On unconstrained higher spins

of any symmetry

Dario Francia

Université Paris VII - APC

4th International Sakharov Conference - Moscow, May 22, 2009

Introduction I

"Problems with higher spins are not problems with free theory"

♋

True!

but still

Free theory not a closed subject

"Canonical" description of *free, symmetric* higher-spin gauge fields via

(Fang-) Fronsdal equations (1978):

- *Bosons* ($\sim$ *spin* $2 \longrightarrow R_{\mu\nu} = 0$) *:*

$$\boxed{\mathcal{F}_{\mu_1 \ldots \mu_s} \equiv \Box \varphi_{\mu_1 \ldots \mu_s} - \partial_{\mu_1} \partial^{\alpha} \varphi_{\alpha \mu_2 \ldots \mu_s} + \ldots + \partial_{\mu_1} \partial_{\mu_2} \varphi^{\alpha}{}_{\alpha \mu_3 \ldots \mu_s} + \ldots = 0}$$

- gauge invariant under $\delta \varphi = \partial \Lambda$ *iff* $\Lambda' (\equiv \Lambda^{\alpha}{}_{\alpha}) \equiv 0$;
- Lagrangian description *iff* $\varphi'' (\equiv \varphi^{\alpha\beta}{}_{\alpha\beta}) \equiv 0\,.$

- *Fermions* ($\sim$ *spin* $\frac{3}{2} \longrightarrow \not{\partial} \psi_{\mu} - \gamma_{\mu} \not{\psi} = 0$) *:*

$$\boxed{\mathcal{S}_{\mu_1 \ldots \mu_s} \equiv i \{ \gamma^{\alpha} \partial_{\alpha} \psi_{\mu_1 \ldots \mu_s} - (\partial_{\mu_1} \gamma^{\alpha} \psi_{\alpha \mu_2 \ldots \mu_s} + \ldots) \} = 0}$$

- gauge invariant under $\delta \psi = \partial \epsilon$ *iff* $\not{\epsilon} \equiv 0$;
- Lagrangian description *iff* $\not{\psi}' (\equiv \not{\psi}^{\alpha}{}_{\alpha}) \equiv 0\,.$

Generalisation to (spinor -) tensors of *any symmetry* type in

Labastida equations (1986 – 1989):

- **Bosons** (2-families: $\varphi_{\mu_1\cdots\mu_s,\,\nu_1\cdots\nu_r} \equiv \varphi_{\mu_s,\,\nu_r}$):

$$\boxed{\mathcal{F}_{\mu_s,\,\nu_r} \equiv \Box\varphi_{\mu_s,\,\nu_r} - \partial_\mu\,\partial^\alpha\,\varphi_{\alpha\mu_{s-1},\,\nu_r} - \partial_\nu\,\partial^\alpha\,\varphi_{\mu_s,\,\alpha\nu_{r-1}} + \partial^2{}_\mu\cdots + \partial^2{}_\nu\cdots + \partial_\mu\,\partial_\nu\cdots = 0}$$

- gauge invariant under

$$\delta\,\varphi_{\mu_s,\,\nu_r} = \partial_\mu\,\Lambda^{(1)}{}_{\mu_{s-1},\,\nu_r} + \partial_\nu\,\Lambda^{(2)}{}_{\mu_s,\,\nu_{r-1}}$$

iff suitable *combinations* of *traces* of $\Lambda^{(1)}$ and $\Lambda^{(2)}$ vanish;

- Lagrangian description *iff* suitable *combinations* of *double traces* of $\varphi_{\mu_s,\,\nu_r}$ vanish.

