Reciprocity in AdS/CFT

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References:

V.F., M. Beccaria arXiv: **0803.3768, 0901.1256**
M. Beccaria, V.F., A. Tirziu, A.A. Tseytlin arXiv: **0809.5234**
M. Beccaria, V.F., T. Lukowski, S. Zieme: **0901.4864**

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Outline

Background & Motivation

- ***AdS/CFT and integrability, twist operators and their large spin expansion***

Reciprocity

- ***At weak coupling***
- ***At strong coupling***

The central role of N=4 SYM

from string theory to strong interactions

$$AdS_5 \times S^5 \longleftrightarrow \begin{array}{c}\mathcal{N}=4 \\ \text{SYM}\end{array} \longrightarrow \begin{array}{c}\mathcal{N}=1,\,2 \\ \text{SYM}\end{array} \longrightarrow \text{QCD}$$

I **II**

I. AdS/CFT duality conjecture **[Maldacena, 97]**

type IIB strings on $AdS_5 x S^5$ ↔ $\mathcal{N}$=4 Super Yang Mills in d=3+1

- Agreement of underlying symmetry PSU(2,2|4)
- Weak/strong coupling duality $\lambda = N\, g_{\rm YM}^2, \qquad \sqrt{\lambda} = \frac{R^2}{\alpha'}, \qquad \frac{4\pi\,\lambda}{N} = g_S$
- Prediction $E_{\rm string} = \Delta_{\rm CFT}$
- Planar limit $N \to \infty \;\Rightarrow\; g_s = 0$ free string. Integrability!

II. Superconformal (β=0) vs. confined (β<0), SU(N) vs SU(3), adjoint vs fundam.

The central role of N=4 SYM
from string theory to strong interactions

energies E $\longleftrightarrow$ scaling dimensions Δ $\longrightarrow$ "splitting functions" $P(x)$

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RECIPROCITY: Large (Lorentz) spin expansion of scaling dimensions and energies = Mellin-space translation of a reciprocity relation for "splitting functions" $P(x) = -x\,P\left(\frac{1}{x}\right)$ *(when* $x \to 1$*)*

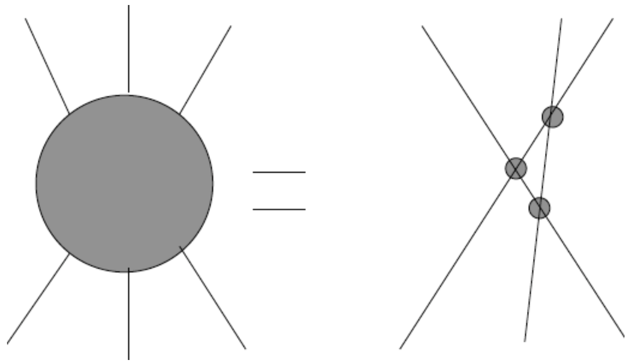
Integrability

Quantitative understanding of the duality: remarkable boost with ***data*** from perturbative gauge theory (4-loop in λ) and perturbative string theory (2-loop in $1/\sqrt{\lambda}$).

Framework: ***integrable structures*** discovered on both sides of AdS/CFT (planar limit!)

Integrable CFT: ***<u>not</u> in the sense of factorised space-time scattering***

[not really "not": <u>Beisert, Gorsky, Lipatov talks</u>]



Observables of the theory: correlation functions of *gauge invariant local composite operators*

$$\mathcal{O} = \mathrm{Tr}(\mathcal{X}\mathcal{Y}\mathcal{Z}\mathcal{F}_{\mu\nu}\Psi(\mathcal{D}_\mu\mathcal{Z})\dots)$$

It is integrable the evolution of the composite operators with the RG scale.

Planar dilatation operator maps to a spin chain Hamiltonian, ***integrable*** = ***solvable*** via Bethe Ansatz.

$$\mathfrak{D}(\lambda) = \mathfrak{D}_0 + \sum_{\ell\geq 1} \lambda^\ell \mathcal{H}^{(\ell)}_{integrable}$$

[Minahan Zarembo 02]

Features not exclusive of N=4 SYM! **[Lipatov 93, Fadeev, Korchemsky 94, Korchemsky 95]**

[Belitsky, Braun, Derkachov, Korchesky, Manashov, 98-99] [Belitsky, Gorsky, Korchemsky, 03]

The spectrum of AdS/CFT

- Easier with S-matrix: constrained by the global symmetry *plus* crossing symmetry and string data ⟶ **All-loop PSU(2,2|4) asymptotic Bethe equations.**

[Beisert, 05][Janik, 06] [Beisert, Staudacher 05][Beisert, Eden, Staudacher 06] *[Zarembo talk]*

$$1=\prod_{j=1}^{K_4}\frac{x^+_{4,j}}{x^-_{4,j}},$$

$$1=\prod_{j=1}^{K_2}\frac{u_{1,k}-u_{2,j}+\frac{i}{2}\eta_1}{u_{1,k}-u_{2,j}-\frac{i}{2}\eta_1}\prod_{j=1}^{K_4}\frac{1-g^2/2x_{1,k}x^{+\eta_1}_{4,j}}{1-g^2/2x_{1,k}x^{-\eta_1}_{4,j}},$$

$$1=\prod_{\substack{j=1\\j\neq k}}^{K_2}\frac{u_{2,k}-u_{2,j}-i\eta_1}{u_{2,k}-u_{2,j}+i\eta_1}\prod_{j=1}^{K_3}\frac{u_{2,k}-u_{3,j}+\frac{i}{2}\eta_1}{u_{2,k}-u_{3,j}-\frac{i}{2}\eta_1}\prod_{j=1}^{K_1}\frac{u_{2,k}-u_{1,j}+\frac{i}{2}\eta_1}{u_{2,k}-u_{1,j}-\frac{i}{2}\eta_1},$$

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Strings!

