Casimir apparatuses in a weak gravitational field

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Abstract

We review and assess a part of the recent work on Casimir apparatuses in the weak gravitational field of the Earth. For a free, real massless scalar field subject to Dirichlet or Neumann boundary conditions on the parallel plates, the resulting regularized and renormalized energy-momentum tensor is covariantly conserved, while the trace anomaly vanishes if the massless field is conformally coupled to gravity. Conformal coupling also ensures a finite Casimir energy and finite values of the pressure upon parallel plates. These results have been extended to an electromagnetic field subject to perfect conductor (hence idealized) boundary conditions on parallel plates, by various authors. The regularized and renormalized energy-momentum tensor has been evaluated up to second order in the gravity acceleration. In both the scalar and the electromagnetic case, studied to first order in the gravity acceleration, the theory predicts a tiny force in the upwards direction acting on the apparatus. This effect is conceptually very interesting, since it means that Casimir energy is indeed expected to gravitate, although the magnitude of the expected force makes it necessary to overcome very severe signal-modulation problems.

I. INTRODUCTION

Ever since Casimir discovered that suitable differences of zero-point energies of the quantized electromagnetic field can be made finite and provide measurable effects [1], several efforts have been produced to understand the physical implications and applications of this property [2]–[6]. In particular, we are here going to review the recent theoretical discovery that Casimir energy gravitates [7]-[10]. In Ref. [9], this was proved as part of an investigation that led, for the first time, to the evaluation of the energy-momentum tensor of a Casimir apparatus in a weak gravitational field (cf. the work in Ref. [11]). In that piece of work, Maxwell theory was quantized via functional integral, with perfect conductor boundary conditions on parallel plates at distance a from each other. On using Fermi–Walker coordinates, where the (x_1, x_2) coordinates span the plates, while the $z = x_3$ axis coincides with the vertical upward direction (so that the plates have equations $z = 0$ and $z = a$, respectively), and working to first order in the constant gravity acceleration g, the spacetime metric reads as [9]

$$ds^2 = -c^2 \left(1 + \varepsilon \frac{z}{a}\right) dt^2 + dx_1^2 + dx_2^2 + dz^2 + \mathrm{O}(|x|^2), \tag{1.1}$$

where $\varepsilon \equiv \frac{2ga}{c^2}$.

Our paper provides a review of some key findings by the authors and by other research groups interested in the same topics. For this purpose, Sec. II studies the Feynman Green function for the scalar wave operator to zeroth and first order in ε, Sec. III obtains the resulting regularized and renormalized energy-momentum tensor while Sec. IV evaluates Casimir energy and pressure upon the plates. All of this with Dirichlet conditions on the plates for the Green function. The case of Neumann boundary conditions is considered in Sec. V, while the electromagnetic analysis is summarized in Sec. VI. Concluding remarks are presented in Sec. VII.

II. FEYNMAN GREEN FUNCTION TO ZEROTH AND FIRST ORDER

To first order in the ε parameter of Sec. I, the only nonvanishing Christoffel symbols associated with the metric (1.1) are

$$\Gamma^0_{\ 30} = \Gamma^0_{\ 03} = \frac{\varepsilon}{2(a + \varepsilon z)} \sim \frac{\varepsilon}{2a} + \mathrm{O}(\varepsilon^2), \ \Gamma^3_{\ 00} \sim \frac{\varepsilon}{2a} + \mathrm{O}(\varepsilon^2). \tag{2.1}$$

We now compute the wave operator $\Box$, the Feynman Green function of the hyperbolic operator $(\Box - \xi R)$, and eventually the Hadamard function and the regularized energy-momentum tensor.

Indeed, a Green function of the scalar wave operator obeys the differential equation

$$(\Box - \xi R)G(x,x') = -\frac{\delta(x,x')}{\sqrt{-g}}. \tag{2.2}$$

The Feynman Green function G_F is the unique symmetric complex-valued Green function which obeys the relation [12]

$$\delta G = G\, \delta F\, G,$$

where F is the invertible operator obtained from variation of the action functional with respect to the field. This definition is well suited for the purpose of defining the Feynman Green function even when asymptotic flatness does not necessarily hold [12].

