Light-Front Holography and Gauge/Gravity Correspondence: Applications to Hadronic Physics

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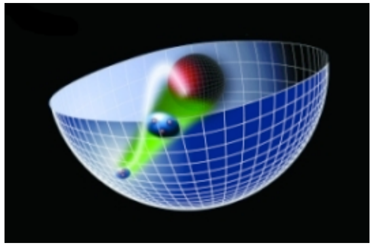

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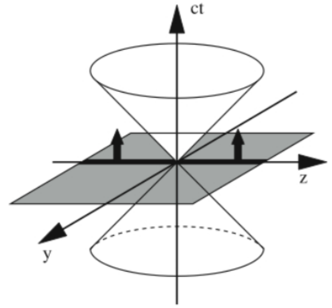
1 Introduction

- Most challenging problem of strong interaction dynamics: determine the composition of hadrons in terms of their fundamental QCD quark and gluon degrees of freedom
- Recent developments inspired by the AdS/CFT correspondence [Maldacena (1998)] between string states in AdS space and conformal field theories in physical space-time have led to analytical insights into the confining dynamics of QCD
- Description of strongly coupled gauge theory using a dual gravity description!
- Strings describe spin-J extended objects (no quarks). QCD degrees of freedom are pointlike particles and hadrons have orbital angular momentum: how can they be related?
- Light-front (LF) quantization is the ideal framework to describe hadronic structure in terms of quarks and gluons: simple vacuum structure allows unambiguous definition of the partonic content of a hadron, exact formulae for form factors, physics of angular momentum of constituents ...
- Frame-independent LF Hamiltonian equation $P_\mu P^\mu|P\rangle = \mathcal{M}^2|P\rangle$ similar structure of AdS EOM
- First semiclassical approximation to the bound-state LF Hamiltonian equation in QCD is equivalent to equations of motion in AdS and can be systematically improved

2 Light Front Dynamics

- Different possibilities to parametrize space-time [Dirac (1949)]
- Parametrizations differ by the hypersurface on which the initial conditions are specified. Each evolve with different "times" and has its own Hamiltonian, but should give the same physical results
- *Instant form*: hypersurface defined by $t = 0$, the familiar one



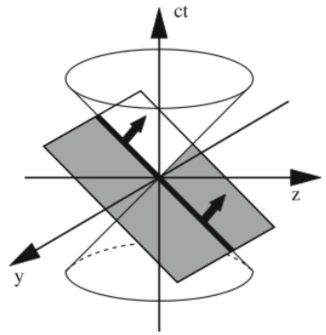
- *Front form*: hypersurface is tangent to the light cone at $\tau = t + z/c = 0$

$x^+ = x^0 + x^3$ light-front time

$x^- = x^0 - x^3$ longitudinal space variable

$k^+ = k^0 + k^3$ longitudinal momentum $(k^+ > 0)$

$k^- = k^0 - k^3$ light-front energy



$$k \cdot x = \tfrac{1}{2}\left(k^+ x^- + k^- x^+\right) - \mathbf{k}_\perp \cdot \mathbf{x}_\perp$$

On shell relation $k^2 = m^2$ leads to dispersion relation $k^- = \frac{\mathbf{k}_\perp^2 + m^2}{k^+}$

- QCD Lagrangian

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4g^2}\mathrm{Tr}\left(G^{\mu\nu}G_{\mu\nu}\right) + i\overline{\psi}D_\mu\gamma^\mu\psi + m\overline{\psi}\psi$$

- LF Momentum Generators $P = (P^+, P^-, \mathbf{P}_\perp)$ in terms of dynamical fields ψ, $\mathbf{A}_\perp$

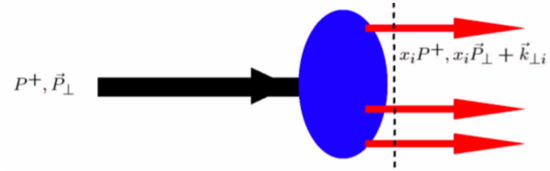
$$\begin{aligned}
P^- &= \frac{1}{2}\int dx^- d^2\mathbf{x}_\perp \overline{\psi}\,\gamma^+ \frac{(i\nabla_\perp)^2 + m^2}{i\partial^+}\psi + \text{interactions} \\
P^+ &= \int dx^- d^2\mathbf{x}_\perp\, \overline{\psi}\,\gamma^+ i\partial^+\psi \\
\mathbf{P}_\perp &= \frac{1}{2}\int dx^- d^2\mathbf{x}_\perp\, \overline{\psi}\,\gamma^+ i\nabla_\perp\psi
\end{aligned}$$

- LF energy P^- generates LF time translations

$$\left[\psi(x), P^-\right] = i\frac{\partial}{\partial x^+}\psi(x)$$

and the generators P^+ and $\mathbf{P}_\perp$ are kinematical

Light-Front Fock Representation



- Dirac field ψ, expanded in terms of ladder operators on the initial surface $x^+ = x^0 + x^3$

$$P^- = \sum_\lambda \int \frac{dq^+ d^2\mathbf{q}_\perp}{(2\pi)^3} \left(\frac{\mathbf{q}_\perp^2 + m^2}{q^+}\right) b_\lambda^\dagger(q) b_\lambda(q) + \text{interactions}$$

Sum over free quanta $q^- = \frac{\mathbf{q}_\perp^2 + m^2}{q^+}$ plus interactions ($m^2 = 0$ for gluons)

- Construct light-front invariant Hamiltonian for the composite system: $H_{LF} = P_\mu P^\mu = P^- P^+ - \mathbf{P}_\perp^2$

$$H_{LC} \mid \psi_H \rangle = \mathcal{M}_H^2 \mid \psi_H \rangle$$

- State $|\psi_H(P)\rangle = |\psi_H(P^+, \mathbf{P}_\perp, J_z)\rangle$ is an expansion in multi-particle Fock eigenstates $\mid n \rangle$ of the free LF Hamiltonian:

$$|\psi_H\rangle = \sum_n \psi_{n/H} |n\rangle$$

- Fock components $\psi_{n/H}(x_i, \mathbf{k}_{\perp i}, \lambda_i^z)$ are independent of P^+ and $\mathbf{P}_\perp$ and depend only on relative partonic coordinates: momentum fraction $x_i = k_i^+/P^+$, transverse momentum $\mathbf{k}_{\perp i}$ and spin λ_i^z

$$\sum_{i=1}^n x_i = 1, \quad \sum_{i=1}^n \mathbf{k}_{\perp i} = 0.$$

- Compute $\mathcal{M}^2$ from hadronic matrix element

$$\langle\psi_H(P')|H_{LF}|\psi_H(P)\rangle = \mathcal{M}_H^2\langle\psi_H(P')|\psi_H(P)\rangle$$

- Find

$$\mathcal{M}_H^2 = \sum_n \int \big[dx_i\big]\big[d^2\mathbf{k}_{\perp i}\big] \sum_\ell \left(\frac{\mathbf{k}_{\perp\ell}^2 + m_\ell^2}{x_q}\right) \big|\psi_{n/H}(x_i, \mathbf{k}_{\perp i})\big|^2 + \text{interactions}$$

- Phase space normalization of LFWFs

$$\sum_n \int \big[dx_i\big]\,\big[d^2\mathbf{k}_{\perp i}\big]\,\big|\psi_{n/h}(x_i, \mathbf{k}_{\perp i})\big|^2 = 1$$