- **Fermions** (2-families: $\psi^a{}_{\mu_1\cdots\mu_s,\,\nu_1\cdots\nu_r} \equiv \psi_{\mu_s,\,\nu_r}$):

$$\boxed{\mathcal{S}_{\mu_s,\,\nu_r} \equiv i\,\{\gamma^\alpha\,\partial_\alpha\,\psi_{\mu_s,\,\nu_r} - \partial_\mu\,\gamma^\alpha\,\psi_{\alpha\,\mu_{s-1},\,\nu_r} - \partial_\nu\,\gamma^\alpha\,\psi_{\mu_s,\,\alpha\,\nu_{r-1}})\} = 0}$$

- similar constraints, but *no Lagrangian description available* for the general case**!**

Constraints

$\leftrightarrow$

keep to a minimum the number of off-shell components

➽ Consider the equations of motion for open String Field Theory

$$\mathcal{Q}\,|\Phi\rangle \;=\; 0\;,$$

where $\mathcal{Q}$ is the BRST charge, and evaluate the limit $\alpha' \to \infty$;
[*Bengtsson, Henneaux-Teitelboim, Lindström, Sundborg, D.F.-Sagnotti, Sagnotti-Tsulaia, Lindström-Zabzine, Bonelli, Savvidy, Buchbinder-Fotopoulos-Tsulaia-Petkou, . . .*]

➽ Actually, by restricting the attention *e. g.* to totally symmetric tensors it is possible to show that this equation splits into a series of *triplet* equations:

$$\begin{aligned} \Box\,\varphi &= \partial C\;, & \delta\,\varphi &= \partial\,\Lambda\,,\\ \Box\,C &= \partial\cdot\varphi \,-\, \partial\,D\;, & \delta\,C &= \Box\,\Lambda\,,\\ \Box\,D &= \partial\cdot C\;, & \delta\,D &= \partial\cdot\Lambda \end{aligned}$$

where φ is the spin-s field, describing the propagation of spins $s, s-2, s-4, \ldots$

with more off-shell components than $\sim \sum$ *(Fronsdal)*.

[Extension of triplets to irreducible spin $s \to$ Buchbinder-Galajinski-Krykhtin 2007*;*
frame-like analysis for reducible & irreducible cases $\to$ Sorokin-Vasiliev 2008*]*

For Maxwell, Yang-Mills (spin 1) and Einstein (spin 2) theories

$$\text{the curvature}: \begin{cases} A_\mu \;\rightarrow\; F_{\mu\nu} \sim \partial A \\ h_{\mu\nu} \;\rightarrow\; \mathcal{R}_{\mu\nu,\rho\sigma} \sim \partial^2 h \end{cases}$$

central to provide a *geometrical understanding of the dynamics*

∼

Do they exist analogous tensors for hsp?

Yes, at least at the linear level.

[*de Wit-Freedman* '80]

$$\varphi_{\mu_1\ldots\mu_s} \quad \rightarrow \quad \mathcal{R}_{\mu_1\ldots\mu_s;\,\nu_1\ldots\nu_s} \sim \partial^s \varphi$$

s.t.

$$\delta\,\mathcal{R}_{\mu_1\ldots\mu_s;\,\nu_1\ldots\nu_s} \equiv 0$$

under $\quad \delta\varphi_{\mu_1\ldots\mu_s} = \partial_{\mu_1}\Lambda_{\mu_2\mu_3\ldots\mu_s} + \partial_{\mu_2}\Lambda_{\mu_1\mu_3\ldots\mu_s} + \ldots$

for <u>unconstrained</u> gauge fields and gauge parameters

Three questions

I. Lagrangian description for fermions of mixed symmetry?

II. Unconstrained Lagrangians for bosons and fermions?

III. Any role for curvatures in the dynamics?

Appendix: *unconstrained Lagrangians & Stueckelberg symmetries*

(Unconstrained) Lagrangians for bosons & fermions of any symmetry

Fronsdal	Unconstrained
$\mathcal{F}$ *s. t.* $\delta\mathcal{F} = 3\,\partial^3\Lambda'$	$\mathcal{A} \equiv \mathcal{F} - 3\,\partial^3\alpha \rightarrow \begin{cases}\delta\,\alpha = \Lambda', \\ \delta\,\mathcal{A} = 0.\end{cases}$
$\mathcal{F} = 0$	$\mathcal{A} = 0$
$\mathcal{L}_{\varphi''\equiv 0} = \frac{1}{2}\varphi\left(\mathcal{F} - \frac{1}{2}\eta\,\mathcal{F}'\right)$	$\mathcal{L} = ?$

