MHV amplitudes in N=4 SUSY Yang-Mills theory and geometry of the momentum space

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Introduction and tree MHV amplitudes

MHV amplitudes $(++---)$ are the simplest objects to discuss within the gauge/string duality

Simplification at large N - MHV amplitudes are described by the single function of the kinematical variables

Properties of the tree amplitudes

- Holomorphy - it depends only on the "'half'' of the momentum variables $p_{\alpha,\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}$
- Fermionic representation (Nair,88) - tree amplitudes are the correlators of the chiral fermions of the sphere

- Tree amplitudes admit the twistor representation(Witten,04). Tree MHV amplitudes are localized on the curves in the twistor space. Twistor space - $CP(3\|4)$
- Twistor space emerges if we make a Fourier transform with respect to the "half" of the momentum variables $\int d\lambda e^{i\mu\lambda} f(\lambda\bar{\lambda})$. Point in the Minkowski space corresponds to the plane in the twistor space
- Localization follows from the holomorphic property of the tree MHV amplitude. Possible link to integrability via fermionic representation
- Stringy interpretation - auxiliary fermions are the degrees of freedom on the D1-D5 open strings ended on the Euclidean D1 instanton.

- The tree MHV amplitude has very simple form

$$A(1^-,2^-,3^+\dots,n^+) = g^{n-2}\frac{<12>^4}{<12><23>\cdots<n1>}$$

- The on-shell momenta of massless particle in the standard spinor notations read as $p_{a\dot{a}} = \lambda_a\tilde{\lambda}_{\dot{a}}$, λ_a and $\tilde{\lambda}_{\dot{a}}$ are positive and negative helicity spinors.
- Inner products in spinor notations $<\lambda_1,\lambda_2>=\epsilon_{ab}\lambda_1^a\lambda_2^b$ and $[\tilde{\lambda}_1\tilde{\lambda}_2]=\epsilon_{\dot{a}\dot{b}}\tilde{\lambda}_1^{\dot{a}}\tilde{\lambda}_2^{\dot{b}}$.

Properties of the loop MHV amplitudes

- Exponentiation of the ratio $\frac{M_{all-loop}}{M_{tree}}$ which contains the IR divergent and finite parts.
- BDS conjecture for the all loop answer

$$\log \frac{M_{all-loop}}{M_{tree}} = (IR_{div} + \Gamma_{cusp}(\lambda) M_{one-loop})$$

- It involves only two main ingredients - one-loop amplitude and all-loop $\Gamma_{cusp}(\lambda)$
- $\Gamma_{cusp}(\lambda)$ obeys the integral equation (Beisert-Eden-Staudacher) and can be derived recursively
- The conjecture fails starting from six external legs at two loops (Bern -Dixon-Kosower, Drummond-Henn-Korchemsky-Sokachev) and at large number of legs at strong coupling(Alday-Maldacena)

- One more all-loop conjecture - $\frac{M_{all-loop}}{M_{tree}}$ coincides with the Wilson polygon built from the external light-like momenta p_i.
- The conjecture was formulated at strong coupling (Alday-Maldacena, 06) upon the T-duality at the worldsheet of the string in the AdS_5 geometry
- Checked at weak coupling (one and two loops) as well (Drummond- Henn- Korchemsky- Sokachev, Bern-Dixon-Kosover, Brandhuber-Heslop-Travagnini 07).
- Important role of Ward identities with respect to the special conformal transformation in determination of the Wilson polygon (Drummond-Henn-Korchemsky-Sokachev). It fixes the form of the amplitudes at small number of legs

- There is no satisfactory stringy explanation of the loop MHV amplitudes and Wilson polygon-amplitude duality. Suspicion - closed string modes contribute (Cachazo-Swrchek-Witten)that is perturbative diagrams in YM theory are sensitive to the gravity degrees of freedom.
- The T-duality in the radial AdS direction supplemented by the fermionic T-duality is the symmetry of the sigma model (Berkovits- Maldacena, Beisert, Tseytlin, Wolf) hence it restricts the amplitudes

Main Questions

- Is there fermionic representation of the loop MHV amplitudes similar to the tree case?
- Is there link with integrability at generic kinematics ? The integrability behind the amplitudes is known at low-loop Regge limit (Lipatov 93, Faddeev-Korchemsky 94) only
- Is there trace of the weak-strong coupling S-suality of N=4 SYM in the amplitudes?
- What is the stringy geometrical origin of the BDS conjecture, if any?
- What is the physical origin of MHV amplitude-Wilson polygon duality?