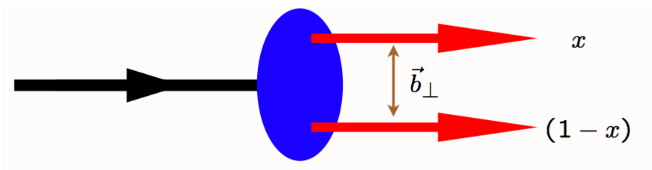
- In terms of $n-1$ independent transverse impact coordinates $\mathbf{b}_{\perp j}$, $j = 1, 2, \ldots, n-1$,

$$\mathcal{M}_H^2 = \sum_n \prod_{j=1}^{n-1} \int dx_j d^2\mathbf{b}_{\perp j} \psi_{n/H}^*(x_i, \mathbf{b}_{\perp i}) \sum_\ell \left(\frac{-\nabla_{\mathbf{b}_{\perp\ell}}^2 + m_\ell^2}{x_q}\right)\psi_{n/H}(x_i, \mathbf{b}_{\perp i}) + \text{interactions}$$

- Normalization

$$\sum_n \prod_{j=1}^{n-1} \int dx_j d^2\mathbf{b}_{\perp j}\, |\psi_n(x_j, \mathbf{b}_{\perp j})|^2 = 1$$

3 Semiclassical Approximation to QCD



- Consider a two-parton hadronic bound state in transverse impact space in the limit $m_q \to 0$

$$\mathcal{M}^2 = \int_0^1 \frac{dx}{1-x} \int d^2\mathbf{b}_\perp \, \psi^*(x, \mathbf{b}_\perp) \big(-\nabla^2_{\mathbf{b}_\perp}\big) \, \psi(x, \mathbf{b}_\perp) + \text{interactions}$$

- Separate angular, transverse and longitudinal modes in terms of boost invariant transverse variable: $\zeta^2 = x(1-x)\mathbf{b}_\perp^2$ (In $\mathbf{k}_\perp$ space key variable is the LF KE $\frac{\mathbf{k}_\perp^2}{x(1-x)}$)

$$\psi(x, \zeta, \varphi) = \frac{\phi(\zeta)}{\sqrt{2\pi\zeta}} \, e^{iM\varphi} f(x)$$

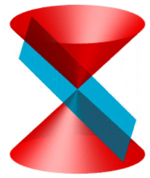
- Find ($L = |M|$)

$$\mathcal{M}^2 = \int d\zeta \, \phi^*(\zeta) \sqrt{\zeta} \left(-\frac{d^2}{d\zeta^2} - \frac{1}{\zeta} \frac{d}{d\zeta} + \frac{L^2}{\zeta^2} \right) \frac{\phi(\zeta)}{\sqrt{\zeta}} + \int d\zeta \, \phi^*(\zeta) \, U(\zeta) \, \phi(\zeta)$$

where the confining forces from the interaction terms is summed up in the effective potential $U(\zeta)$

- Ultra relativistic limit $m_q \to 0$ longitudinal modes decouple and LF eigenvalue equation $H_{LF}|\phi\rangle = \mathcal{M}^2|\phi\rangle$ is a LF wave equation for ϕ

$$\boxed{\Big(\underbrace{-\frac{d^2}{d\zeta^2} - \frac{1-4L^2}{4\zeta^2}}_{kinetic\ energy\ of\ partons} + \underbrace{U(\zeta)}_{confinement}\Big)\phi(\zeta) = \mathcal{M}^2\phi(\zeta)}$$



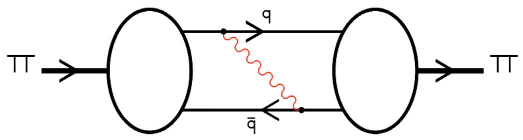
- Effective light-front Schrödinger equation: relativistic, frame-independent and analytically tractable

- Eigenmodes $\phi(\zeta)$ determine the hadronic mass spectrum and represent the probability amplitude to find n-massless partons at transverse impact separation ζ within the hadron at equal light-front time

- LF modes $\phi(\zeta) = \langle\zeta|\phi\rangle$ are normalized by

$$\langle\phi|\phi\rangle = \int d\zeta \, |\langle\zeta|\phi\rangle|^2 = 1$$

- Semiclassical approximation to light-front QCD does not account for particle creation and absorption but can be implemented in the LF Hamiltonian EOM



Hard-Wall Model

- Consider the potential (hard wall)

$$U(\zeta) = \begin{cases} 0 & \text{if} \quad \zeta \leq \frac{1}{\Lambda_{\rm QCD}} \\ \infty & \text{if} \quad \zeta > \frac{1}{\Lambda_{\rm QCD}} \end{cases}$$

- If $L^2 \geq 0$ the Hamiltonian is positive definite $\langle \phi \left| H^L_{LF} \right| \phi \rangle \geq 0$ and thus $\mathcal{M}^2 \geq 0$
- If $L^2 < 0$ the Hamiltonian is not bounded from below (“Fall-to-the-center” problem in Q.M.)
- Critical value of the potential corresponds to $L = 0$, the lowest possible stable state
- Solutions:

$$\phi_L(\zeta) = C_L \sqrt{\zeta} J_L \left(\zeta \mathcal{M}\right)$$

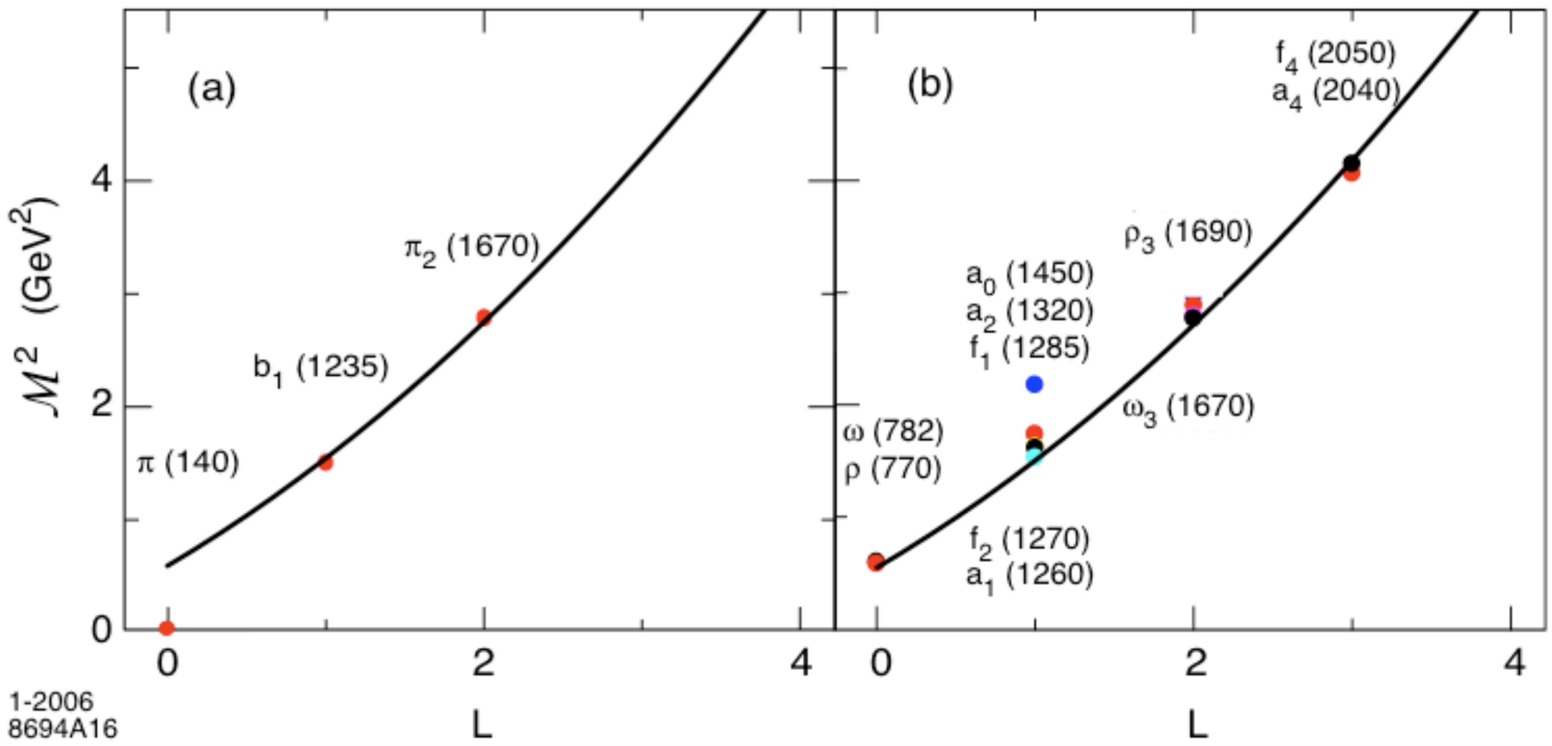
- Mode spectrum from boundary conditions