$$1=\left(\frac{x^-_{4,k}}{x^+_{4,k}}\right)^L\prod_{\substack{j=1\\j\neq k}}^{K_4}\left(\frac{x^{+\eta_1}_{4,k}-x^{-\eta_1}_{4,j}}{x^{-\eta_2}_{4,k}-x^{+\eta_2}_{4,j}}\,\frac{1-g^2/2x^+_{4,k}x^-_{4,j}}{1-g^2/2x^-_{4,k}x^+_{4,j}}\,\sigma^2(x_{4,k},x_{4,j})\right)$$
$$\times\prod_{j=1}^{K_1}\frac{1-g^2/2x^{-\eta_1}_{4,k}x_{1,j}}{1-g^2/2x^{+\eta_1}_{4,k}x_{1,j}}\prod_{j=1}^{K_3}\frac{x^{-\eta_1}_{4,k}-x_{3,j}}{x^{+\eta_1}_{4,k}-x_{3,j}}\prod_{j=1}^{K_5}\frac{x^{-\eta_2}_{4,k}-x_{5,j}}{x^{+\eta_2}_{4,k}-x_{5,j}}\prod_{j=1}^{K_7}\frac{1-g^2/2x^{-\eta_2}_{4,k}x_{7,j}}{1-g^2/2x^{+\eta_2}_{4,k}x_{7,j}}$$

$$1=\prod_{j=1}^{K_6}\frac{u_{5,k}-u_{6,j}+\frac{i}{2}\eta_2}{u_{5,k}-u_{6,j}-\frac{i}{2}\eta_2}\prod_{j=1}^{K_4}\frac{x_{5,k}-x^{+\eta_2}_{4,j}}{x_{5,k}-x^{-\eta_2}_{4,j}},$$

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$$Q_r=\frac{1}{r-1}\sum_{j=1}^{K_4}\left(\frac{i}{(x^+_{4,j})^{r-1}}-\frac{i}{(x^-_{4,j})^{r-1}}\right),\qquad \delta D=g^2Q_2=g^2\sum_{j=1}^{K_4}\left(\frac{i}{x^+_{4,j}}-\frac{i}{x^-_{4,j}}\right).$$

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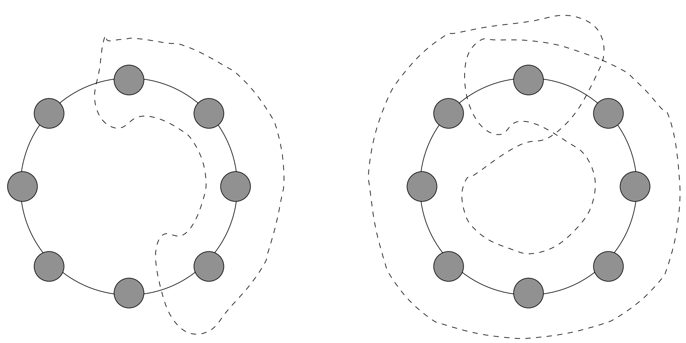
- Easier with S-matrix: constrained by the global symmetry *plus* crossing symmetry and string data → **All-loop PSU(2,2|4) asymptotic Bethe equations.**

[Beisert, 05][Janik, 06] [Beisert, Staudacher 05][Beisert, Eden, Staudacher 06] [Zarembo talk]

CAVEAT: wrapping!



Bethe eqs correct up to $O(\lambda^L)$
(L: length of operators)

[Kotikov, Lipatov, Rej, Staudacher, Velizhanin, 07]

$$1=\prod_{j=1}^{K_4}\frac{x_{4,j}^+}{x_{4,j}^-},$$

$$1=\prod_{j=1}^{K_2}\frac{u_{1,k}-u_{2,j}+\frac{i}{2}\eta_1}{u_{1,k}-u_{2,j}-\frac{i}{2}\eta_1}\prod_{j=1}^{K_4}\frac{1-g^2/2x_{1,k}x_{4,j}^{+\eta_1}}{1-g^2/2x_{1,k}x_{4,j}^{-\eta_1}},$$

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$$1=\left(\frac{x_{4,k}^-}{x_{4,k}^+}\right)^L\prod_{\substack{j=1\\j\neq k}}^{K_4}\left(\frac{x_{4,k}^{+\eta_1}-x_{4,j}^{-\eta_1}}{x_{4,k}^{-\eta_2}-x_{4,j}^{+\eta_2}}\,\frac{1-g^2/2x_{4,k}^+x_{4,j}^-}{1-g^2/2x_{4,k}^-x_{4,j}^+}\,\sigma^2(x_{4,k},x_{4,j})\right)$$
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$$Q_r=\frac{1}{r-1}\sum_{j=1}^{K_4}\left(\frac{i}{(x_{4,j}^+)^{r-1}}-\frac{i}{(x_{4,j}^-)^{r-1}}\right),\qquad \delta D=g^2Q_2=g^2\sum_{j=1}^{K_4}\left(\frac{i}{x_{4,j}^+}-\frac{i}{x_{4,j}^-}\right).$$

- *Inspiration* from relativistic th. + *data* from Thermodynamic Bethe Ansatz = finite size effects in terms of infinite volume data!

Generalized Lüscher corrections [Bajnok, Janik, Lukowski 08]

→ **TBA** [Arutyunov Frolov 08,09] [S.Frolov talk]

Y-system [Kazakov Gromov Vieira 09]

Unifying asymptotic and wrapping spectrum

[V.Kazakov talk]

The remarkable outcome: cusp anomaly

A spectacular example of ***interpolation between weak and strong coupling***.

- **QCD**: ***logarithmic scaling*** in leading ***twist operators*** at large spin S (x→1)

$$\gamma(S) = 2\,\Gamma_{\rm cusp}(\alpha)\,\log S + \mathcal{O}(S^0) \qquad\qquad \mathcal{O}_S = \bar{q}(\gamma_+\mathcal{D}^+)^S q$$

- **N=4 SYM**: Twist two operators in sl(2) ⊂ psu(2,2|4)

$$\gamma(S) = f(\lambda)\,\log S + O(S^0) \qquad\qquad \mathcal{O}_S = \mathrm{Tr}(\varphi D^S \varphi)$$

Integral equation from the Bethe Ansatz. **[Beisert, Eden, Staudacher 06]**

- At weak coupling $f(\lambda) = \frac{\lambda}{2\pi^2} - \frac{\lambda^2}{96\pi^2} + \frac{11\lambda^3}{23040\pi^2} - 16\left(\frac{73}{630} + 4\zeta_3^2\right)\frac{\lambda^4}{4096\pi^8} + \ldots$