In our first-order expansion in the ε parameter, the scalar curvature gives vanishing contribution to Eq. (2.2), which therefore takes the form (hereafter $\Box^0 \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$)

$$\left(\Box^0 + \frac{\varepsilon z}{(a+\varepsilon z)}\frac{\partial^2}{\partial t^2} + \Gamma^3{}_{00}\frac{a}{(a+\varepsilon z)}\frac{\partial}{\partial z}\right) G(x,x') = -\frac{\delta(x,x')}{\sqrt{-g}}. \tag{2.3}$$

We now follow our work in Ref. [9] and assume that the Feynman Green function admits the asymptotic expansion

$$G_F(x,x') \sim G^{(0)}(x,x') + \varepsilon G^{(1)}(x,x') + \mathrm{O}(\varepsilon^2). \tag{2.4}$$

Its existence is proved by the calculations described hereafter. Indeed, by insertion of (2.4) into (2.3) we therefore obtain, picking out terms of zeroth and first order in ε, the pair of differential equations

$$\Box^0 G^{(0)}(x,x') = J^{(0)}(x,x'), \tag{2.5}$$

$$\Box^0 G^{(1)}(x,x') = J^{(1)}(x,x'), \tag{2.6}$$

having set

$$J^{(0)}(x,x') \equiv -\delta(x,x'), \tag{2.7}$$

$$J^{(1)}(x,x') \equiv \frac{z}{2a}\delta(x,x') - \left(\frac{z}{a}\frac{\partial^2}{\partial t^2} + \frac{1}{2a}\frac{\partial}{\partial z}\right) G^{(0)}(x,x'). \tag{2.8}$$

Our boundary conditions are Dirichlet in the spatial variable z. Since the full Feynman function $G_F(x,x')$ is required to vanish at $z = 0, a$, this implies the following homogeneous

Dirichlet conditions on the zeroth and first-order terms:

$$G^{(0)}(x,x')\Big|_{z=0,a}=0, \tag{2.9}$$

$$G^{(1)}(x,x')\Big|_{z=0,a}=0. \tag{2.10}$$

To solve Eqs. (2.5) and (2.6), we perform a Fourier analysis of $G^{(0)}$ and $G^{(1)}$, which remains meaningful in a weak gravitational field [9], by virtue of translation invariance. In such an analysis we separate the z variable, i.e. we write (cf. [9])

$$G^{(0)}(x,x')=\int \frac{dk^0 d\vec{k}_\perp}{(2\pi)^3}\gamma^{(0)}(z,z')e^{i\vec{k}_\perp\cdot(\vec{x}_\perp-\vec{x}'_\perp)-ik^0(x_0-x'_0)}, \tag{2.11}$$

and similarly for $G^{(1)}(x,x')$, with a "reduced Green function" $\gamma^{(1)}(z,z')$ in the integrand as a counterpart of the zeroth-order Green function $\gamma^{(0)}(z,z')$ in (2.9). Equations (2.3) and (2.4) lead therefore to the following equations for reduced Green functions (hereafter $\lambda \equiv \sqrt{k_0^2-k_\perp^2}$):

$$\left(\frac{\partial^2}{\partial z^2}+\lambda^2\right)\gamma^{(0)}(z,z')=-\delta(z,z'), \tag{2.12}$$

$$\left(\frac{\partial^2}{\partial z^2}+\lambda^2\right)\gamma^{(1)}(z,z')=\frac{z}{2a}\delta(z,z')+\left(\frac{z}{a}k_0^2-\frac{1}{2a}\frac{\partial}{\partial z}\right)\gamma^{(0)}(z,z'). \tag{2.13}$$