$$\phi\left(\zeta = \frac{1}{\Lambda_{\rm QCD}}\right) = 0$$

Thus

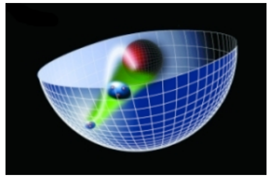
$$\mathcal{M}^2 = \beta_{Lk} \Lambda_{\rm QCD}$$

- Excitation spectrum hard-wall model: $\mathcal{M}_{n,L} \sim L + 2n$



Light-meson orbital spectrum $\Lambda_{QCD} = 0.32$ GeV

Holographic Mapping



- Holographic mapping found originally by matching expressions of EM and gravitational form factors of hadrons in AdS and LF QCD [Brodsky and GdT (2006, 2008)]

- Substitute $\Phi(\zeta) \sim \zeta^{3/2}\phi(\zeta), \quad \zeta \to z$ in the conformal LFWE

$$\left(-\frac{d^2}{d\zeta^2} - \frac{1-4L^2}{4\zeta^2}\right)\phi(\zeta) = \mathcal{M}^2\phi(\zeta)$$

- Find:

$$\left[z^2\partial_z^2 - 3z\,\partial_z + z^2\mathcal{M}^2 - (\mu R)^2\right]\Phi(z) = 0$$

with $(\mu R)^2 = -4 + L^2$, the wave equation of string mode in AdS$_5$!

- Isomorphism of $SO(4,2)$ group of conformal QCD with generators $P^\mu, M^{\mu\nu}, D, K^\mu$ with the group of isometries of AdS$_5$ space: $x^\mu \to \lambda x^\mu, \; z \to \lambda z$

$$ds^2 = \frac{R^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2)$$

- AdS Breitenlohner-Freedman bound $(\mu R)^2 \geq -4$ equivalent to LF QM stability condition $L^2 \geq 0$

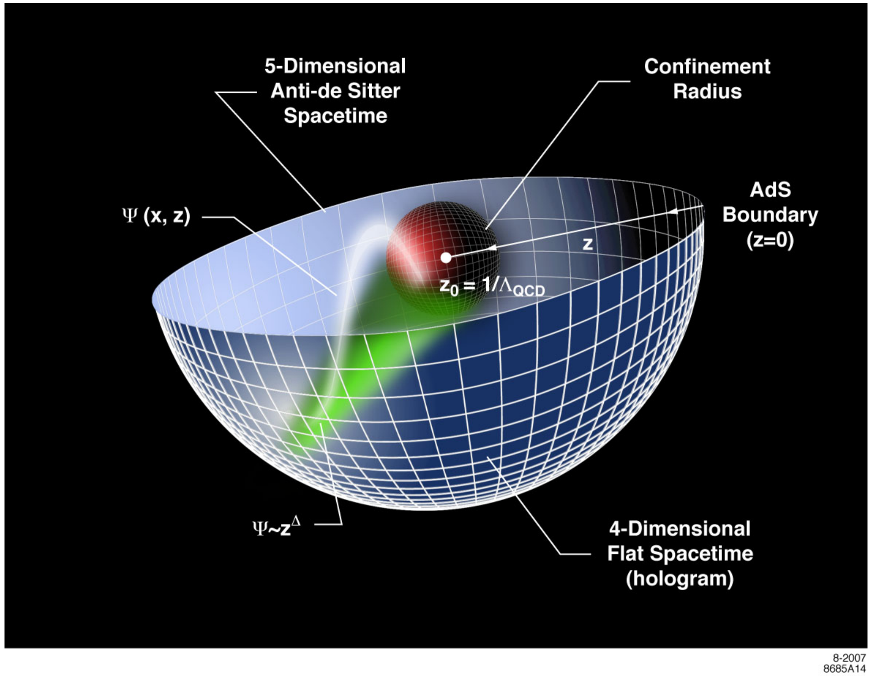
- Conformal dimension Δ of AdS mode Φ given in terms of 5-dim mass by $(\mu R)^2 = \Delta(\Delta - 4)$. Thus $\Delta = 2 + L$ in agreement with the twist scaling dimension of a two parton object in QCD

- AdS$_5$ metric:

$$\underbrace{ds^2}_{L_{\rm AdS}} = \frac{R^2}{z^2}\big(\underbrace{\eta_{\mu\nu}dx^\mu dx^\nu}_{L_{\rm Minkowski}} - dz^2\big)$$

- A distance $L_{\rm AdS}$ shrinks by a warp factor as observed in Minkowski space $(dz = 0)$:

$$L_{\rm Minkowski} \sim \frac{z}{R}\, L_{\rm AdS}$$



- Different values of z correspond to different scales at which the hadron is examined
- Since $x^\mu \to \lambda x^\mu$, $z \to \lambda z$, short distances $x_\mu x^\mu \to 0$ maps to UV conformal AdS$_5$ boundary $z \to 0$, which corresponds to the $Q \to \infty$ UV zero separation limit
- Large confinement dimensions $x_\mu x^\mu \sim 1/\Lambda^2_{\rm QCD}$ maps to large IR region of AdS$_5$, $z \sim 1/\Lambda_{\rm QCD}$, thus there is a maximum separation of quarks and a maximum value of z at the IR boundary
- Local operators like $\mathcal{O}$ and $\mathcal{L}_{\rm QCD}$ defined in terms of quark and gluon fields at the AdS$_5$ boundary
- Use the isometries of AdS to map the local interpolating operators at the UV boundary of AdS into the modes propagating inside AdS