Basic ingredient: *the Bianchi identity*:

$$\partial\cdot\mathcal{A} - \frac{1}{2}\partial\,\mathcal{A}' \equiv -\frac{3}{2}\partial^3\underbrace{(\varphi'' - \partial\cdot\alpha - \partial\,\alpha')}_{\equiv\,\mathcal{C}}$$

compare with gravity

$$\partial^\alpha\mathcal{R}_{\alpha\mu} - \frac{1}{2}\partial_\mu\mathcal{R} \equiv 0$$

$$\mathcal{L}\,(\varphi,\alpha,\beta) = \frac{1}{2}\varphi\left(\mathcal{A} - \frac{1}{2}\,\eta\,\mathcal{A}'\right) - \frac{3}{4}\binom{s}{3}\alpha\,\partial\cdot\mathcal{A}' - 3\binom{s}{4}\beta\,\mathcal{C}\,,$$

unconstrained Lagrangians for any spin s [*D. F. - A. Sagnotti* 2005, 2006]

Generalisation to *(A)dS*: [*A. Sagnotti - M. Tsulaia* '03; *D. F. - J. Mourad - A. Sagnotti*, '07]

Unconstrained hsp III: local theory - mixed symmetry bosons

[*A. Campoleoni - D. F. - J. Mourad - A. Sagnotti, 2008*]

Here: Two-family fields $\varphi_{\mu_1 \dots \mu_{s_1} ; \nu_1 \dots \nu_{s_2}}$

Notation:
$$\begin{cases} \varphi_{\mu_1 \dots \mu_{s_1} ; \nu_1 \dots \nu_{s_2}} & \to \quad \varphi \,, \\ \partial_{(\, \mu_1^i |} \, \varphi_{\dots ; | \, \mu_2^i \dots \mu_{s_i+1}^i) \, ; \dots} & \to \quad \partial^{\,i} \varphi \,, \\ \partial^{\,\lambda} \varphi_{\dots ; \lambda \, \mu_2^i \dots \mu_{s_i}^i \, ; \dots} & \to \quad \partial_i \, \varphi \,, \\ \varphi_{\dots ; \; {}^{\lambda}{}_{\mu_2^i \dots \mu_{s_i}^i \, ; \dots ; \lambda \, \mu_2^j \dots \mu_{s_j}^j \, ; \dots}} & \to \quad T_{ij} \, \varphi \,. \end{cases}$$

upper indices $\leftrightarrow$ *added indices*

lower indices $\leftrightarrow$ *removed indices*

Families of *symmetric* indices $\longrightarrow$ *reducible* $gl\,(D)$ tensors

$\sim$

Basic *constrained* theory: [*Labastida* 1986, 1989]

$$\mathcal{F} \, = \, \Box \, \varphi \, - \, \partial^{\,i} \partial_i \, \varphi + \frac{1}{2} \, \partial^{\,i} \partial^{\,j} \, T_{ij} \, \varphi \, = \, 0 \,,$$

- gauge invariant under $\delta \, \varphi = \partial^{\,i} \Lambda_i$ *iff* $T_{(\, ij} \, \Lambda_{k\,)} \equiv 0;$
- Lagrangian description *iff* $T_{(\, ij} \, T_{kl\,)} \, \varphi = 0 \,.$

$\rightarrow$ *not all traces vanish;*

$\rightarrow$ *the constraints are not independent.*

Basic unconstrained kinetic tensor:

$$\boxed{\mathcal{A} = \mathcal{F} - \frac{1}{2}\partial^i\partial^j\partial^k \alpha_{ijk}},$$

But, due to linear dependence of constraints

$$\begin{cases}\alpha_{ijk} \equiv \alpha_{ijk}(\Phi) = \frac{1}{3} T_{(ij}\,\Phi_{k)}\,,\\ \delta\,\Phi_k = \Lambda_k\,.\end{cases}$$

