text{anomalies}\,,\\
H^{-1}(s) &- \text{conserved currents (inequivalent global symmetries)}\,,\\
\ldots &- \quad \ldots
\end{aligned}$$

In the non-Lagrangian case (i.e. if only s is given)

The cohomology of the adjoined action $s_E = [s, \cdot]$ in the space of evolutionary vector fields

$$
\begin{aligned}
H^1(s_E) &- \text{1st order consistent deformations}\,,\\
H^0(s) &- \text{inequivalent global symmetries}\,,\\
\ldots &- \quad \ldots
\end{aligned}
$$

In the same way one can extend s_E to functional multivectors.
The respective local BRST cohomology groups are also relevant.

Alexandrov-Kontsevich-Schwartz-Zaboronsky (AKSZ) type sigma models

Instead of constructing s from the initial data (equations, gauge generators, reducibility relations, $\ldots$) in some interesting cases it can be extremely useful to define theory in terms of the BRST differential.

For instance, if $\widehat{\Omega}$ is a BRST operator of a constrained system then

$$S = \frac{1}{2}\langle\Psi, \widehat{\Omega}\Psi\rangle \qquad \text{- BV master action}$$

Here $\Psi = \ldots + \Psi_{-1} + \Psi_0 + \Psi_1 + \ldots$.
Ψ_0 – physical fields, Ψ_{-1}-ghosts, Ψ_1-antifields, $\ldots$

Well known example – open SFT:

$$S = \frac{1}{2}\langle\Psi, \widehat{\Omega}\Psi\rangle + \frac{1}{6}\langle\Psi, \Psi * \Psi\rangle$$

$*$ – *Witten* star-product

Another example: higher spin field Lagrangians can be represented in the form $\langle \Psi, \widehat{\Omega}\Psi \rangle$ for some first-quantized "higher spin particle model" *Ouvry, Stern (1986) , Bengtsson (1986)*

AKSZ sigma model:

Consider two Q-manifolds: *A.Schwartz*

Target space $\mathcal{M}$, degree $\mathrm{gh}_{\mathcal{M}}$, nilpotent vector field Q

$$Q^2 = 0\,, \qquad \mathrm{gh}_{\mathcal{M}}(Q) = 1$$

Space-time $\mathcal{X}$, degree $\mathrm{gh}_{\mathcal{X}}$, $\boldsymbol{d}$, $\mathrm{gh}_{\mathcal{X}}(\boldsymbol{d}) = 1$, $\boldsymbol{d}^2 = 0$,
$\boldsymbol{d}$-invariant volume form $d\mu$

Typical example: $\mathcal{X} = \Pi T\mathcal{X}_0$, coordinates x^μ, θ^μ, $n = \dim \mathcal{X}_0$

$$\boldsymbol{d} = \theta^\mu \frac{\partial}{\partial x^\mu}\,, \qquad d\mu = dx^0 \dots dx^{n-1} d\theta^{n-1} \dots d\theta^0 \equiv d^n x d^n \theta$$

Supermanifold of maps ($\mathcal{M}$-valued fields on $\mathcal{X}$): BRST differential:

$$s = \int_{\mathcal{X}} d^n x d^n \theta \left[\boldsymbol{d}\Psi^A(x,\theta) + Q^A(\Psi(x,\theta))\right] \frac{\delta}{\delta \Psi^A(x,\theta)}$$

total ghost degree: $\qquad \mathrm{gh}(A) = \mathrm{gh}_{\mathcal{M}}(A) + \mathrm{gh}_{\mathcal{X}}(A)$

Because $s^2 = 0$, $\mathrm{gh}(s) = 1 \quad \Longrightarrow$
Nonlagrangian local gauge theory

If in addition (odd) symplectic structure is defined on $\mathcal{M}$ such that

$$Q = \{S, \cdot\}_{\mathcal{M}} \,, \qquad \frac{1}{2}\{S, S\}_{\mathcal{M}} = 0$$

$\{\cdot\,,\,\cdot\}_{\mathcal{M}}$ – respective (odd) Poisson bracket on $\mathcal{M}$.