4 Higher-Spin Bosonic Modes

Hard-Wall Model

- AdS$_{d+1}$ metric $x^\ell = (x^\mu, z)$:

$$ds^2 = g_{\ell m} dx^\ell dx^m = \frac{R^2}{z^2}\left(\eta_{\mu\nu} dx^\mu dx^\nu - dz^2\right)$$

- Action for gravity coupled to scalar field in AdS$_{d+1}$

$$S = \int d^{d+1}x \sqrt{g}\, \Big(\underbrace{\frac{1}{\kappa^2}\left(\mathcal{R} - 2\Lambda\right)}_{S_G} + \underbrace{\frac{1}{2}\left(g^{\ell m}\partial_\ell \Phi \partial_m \Phi - \mu^2 \Phi^2\right)}_{S_M} \Big)$$

- Equations of motion for S_M

$$z^3 \partial_z \left(\frac{1}{z^3}\partial_z \Phi\right) - \partial_\rho \partial^\rho \Phi - \left(\frac{\mu R}{z}\right)^2 \Phi = 0$$

- Physical AdS modes $\Phi_P(x, z) \sim e^{-iP\cdot x}\, \Phi(z)$ are plane waves along the Poincaré coordinates with four-momentum P^μ and hadronic invariant mass states $P_\mu P^\mu = \mathcal{M}^2$

- Factoring out dependence of string mode $\Phi_P(x, z)$ along x^μ-coordinates

$$\left[z^2 \partial_z^2 - (d-1) z\, \partial_z + z^2 \mathcal{M}^2 - (\mu R)^2\right] \Phi(z) = 0$$

- Spin J-field on AdS represented by rank-J totally symmetric tensor field $\Phi(x,z)_{\ell_1\cdots\ell_J}$ [Fronsdal; Fradkin and Vasiliev]

- Action in AdS$_{d+1}$ for spin-J field

$$S_M = \frac{1}{2}\int d^{d+1}x\sqrt{g}\,\left(\partial_\ell \Phi_{\ell_1\cdots\ell_J}\partial^\ell \Phi^{\ell_1\cdots\ell_J} - \mu^2 \Phi_{\ell_1\cdots\ell_J}\Phi^{\ell_1\cdots\ell_J} + \ldots\right)$$

- Each hadronic state of total spin J is dual to a normalizable string mode

$$\Phi_P(x,z)_{\mu_1\cdots\mu_J} = e^{-iP\cdot x}\,\Phi(z)_{\mu_1\cdots\mu_J}$$

  with four-momentum P_μ, spin polarization indices along the 3+1 physical coordinates and hadronic invariant mass $P_\mu P^\mu = \mathcal{M}^2$

- For string modes with all indices along Poincaré coordinates, $\Phi_{z\mu_2\cdots\mu_J} = \Phi_{\mu_1 z\cdots\mu_J} = \cdots = 0$ and appropriate subsidiary conditions system of coupled differential equations from S_M reduce to a homogeneous wave equation for $\Phi(z)_{\mu_1\cdots\mu_J}$

- Obtain spin-J mode $\Phi_{\mu_1 \cdots \mu_J}$ with all indices along 3+1 coordinates from Φ by shifting dimensions

$$\Phi_J(z) = \left(\frac{z}{R}\right)^{-J} \Phi(z)$$

- Normalization [Hong, Yoon and Strassler (2006)]

$$R^{d-2J-1} \int_0^{z_{max}} \frac{dz}{z^{d-2J-1}} \, \Phi_J^2(z) = 1$$

- Substituting in the AdS scalar wave equation for Φ

$$\left[z^2 \partial_z^2 - (d-1-2J) z \, \partial_z + z^2 \mathcal{M}^2 - (\mu R)^2\right] \Phi_J = 0$$

upon fifth-dimensional mass rescaling $(\mu R)^2 \to (\mu R)^2 - J(d-J)$

- Conformal dimension of J-mode

$$\Delta = \frac{1}{2}\left(d + \sqrt{(d-2J)^2 + 4\mu^2 R^2}\right)$$

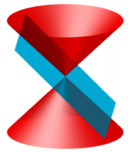
and thus $(\mu R)^2 = (\Delta - J)(\Delta - d + J)$

- Upon substitution $z \to \zeta$ and

$$\phi_J(\zeta) \sim \zeta^{-3/2+J} \Phi_J(\zeta)$$

we recover the QCD LF wave equation ($d = 4$)

$$\boxed{\left(-\frac{d^2}{d\zeta^2} - \frac{1-4L^2}{4\zeta^2}\right) \phi_{\mu_1 \cdots \mu_J} = \mathcal{M}^2 \phi_{\mu_1 \cdots \mu_J}}$$



with $(\mu R)^2 = -(2-J)^2 + L^2$

- J-decoupling in the HW model

- For $L^2 \geq 0$ the LF Hamiltonian is positive definite $\langle \phi_J | H_{LF} | \phi_J \rangle \geq 0$ and we find the stability bound $(\mu R)^2 \geq -(2-J)^2$

- The scaling dimensions are $\Delta = 2 + L$ independent of J in agreement with the twist scaling dimension of a two parton bound state in QCD

Note: p-forms

- In tensor notation EOM for a p-form in AdS$_{d+1}$are $p+1$ coupled differential equations [I'Yi (1998)]

$$\left[z^2\partial_z^2 - (d+1-2p)z\,\partial_z - z^2\mathcal{M}^2 - (\mu R)^2 + d + 1 - 2p\right]\Phi_{z\alpha_2\cdots\alpha_p} = 0$$

$$\cdots$$

$$\left[z^2\partial_z^2 - (d-1-2p)z\,\partial_z - z^2\mathcal{M}^2 - (\mu R)^2\right]\Phi_{\alpha_1\alpha_2\cdots\alpha_p}$$
$$= 2z\left(\partial_{\alpha_1}\Phi_{z\alpha_2\cdots\alpha_p} + \partial_{\alpha_2}\Phi_{\alpha_1 z\cdots\alpha_p} + \cdots\right)$$

- For modes with all indices along the Poincaré coordinates $\Phi_{z\alpha_2\cdots\alpha_p} = \Phi_{\alpha_1 z\cdots\alpha_p} = \cdots = 0$

$$\left[z^2\partial_z^2 - (d-1-2p)z\,\partial_z + z^2\mathcal{M}^2 - (\mu R)^2\right]\Phi_{\alpha_1\cdots\alpha_p} = 0$$

with $(\mu R)^2 = (\Delta - p)(\Delta - d + p)$

- Identical with spin-J solution from shifting dimensions

Note: Algebraic Construction

- If $L^2 > 0$ the LF Hamiltonian, H_{LF}, is written as a bilinear form [Bargmann (1949)]

$$H_{LF}^{L}(\zeta) = \Pi_L^\dagger(\zeta)\Pi_L(\zeta)$$

in terms of the operator

$$\Pi_L(\zeta) = -i\left(\frac{d}{d\zeta} - \frac{L+\frac{1}{2}}{\zeta}\right)$$

and its adjoint

$$\Pi_L^\dagger(\zeta) = -i\left(\frac{d}{d\zeta} + \frac{L+\frac{1}{2}}{\zeta}\right)$$

with commutation relations

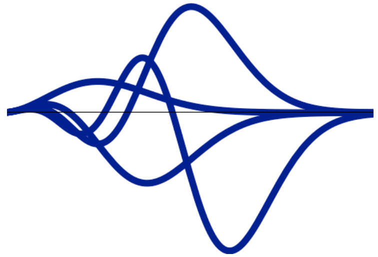
$$\left[\Pi_L(\zeta), \Pi_L^\dagger(\zeta)\right] = \frac{2L+1}{\zeta^2}$$