✓ Agreement with MHV 4-point gluon amplitudes of N=4 SYM at four loops

$$\log\mathcal{A} = \frac{f(\lambda)}{\epsilon^2} + \ldots$$

[Bern, Czakon, Dixon, Kosower, Smirnov, 06]

- At strong coupling $f(\lambda) = \frac{\sqrt{\lambda}}{\pi}\left[1 - \frac{3\,\ln 2}{\sqrt{\lambda}} - \frac{\mathrm{K}}{(\sqrt{\lambda})^2} + \ldots\right]$ **[Basso, Korchemsky 07]**

- **STRINGS**: Agreement with two loops string calculations ! **[Roiban, Tseytlin 07]**

Motivation of our work

Large spin expansion ***BEYOND the leading order***

$$\gamma_{\mathcal{O}_\mathcal{S}}(S) = f(\lambda)\,\log S + B(\lambda) + \ldots$$

First subleading order

- Relation between B and $g(\lambda)$ $\quad \log \mathcal{A} = \dfrac{f(\lambda)}{\epsilon^2} + \dfrac{g(\lambda)}{\epsilon} + \ldots$ **[Dixon, Magnea, Sterman 08]**

  confirmed at strong coupling! **[Alday 09]**

- Integral equation for B **[Freyhult, Zieme 09]**

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Further subleading orders

<u>*need closed formulas as functions of the spin S*</u>

- Most powerful tool: **Bethe equations** (+ Lüscher corrections)

  Solvable only **numerically with QCD-inspired** Ansätze **...**

N=4 SYM and QCD interplay: I

Maximum Transcendentality Principle (MTP)

The N=4 (universal) twist two anomalous dimension at n loops is a linear combination of harmonic sums of transcendentality $2n-1$.

[Kotikov, Lipatov, Onishchenko, Velizhanin, 04]

$$\gamma^{(1)}(S) = \sum_{|\tau|=1} c_\tau S_\tau(S) = c\, S_1(S)$$

$$\gamma^{(2)}(S) = \sum_{|\tau|=3} c_\tau S_\tau(S)$$

$$\gamma^{(3)}(S) = \sum_{|\tau|=5} c_\tau S_\tau(S)$$

(Nested) harmonic sums

$$S_a(S) = \sum_{n=1}^{S} \frac{(\mathrm{sign(a)})^n}{n^{|a|}}$$

$$S_{a,\mathbf{b}}(S) = \sum_{n=1}^{S} \frac{(\mathrm{sign}(a))^n}{n^{|a|}} S_{\mathbf{b}}(n)$$

Three-loop twist two anomalous dimension extracted from the "most complicated terms" of the QCD result of **[Moch, Vermaresen, Vogt, 04]** with C_F=C_A=N_C

- MTP confirmed in **[Kotikov, Rej, Zieme, 08]** and <u>generalized</u> in **[Beccaria, Forini 08]**

$$\boxed{\gamma_\tau(n) = \sum_{k+\ell=\tau} \frac{H_{\tau,\ell}(n)}{(n+1)^k}}$$

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N=4 SYM and QCD interplay: II

x $\to$ 0, *negative* spin *S* (BFKL equation)

Prediction for poles arising for negative values of the spin [Balitsky, Fadin, Kuraev, Lipatov 77]

➡ Breakdown of the Bethe eqs. at four loops [Kotikov, Lipatov, Rej, Staudacher, Velizhanin, 07]

➡ Correctness of the full twist two anomalous dimension! [Bajnok, Janik Lukowsky, 08]

Large spin expansion of closed formulas

Experiments

Ex.1 One-loop anomalous dimension (N=4 SYM twist two)

$$\gamma_1(S) = 8\,S_1(S) = 8\,\ln\bar{S} + \frac{4}{S} - \frac{3}{2\,S^2} + \frac{1}{15\,S^4} - \frac{2}{63\,S^6}, \qquad \bar{S} = S + \gamma_E$$

Express in terms of $C^2 = S(S+1)$

$$\gamma_1(C) \sim 8\ln C + \frac{4}{3\,C^2} - \frac{4}{15\,C^4} + \frac{32}{315\,C^6} + \dots$$

Only *even* powers of C^{-1}

Ex.2 Two-loops

$$\begin{aligned}\gamma_2(S) &= -16\Big[S_3 + S_{-3} - 2S_{-2,1} + 2S_1(S_2 + S_{-2})\Big] \\ &\sim -\frac{8}{3}\pi^2\ln S - 24\zeta_3 + \frac{1}{S}\Big(32\ln S - \frac{4\pi^2}{3}\Big) + \dots\end{aligned}$$

All powers of C^{-1}, but odd powers of S^{-1} are not independent!

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- QCD twist operators up to three loops **[Moch, Vermaresen, Vogt, 04]**

$$\gamma(S) = A\,\log S + B + C\,\frac{\log S}{S} + \ldots$$

$$C = \frac{A^2}{2}$$

$\longrightarrow$ Non trivial! A, B, C depend on the coupling!

Generalized Reciprocity in (QCD and) N=4 SYM

Rephrase the large *S* expansion of γ in terms of another function f

$$\gamma = \mathrm{f}\left(S + \tfrac{1}{2}\gamma\right)$$

the ***evidence*** is that f has a (large S) ***parity invariant*** expansion of the form

$$\mathrm{f}\left(S\right) = \sum_n \frac{a_n(\ln C)}{C^{2n}} \qquad\qquad C^2 = S(S+1)$$

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In Mellin space, ***parity invariance*** becomes

$$\mathrm{F}(x) = -x\,\mathrm{F}\left(\frac{1}{x}\right) \qquad \text{where} \qquad \mathrm{f}(x) = \int_0^1 dx\, x^{S-1}\,\mathrm{F}(x)$$

or a ***generalized*** *(Gribov-Lipatov)* <u>***reciprocity***</u> .