MHV amplitude - Wilson loop correspondence

- It is possible to derive the correspondence at one-loop level (A.Zhiboedov, A.G.)
- Since one-loop answer is expressed it terms of 2 mass easy diagrams it is sufficient to get it for 2me box
- Start with D=4 2me box → introduce Feynman parametrization → integrate over momentum in the loop → make a change of variables→ Wilson polygon in D=6
- The second ingredient of derivation - relation between D=6 and D=4 integrals (Tarasov, Bern-Dixon, Nizic)

- The change of variables is quite simple

$$
\begin{aligned}
x_1 &= \sigma_1(1-\tau_1) \\
x_2 &= \sigma_1\tau_1 \\
x_3 &= \sigma_2\tau_2 \\
x_4 &= \sigma_2(1-\tau_2) \\
\left|\frac{\partial(x_i)}{\partial(\sigma_i,\tau_i)}\right| &= \sigma_1\sigma_2
\end{aligned}
\tag{1}
$$

- IR divergence in the amplitude explicitly get mapped into UV divergence of the Wilson polygons

$$
\begin{aligned}
d_{UV} + d_{IR} &= 10 \\
\epsilon_{IR} &= -\epsilon_{UV} \\
(\mu_{UV}^2\pi)^{\epsilon_{UV}} &= (\mu_{IR}^2)^{\epsilon_{IR}}
\end{aligned}
\tag{2}
$$

- Integrals over σ get factorized and two integrals over τ yield the integration in the one-loop Wilson polygon
- It is possible to get the duality between three-point function and Wilson triangle

$$
\begin{aligned}
d_{UV} + d_{IR} &= 8 \\
\epsilon_{IR} &= -\epsilon_{UV} \\
(\mu_{UV}^2 \pi)^{\epsilon_{UV}} &= (\mu_{IR}^2)^{\epsilon_{IR}}
\end{aligned}
\tag{3}
$$

- In this case we have analogue of 2mass hard diagram and on the Wilson polygon side we have insertion of the particular vertex operator

$$
< Tr\mathcal{P} q^\mu A_\mu(x_b) \exp[ig \oint_{\mathcal{C}} d\tau \dot{x}^\mu(\tau) A_\mu(x(\tau))] > \tag{4}
$$

where q^μ can be chosen as be arbitrary vector which is not orthogonal to p_3 in Minkowski sense, $(p_3 q) \neq 0$.

BDS anzatz and fermionic representation of amplitudes

- Important observation-one-loop box can be identified with the volume of the ideal hyperbolic tetrahedron in the space of Feynman parameters(Davydychev-Delbourgo,98). Good starting point for all-loop generalization
- Natural framework -topological strings and effective gravity in the target space description. Effective target space description - fermions on the Riemann surface.
- The "fermions" represent the proper branes. Lagrangian branes in the Kahler gravity description of A-model. Noncompact branes in the Kodaira-Spencer description of B-model.

- The generating function for the amplitude is expected to have the structure

$$\tau(t_k) =< 0|exp(\sum t_k V_k)exp \int (\bar{\psi} A \psi) exp(\sum t_{-k} V_{-k})|0 > \quad (5)$$

- That is scattering amplitudes can be described in terms of the fermionic currents on the Fermi surface
- Riemann Fermi surface reflects the hidden moduli space of the theory (chiral ring) and it gets quantized. Equation of the Riemann surface becomes the operator acting on the wave function(the analogue of the secondary quantization). The following commutation relation is implied

$$[x, y] = i\hbar$$

- This procedure of the quantization of the Riemann surface is familiar in the theory of integrable systems. Quantum Riemann surface =so-called Baxter eqution
- Degrees of freedom on the Riemann surface - Kodaira-Spencer gravity reduced to two dimensions (Dijkgraaf-Vafa,07)
- Solution to the Baxter equation - wave function of the single separated variable - Lagrangian brane or Lagrangian branes intersection (Nekrasov-Rubtsov-A.G. 2000)
- Polynomial solution to the Baxter equation - Bethe equations for the roots

- Why moduli space? Naively we have set of external momenta which yield a set of points in the momentum space. These set of points provides the moduli space of the complex structures. More carefully - the marked points in the rapidity space yield the desired moduli space i the B model. Kahler modulus of the ideal tetrahedron-A model
- From the Feynman diagrams - integration over the loop Schwinger parameters in the first quantized language amounts to the integration over $M_{0,n}$ (Gopakumar. Aharony et.al)
- At strong coupling. To have the proper interpretation of the Wilson loop as the wave function the integration over the diffeomorphisms $F(s)$ of the contour is necessary (Polyakov). In the amplitude case infinite dimensional integral over F is reduced to finite dimensional integration at the vertexes.