- If $L^2 \geq 0$ the LF Hamiltonian is positive definite

$$\langle\phi\left|H_{LF}^{L}\right|\phi\rangle = \int d\zeta\, |\Pi_L \phi(z)|^2 \geq 0$$

- Higher-spin fields in AdS [Metsaev (1998) and (1999)]

Soft-Wall Model



- Soft-wall model [Karch, Katz, Son and Stephanov (2006)] retain conformal AdS metrics but introduce smooth cutoff wich depends on the profile of a dilaton background field $\varphi(z) = \pm\kappa^2 z^2$

$$S = \int d^d x \, dz \, \sqrt{g} \, e^{\varphi(z)} \mathcal{L},$$

- Equation of motion for scalar field $\mathcal{L} = \frac{1}{2}\left(g^{\ell m}\partial_\ell \Phi \partial_m \Phi - \mu^2 \Phi^2\right)$

$$\left[z^2 \partial_z^2 - \left(d - 1 \mp 2\kappa^2 z^2\right) z \, \partial_z + z^2 \mathcal{M}^2 - (\mu R)^2\right] \Phi(z) = 0$$

with $(\mu R)^2 \geq -4$. See also [Metsaev (2002), Andreev (2006)]

- LH holography requires 'plus dilaton' $\varphi = +\kappa^2 z^2$. Lowest possible state $(\mu R)^2 = -4$

$$\mathcal{M}^2 = 4\kappa^2 n, \quad \Phi_n(z) \sim z^2 e^{-\kappa^2 z^2} L_n(\kappa^2 z^2)$$

$\Phi_0(z)$ a chiral symmetric bound state of two massless quarks with scaling dimension 2: the pion

- Action in AdS$_{d+1}$ for spin J-field

$$S_M = \frac{1}{2}\int d^d x\, dz\, \sqrt{g}\, e^{\kappa^2 z^2} \left(\partial_\ell \Phi_{\ell_1 \cdots \ell_J} \partial^\ell \Phi^{\ell_1 \cdots \ell_J} - \mu^2 \Phi_{\ell_1 \cdots \ell_J} \Phi^{\ell_1 \cdots \ell_J} + \dots\right)$$

- Obtain spin-J mode $\Phi_{\mu_1 \cdots \mu_J}$ with all indices along 3+1 coordinates from Φ by shifting dimensions

$$\Phi_J(z) = \left(\frac{z}{R}\right)^{-J} \Phi(z)$$

- Normalization

$$R^{d-2J-1} \int_0^\infty \frac{dz}{z^{d-2J-1}}\, e^{\kappa^2 z^2}\, \Phi_J^2(z) = 1.$$

- Substituting in the AdS scalar wave equation for Φ

$$\left[z^2 \partial_z^2 - \left(d-1-2J - 2\kappa^2 z^2\right) z\, \partial_z + z^2 \mathcal{M}^2 - (\mu R)^2\right] \Phi_J = 0$$

upon mass rescaling $(\mu R)^2 \to (\mu R)^2 - J(d-J)$ and $\mathcal{M}^2 \to \mathcal{M}^2 - 2J\kappa^2$

- Upon substitution $z \to \zeta \;\; (J_z = L_z + S_z)$ we find for $d = 4$

$$\phi_J(\zeta) \sim \zeta^{-3/2+J} e^{\kappa^2 \zeta^2/2}\, \Phi_J(\zeta), \qquad (\mu R)^2 = -(2-J)^2 + L^2$$

$$\boxed{\left(-\frac{d^2}{d\zeta^2} - \frac{1-4L^2}{4\zeta^2} + \kappa^4 \zeta^2 + 2\kappa^2 (L+S-1)\right) \phi_{\mu_1 \cdots \mu_J} = \mathcal{M}^2 \phi_{\mu_1 \cdots \mu_J}}$$

- Eigenfunctions

$$\phi_{nL}(\zeta) = \kappa^{1+L} \sqrt{\frac{2n!}{(n+L)!}}\, \zeta^{1/2+L} e^{-\kappa^2 \zeta^2/2} L_n^L(\kappa^2 \zeta^2)$$