$\sim$

To construct the Lagrangian $\rightarrow$ resort to *Bianchi identity*:

$$\partial_i\,\mathcal{A} - \frac{1}{2}\partial^j\,T_{ij}\,\mathcal{A} = -\frac{1}{4}\partial^j\partial^k\partial^l\,\mathcal{C}_{ijkl}$$

$$\mathcal{C}_{ijkl} = T_{(ij}\,T_{kl)}\,\varphi + \mathcal{C}_{ijkl}(\alpha)$$

As for symm case, take care of terms in $\propto \mathcal{C}_{ijkl}$ via a *Lagrange multiplier β*:

$$\mathcal{L} = \frac{1}{2}\,\langle\varphi\,,E_\varphi\rangle + \frac{1}{2}\,\langle\Phi_i\,,(E_\Phi)_i\rangle + \frac{1}{2}\,\langle\beta_{ijkl}\,,(E_\beta)_{ijkl}\rangle$$

where in particular the e.o.m. for φ, gauge fixing $\alpha_{ijk} = \frac{1}{3} T_{(ij}\,\Phi_{k)}$ to zero, is

$$E_\varphi = \mathcal{E}_\varphi + \frac{1}{2}\,\eta^{ij}\,\eta^{kl}\,\mathcal{B}_{ijkl} = 0\,,$$

$$\boxed{\mathcal{E}_\varphi = \mathcal{F} - \frac{1}{2}\eta^{ij}\,T_{ij}\,\mathcal{F} + \frac{1}{36}\eta^{ij}\,\eta^{kl}\left(2\,T_{ij}\,T_{kl} - T_{i(k}\,T_{l)j}\right)\mathcal{F}}.$$

[A. Campoleoni - D. F. - J. Mourad - A. Sagnotti, 2009]

The basic kinematical setting of Labastida [1987]

$$\begin{cases} \mathcal{S} \,=\, i\left(\not{\partial}\,\psi \,-\, \partial^{\,i}\not{\psi}_i\right) \,=0\,, \\ \delta\,\psi \,=\, \partial^{\,i}\,\epsilon_{\,i}\,, \\ T_{(\,ij}\psi_{k\,)} \,=\, 0\,;\; \gamma_{(\,i}\,\epsilon_{\,j\,)} \,=\, 0\,, \end{cases}$$

can be easily turned to its *unconstrained* counterpart:

$$\begin{cases} \mathcal{W} \,=\, \mathcal{S} \,+\, i\,\partial^{\,i}\,\partial^{\,j}\,\xi_{ij} \,=\, 0\,, \\ \delta\,\psi \,=\, \partial^{\,i}\,\epsilon_{\,i}\,, \\ \xi_{\,ij}\,(\Psi) \,=\, \frac{1}{2}\gamma_{(\,i}\,\Psi_{\,j\,)}\,, \\ \delta\,\Psi_{\,i} \,=\, \epsilon_{\,i}\,, \end{cases}$$

BUT, in the constrained setting, *no Lagrangian available for fermions;*

- Using the Bianchi identity (here constrained theory, for simplicity)

$$\partial_i\,\mathcal{S} \,-\, \frac{1}{2}\,\not{\partial}\,\gamma_i\,\mathcal{S} \,-\, \frac{1}{2}\,\partial^{\,j}\,T_{ij}\,\mathcal{S} \,-\, \frac{1}{6}\,\partial^{\,j}\,\gamma_{ij}\,\mathcal{S} \,=\, \frac{i}{2}\,\partial^{\,j}\partial^{\,k}\,T_{(\,ij}\,\gamma_{\,k\,)}\,\psi$$

it is possible to find the complete Lagrangian, for N-family fields, in the form

$$\begin{cases} \mathcal{L} \,=\, \frac{1}{2}\,\langle\,\bar{\psi}\,, \sum_{p\,,\,q\,=\,0}^{N}\,k_{\,p\,,\,q}\;\eta^{\,p}\,\gamma^{\,q}\,(\,\gamma^{\,[q]}\,\mathcal{S}^{\,[p]}\,)\,\rangle \,+\, \text{h.c.}\,, \\ \\ k_{\,p\,,\,q} \,=\, \frac{(-1)^{\,p+\,\frac{q\,(q+1)}{2}}}{p\,!\,q\,!\,(\,p+q+1\,)\,!}\;. \end{cases}$$