By virtue of the Dirichlet conditions (2.9), $\gamma^{(0)}$ reads as

$$\gamma^{(0)}(z,z')=-\frac{\sin(\lambda z_<)\sin(\lambda(z_>-a))}{\lambda\sin(\lambda a)}, \tag{2.14}$$

where $z_< \equiv \min(z,z')$, $z_> \equiv \max(z,z')$. The evaluation of the reduced Green function $\gamma^{(1)}$ is slightly more involved. For this purpose, we distinguish the cases $z<z'$ and $z>z'$, and find the two equations

$$\left(\frac{\partial^2}{\partial z^2}+\lambda^2\right)\gamma_\pm^{(1)}(z,z')=j_\pm^{(1)}(z,z'), \tag{2.15}$$

where

$$j_-^{(1)}=\frac{1}{2a}\frac{\lambda\cos(\lambda z)-2zk_0^2\sin(\lambda z)}{\lambda\sin(\lambda a)}\sin(\lambda(z'-a)) \text{ if } z<z', \tag{2.16}$$

$$j_+^{(1)}=\frac{1}{2a}\frac{\lambda\cos(\lambda(z-a))-2zk_0^2\sin(\lambda(z-a))}{\lambda\sin(\lambda a)}\sin(\lambda z') \text{ if } z>z'. \tag{2.17}$$

We have therefore two different solutions in the intervals $z<z'$ and $z>z'$. In this case the differential equation (2.15) is solved by imposing the matching condition

$$\gamma_-^{(1)}(z',z')=\gamma_+^{(1)}(z',z') \tag{2.18}$$

jointly with the jump condition

$$\frac{\partial}{\partial z}\gamma_{+}^{(1)}\Big|_{z=z'} - \frac{\partial}{\partial z}\gamma_{-}^{(1)}\Big|_{z=z'} = \frac{z'}{2a}. \tag{2.19}$$

Equation (2.18) is just the continuity requirement of the reduced Green function $\gamma^{(1)}(z,z')$ at $z=z'$, while Eq. (2.19) can be obtained by integrating Eq. (2.13) in a neighborhood of z', since

$$\lim_{\epsilon\to 0}\frac{\partial}{\partial z}\gamma^{(1)}\Big|_{z'-\epsilon}^{z'+\epsilon} = \lim_{\epsilon\to 0}\int_{z'-\epsilon}^{z'+\epsilon}\frac{z}{2a}\delta(z,z')dz = \frac{z'}{2a}. \tag{2.20}$$

Bearing in mind Eq. (2.14) we can therefore write, for all z, z',

$$\begin{aligned}\gamma^{(1)}(z,z') &= \frac{1}{4a\lambda^2}\Bigg\{\left[(k_0^2-\lambda^2)(z+z') - k_0^2\left(z^2\frac{\partial}{\partial z} + z'^2\frac{\partial}{\partial z'}\right)\right]\gamma^{(0)}(z,z')\\ &- k_0^2 a^2\frac{\sin(\lambda z)\sin(\lambda z')}{\sin^2(\lambda a)}\Bigg\}.\end{aligned} \tag{2.21}$$

III. REGULARIZED AND RENORMALIZED ENERGY-MOMENTUM TENSOR

In the previous section we have focused on the Feynman Green function G_F because it is then possible to develop a recursive scheme for the evaluation of its asymptotic expansion at small ε. However, we eventually need the Hadamard function $H(x,x')$, which is obtained as [9]

$$H(x,x') \equiv 2\mathrm{Im}G_F(x,x') \sim 2\mathrm{Im}(G^{(0)}(x,x') + \varepsilon G^{(1)}(x,x')) + \mathrm{O}(\varepsilon^2). \tag{3.1}$$