Original GL reciprocity formulated for splitting functions. **[Gribov, Lipatov, 72]**
<u>Broken</u> in QCD beyond leading order. **[Curci, Furmansky, Petronzio, 80]**

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$\star$ In QCD $\quad \gamma(S) = f(S + \frac{1}{2}\gamma(S) - \frac{1}{2}\beta)$ **[Basso, Korchemsky 06]**

Interpretation I: revisiting parton evolution

- The ***DGLAP evolution equations*** for the parton distribution function

$$\frac{df_\sigma(x,Q^2)}{d\log Q^2} = \int_x^1 \frac{dz}{z} P_\sigma(z,\alpha_s) f_\sigma\left(\frac{x}{z},Q^2\right)$$

$\sigma = S, T$ space-like (DIS) time-like (e^+e^- ann)

or, in Mellin space,

$$\frac{df_\sigma(S,Q^2)}{d\log Q^2} = -\frac{1}{2}\gamma_\sigma(S,\alpha_s(Q^2))\; f_\sigma(S,Q^2)$$

$$\gamma_\sigma(S,Q^2) = -\frac{1}{2}\int_0^1 \frac{dx}{x}\, x^S\, P_\sigma(x,\alpha_s(Q^2))$$

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space-like (DIS)
time-like (e^+e^- ann)

or, in Mellin space,

$$\frac{df_\sigma(S,Q^2)}{d\log Q^2} = -\frac{1}{2}\gamma_\sigma(S,\alpha_s(Q^2))\ f_\sigma(S,Q^2) \qquad \gamma_\sigma(S,Q^2) = -\frac{1}{2}\int_0^1 \frac{dx}{x}\, x^S\, P_\sigma(x,\alpha_s(Q^2))$$

- Proposal: ***Reciprocity Respecting Evolution Equations*** **[Dokshitzer, Marchesini, Salam, 05]**

$$\frac{df_\sigma(x,Q^2)}{d\log Q^2} = \int_x^1 \frac{dz}{z} \mathcal{P}(z,\alpha_s) f_\sigma\left(\frac{x}{z}, z^\sigma Q^2\right) \qquad \sigma = \mp$$

with a universal kernel $\mathcal{P}$ (identical in for space-like and time-like evolution)

[Gribov, Lipatov, 72]

Interpretation I: revisiting parton evolution

- The ***DGLAP evolution equations*** for the parton distribution function

$$\frac{df_\sigma(x,Q^2)}{d\log Q^2} = \int_x^1 \frac{dz}{z} P_\sigma(z,\alpha_s) f_\sigma\left(\frac{x}{z},Q^2\right) \qquad \sigma = S,T$$

space-like (DIS)
time-like (e^+e^- ann)

or, in Mellin space,

$$\frac{df_\sigma(S,Q^2)}{d\log Q^2} = -\frac{1}{2}\gamma_\sigma(S,\alpha_s(Q^2))\ f_\sigma(S,Q^2) \qquad \gamma_\sigma(S,Q^2) = -\frac{1}{2}\int_0^1 \frac{dx}{x}\, x^S\, P_\sigma(x,\alpha_s(Q^2))$$

- Proposal: ***Reciprocity Respecting Evolution Equations*** **[Dokshitzer, Marchesini, Salam, 05]**

$$\frac{df_\sigma(x,Q^2)}{d\log Q^2} = \int_x^1 \frac{dz}{z}\mathcal{P}(z,\alpha_s) f_\sigma\left(\frac{x}{z}, z^\sigma Q^2\right) \qquad \sigma = \mp$$

with a universal kernel $\mathcal{P}$ (identical in for space-like and time-like evolution)
[Gribov, Lipatov, 72]

Evolution is now solved by the ***non-linear relation***

$$\gamma_\sigma = \mathcal{P}\left(S - \frac{1}{2}\sigma\gamma_\sigma(S)\right)$$

Assuming $\mathcal{P}(x) = -x\,\mathcal{P}\left(\frac{1}{x}\right)$ MVV relations satisfied ! (up to a term..)

Intepretation II: conformal symmetry

[Basso Korchemsky 06]

Operators $\mathcal{O} = \mathrm{Tr}\{D_+^{\kappa_1} X \ldots D_+^{\kappa_J} X\}$: reprs of the ***collinear SL(2; R) subgroup*** of SO(2, 4)

[Ohrndorf 82]

Different SL(2; R) multiplets cannot mix under renormalization: their anomalous dimension depends on the ***conformal SL(2; R) spin s***

$$s = \frac{(S + \Delta)}{2}$$

S = Lorentz spin Δ = scaling dimension

The scaling dimension receives anomalous contribution due to interaction

$$\Delta(\lambda) = S + J + \gamma_\lambda(S)$$

⇒ the *conformal spin gets modified in higher loops* (e.g. X = scalars) [Belitsky Mueller 98]

$$s(\lambda = 0) = S + \frac{J}{2} \longrightarrow s(\lambda) = S + \frac{J}{2} + \frac{1}{2}\gamma(S)$$

or: *The anomalous dimension is a* (twist-dep) *function of the conformal spin*

$$\boxed{\gamma(S) = \mathrm{f}\left(S + \frac{1}{2}\gamma(S)\right)}$$

$$\boxed{\text{In QCD} \quad \gamma(S) = f(S + \tfrac{1}{2}\gamma(S) - \tfrac{1}{2}\beta)}$$

The evidence of reciprocity

[Basso, Korchemsky 06]

The ***parity invariance*** ***(reciprocity respecting*** RR) of f

$$\mathrm{f}(S) = \sum_n \frac{a_n(\ln C)}{C^{2n}}$$

is verified for a large class of operators $\mathcal{O} = \mathrm{Tr}\{D^{k_1}X...D^{k_J}X\}$ $\quad k_1 + ... + k_J = S$

with C^2 the ***Casimir of*** SL(2,R) $\subset$ SO(4,2)

$$C^2 = (S + J\,\ell)(S + J\,\ell - 1)$$

J : twist $\quad \ell$:

φ	λ	A
$\frac{1}{2}$	1	$\frac{3}{2}$

✓ All twist-2 anomalous dimensions in QCD and $\mathcal{N}$=4 SYM (3 loops) [Basso, Korchemsky 06]

✓ Twist-2 at four loops (*including wrapping*) [Beccaria Forini 09]

✓ Twist-3 scalars, gauginos, gluons in $\mathcal{N}$=4 SYM

[Beccaria, Marchesini, Dokshitzer 07] [Beccaria 07] [Beccaria Forini 08]

[Notice: for twist-J > 2, anomalous dimensions, as functions of S, occupy a *band*!
Reciprocity has been only detected for the ***minimal energy***, lower edge of the band]

Reciprocity and AdS / CFT?