Unconstrained higher spins & geometry

♋

Generalisation of geometric equations for spin 1 et spin 2:

[*D.F. - A. Sagnotti, 2002, D.F. - J. Mourad - A. Sagnotti, 2007*]

spin 1 (Maxwell) : $\partial^{\alpha} F_{\alpha,\mu} = 0$

spin 2 (Einstein) : $\eta^{\alpha\beta} \mathcal{R}_{\alpha\mu,\beta\nu} = 0$

spin 3 : $\boxed{\mathcal{A}_{\varphi} \equiv \frac{1}{\Box} \partial^{\alpha} \mathcal{R}^{\beta}{}_{\beta\alpha,\,\mu\nu\rho} = 0}$

- (*Consistency :*) the equation $\mathcal{A}_{\varphi} = 0$ *always* implies the *compensator equation*

$$\mathcal{A}_{\varphi} = 0 \rightarrow \mathcal{F} - 3\,\partial^{3} \alpha_{\varphi} = 0, \quad \text{with} \quad \delta\,\alpha_{\varphi} = \Lambda'$$

- (*Lagrangian :*) $\forall$ "Ricci tensor" $\mathcal{A}_{\varphi}(\{a_k\})$ *identically divergenceless Einstein tensors* $\mathcal{E}_{\varphi}(\{a_k\})$ s.t.

$$\mathcal{L} = \frac{1}{2}\varphi\,\mathcal{E}_{\varphi}(\{a_k\}) \quad \longrightarrow \quad \mathcal{E}_{\varphi}(\{a_k\}) = 0 \quad \longrightarrow \quad \mathcal{A}_{\varphi}(\{a_k\}) = 0\,,$$

Spin 2: massive theory as

quadratic deformation of the geometric theory:

- *Spin* 2 [*Fierz-Pauli*]

$$\mathcal{L}(m=0) = \frac{1}{2}h_{\mu\nu}\,(\mathcal{R}^{\mu\nu} - \frac{1}{2}\,\eta^{\mu\nu}\,\mathcal{R})$$

$$\mathcal{L}(m) = \frac{1}{2}h_{\mu\nu}\,\{\underbrace{(\mathcal{R}^{\mu\nu} - \frac{1}{2}\,\eta^{\mu\nu}\,\mathcal{R})}_{\partial\cdot\mathcal{E}_{s=2}\equiv 0} - m^2\,\underbrace{(h^{\mu\nu} - \eta^{\mu\nu}\,h^{\alpha}{}_{\alpha})}_{Fierz-Pauli\ mass\ term}\,\}$$

- *Spin s:* $\boxed{\text{General idea: } \textit{higher traces} \text{ should appear in the mass term}}$, s.t.

$$\mathcal{L} = \frac{1}{2}\,\varphi\,\{\mathcal{E}_{\varphi}\,(a_1,\ldots a_k,\ldots)\; - \; m^{2}\,M_{\varphi}\} \qquad \text{where} \qquad \underbrace{M_{\varphi} = \sum \lambda_k\,\eta^{\,k}\,\varphi^{\,[k]}\,,}_{generalised\ FP\ mass\ term}$$

- Fronsdal : $\partial\cdot\{\mathcal{F} - \frac{1}{2}\eta\mathcal{F}'\} \neq 0 \;\Rightarrow$ need for *auxiliary fields*;
- Differently, for *all* geometric Einstein tensors $\mathcal{E}_{\varphi}$ we have $\partial\cdot\mathcal{E}_{\varphi} \equiv 0$!
- Indeed it is possible to define a consistent massive theory with

$$M_{\varphi} = \varphi - \eta\,\varphi' - \eta^{2}\,\varphi'' - \frac{1}{3}\eta^{3}\,\varphi''' - \,\cdots\, - \frac{1}{(2\,n\,-\,3)\,!!}\,\eta^{\,n}\,\varphi^{\,[n]}\,.$$