The coincidence limits in the formula of the regularized and renormalized energy-momentum tensor make it necessary to perform the replacements

$$H_{;\mu'\nu} + H_{;\mu\nu'} \to P_{\mu}^{\ \mu'}H_{;\mu'\nu} + P_{\nu}^{\ \nu'}H_{;\mu\nu'},\ H_{;\sigma}^{\ \ \sigma'} \to g^{\sigma\rho}P_{\rho}^{\ \rho'}H_{;\sigma\rho'},\ H_{;\mu'\nu'} \to P_{\mu}^{\ \mu'}P_{\nu}^{\ \nu'}H_{;\mu'\nu'}, \tag{3.2}$$

where $P^{\mu}_{\ \nu'}$ is the parallel displacement bivector [13]

$$P^{\mu}_{\ \nu'} \sim \mathrm{diag}\left(1 + \frac{\varepsilon}{2a}(z'-z), 1, 1, 1\right) + \mathrm{O}(\varepsilon^2). \tag{3.3}$$

Hence we get the asymptotic expansion at small ε of the regularized energy-momentum tensor according to (hereafter we evaluate its covariant, rather than contravariant, form)

$$\langle T_{\mu\nu}\rangle \sim \langle T_{\mu\nu}^{(0)}\rangle + \varepsilon\langle T_{\mu\nu}^{(1)}\rangle + \mathrm{O}(\varepsilon^2), \tag{3.4}$$

where, on defining $s \equiv \pi z/a,\ s' \equiv \pi z'/a$, we find

$$
\begin{aligned}
\langle T^{(0)}_{\mu\nu} \rangle \ &= \ \left[-\frac{\pi^2}{1440a^4} - \lim_{s' \to s} \frac{\pi^2}{2a^4(s-s')^4} \right] \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \\
&+ \ \left(\xi - \frac{1}{6} \right) \frac{\pi^2}{8a^4} \left[\frac{3 - 2\sin^2 s}{\sin^4 s} \right] \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\end{aligned} \tag{3.5}
$$

and

$$
\begin{aligned}
\langle T^{(1)}_{00} \rangle \ &= \ \frac{\pi}{1440a^4 \sin^4 s} \left[\frac{311}{40}\pi - \frac{637}{40}s + \frac{1}{10}(43\pi - 81s)\cos 2s \right. \\
&+ \ \left. \frac{s - 3\pi}{40} \cos 4s + 5 \sin 2s + 2(\pi - s)s(\sin 2s - 6\cot s) \right] \\
&+ \ \left(\xi - \frac{1}{6} \right) \frac{\pi}{48a^4 \sin^4 s} \left[2(\pi + s)(2 + \cos 2s) + \frac{5}{2}\sin 2s \right. \\
&+ \ \left. (\pi - s)s(\sin 2s - 6\cot s) \right] - \lim_{s' \to s} \frac{\pi s}{2a^4(s-s')^4},
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
\langle T^{(1)}_{11} \rangle \ &= \ \frac{\pi}{7200a^4} \left[\pi - 2s + \frac{5}{\sin^2 s} \left(2(\pi - 2s) \left(-2 + \frac{3}{\sin^2 s} \right) \right. \right. \\
&+ \ \left. \left. \cot s \left(5 + 2(\pi - s)s - 6(\pi - s) \frac{s}{\sin^2 s} \right) \right) \right] \\
&+ \ \left(\xi - \frac{1}{6} \right) \frac{\pi}{96a^4 \sin^5 s} \Big[(11(\pi - s)s - 1)\cos s \\
&+ \ ((\pi - s)s + 1)\cos 3s - 2(\pi - 2s)(3\sin s + \sin 3s) \Big],
\end{aligned} \tag{3.7}
$$

$$
\langle T^{(1)}_{22} \rangle = \langle T^{(1)}_{11} \rangle, \tag{3.8}
$$

$$
\langle T^{(1)}_{33} \rangle = -\frac{\pi^2}{1440a^4} + \frac{\pi s}{720a^4} + \left(\xi - \frac{1}{6} \right) \frac{\pi}{16a^4} \frac{\cos s}{\sin^3 s}. \tag{3.9}
$$

The next step of our analysis is the renormalization of the regularized energy-momentum tensor. For this purpose, following our work in Ref. [9], we subtract the energy-momentum tensor evaluated in the absence of bounding plates, i.e.