- The moduli space and more precisely Teichmuller space is closely related to the Liouville theory. Classically the universal Teichmueller space is the coadjoint Virasoro orbit. On the other hand Liouville Lagrangian is nothing but free PdQ system on this manifold.
- The discrete Liouville system is related to the Teichmueller space of the disc like surface with n-points at the boundary. The mapping class group generator is identified with the Hamiltonian of the discrete Liouville system (Faddeev-Kashaev).
- Hence we can claim that the transition from the tree to loop amplitude involves the proper dressing by the discrete Liouville modes

- Consider the moduli space of the complex structures for genus zero surface with n marked points, $M_{0,n}$. Inequivalent triangulations of the surface can be mapped into set of geodesics on the upper half-plane
- This manifold has the Poisson structure and can be quantized in the different coordinates (Kashaev-Fock-Chekhov, 97-01). The generating function of the special canonical transformations (flip) on this symplectic manifold is provided by $Li_2(z)$ where z- is so-called shear coordinate related to the conformal cross-ratio of four points on the real axe

$$exp(z) = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_3)(x_2 - x_4)}$$

- The natural objects geodesics can be determined in terms of shear variables z_a
- The symplectic structure in terms of these variables is simple $\sum_a dz_a \wedge dz_b$ where a corresponds to oriented edge and b is edge next to the right
- Upon quantization

$$[Z_a, Z_b] = 2\pi\hbar\{z_a, z_b\}$$

- Quantum mechanically there is operator of the "'duality"' K acting on this phase space with the property $\hat{K}^5 = 1$. It is the analogue of the Q-operator in the theory of the integrable systems since it is build from the eigenfunction of the "'quantum spectral curve operator"' Classically this curve looks as

$$e^u + e^v + 1 = 0$$

and gets transformed quantum mechanically into the Baxter equation

$$(e^{i\hbar\partial_v} + e^v + 1)Q(v) = 0$$

- The pair of Baxter equations for the discrete Liouville reads as (Kashaev)

$$Q(x + ib^{\pm}/2) + (1 - e^{4\pi x b^{\pm}})^N Q(x - ib^{\pm}/2) = t(x)Q(x)$$

- Let us use the representation for the finite part of the one-loop amplitude as the sum of the following dilogariphms. The whole amplitude is expressed in terms of the sums of the so-called two easy-mass box functions

$$\sum_i \sum_r Li_2\left(1 - \frac{x_{i,i+r}^2 x_{i+1,i+r+1}^2}{x_{i,i+r+1}^2 x_{i-1,i+r}^2}\right)$$

$$x_{i,k} = p_i - p_k$$

where p_i are the external on-shell momenta of gluons

- One-loop amplitude with n-gluons is described in terms of the "fermions" living on the spectral curve=Fermi surface which is embedded into the four dimensional complex space! MHV loop amplitude - fermionic current correlator on the spectral curve. Fermi surface lives in the space $T^*M_{0,4}$
- BDS conjecture for all-loop answer=quasiclassics of the fermionic correlator with the identification

$$\hbar^{-1} = \Gamma_{cusp}(\lambda)$$

- Is any ground behind this identification?
- In the limit describing the operators with large Lorentz spin the ground state energy of the corresponding string O(6)(Alday-Maldacena) behaves as

$$E \propto \Gamma_{cusp} logS \propto TL$$

that is Γ_{cusp} plays the role of the effective tension when the boundary of the string worldsheet is light-like

- For the Wilson loop with cusps and without self-intersections the loop equations reads as

$$\Delta W(c) = \sum_i \Gamma_{cusp}(\lambda, \theta_i) W(c)$$

that is Γ_{cusp} plays the same role

- "Fermions" on Fermi surface represent the noncompact branes (IR regulator) in the B model . In the mirror dual A model geometry fermions represent Lagrangian branes. Arguments of the brane wave-functions are the points on the moduli space of the complex structures. Fermions are transformed nontrivially on the Fermi surface because of its quantum nature
- Geometry: The spectral curve is embedded as the holomorphic surface in the internal 4-dimensional complex space

$$xy = e^u + e^v + 1$$

- Two branes in C^4 have the geometry

$$x = 0 \quad e^u + e^v + 1 = 0$$

and

$$y = 0 \quad e^u + e^v + 1 = 0$$

They can be identified with D3 branes in B model. There are also D1 (regulator=instanton) branes which are classically localized on the Riemann surface.

- Classically we have degrees of freedom on the intersection of the Lagrangian branes. There are also open strings, representing gluons with the disk geometry ending on the noncompact branes. These strings correspond to the external gluons.
- The tree amplitudes are localized at the points in the Minkowski space. Is there similar "localization" of the loop amplitudes? The suggestive relation - $Gr(2,4)//T = \bar{M}_{0,4}$ where T-maximal torus. The complexified Minkowski space M_c is $Gr(2,4)$ that is localization at points in $M_{0,4}$ can be considered as a kind of localization at the submanifold in M_c.
- The space where the string propagates is essentially noncommutative because of the conventional Planck constant. This is essential when the loop effects in the gauge theory are calculated.