Anomalous dimensions of operators in N=4 SYM ⟷ Energies of semiclassical strings in $AdS_5 \times S^5$

Bosonic $AdS_5 \times S^5$ σ-model $$I = \frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma\, G_{\mu\nu}(X)\,\partial_a X^\mu \partial^a X^\nu + \ldots$$

states belonging to reprs of $SO(2,4)$ x $SO(6)$ ➡ Classified by $(E, S_1, S_2\,; J_1, J_2, J_3)$.

- Operators with large Lorentz spin and minimal energy: *folded* strings rotating in AdS_3

$$E = S + \frac{\sqrt{\lambda}}{\pi}\log\frac{S}{\sqrt{\lambda}} + \ldots \qquad S \gg \sqrt{\lambda}$$

[Gubser, Klebanov, Polyakov 02]

- Classical energy $E = \sqrt{\lambda}\,\mathcal{E}(\omega)$ and classical spins $S = \sqrt{\lambda}\,\mathcal{S}(\omega)$, $J = \sqrt{\lambda}\,\mathcal{J}(\omega)$

  The energy organizes in the semiclassical expansion

$$E = \sqrt{\lambda}\left[E_0 + \frac{E_1}{\sqrt{\lambda}} + \frac{E_2}{(\sqrt{\lambda})^2} + \ldots\right]$$

- Logarithmic behavior confirmed at one and two loops

[Frolov, Tseytlin 02]
[Roiban, Tseytlin 07]

Folded string in AdS$_{3:}$ I

S^5 momentum J ignored $\longleftrightarrow$ twist of the operator *small* compared to Lorentz spin.

- Ansatz for a stationary solution rotated and boosted $(0 \leq \rho(\sigma) \leq \rho_0)$

$$\rho = \rho(\sigma) \qquad t = \kappa\, \tau \qquad \phi = \omega\, \tau \qquad \coth^2 \rho_0 = 1 + \eta$$

- Exact solution

$$\sinh \rho = \frac{1}{\sqrt{\eta}}\, \mathrm{sn}\left[\kappa \sqrt{\eta}\, \sigma, -\frac{1}{\eta}\right], \qquad 0 \leq \sigma \leq \frac{\pi}{2}$$

- Integrals of motion: energy and spin

$$E = P_t = \sqrt{\lambda}\, \kappa \int_0^{2\pi} \frac{d\sigma}{2\pi} \cosh^2 \rho \equiv \sqrt{\lambda}\, \mathcal{E} \qquad\qquad S = P_\phi = \sqrt{\lambda}\, \omega \int_0^{2\pi} \frac{d\sigma}{2\pi} \sinh^2 \rho \equiv \sqrt{\lambda}\, \mathcal{S}$$

In parametric form

$$\mathcal{E} = \frac{2}{\pi \sqrt{\eta}}\, \mathbb{E}\left(-\frac{1}{\eta}\right) \qquad \mathcal{S} = \frac{2}{\pi \sqrt{\eta}} \left[\mathbb{E}\left(-\frac{1}{\eta}\right) - \mathbb{K}\left(-\frac{1}{\eta}\right)\right]$$

▶ In the "long string" limit $\eta \to 0$

✔ *Structure!* $$\mathcal{E} = \mathcal{S} + \frac{\ln \bar{\mathcal{S}} - 1}{\pi} + \frac{\ln \bar{\mathcal{S}} - 1}{2\,\pi^2\,\mathcal{S}} - \frac{2\ln^2 \bar{\mathcal{S}} - 9\ln \bar{\mathcal{S}} + 5}{16\,\pi^3 \mathcal{S}^2} + \ldots \qquad \bar{\mathcal{S}} \equiv 8\,\pi\,\mathcal{S}$$

✔ *Reciprocity!* the function f runs in even negative powers of the Casimir $\mathcal{C} \equiv \mathcal{S}$

$$\tilde{\mathrm{f}}(\mathcal{S}) \equiv \frac{\mathrm{f}}{\sqrt{\lambda}} = \frac{1}{\pi}\Big[\ln \bar{\mathcal{S}} - 1 + \frac{\ln \bar{\mathcal{S}} + 1}{16\pi^2 \mathcal{S}^2} + \mathcal{O}\left(\frac{1}{\mathcal{S}^4}\right)\Big] + \mathcal{O}(\frac{1}{\sqrt{\lambda}})$$

Folded string in AdS_3 x S^1

▶ Add $J = \sqrt{\lambda}\,\mathcal{J}$ and consider the limit $\mathcal{S} \gg \mathcal{J}$

▶ "Slow" long strings $\mathcal{J} \ll \ln \mathcal{S}$

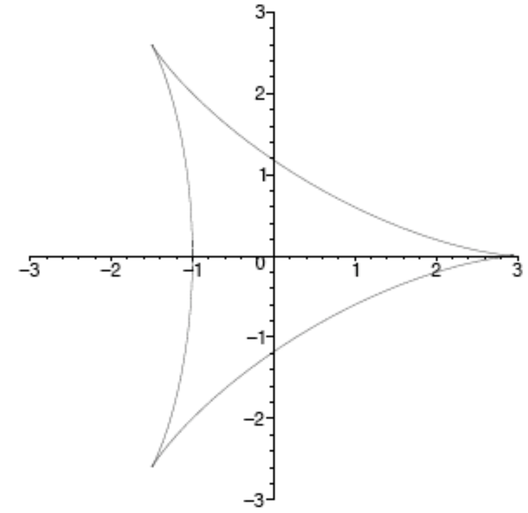
Structure

$$\mathcal{E} - \mathcal{S} - \mathcal{J} \approx \frac{1}{\pi}(\ln \bar{\mathcal{S}} - 1) + \frac{\pi\,\mathcal{J}^2}{2\,\ln \bar{\mathcal{S}}} - \frac{\pi^3\,\mathcal{J}^4}{8\,\ln^3 \bar{\mathcal{S}}}(1 - \frac{1}{\ln \bar{\mathcal{S}}}) + \ldots$$
$$+ \frac{4}{\bar{\mathcal{S}}}\Big[\frac{1}{\pi}(\ln \bar{\mathcal{S}} - 1) + \frac{\pi\,\mathcal{J}^2}{2\ln^2 \bar{\mathcal{S}}} - \frac{3\pi^3\,\mathcal{J}^4}{4\ln^4 \bar{\mathcal{S}}}(1 - \frac{2}{3\,\ln \bar{\mathcal{S}}}) + \ldots\Big] + \ldots$$

✔ *Reciprocity!* f runs in even negative powers of the Casimir $\mathcal{C} \equiv \mathcal{S} + \frac{1}{2}\mathcal{J}$

▶ "Fast" long strings $\ln \mathcal{S} \ll \mathcal{J} \ll \mathcal{S}$: again reciprocity respecting .