Then

$$s = \{\mathbf{S}, \cdot\} \,, \qquad \frac{1}{2}\{\mathbf{S}, \mathbf{S}\} = 0\,,$$

with

$$\mathbf{S}[\Psi] = \int d^n x d^n \theta \left[\big(\boldsymbol{d}\Psi^A(x,\theta)\big) V_A(\Psi(x,\theta)) + S\big(\Psi(x,\theta)\big) \right]$$

$$\{F, G\} = \pm \int d^n x d^n \theta \, \frac{\delta^R F}{\delta \Psi^A(x,\theta)} \left\{\Psi^A, \Psi^B\right\}_{\mathcal{M}} (\Psi(x,\theta)) \frac{\delta G}{\delta \Psi^B(x,\theta)}$$

Note that $\{,\}_{\mathcal{M}}$ and $\{,\}$ have different parities for odd n.

For $\mathbf{S}$ even and $\mathrm{gh}(\mathbf{S}) = 0$ – Lagrangian AKSZ sigma model

Comments:

- If $\mathrm{gh}(\Psi^A) \geqslant 0$ then equations of motion for physical fields encoded in s defined the Free Differential Algebra
Sullivan (1977), *d'Auria, P. Fre (1982)*

- A closely related approach is the nonlinear unfolded formalism developed in the context of higher spin gauge theories by M.Vasiliev. The nonlinear theory of higher spins is naturally formulated in terms of this approach *Vasiliev (1990)*.

- Non-Lagrangan AKSZ approach can be seen as a BRST extension of the nonlinear unfolded formalism *Barnich, M.G. (2005)*

Examples:

Chern-Simons theory

$\mathcal{M} = \Pi\mathcal{G}$ with coordinates c^a, $\mathrm{gh}(c^a) = 1$, $\mathcal{G}$-Lie algebra

$$Q = \frac{1}{2}\,c^a c^b U^c_{ab}\frac{\partial}{\partial c^c} \qquad \text{Lie algebra differential}$$

Fields $\Psi^a = c^a + \theta^\mu A^a_\mu + \ldots$, equations of motion and gauge symmetries

$$dA + \frac{1}{2}[A, A] = 0\,. \qquad \delta A_\lambda = d\lambda$$

If in addition $\mathcal{G}$ is equipped with invariant metric g_{ab} and $\dim \mathcal{X}_0 = 3$ then

$$\left\{c^a, c^b\right\}_{\mathcal{M}} = g^{ab}\,, \qquad Q = \left\{\frac{1}{6}\,U_{abc}c^a c^b c^c,\ \cdot\ \right\}$$

The action and the BV action

$$S_0 = \int_{\mathcal{X}} \frac{1}{2}\,A\boldsymbol{d}A + \frac{1}{6}\,\langle A, [A, A]\rangle\,, \quad \mathbf{S} = \int_{\mathcal{X}} \frac{1}{2}\,\Psi\boldsymbol{d}\Psi + \frac{1}{6}\,\langle \Psi, [\Psi, \Psi]\rangle$$

Note: S_0 and $\mathbf{S}$ have the same structure

Hamiltonian BFV systems with vanishing Hamiltonian

$\mathcal{M}$ – extended phase space of the Hamiltonian BFV formulation

Ω – BRST charge , $\{\cdot\,,\,\cdot\}$ – Extended Poisson bracket

So that

$Q = \{\Omega,\,\cdot\,\}$, $\mathrm{gh}(\cdot)$ –usual BFV ghost degree

The associated BV formulation

Fisch, Henneaux (1989), Batalin, Fradkin (1988), Siegel (1989)

can be represented as AKSZ sigma model *M.G., Damgaard (1999)*

$$\mathbf{S} = \int dt d\theta \Big[\big(\boldsymbol{d}\Psi^A(t,\theta)\big) V_A(\Psi(t,\theta)) + \Omega\big(\Psi(t,\theta)\big) \Big] ,$$

$\boldsymbol{d} = \theta \frac{\partial}{\partial t}$; BV antibracket $\big(\cdot,\cdot\big) = \int dt d\theta \, \{\,\cdot\,,\,\cdot\,\}_{\mathcal{M}}$

A general AKSZ sigma model appears as a multi-dimensional generalization of Hamiltonian description for reparametrization invariant systems

The isomorphism

For a general AKSZ model:

$$\mathcal{I} : C^\infty(\mathcal{M}) \;\rightarrow\; \text{local functionals}$$

$$f(\Psi) \mapsto F[\Psi] = \int d^n x d^n \theta f(\Psi(x,\theta))$$

One has

$$s\mathcal{I} = \mathcal{I}Q \qquad \text{– map of complexes}$$

Moreover, *M.G., Damgaard (1999)*

$$\mathcal{I}(\{f,g\}_{\mathcal{M}}) = \{\mathcal{I}(f),\mathcal{I}(g)\} \qquad \text{– Lie algebra homomorphism}$$