No auxiliary fields are needed [*D.F.*, 2007, 2008]

We found consistent formulations for unconstrained hsp

♋

on the other hand:

- Using curvatures $\rightarrow$ *non-localities*;

- Minimal local Lagrangians $\rightarrow$ *higher-derivatives*: $\sim \alpha \, \Box^{\,2} \alpha$

- BRST approach(*): to describe spin $s \rightarrow \mathcal{O}(s)$ *auxiliary fields*

$\rightarrow$

intrinsic complication of the unconstrained approach?

(*)[*Pashnev - Tsulaia - Buchbinder et al.* 1997, ...]

There is a simple, alternative interpretation of the minimal local Lagrangians:

- Consider the Fronsdal Lagrangian, together with a multiplier for ϕ'':

$$\mathcal{L} = \phi(\mathcal{F} - \frac{1}{2}\eta\mathcal{F}') + \beta\phi''$$

$\mathcal{L}$ is gauge-invariant under $\delta\varphi = \partial\lambda, \ \delta\beta = \partial\cdot\partial\cdot\partial\cdot\lambda$, with $\lambda' = 0$

- Perform the *Stueckelberg substitution*

$$\phi \quad \rightarrow \quad \varphi - \partial\,\theta$$

obtaining an *unconstrained* Lagrangian, gauge invariant under

$$\delta\,\varphi = \partial\,\Lambda\,; \qquad\qquad \delta\,\theta = \Lambda$$

with an *unconstrained* parameter Λ.

- Only the trace of θ appears in $\mathcal{L}$ (after a redefinition of β)so that, defining $\theta' \equiv \alpha$ we recover the minimal Lagrangian

$$\mathcal{L}\,(\varphi,\alpha,\beta) = \frac{1}{2}\varphi\left(\mathcal{A} - \frac{1}{2}\,\eta\,\mathcal{A}'\right) - \frac{3}{4}\binom{s}{3}\alpha\,\partial\cdot\mathcal{A}' - 3\binom{s}{4}\beta\,\mathcal{C}$$

Two basic observations:

- higher-derivative terms are simply due to the different dimensions of θ w.r.t. φ in $\phi \;\rightarrow\; \varphi - \partial\,\theta$;

- Under this substitution *any* function of ϕ would be (trivially) gauge-invariant.

This is too much!

What we want is to *extend* to the unconstrained level

a constrained gauge symmetry already present in the Lagrangian

♋

In this sense, maybe it is possible to improve the Stueckelberg idea.

Appendix III: Unconstrained hsp without higher derivatives

[*see also* Buchbinder, Galajinsky, Krykhtin '07]

- In $\delta\phi = \partial\Lambda$ separate *traceless* and *trace* parts of the parameter Λ:

$$\Lambda = \Lambda^{t} + \eta\,\Lambda^{p}\,,$$
$$\Lambda^{p} : \Lambda' = (\eta\,\Lambda^{p})'$$

- introduce a new compensator θ_p, s.t. $\delta\,\theta_p = \partial\,\Lambda^p$ (so θ_p *is not* pure gauge)

- perform in $\mathcal{L}$ the substitution

$$\phi \;\rightarrow\; \varphi - \eta\,\theta_p$$

where $\varphi - \eta\,\theta_p$ *transforms as the 'old' Fronsdal field*.