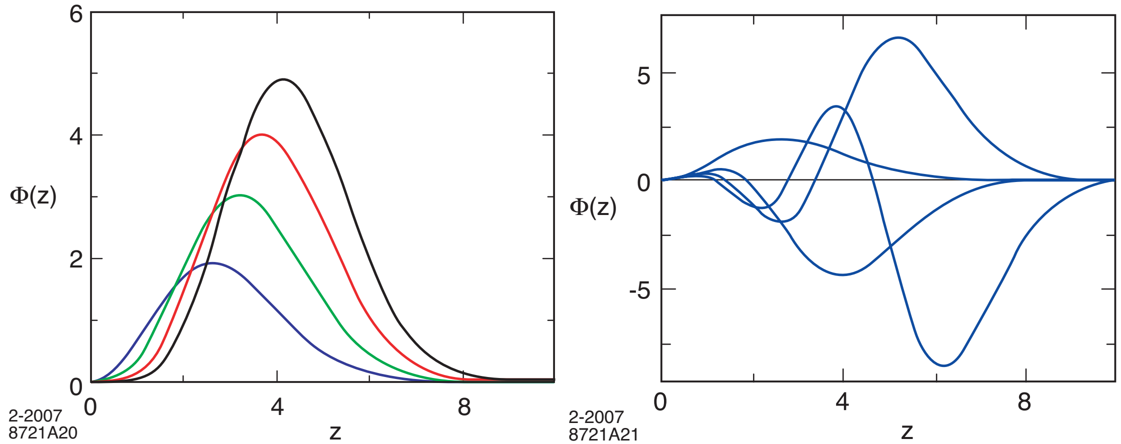
- Eigenvalues

$$\mathcal{M}^2_{n,L,S} = 4\kappa^2 \left(n + L + \frac{S}{2}\right)$$

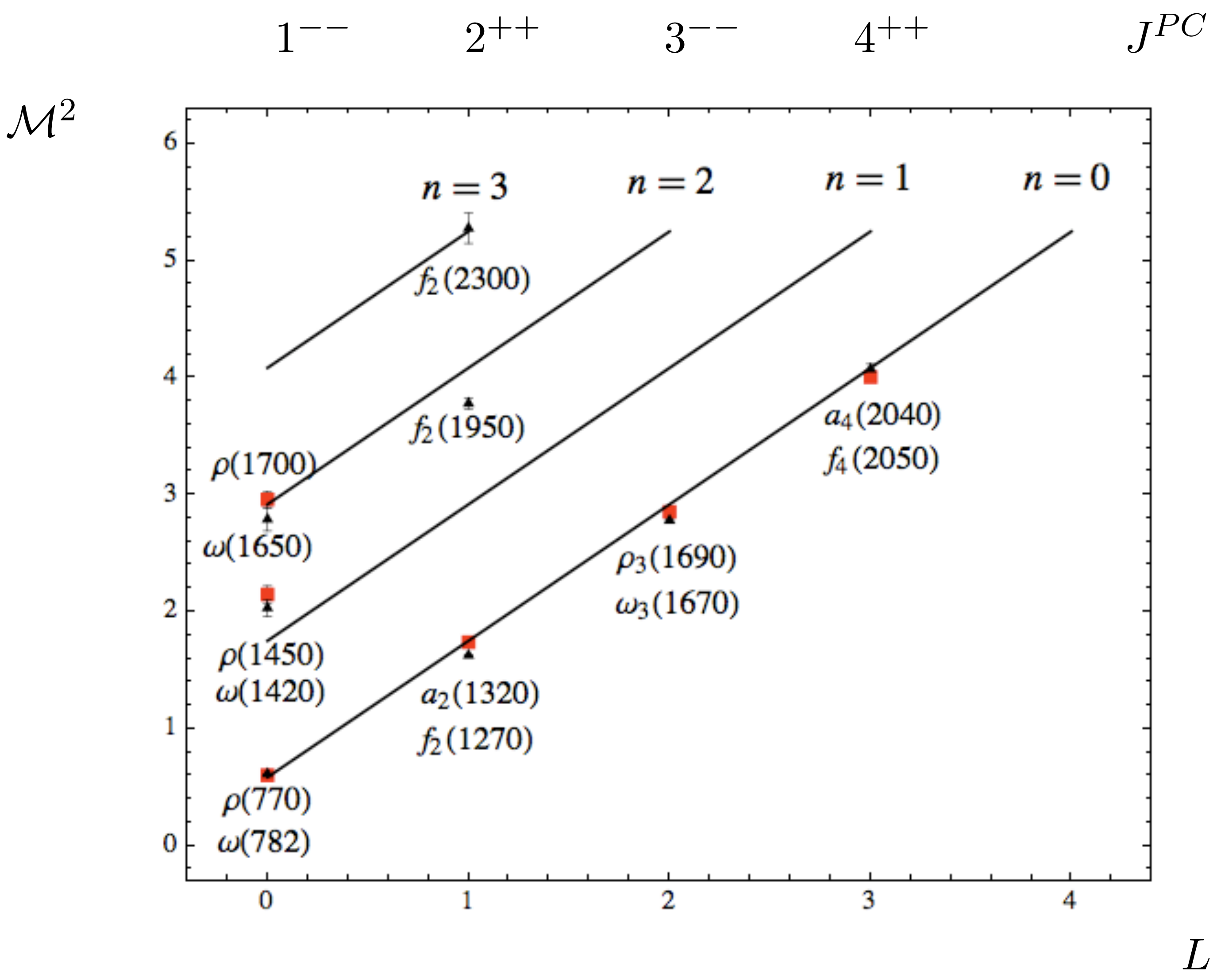
$4\kappa^2$ for $\Delta n = 1$

$4\kappa^2$ for $\Delta L = 1$

$2\kappa^2$ for $\Delta S = 1$



Orbital and radial states: $\langle \zeta \rangle$ increase with L and n



Parent and daughter Regge trajectories for the $I = 1$ ρ-meson family (red) and the $I = 0$ ω-meson family (black) for $\kappa = 0.54$ GeV

Note: Algebraic Construction

- Write the LF Hamiltonian, H_{LF}

$$H^L_{LF}(\zeta) = \Pi^\dagger_L(\zeta)\Pi_L(\zeta) + C$$

in terms of the operator

$$\Pi_L(\zeta) = -i\left(\frac{d}{d\zeta} - \frac{L+\frac{1}{2}}{\zeta} - \kappa^2\zeta\right)$$

and its adjoint

$$\Pi^\dagger_L(\zeta) = -i\left(\frac{d}{d\zeta} + \frac{L+\frac{1}{2}}{\zeta} + \kappa^2\zeta\right)$$

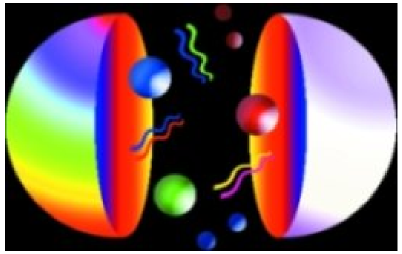
with commutation relations

$$\left[\Pi_L(\zeta), \Pi^\dagger_L(\zeta)\right] = \frac{2L+1}{\zeta^2} - 2\kappa^2$$

- The LF Hamiltonian is positive definite, $\langle\phi\left|H^L_{LF}\right|\phi\rangle \geq 0$, for $L^2 \geq 0$ and $C \geq -4\kappa^2$

5 Higher-Spin Fermionic Modes

Hard-Wall Model



From Nick Evans

- Action for massive fermionic modes on AdS$_{d+1}$:

$$S[\overline{\Psi}, \Psi] = \int d^d x \, dz \, \sqrt{g} \, \overline{\Psi}(x,z) \left(i\Gamma^\ell D_\ell - \mu \right) \Psi(x,z)$$

- Equation of motion: $\left(i\Gamma^\ell D_\ell - \mu \right) \Psi(x,z) = 0$

$$\left[i \left(z\eta^{\ell m}\Gamma_\ell \partial_m + \frac{d}{2}\Gamma_z \right) + \mu R \right] \Psi(x^\ell) = 0$$

- Solution $(\mu R = \nu + 1/2, \, d = 4)$

$$\Psi(z) = C z^{5/2} \left[J_\nu(z\mathcal{M}) u_+ + J_{\nu+1}(z\mathcal{M}) u_- \right]$$

- Hadronic mass spectrum determined from IR boundary conditions $\psi_\pm \left(z = 1/\Lambda_{\rm QCD} \right) = 0$