Spiky strings
vs. higher twist on excited trajectories



Rigidly rotating , n cusps or spikes.

Same asymptotic log BUT proportional to # spikes
$n > 2 \Rightarrow$ higher energy for given spin

[Belitsky, Gorsky, Korchemsky, 03]
[Kruczenski 04]
[Dorey 07]

$$\mathcal{E} - \mathcal{S} = \frac{n}{2\,\pi} \ln \frac{16\,\pi\,\mathcal{S}}{n} + \ldots$$

Beyond the leading large spin limit: the ends of the spikes do not approach the bndry

✓ ***~~Reciprocity:~~ NO!***

$$\tilde{\mathrm{f}}(\mathcal{S}) \quad = \quad \frac{n}{2\pi}\left[\ln\bar{\mathcal{S}} + q_1 + \frac{q_2}{\bar{\mathcal{S}}} + \ldots\right]$$

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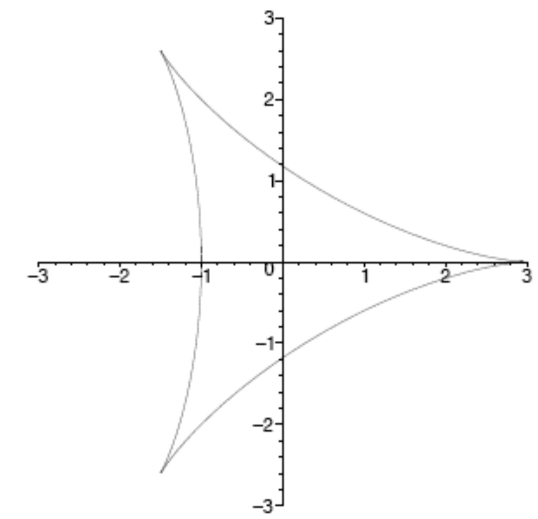

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[Dorey 07]

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✓ ***~~Reciprocity:~~ NO!***

$$\tilde{\mathrm{f}}(\mathcal{S}) \quad = \quad \frac{n}{2\pi}\left[\ln \bar{\mathcal{S}} + q_1 + \frac{q_2}{\bar{\mathcal{S}}} + \ldots\right]$$

However

$$\mathcal{E} - \mathcal{S} = \frac{n}{2\pi} \ln \mathcal{S} + \frac{n^2}{8\,\pi^2 \mathcal{S}} \ln \mathcal{S} - \frac{n^3}{64\,\pi^3 \mathcal{S}^2} \ln^2 \mathcal{S} + \ldots$$

is consistent with the functional relation

$$E - S = \frac{\sqrt{\lambda}\,n}{2\pi} \ln\left(S + \frac{1}{2}\frac{\sqrt{\lambda}\,n}{2\pi}\ln S + \ldots\right) + \ldots \qquad E - S = \mathrm{f}\left(S + \frac{1}{2}(E - S)\right)$$

✓ Exactly as it happens at weak coupling! **[Belitsky, Korchemsky, Pasechnik 08]**
For corresponding to ***higher twist*** $J=n$ operators with ***non minimal*** anomalous dim.

String perturbation theory: I

- With the 1-loop corrections included:
  the structure of the large spin expansion remains the same?
  are the "MVV" constraints still satisfied?

Gauge and string perturbative expansions are different!

Gauge theory	$\lambda \ll 1$	$S =$	fixed and then	$S \gg 1$
String theory	$\sqrt{\lambda} \gg 1$	$\frac{S}{\sqrt{\lambda}} =$	fixed and then	$\frac{S}{\sqrt{\lambda}} \gg 1$

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String theory	$\sqrt{\lambda} \gg 1$	$\frac{S}{\sqrt{\lambda}} =$	fixed and then	$\frac{S}{\sqrt{\lambda}} \gg 1$

- ***Standard procedure*** for leading (1 loop) quantum corrections - folded string

Fluctuation lagrangean $\tilde{L}$ $\longrightarrow$ 2-d Effective action Γ_1

$$\xrightarrow{\text{Euclidean}} \qquad E_1 = \frac{\Gamma_1}{\kappa \mathcal{T}} \,, \qquad \mathcal{T} \equiv \int d\tau \to \infty$$

String perturbation theory II: details

[Foerste, Ghoshal, Theisen 00]

[Drukker, Gross, Tseytlin 00]

[Frolov, Tseytlin 02]

- 1-loop correction to the effective action (conformal gauge)

$$\boxed{\begin{aligned}\Gamma_1 = \quad & \frac{\mathcal{T}}{4\pi}\int_{-\infty}^{\infty} d\omega \Big[-8\ln\frac{\det[-\partial_1^2+\omega^2+\rho'^2]}{\det[-\partial_1^2+\omega^2+\kappa_0^2]} + 2\ln\frac{\det[-\partial_1^2+\omega^2+2\rho'^2]}{\det[-\partial_1^2+\omega^2+2\kappa_0^2]} \\ & - \ln\frac{\det^8[-\partial_1^2+\omega^2+\kappa_0^2]}{\det^2[-\partial_1^2+\omega^2+2\kappa_0^2]\ \det^6[-\partial_1^2+\omega^2]} + \ln\frac{\det Q_\omega}{\det Q_\omega^{(0)}} - \ln\frac{\det P_\omega}{\det Q_\omega^{(0)}}\Big]\end{aligned}}$$

Q_ω: quadratic fluctuation operator $\quad Q_\omega = Q_\omega^{(0)} + \eta\, Q_\omega^{(1)} + \dots$

- We calculate the $\mathcal{O}(\eta)$ correction $\qquad \Gamma_1 = \Gamma_1^{(0)} + \eta\,\Gamma_1^{(1)} + \mathcal{O}(\eta^2)$

- Order zero contribution (constant mass relativistic fields on the cilinder!)