- The corresponding "Ricci tensor" (and generalisations thereof)

$$\mathcal{A}_{\varphi,\theta} = \mathcal{F} - (D + 2s - 6)\,\partial^{2}\theta - \eta\,\mathcal{F}_{\theta}\,,$$

is the building-block of *unconstrained Lagrangians*, with a *minimal* content of auxiliary fields and *no higher-derivatives*

for bosons and fermions of any symmetry type

[*D. F. 2007; A. Campoleoni - D. F. - J. Mourad - A. Sagnotti; 2008; 2009*]

∼ *Perspectives* ∼

Still open issues on the *free theory :*

- hsp supersymmetry multiplets;
- Dualities;
- Quantization;
- . . .

whether or not allowing for a wider gauge symmetry might prove to be truly important, only a deeper insight into interactions will tell

still, unconstrained formulation is technically simpler (no need to project*), and more flexible* (more gauge fixings allowed*)*

♋

To go beyond

Quartic interactions :

- For *spin* 1 (YM) and *spin* 2 (EH) *cubic vertex* implies *full Lagrangian*;
- for higher spins *nothing known about quartic couplings*; *but* “proper” hsp features from quartic coupling onwards:

maybe worth the effort to try and overcome the “cubic” barrier

Appendix: Hsp geometry & current exchanges, $m = 0$

Are all the geometrical Einstein tensors really equivalent?

- *Propagator* from Lagrangian equation with an external current:

$$\mathcal{E}_{\varphi}(a_1, \ldots a_k \ldots) = \mathcal{J} \quad \Rightarrow \quad \varphi = \mathcal{G}(a_1, \ldots a_k \ldots) \cdot \mathcal{J}$$

- *Current exchange* $\mathcal{J} \cdot \varphi = \mathcal{J} \cdot \mathcal{G} \cdot \mathcal{J} \rightarrow$ consistency conditions on the *polarisations flowing*:

almost all geometric theories give the wrong result, <u>but one</u>.

The correct theory has a simple structure:

- The 'Ricci' tensor has the compensator form $\mathcal{A}_{\varphi} = \mathcal{F} - 3\,\partial^3 \gamma_{\varphi}$;
- It satisfies the identities : $\begin{cases} \partial \cdot \mathcal{A}_{\varphi} - \frac{1}{2}\partial \mathcal{A}'_{\varphi} \equiv 0 \\ \mathcal{A}''_{\varphi} \equiv 0 \end{cases}$, and the Lagrangian is

$$\boxed{\mathcal{L} = \frac{1}{2}\varphi\left(\mathcal{A}_{\varphi} - \frac{1}{2}\eta\,\mathcal{A}'_{\varphi} + \eta^2\,\mathcal{B}_{\varphi}\right) - \varphi \cdot \mathcal{J}}$$

[*D.F. - J. Mourad - A. Sagnotti, 2007*]

Appendix: Hsp geometry & current exchanges, $m \neq 0$

- Consider the family of Lagrangians, for spin 4: [D.F. 2007, 2008]

$$\mathcal{L}\,(m) \,=\, \frac{1}{2}\,\varphi\,\{\mathcal{E}_{\varphi}\,(a_1,\,a_2)\,-\,m^2\,M_{\varphi}\}\,-\,\varphi\,\cdot\,\mathcal{J}\,,$$

where $\mathcal{J}$ is a *conserved* current: $\partial\cdot\mathcal{J}\,=\,0\,.$

- The divergence of the eom

$$\partial\cdot\{\mathcal{E}_{\varphi}\,(a_1,\,a_2)\,-\,m^2\,(\varphi\,-\,\eta\,\varphi'\,-\,\eta^2\,\varphi'')\}\,=\,\partial\cdot\mathcal{J}\,=\,0\,,$$

implies the *same consequences as in the absence of* $\mathcal{J}$.

- Actually, $\underline{\forall a_1, a_2}$ the eom reduce to

$$\Box\,\varphi\,-\,\tfrac{\partial^2}{\Box}\,\varphi'\,-\,3\,\tfrac{\partial^4}{\Box^2}\,\varphi''\,-\,m^2\,(\varphi-\eta\,\varphi'\,-\,\eta^2\,\varphi'')\,=\,\mathcal{J}\,,$$

- where a_1, a_2 disappeared; computing the product $\mathcal{J}\cdot\mathcal{J}$:

  (1) *only surviving contribution from the family of Einstein tensors is* $\Box\,\varphi$

  (2) *full structure of the propagator encoded in the coefficients of* M_{φ}

- Inverting the equation of motion we find the correct result

$$\mathcal{J}\cdot\varphi\,=\,\frac{1}{p^2\,-\,m^2}\,\{\mathcal{J}\cdot\mathcal{J}\,-\,\frac{6}{D+3}\;J'\cdot J'\,+\,\frac{3}{(D+1)(D+3)}\;J''\cdot J''\}$$

Appendix: Hsp geometry: uniqueness of mass deformation

The same mass term M_φ generates *infinitely many* consistent massive theories.