$$\langle \widetilde{T}^{(0)}_{\mu\nu} \rangle = - \lim_{s' \to s} \frac{\pi^2}{2a^4(s-s')^4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \tag{3.10}$$

and

$$\langle \widetilde{T}^{(1)}_{\mu\nu} \rangle = - \lim_{s' \to s} \frac{\pi s}{2a^4(s-s')^4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.11}$$

To test consistency of our results we should now check whether our regularized and renormalized energy-momentum tensor is covariantly conserved, since otherwise we would be outside the realm of quantum field theory in curved spacetime, which would be unacceptable. Indeed, the condition

$$\nabla^\mu \langle T_{\mu\nu} \rangle = 0 \tag{3.12}$$

yields, working up to first order in ε, the pair of equations

$$\frac{\partial}{\partial z} \langle T^{(0)}_{33} \rangle = 0, \ \ (\varepsilon^0 \text{ term}) \tag{3.13}$$

$$\frac{\partial}{\partial z} \langle T^{(1)}_{33} \rangle + \frac{1}{2a} \Big(\langle T^{(0)}_{00} \rangle + \langle T^{(0)}_{33} \rangle \Big) = 0 \ (\varepsilon^1 \text{ term}), \tag{3.14}$$

which are found to hold identically for all values of ξ in our problem.

The trace of $\langle T_{\mu\nu} \rangle$ is obtained as

$$\tau \equiv g^{\mu\nu} \langle T_{\mu\nu} \rangle \sim \eta^{\mu\nu} \langle T^{(0)}_{\mu\nu} \rangle + \varepsilon \Big[\eta^{\mu\nu} \langle T^{(1)}_{\mu\nu} \rangle + \frac{z}{a} \langle T^{(0)}_{00} \rangle \Big] + \mathrm{O}(\varepsilon^2), \tag{3.15}$$

from which we find a ξ-dependent part

$$\begin{aligned} \tau_\xi \ = \ & \left(\xi - \frac{1}{6} \right) \Big\{ - \frac{3\pi^2 (2 + \cos 2s)}{8a^4 \sin^4 s} - \varepsilon \frac{\pi}{32 a^4 \sin^5 s} \Big[(1 - 11(\pi - s)s) \cos s \\ & - \ (1 + (\pi - s)s) \cos 3s + 2(\pi - 2s)(3 \sin s + \sin 3s) \Big] \Big\}. \end{aligned} \tag{3.16}$$

Interestingly, the value $\xi = \frac{1}{6}$ which yields conformal invariance of the classical action is the same as the value of ξ yielding no trace anomaly [13].

IV. CASIMIR ENERGY AND PRESSURE

In order to evaluate the energy density ρ of our "scalar" Casimir apparatus, we project the regularized and renormalized energy-momentum tensor along a unit timelike vector $u^\mu = \left(-\frac{1}{\sqrt{-g_{00}}}, 0, 0, 0\right)$. This yields

$$\begin{aligned} \rho &= \langle T_{\mu\nu} \rangle u^\mu u^\nu = -\frac{\pi^2}{1440a^4} + \frac{\pi}{7200a^4}\Bigg[-3\pi + 6s + \frac{10}{\sin^2 s}\Big(2(\pi - 2s) \\ &\times \left(-2 + \frac{3}{\sin^2 s}\right) + \cot s \left((5 + 2(\pi - s)s + 6\frac{s(-\pi + s)}{\sin^2 s}\right)\Big)\Bigg]\varepsilon \\ &+ \left(\xi - \frac{1}{6}\right)\Bigg\{\frac{\pi^2(2 + \cos 2s)}{8a^4 \sin^4 s} - \frac{\pi}{192a^4 \sin^5 s}\Bigg[\Big(-5 + 22(\pi - s)s\Big)\cos s \\ &+ \Big(5 + 2(\pi - s)s\Big)\cos 3s - 4(\pi - 2s)(3\sin s + \sin 3s)\Bigg]\varepsilon\Bigg\}. \end{aligned} \tag{4.1}$$