$$\Gamma_1^{(0)} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\Big[2\sqrt{n^2+2\kappa_0^2} + \sqrt{n^2+4\kappa_0^2} + 5\sqrt{n^2} - 8\sqrt{n^2+\kappa_0^2}\Big]$$ [Frolov, Tseytlin 02]

- Order eta contribution

$$\Gamma_1^{(1)} = -\frac{\mathcal{T}\eta}{4\pi}\sum_{n=-\infty}^{\infty}\Big[\frac{8\kappa_0}{\sqrt{n^2+\kappa_0^2}} - \frac{4\kappa_0}{\sqrt{n^2+2\kappa_0^2}} - \frac{4\kappa_0}{\sqrt{n^2+4\kappa_0^2}}\Big]$$

String perturbation theory: III Final result

[Beccaria, VF, Tirziu, Tseytlin 08]

▶ The 1-loop correction to the energy up to order $1/S$ is

$$E_1 = b_0 \ln \mathcal{S} + b_c + \frac{b_{11} \ln \mathcal{S} + b_{10}}{\mathcal{S}} + \mathcal{O}\left(\frac{\ln^2 \mathcal{S}}{\mathcal{S}^2}\right)$$

c: sensitive to turning point contributions

$$b_0 = -\frac{3 \ln 2}{\pi} \ , \qquad b_c = \frac{1}{\pi}\big(- 3 \ln 2 \ \ln 8\pi + \mathrm{c}\big), \qquad b_{11} = -\frac{3 \ln 2}{\pi^2}, \qquad b_{10} = \frac{1}{2\pi^2}\Big[- 6 \ln 2 \ \big(\ln 8\pi - \frac{1}{2}\big) + \mathrm{c}\Big]$$

✔ *The* ***structure*** *is identical to the one at weak coupling!*
Despite the different order of limits...

✔ ***Coefficients*** *in the large S expansion* ***related by reciprocity!***

$$b_{11} = a_0 b_0 \ , \qquad b_{10} = \frac{1}{2}(a_0 b_c + b_0 a_c)$$

where $\mathcal{E} - \mathcal{S} = a_0 \ \ln \mathcal{S} + a_c + \ldots$ is the expansion of the classical part.

Strong **indication that reciprocity holds also at strong coupling**

Uses of reciprocity

you can't explain it...just assume it!

- **Guiding principle** for further orders (string sigma model) calculations
- **Tool** that drastically simplify calculations of multiloop anomalous dimensions

Example: twist three at 5 loops. $\boxed{\mathrm{Tr}\,(\mathcal{D}^{s_1} Z\, \mathcal{D}^{s_2}\, Z\, \mathcal{D}^{s_3} Z) \quad \text{with} \quad S = s_1 + s_2 + s_3}$

Maximum Transcendentality Principle: 256 terms! Numerically challenging

1. Look for a reciprocity respecting basis of (complementary) harmonic sums

$$\boxed{\begin{array}{ll} \text{Map:} & \underline{\omega}_a(\underline{S_{b,\mathbf{c}}}) = \underline{S_{a,b,\mathbf{c}}} - \frac{1}{2}\,\underline{S_{a+b,\mathbf{c}}}. \\ \text{Combinations:} & \underline{\Omega}_{a_1,\dots,a_d} \;=\; \underline{\omega}_{a_1}\,(\Omega_{a_2,\dots,a_d}) \end{array}}$$

[Beccaria, VF 09]

2. Write the MTP ansatz in terms of the RR basis $(-1)^{a_1+\ldots+a_d} = (-1)^d$
3. List of values from Bethe Ansatz (<u>now</u> an over-determined set of equations!)
4. Evaluate wrapping corrections from Luescher formulas