$\longrightarrow$

issue of uniqueness

I. ➽ Origin of the Fierz-Pauli mass-term, for $s = 2$: KK reduction ($\Box \to \Box - m^2$):

$$\mathcal{R}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\mathcal{R} \sim \Box(h - \eta h') + \ldots ,$$

A similar mechanism for M_φ?

➽ For each Einstein tensor $\mathcal{E}_\varphi(a_1, \ldots, a_k)$ it is unambiguously defined the "pure massive" contribution of the reduction, neglecting singularities from $\frac{1}{\Box} \to \frac{1}{\Box - m^2}$:

$$\mathcal{E}_\varphi(a_1, \ldots, a_k) \sim \Box(\varphi + k_1 \eta \varphi' + k_2 \eta^2 \varphi'' + \ldots) + \ldots ,$$

where $k_i = k_i(a_1, \ldots, a_k)$.

➽ Is it possible to find a geometric theory whose "box" term encodes the coefficients of the generalised FP mass term M_φ?

Yes! Up to spin 11 *(at least) it is just the unique theory with the correct current exchange*.

II. ➽ Why the mass term works well with *all* geometric Einstein tensors? Not too strange, also true for spin 2: the non-local (wrong!) theory defined by the eom

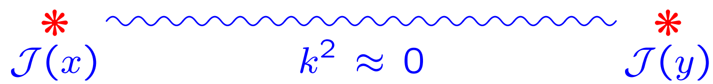
$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\mathcal{R} + \lambda(\eta - \frac{\partial^2}{\Box})\mathcal{R} - m^2(h - \eta h') = T_{\mu\nu} ,$$

with $T_{\mu\nu}$ *conserved*, reduces to the Fierz system, and gives the correct current exchange!

Appendix: Massive theory & Current exchanges

❈ Massive Lagrangians from massless ones → K-K reduction from $D+1$ to D

❈ Response of the theory to the presence of an external source $\mathcal{J}$; *unitarity* : only transverse, on-shell polarisations mediate the interaction between distant sources:



tantamount to computing the *propagator*

➽ Straightforward in flat bkg;

$$s = 3: \quad \begin{cases} p^2\, \mathcal{J}\cdot\varphi = \mathcal{J}\cdot\mathcal{J} - \frac{3}{D}\,\mathcal{J}'\cdot\mathcal{J}' & m = 0 \\ (p^2 - m^2)\,\mathcal{J}\cdot\varphi = \mathcal{J}\cdot\mathcal{J} - \frac{3}{D+1}\,\mathcal{J}'\cdot\mathcal{J}' & m \neq 0 \end{cases}$$

(generalisation to hsp of the vDVZ discontinuity)

➽ Less direct to describe (partially) massive (A)dS fields$^{(*)}$;

$$s = 3: \quad \begin{cases} P_L^2\, \mathcal{J}\cdot\varphi = \mathcal{J}\cdot\mathcal{J} - \frac{3}{D}\,\mathcal{J}'\cdot\mathcal{J}' & m = 0 \\ (P_L^2 - m^2)\,\mathcal{J}\cdot\varphi = \mathcal{J}\cdot\mathcal{J} - 3\,\frac{m^2L^2+D+1}{(D+1)\,(m^2L^2+D)}\,\mathcal{J}'\cdot\mathcal{J}' & m \neq 0 \end{cases}$$

(*no vDVZ discontinuity* for hsp on (A)dS)

$^{(*)}P_L^2 = \Box_L - 4\frac{D}{L^2}$ [*D.F. - J. Mourad - A. Sagnotti, '07, '08*]