Main statement:

Proposition 0.1. *Locally in $\mathcal{X}$ and $\mathcal{M}$ map $\mathcal{I}$ is an isomorphism in cohomology (local BRST cohomology of AKSZ sigma model is isomorphic to the target space Q-cohomology).*

Comments:

– for the particular case of Chern-Simons the statement is known from *Delduc, Blasi, Lucchesi, Piguet, Sorella, (1990)*

– that Q-cohomology determines physically relevant invariants like actions and conserved charges was stressed in *Vasiliev, (2005)*

– in the case of the 1-dimensional AKSZ sigma models associated with Hamiltonian BFV systems with vanishing Hamiltonian, proposition states that the Poisson algebra of Hamiltonian BRST cohomology and the antibracket algebra of Lagrangian BV cohomology in the space of local functionals are locally isomorphic. This was originally derived in *Barnich, Henneaux (1996)*

Idea of the proof:

First compute cohomology for s_0 defined by $s_0\Psi = \boldsymbol{d}\Psi$. All variables are contractible pairs except for 0-forms Ψ^A *Piguet, Sorella (1992) and Henneaux, Knaepen (1998)*

Then compute local BRST cohomology of s_0 using the descent equation.

Finally, obtain local s-cohomology by a suitable spectral sequence.

Cohomology for functional multivectors

The usual way to describe (functional) multivectors is to introduce momenta π_α for each filed z^α

graded symmetric functional k-vector $\longrightarrow$ local functional of homogeneity k in π and $\pi_{;\mu\dots}$

For instance, for the BRST differential

$$\Omega_0 = -\int d^n x \, s^\alpha[z]\pi_\alpha, \qquad \mathrm{gh}(\Omega_0) = 1\,, \quad s z^\alpha = s^\alpha[z]$$

A map from local functionals to evolutionary vector fields:

$$\{F[z,\pi],\,\cdot\,\}_E$$

gives a functional Poisson bracket for local functionals.

Defines BRST differential for functional multivectors:

$$s_E = \{\Omega_0,\,\cdot\,\}\,, \qquad \frac{1}{2}\{\Omega_0,\Omega_0\}_E = 0$$

Functionals in z,π with s_E – extended complex

For AKSZ model the extended complex is again of AKSZ type

Indeed, introduce target space variables Π_A with

$$\mathrm{gh}(\Pi_A) = -\mathrm{gh}(\Psi^A) + n\,, \,, \qquad \{\Pi_B, \Psi^A\}_{\mathcal{M}_E} = -\delta^A_B$$

Then

$$\Omega_0 = -\int d^n x\, s^\alpha[z]\pi_\alpha = -\int d^n x d^n\theta\, s\Psi^A\Pi_A = \\ -\int d^n x d^n\theta\left[d\Psi^A\Pi_A + Q^A(\Psi)\Pi_A\right].$$

Again AKSZ type sigma model with

$$\mathcal{M}_E = (\Pi)T^*\mathcal{M}\,, \qquad Q_E = \{Q^A\Pi_A, \,\cdot\,\}_{\mathcal{M}_E}$$

Because Q_E defines the Q-cohomology in the target-space multivectors

Proposition 0.2. *Local BRST cohomology in the space of functional multivectors is isomorphic to Q-cohomology in the space of target-space multivectors*

Example:

If $\{\,\cdot\,,\,\cdot\,\}_{\mathcal{M}}$ is an (odd) bracket in $\mathcal{M}$, $\omega = \omega^{AB}\Pi_A\Pi_B$ associated bivector then

$$\mathcal{I}_E\omega = \int d^nx d^n\theta\omega$$

determines familiar functional bracket:

$$\{F,G\} = \pm\int d^nx d^n\theta \left(\frac{\delta^R F}{\delta\Psi^A(x,\theta)}\,\omega^{AB}(\Psi(x,\theta))\,\frac{\delta G}{\delta\Psi^B(x,\theta)}\right).$$

Corolary: nontrivial s_E-invariant brackets originates from the target space brackets

Applying this to the 1-dimensional AKSZ model associated to the Hamiltonian constrained system one gets the "Isomorphisms between the Batalin-Vilkovisky antibracket and the Poisson bracket" *Barnich, Henneaux (1996)*

Inverse problem of the calculus of variation

General setting:

Standard: given equations of motion $T_i[\phi]$ the problem is whether they derive from a Lagrangian i.e. $T_i = \dfrac{\delta L}{\delta \phi^i}$ for some $L[\phi]$.