The energy E stored within our Casimir cavity is given by

$$E = \int_{V_c} d^3\Sigma \sqrt{-g}\rho, \tag{4.2}$$

where $d^3\Sigma$ is the volume element of an observer with four-velocity u^μ, and V_c is the volume of the cavity. The integration used here requires the use of approximating domains, i.e. the z-integration is performed in the interval $(\zeta, a - \zeta)$, corresponding to $\frac{\pi}{a}(\zeta, a - \zeta)$ in the s variable, taking eventually the $\zeta \to 0$ limit. We thus obtain [13]

$$E_\xi = -\frac{\pi^2 A}{1440a^3} - \frac{\pi^2 A\varepsilon}{5760a^3} + \left(\xi - \frac{1}{6}\right)\frac{\pi A}{4a^3}\left(1 + \frac{\varepsilon}{4}\right)\lim_{\zeta \to 0}\frac{\cos\zeta}{\sin^3\zeta}, \tag{4.3}$$

where A is the area of parallel plates. Note that the conformal coupling value $\xi = \frac{1}{6}$ is picked out as the only value of ξ for which the Casimir energy remains finite. In this case, reintroducing the constants $\hbar, c$ and writing explicitly ε, we find [13]

$$E_c = -\frac{\hbar c\pi^2}{1440}\frac{A}{a^3}\left(1 + \frac{1}{2}\frac{ga}{c^2}\right). \tag{4.4}$$

In the same way, the pressure P_ξ on the parallel plates is found to be [13]

$$P_\xi(z = 0) = \frac{\pi^2}{480a^4} + \frac{\pi^2\varepsilon}{1440a^4} - \left(\xi - \frac{1}{6}\right)\frac{\pi\varepsilon}{16a^4}\lim_{s \to 0}\frac{\cos s}{\sin^3 s}, \tag{4.5}$$

$$P_\xi(z = a) = -\frac{\pi^2}{480a^4} + \frac{\pi^2\varepsilon}{1440a^4} + \left(\xi - \frac{1}{6}\right)\frac{\pi\varepsilon}{16a^4}\lim_{s \to \pi}\frac{\cos s}{\sin^3 s}. \tag{4.6}$$

Once again, one can get rid of divergent terms by setting $\xi = \frac{1}{6}$, which yields, upon reintroducing $\hbar, c$ [13]

$$P_c(z=0) = \frac{\pi^2}{480}\frac{\hbar c}{a^4}\left(1 + \frac{2}{3}\frac{ga}{c^2}\right), \tag{4.7}$$

$$P_c(z=a) = -\frac{\pi^2}{480}\frac{\hbar c}{a^4}\left(1 - \frac{2}{3}\frac{ga}{c^2}\right). \tag{4.8}$$

To obtain the resulting force one has to multiply each of these pressures by the redshift r of the point where they act, relative to the point where they are added [10], i.e.,

$$r = \sqrt{\frac{|g_{00}(P_{\rm act})|}{|g_{00}(P_{\rm added})|}} \approx 1 + \frac{g}{c^2}(z - z_Q), \tag{4.9}$$

to leading order in $\frac{gz}{c^2}$. Thus, a net force is obtained of magnitude

$$F = -\frac{\pi^2\hbar c}{a^4}A\left[\frac{g}{480c^2}(z_2 - z_1) - \frac{4g}{1440c^2}(z_2 - z_1)\right] = \frac{\pi^2}{1440}\frac{A\hbar g}{ca^3} = \frac{|E_C^0|}{c^2}g, \tag{4.10}$$

having defined $E_C^0 \equiv -\frac{\pi^2}{1440}\hbar c\frac{A}{a^3}$, which points upwards along the z-axis and is in full agreement with the equivalence principle.