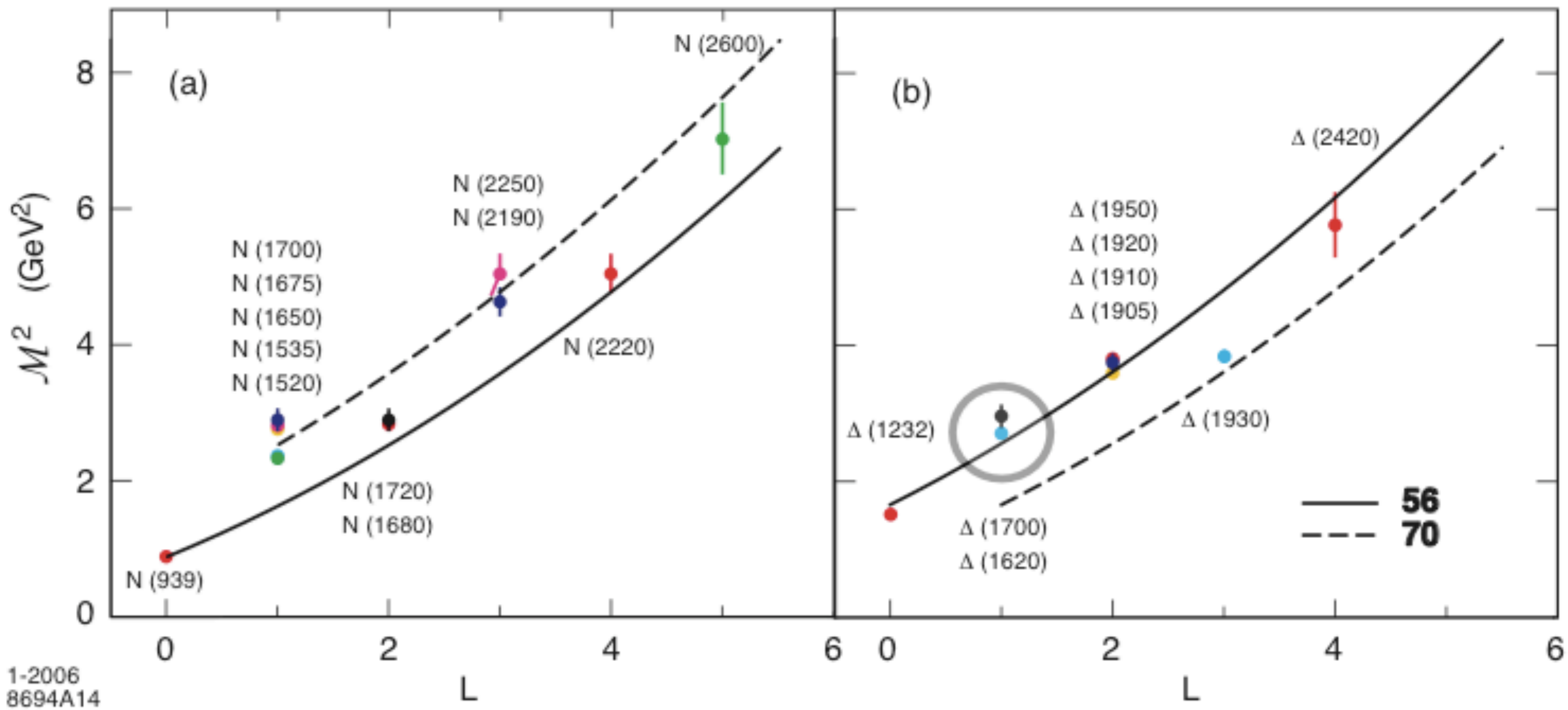
$$\mathcal{M}^+ = \beta_{\nu,k} \, \Lambda_{\rm QCD}, \quad \mathcal{M}^- = \beta_{\nu+1,k} \, \Lambda_{\rm QCD}$$

with scale independent mass ratio

- Obtain spin-J mode $\Phi_{\mu_1 \cdots \mu_{J-1/2}}$, $J > \frac{1}{2}$, with all indices along 3+1 from Ψ by shifting dimensions

SU(6)	S	L	Baryon State
56	$\frac{1}{2}$	0	$N\frac{1}{2}^+(939)$
	$\frac{3}{2}$	0	$\Delta\frac{3}{2}^+(1232)$
70	$\frac{1}{2}$	1	$N\frac{1}{2}^-(1535)$ $N\frac{3}{2}^-(1520)$
	$\frac{3}{2}$	1	$N\frac{1}{2}^-(1650)$ $N\frac{3}{2}^-(1700)$ $N\frac{5}{2}^-(1675)$
	$\frac{1}{2}$	1	$\Delta\frac{1}{2}^-(1620)$ $\Delta\frac{3}{2}^-(1700)$
56	$\frac{1}{2}$	2	$N\frac{3}{2}^+(1720)$ $N\frac{5}{2}^+(1680)$
	$\frac{3}{2}$	2	$\Delta\frac{1}{2}^+(1910)$ $\Delta\frac{3}{2}^+(1920)$ $\Delta\frac{5}{2}^+(1905)$ $\Delta\frac{7}{2}^+(1950)$
70	$\frac{1}{2}$	3	$N\frac{5}{2}^-$ $N\frac{7}{2}^-$
	$\frac{3}{2}$	3	$N\frac{3}{2}^-$ $N\frac{5}{2}^-$ $N\frac{7}{2}^-(2190)$ $N\frac{9}{2}^-(2250)$
	$\frac{1}{2}$	3	$\Delta\frac{5}{2}^-(1930)$ $\Delta\frac{7}{2}^-$
56	$\frac{1}{2}$	4	$N\frac{7}{2}^+$ $N\frac{9}{2}^+(2220)$
	$\frac{3}{2}$	4	$\Delta\frac{5}{2}^+$ $\Delta\frac{7}{2}^+$ $\Delta\frac{9}{2}^+$ $\Delta\frac{11}{2}^+(2420)$
70	$\frac{1}{2}$	5	$N\frac{9}{2}^-$ $N\frac{11}{2}^-(2600)$
	$\frac{3}{2}$	5	$N\frac{7}{2}^-$ $N\frac{9}{2}^-$ $N\frac{11}{2}^-$ $N\frac{13}{2}^-$

- Excitation spectrum for baryons in the hard-wall model: $\mathcal{M} \sim L + 2n$



Light baryon orbital spectrum for Λ_{QCD} = 0.25 GeV in the HW model. The $\mathbf{56}$ trajectory corresponds to L even $P = +$ states, and the $\mathbf{70}$ to L odd $P = -$ states: (a) $I = 1/2$ and (b) $I = 3/2$

Note: Algebraic Construction

- Fermionic modes are solutions of the light-front Dirac equation for $\psi(\zeta) \sim \zeta^{-2}\Psi(\zeta)$

$$\alpha \Pi(\zeta)\psi(\zeta) = \mathcal{M}\psi(\zeta)$$

where $\alpha^\dagger = \alpha, \ \alpha^2 = 1, \ \{\alpha, \gamma_5\} = 0$

- The operator

$$\Pi_\nu(\zeta) = -i\left(\frac{d}{d\zeta} - \frac{\nu + \frac{1}{2}}{\zeta}\gamma_5\right)$$

and its adjoint

$$\Pi_\nu^\dagger(\zeta) = -i\left(\frac{d}{d\zeta} + \frac{\nu + \frac{1}{2}}{\zeta}\gamma_5\right)$$

satisfy the commutation relations

$$\left[\Pi_\nu(\zeta), \Pi_\nu^\dagger(\zeta)\right] = \frac{2\nu + 1}{\zeta^2}\gamma_5$$

Soft-Wall Model

- Equivalent to Dirac equation in presence of a holographic linear confining potential

$$\left[i\left(z\eta^{\ell m}\Gamma_\ell\partial_m + \frac{d}{2}\Gamma_z\right) + \mu R + \kappa^2 z\right]\Psi(x^\ell) = 0.$$

- Solution $(\mu R = \nu + 1/2,\, d = 4)$

$$\begin{aligned}\Psi_+(z) &\sim z^{\frac{5}{2}+\nu} e^{-\kappa^2 z^2/2} L_n^\nu(\kappa^2 z^2)\\ \Psi_-(z) &\sim z^{\frac{7}{2}+\nu} e^{-\kappa^2 z^2/2} L_n^{\nu+1}(\kappa^2 z^2)\end{aligned}$$

- Eigenvalues

$$\mathcal{M}^2 = 4\kappa^2(n + \nu + 1)$$

- Obtain spin-J mode $\Phi_{\mu_1\cdots\mu_{J-1/2}}$, $J > \frac{1}{2}$, with all indices along 3+1 from Ψ by shifting dimensions