- The origin of the Riemann surface. It corresponds to the summation of all anomalous relations in the gauge theory. Nontrivial effect of regulator degrees of freedom.
- Similar emergence of the Riemann surfaces. N=2 SYM theory-surface follows from the summation of the infinite number of the instantons. N=1 SYM- the surface is the result of the account of all generalized Konishi anomalies under the transformations $\Phi \to F(\Phi)$.

- Quantization of the Fermi surface involves the YM coupling constant

$$\frac{1}{g_{YM}^2} = \frac{\int B_{NS-NS}}{g_s}$$

Usually it is assumed that g_s yields the "Planck constant" for the quantization of the moduli space of the complex structures in the Kodaira-Spenser gravity. However equally some function of Yang-Mills coupling can be considered as the quantization parameter.

- The YM coupling constant yields the quantization of the gravity degrees of freedom in the box diagram (light-on-light scattering)

- Quasiclassics for the solution to the equation of the quantized Fermi surface

$$\Psi(z,\hbar)=\int\frac{e^{ipz}}{p\times sinh(\pi p)sinh(\pi\hbar p)}dp$$

reduces to

$$\Psi(x)\rightarrow exp(\hbar^{-1}Li_2(x)+...)$$

- Arguments of the Li_2 in the expression for the amplitudes correspond to the shear coordinates on the moduli space.
- The quantum dilogariphm has the dual-symmetric form

$$\Psi(z,\hbar)=\frac{e_q(\omega)}{e_{\tilde{q}}(\tilde{\omega})}$$

where $e_q(z)=\prod(1-zq^n)$

- It can be vizualized as two "left" and "right" lattices

- The one-loop MHV amplitude can be presented in the following form

$$M_{one-loop} \propto < 0|J(z_1)...J(z_n)exp(\psi_k A_{nk} \psi_n)|0 >$$

- The variables ψ_k are the modes of the fermion on the spectral curve and $J(z)$ is the fermionic current. The matrix $A_{n,k}$ for the corresponding spectral curve is known (Aganagic-Vafa-Klemm-Marino 03)

Towards the Regge limit

- From the worldsheet viewpoint one considers the discretization of the Liuville mode and the Faddeev-Volkov model yields the good candidate for the correct S-matrix. In the target space the natural integrable system is described by the model with the universal R-matrix based on the modular double

$$D = U_q(SL(2,R)) \otimes U_{\tilde{q}}(SL(2,R))$$

- Candidates for reggeons - open strings between the IR regulator branes. These states have momenta depending masses
- Possible link with the Reggeon field theory

$$L_{int} = -\frac{1}{g}\partial_{+} Pexp(-\frac{g}{2}\int_{-\infty}^{x^{+}} A_{+} dx_{-})\partial^2 V_{-}$$

$$-\frac{1}{g}\partial_{-} Pexp(-\frac{g}{2}\int_{-\infty}^{x^{-}} A_{-} dx_{+})\partial^2 V_{+}$$

where x_{+}, x_{-} are the light-cone coordinates and A is the conventional gluon field. Reggeons are the sources for the Wilson lines in accordance with holographic approach.

Conclusion

- The representation of the loop MHV amplitude as the correlator of the fermionic currents representing regulator degrees of freedom on the quantized Fermi surface is suggested. Nontrivial effect of closed string degrees of freedom(Kodaira-Spencer gravity) in the box diagram
- Link to the integrability behind generic MHV amplitudes via fermionic= IR brane representation. Particular solutions to 3-KP integrable system which correspond to the Faddeev-Volkov model of the discrete conformal mappings (discrete Lioville) with the good S-duality properties. The corresponding statistical model with the positive weights is Bazhanov-Mangazeev-Segreev one (2007)

- BDS conjecture can be reformulated in terms of the quantum geometry of the momentum space with $\Gamma_{cusp}(\lambda)$ as the quantization parameter. Way to improve-take into account the cubic vertex (screening operator) on the world-sheet in the Kodaira-Spencer gravity and loops in the 2d theory. Hopefully this improves the matching with the Regge limit of the amplitudes lost in BDS anzatz

- The degrees of freedom responsible for the dual description of the gluon amplitudes - IR branes=hypersurfaces in the "momentum" space. They are analogue of the D1-instantons (Witten) or IR branes of (Alday-Maldacena)
- Positions of the branes are fixed by the Bethe anzatz equations. Similarly extremization of the superpotential in the brane worldvolume theory yields their positions in the embedding space
- There are some candidates for the "reggeon" degrees of freedom - open strings between two regulator branes. They are analogue of "W-bosons" with masses depending on the momenta. This could explain the same universality class of the N=2 SQCD at $N_f = 2N_c$ and Reggeon Hamiltonian. The brane geometry is similar.