V. NEUMANN BOUNDARY CONDITIONS

When the reduced Green functions obey instead Neumann boundary conditions on parallel plates, i.e.

$$\left.\frac{\partial\gamma^{(i)}}{\partial z}\right|_{z=0} = \left.\frac{\partial\gamma^{(i)}}{\partial z}\right|_{z=a} = 0, \ \forall i = 0, 1, \tag{5.1}$$

our work in Ref. [14] has found, by an analogous procedure, the regularized and renormalized energy-momentum tensor to first order in ε, with trace

$$\begin{aligned} \tau_\xi &\equiv g^{\mu\nu}\langle T_{\mu\nu}\rangle \\ &= \left(\xi - \frac{1}{6}\right)\frac{\pi}{32a^4}\frac{1}{\sin^5 s}\Big\{6\pi(3\sin s + \sin 3s) \\ &- \varepsilon\Big[(1 + 11(\pi - s)s)\cos s - (1 - (\pi - s)s)\cos 3s \\ &- 2(\pi - 2s)(3\sin s + \sin 3s)\Big]\Big\}, \end{aligned} \tag{5.2}$$

which vanishes in the case of conformal coupling, as with Dirichlet boundary conditions. Moreover, the Casimir energy stored between the plates, the pressure on parallel plates and the net force are then found to agree completely with (4.4), (4.5)–(4.6) and (4.10), respectively.

VI. ELECTROMAGNETIC FIELD

The work in Ref. [15] has instead exploited the fact that, to first order in the small quantity gz, the line element (1.1) coincides with the Rindler metric

$$ds^2 = -\left(\frac{\xi}{\xi_1}\right)^2 dt^2 + d\xi^2 + dx_\perp^2, \tag{6.1}$$

where $\xi \equiv \frac{1}{g}+z \equiv \xi_1+z$. In a fully covariant analysis of Feynman Green functions in Rindler spacetime, the components of the Maxwell energy-momentum tensor have been evaluated up to second order in g with perfect conductor boundary conditions, finding that, as $z \rightarrow 0$,

$$\langle 0|T_t^{\ t}|0\rangle \sim \frac{g}{30\pi^2 z^3} + \mathrm{O}(z^{-2}), \tag{6.2}$$

$$\langle 0|T_z^{\ z}|0\rangle \sim -\frac{g^2}{60\pi^2 z^2} + \mathrm{O}(z^{-1}), \tag{6.3}$$

$$\langle 0|T_x^{\ x}|0\rangle = \langle |T_y^{\ y}|0\rangle \sim -\frac{g}{60\pi^2 z^3} + \mathrm{O}(z^{-2}). \tag{6.4}$$

Since T_{zz} is now found to diverge on approaching the plates, no definite meaning can be given to the gravitational correction to the Casimir pressure. Moreover, the divergences in $T_t^{\ t}$ are such that the resulting correction to the total mass energy of the cavity is infinite, even on taking the principal-value integral of $T_t^{\ t}$.

VII. CONCLUDING REMARKS

The literature on the behaviour of rigid Casimir cavities in a weak gravitational field predicts, on theoretical ground, that Casimir energy obeys exactly the equivalence principle, and hence the Casimir apparatus should experience a tiny push in the upward direction. The formula for the push has been obtained in three different ways, i.e. a heuristic summation over modes [16, 17], or a variational approach [7, 8], or an energy-momentum analysis [9, 15]. Moreover, the work in Ref. [18] has shown that Casimir energy for a configuration of parallel plates gravitates according to the equivalence principle both for the finite and divergent parts. This suggests that such divergent parts can be absorbed by a process of renormalization [18].

It now remains to be seen whether this interpretation is viable in all configurations of physical interest. Moreover, on the experimental side, the signal-modulation problems first discussed in Refs. [16, 17] remain, to our knowledge, unsolved, while being of extreme importance on studying the feasibility of the experiment.

Last, but not least, our findings should be compared with those in Ref. [19], where the authors consider the cosmological evolution in a recently suggested new model of quantum initial conditions for the Universe. They find that the effective Friedmann equation incorporates the effect of the conformal anomaly of quantum fields, and shows that their vacuum Casimir energy is completely screened and *does not gravitate.*

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