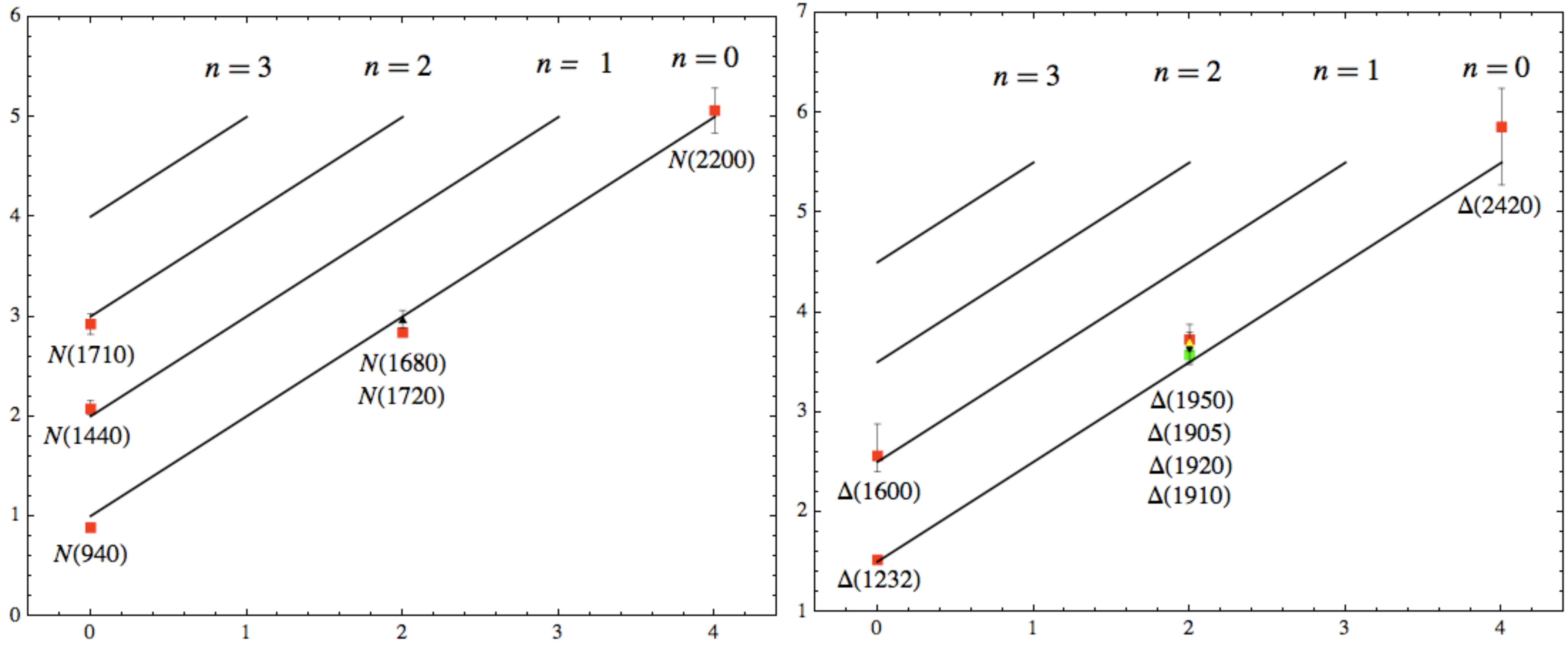
Glazek and Schaden [Phys. Lett. B **198**, 42 (1987)]: $(\omega_B/\omega_M)^2 = 5/8$

$4\kappa^2$ for $\Delta n = 1$

$4\kappa^2$ for $\Delta L = 1$

$2\kappa^2$ for $\Delta S = 1$

$\mathcal{M}^2$



L

Parent and daughter **56** Regge trajectories for the N and Δ baryon families for $\kappa = 0.5$ GeV

Note: Algebraic Construction

- Fermionic modes are solutions of the light-front Dirac equation for $\psi(\zeta) \sim \zeta^{-2}\Psi(\zeta)$

$$\alpha \Pi(\zeta)\psi(\zeta) = \mathcal{M}\psi(\zeta)$$

  where $\alpha^\dagger = \alpha, \ \alpha^2 = 1, \ \{\alpha, \gamma_5\} = 0$

- The operator

$$\Pi_\nu(\zeta) = -i\left(\frac{d}{d\zeta} - \frac{\nu + \frac{1}{2}}{\zeta}\gamma_5 - \kappa^2 \zeta \gamma_5\right)$$

  and its adjoint

$$\Pi_\nu^\dagger(\zeta) = -i\left(\frac{d}{d\zeta} + \frac{\nu + \frac{1}{2}}{\zeta}\gamma_5 - \kappa^2 \zeta \gamma_5\right)$$

  satisfy the commutation relations

$$\left[\Pi_\nu(\zeta), \Pi_\nu^\dagger(\zeta)\right] = \left(\frac{2\nu + 1}{\zeta^2} - 2\kappa^2\right)\gamma_5$$

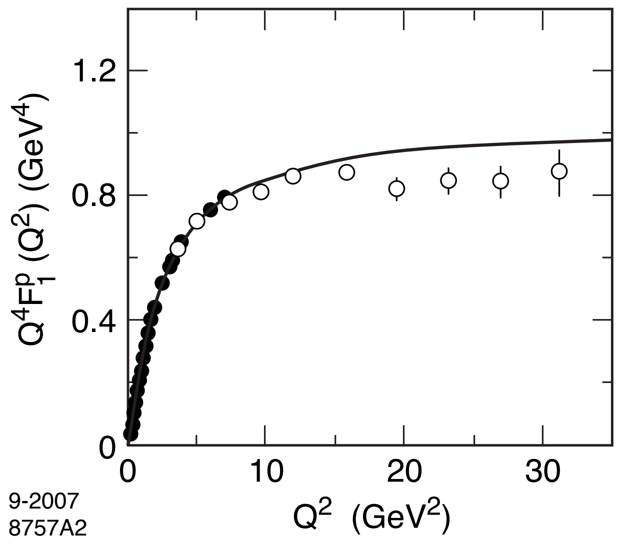
- Supersymmetric QM at equal LF time?

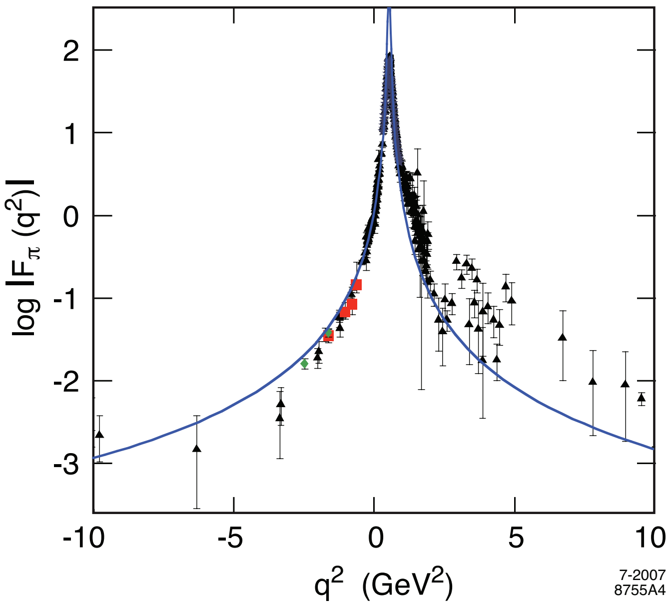
Other Applications of Light-Front Holography

- Nucleon form-factors: space-like region
- Pion form-factors: space and time-like regions
- Gravitational form-factors of composite hadrons

Future Applications

- Pauli Form Factor
- Introduction of massive quarks
- Systematic improvement (QCD Coulomb forces . . .)





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GdT and SJB, arXiv:0809.489