In this form it is too restrictive as one can e.g. allow for integrating multipliers i.e. $T_i \to T'_i = \lambda_i^j[\phi]T_j$.

Usually, one is also allowed to add/eliminate auxiliary fields. Moreover, add/eliminate pure gauge variables. For instance: spin-s Lagrangians *Fierz, Pauli (1939), Singh, Hagen (1974), Fronsdal (1978)*

General point of view: being Lagrangian or not is a property of equivalence classes of equations of motion under addition/elimination of generalized auxiliary fields

Barnich, M.G., Semikhatov, Tipunin (2004)

Generalized auxiliary fields at the level of equations of motion :

$$\phi^A = (\phi^i, w^a, v^a) \qquad sw^a|_{w=0} = 0 \Leftrightarrow v^a = V^a[\phi^i]$$

Lagrangian version – auxiliary fields for BV master action

Henneaux (1990)

A natural framework to study existence of a Lagrangian

– Lagrange structure *Kazinski, Lyakhovich, Sharapov (2005)*

This can be seen as a Lagrangian counterpart of a possibly weak and degenerate Poisson structure of the Hamiltonian formalism.

In the context of local field theory: Lagrange structure can be defined as a deformation

$$\Omega = \Omega_0 + \Omega_1 + \Omega_2 + \dots\,, \qquad \Omega_0 = -\int d^n x s^\alpha \pi_\alpha\,, \quad \mathrm{gh}(\Omega) = 1$$

(Ω_k – local functional homogeneous of degree $k+1$ in π_α) satisfying the compatibility condition

$$\frac{1}{2}\left\{\Omega, \Omega\right\}_E = 0 \Longleftrightarrow \quad \left\{\begin{array}{c} s_E\,\Omega_1 = 0\,, \\ \frac{1}{2}\left\{\Omega_1, \Omega_1\right\}_E + s_E\,\Omega_2 = 0\,, \\ \left\{\Omega_1, \Omega_2\right\}_E + s_E\,\Omega_3 = 0\,, \\ \vdots \end{array}\right. \qquad (1)$$

Two such deformations Ω and Ω' are considered equivalent if there exists a local functional $\Xi = \sum_{k\geqslant 1} \Xi_k$ such that $\Omega' = \exp\left\{\Omega_0,\,\cdot\,\right\}_E \Xi$, where Ξ_k is homogeneous of degree $k+1$ in π_α.

If defined in this way the Lagrange structure is invariant under elimination of the generalized auxiliary fields

Indeed: the deformation theory is controled by the local BRST cohomology that are invariant *Barnich, M.G., Semikhatov, Tipunin (2004)*

Conclusion: local BRST cohomology in the space of functional bivectors and higher multivectors control the inverse variational problem for gauge theories.

For AKSZ sigma model

Usual deformation theory arguments imply that any Lagrange structure is equivalent to

$$\Omega = \Omega_0 + \mathcal{I}_E(\omega) = \Omega_0 + \int_{\mathcal{X}} d^n x d^n \theta \, (\omega_1 + \omega_2 + \ldots) \, .$$

Can be studied in the target space. Substantial simplification.

In other words – the brackets can be assumed not to contain space-time derivatives

Example: Lagrange structure for Chern-Simons theory

Extended model: variables c^a, $\mathrm{gh}(c^a)=1$ and π_a, $\mathrm{gh}(\pi_a)=2$,

$$\{\pi_a, c^b\}_{\mathcal{M}_E} = -\delta^b_a\,, \qquad Q_E = -\left\{\frac{1}{2}\,c^a c^b f^c_{ab}\pi_c,\ \cdot\ \right\}_{\mathcal{M}_E}$$

In the target space $\mathrm{gh}(\omega_k)=4$ so that $\omega=\omega_1=g^{ab}\pi_a\pi_b$.

For $\mathcal{G}$ simple g^{ab} is unique and invertible leading to the standard Lagrangian.

For $\mathcal{G}$ simple:

Whitehead Lemma $\Longrightarrow$ cohomology in vector fields trivial

$\Longrightarrow$ Theory is rigid and no nontrivial global symmetries

at the level of equations of motion as well.

Thanks!