$$
\begin{aligned}
\gamma_5^{\text{asymp}} &= 136S_9 + 368S_{1,8} + 2832S_{2,7} + 4272S_{3,6} + 848S_{4,5} - 3024S_{5,4} - 2736S_{6,3} - 1168S_{7,2} \\
&-496S_{8,1} - 5376S_{1,1,7} - 12352S_{1,2,6} - 8832S_{1,3,5} + 1600S_{1,4,4} + 3968S_{1,5,3} - 64S_{1,6,2} \\
&-1344S_{1,7,1} - 12352S_{2,1,6} - 13760S_{2,2,5} - 2112S_{2,3,4} + 4288S_{2,4,3} - 960S_{2,5,2} - 5440S_{2,6,1} \\
&-9088S_{3,1,5} - 2432S_{3,2,4} + 5120S_{3,3,3} + 2688S_{3,4,2} - 4160S_{3,5,1} + 1280S_{4,1,4} + 5824S_{4,2,3} \\
&+6400S_{4,3,2} + 2112S_{4,4,1} + 5120S_{5,1,3} + 6208S_{5,2,2} + 5312S_{5,3,1} + 3904S_{6,1,2} + 3904S_{6,2,1} \\
&+1728S_{7,1,1} + 21504S_{1,1,1,6} + 22784S_{1,1,2,5} + 5632S_{1,1,3,4} - 1280S_{1,1,4,3} + 6912S_{1,1,5,2} \\
&+11520S_{1,1,6,1} + 22784S_{1,2,1,5} + 9088S_{1,2,2,4} - 1024S_{1,2,3,3} + 6784S_{1,2,4,2} + 17152S_{1,2,5,1} \\
&+5504S_{1,3,1,4} - 3456S_{1,3,2,3} - 1536S_{1,3,3,2} + 7680S_{1,3,4,1} - 4480S_{1,4,1,3} - 6272S_{1,4,2,2} \\
&-3584S_{1,4,3,1} - 3840S_{1,5,1,2} - 3840S_{1,5,2,1} + 768S_{1,6,1,1} + 22784S_{2,1,1,5} + 9088S_{2,1,2,4} \\
&-1024S_{2,1,3,3} + 6784S_{2,1,4,2} + 17152S_{2,1,5,1} + 9088S_{2,2,1,4} - 2688\,S_{2,2,2,3} + 640\,S_{2,2,3,2} \\
&+13440S_{2,2,4,1} - 3456S_{2,3,1,3} - 7040S_{2,3,2,2} - 768S_{2,3,3,1} - 4480S_{2,4,1,2} - 4480S_{2,4,2,1} \\
&+2816S_{2,5,1,1} + 6272S_{3,1,1,4} - 2944S_{3,1,2,3} - 1536S_{3,1,3,2} + 7936S_{3,1,4,1} - 2944S_{3,2,1,3} \\
&-7296S_{3,2,2,2} - 768S_{3,2,3,1} - 6656S_{3,3,1,2} - 6656S_{3,3,2,1} - 1024S_{3,4,1,1} - 3968S_{4,1,1,3} \\
&-6528S_{4,1,2,2} - 3584S_{4,1,3,1} - 6528S_{4,2,1,2} - 6528S_{4,2,2,1} - 4864S_{4,3,1,1} - 5376S_{5,1,1,2} \\
&-5376S_{5,1,2,1} - 5376S_{5,2,1,1} - 4608S_{6,1,1,1} - 32768S_{1,1,1,1,5} - 10240S_{1,1,1,2,4} - 3072S_{1,1,1,3,3} \\
&-17920S_{1,1,1,4,2} - 30720S_{1,1,1,5,1} - 10240S_{1,1,2,1,4} - 8704S_{1,1,2,3,2} - 24064S_{1,1,2,4,1} \\
&+1024S_{1,1,3,1,3} + 2560S_{1,1,3,2,2} - 4096S_{1,1,3,3,1} - 512S_{1,1,4,1,2} - 512S_{1,1,4,2,1} - 10240S_{1,1,5,1,1} \\
&-10240S_{1,2,1,1,4} - 8704S_{1,2,1,3,2} - 24064S_{1,2,1,4,1} + 3072S_{1,2,2,2,2} - 6656S_{1,2,2,3,1} \\
&+512S_{1,2,3,1,2} + 512S_{1,2,3,2,1} - 10752S_{1,2,4,1,1} + 1024S_{1,3,1,1,3} + 3072S_{1,3,1,2,2} - 3584S_{1,3,1,3,1} \\
&+3072S_{1,3,2,1,2} + 3072S_{1,3,2,2,1} - 2560S_{1,3,3,1,1} + 3072S_{1,4,1,1,2} + 3072S_{1,4,1,2,1} + 3072S_{1,4,2,1,1} \\
&+3072S_{1,5,1,1,1} - 10240S_{2,1,1,1,4} - 8704S_{2,1,1,3,2} - 24064S_{2,1,1,4,1} + 3072S_{2,1,2,2,2} \\
&-6656S_{2,1,2,3,1} + 512S_{2,1,3,1,2} + 512S_{2,1,3,2,1} - 10752S_{2,1,4,1,1} + 3072S_{2,2,1,2,2} - 6656S_{2,2,1,3,1} \\
&+3072S_{2,2,2,1,2} + 3072S_{2,2,2,2,1} - 5632S_{2,2,3,1,1} + 3072S_{2,3,1,1,2} + 3072S_{2,3,1,2,1} + 3072S_{2,3,2,1,1} \\
&+3072S_{2,4,1,1,1} + 3072S_{3,1,1,2,2} - 4096S_{3,1,1,3,1} + 3072S_{3,1,2,1,2} + 3072S_{3,1,2,2,1} - 2560S_{3,1,3,1,1} \\
&+3072S_{3,2,1,1,2} + 3072S_{3,2,1,2,1} + 3072S_{3,2,2,1,1} + 4608S_{3,3,1,1,1} + 3072S_{4,1,1,1,2} + 3072S_{4,1,1,2,1} \\
&+3072S_{4,1,2,1,1} + 3072S_{4,2,1,1,1} + 3072S_{5,1,1,1,1} + 16384S_{1,1,1,1,3,2} + 32768S_{1,1,1,1,4,1} \\
&+8192S_{1,1,1,2,3,1} + 4096S_{1,1,1,3,1,2} + 4096S_{1,1,1,3,2,1} + 20480S_{1,1,1,4,1,1} + 8192\,S_{1,1,2,1,3,1} \\
&+12288S_{1,1,2,3,1,1} + 8192S_{1,2,1,1,3,1} + 12288S_{1,2,1,3,1,1} + 8192S_{2,1,1,1,3,1} + 12288S_{2,1,1,3,1,1} \\
&-16384S_{1,1,1,1,3,1,1} + \zeta_3\,(896S_6 - 2304S_{1,5} - 1792S_{2,4} - 768S_{3,3} - 1792S_{4,2} - 2304S_{5,1} \\
&+2560S_{1,1,4} + 512S_{1,2,3} + 1536S_{1,3,2} + 3584S_{4,1} + 512S_{2,1,3} + 1536S_{2,3,1} + 512S_{3,1,2} \\
&+512S_{3,2,1} + 2560S_{4,1,1} - 2048S_{1,1,3,1} - 2048S_{1,3,1,1}) + 1280\,\zeta_5\,(S_{1,3} + S_{3,1} - S_4)
\end{aligned}
$$

$$
\begin{aligned}
\gamma^{\text{wrapping}} &= -64\,g^{10}\,S_1^2\big(35\zeta_7 - 40S_2\zeta_5 + (-8S_4 + 16S_{2,2})\zeta_3 \\
&\quad +2S_7 - 4S_{2,5} - 2S_{3,4} - 4S_{4,3} - 2S_{6,1} + 8S_{2,2,3} + 4S_{3,3,1}\big)
\end{aligned}
$$

- Correct asymptotics:
  leading (cusp anomaly)
  subleading (virtual function)

- Analytical continuation (no BFKL!)
  pole structure agrees with conjecture

[Kotikov, Lipatov, Rej, Staudacher, Velizhanin, 07]

- The full result is
  <u>reciprocity respecting</u>

[Beccaria, VF, Lukowski, Zieme 09]

Conclusive remarks

- Reciprocity: empirical evidence of N=4 SYM at weak coupling,
  built-in property of Bethe equations and of the known examples of wrapping corrs.

- Strong evidence also at strong coupling.

- Guiding principle to get closed formulas for anomalous dimensions.

- Not strictly related to the planar limit, or to SUSY (QCD!)

Outlook 1: Strings and further 1/S orders, generalizations to (S,J)

[work in progress!]

Outlook 2: Connection with Amplitudes/or? in N=4 SYM

$$\log \mathcal{A} = \frac{f(\lambda)}{\epsilon^2} + \frac{g(\lambda)}{\epsilon} + \ldots ?$$

Look at other IR safe quantities.

[work in progress!]