# High Energy Plasmas, General Relativity, Collective Modes and Black Holes

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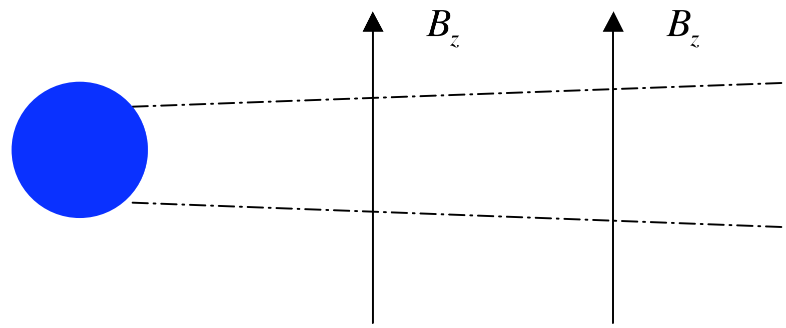
# "Stripped" Plasma Disks




* Axisymmetric, $\mathbf{V} = \mathrm{V}_\phi \mathbf{e}_\phi$
* Imbedded in a Vertical Magnetic Field $B_z$
* Non-accreting

Described by two fluid equations involving the fields $E_z$ and $E_R$ as

$$p = p_e + p_i \ , \tag{1}$$

$$-\frac{\partial p_e}{\partial z} - enE_z = 0\,, \tag{2}$$

$$-\frac{\partial p_i}{\partial z} + enE_z - z\rho\Omega_k^2 = 0\,, \tag{3}$$

$$E_R + \frac{1}{c}\mathrm{V}_\phi B_z = 0 \ . \tag{4}$$

Magnetic surfaces are concentric cylinders $\left(R = \text{const.}\right)$,

$\mathrm{V}_\phi = \Omega\left(R\right)R$, $\Omega \simeq \Omega_k$ ,

$$\Omega_k = \sqrt{\frac{GM_*}{R^3}} = \frac{c}{R}\sqrt{\frac{R_G}{R}} \quad , \qquad\qquad R_G \equiv \frac{GM_*}{c^2} .$$

Moreover,

$$2T = \frac{p}{n}, \;\; T_e \simeq T_i, \;\; \mathrm{v}_{thi}^2 = \frac{2T_i}{m_i}, \;\; \rho \simeq n m_i$$

and

$$\rho\left(z^2, R_0\right) \simeq \rho_0\left(1 - \frac{z^2}{H_0^2}\right), \;\; \text{for} \;\; z^2 \ll H_0^2 \simeq \frac{2\mathrm{v}_{thi}^2}{\Omega_k^2} .$$

The parameter

$$\eta_T \equiv \frac{1}{T}\frac{dT}{dz} \Big/ \frac{1}{n}\frac{dn}{dz}$$

characterizing the relative temperature gradient is particularly important for the kind of modes that can be excited.

We assume, for simplicity, that

$$\beta \equiv \frac{8\pi p}{B^2} > 1 \;\; .$$

# II. Spectrum of Modes Excited in a Stripped Disk

i) Axisymmetric modes

ii) 3-D Standing Spirals

iii) Driving factors $\frac{d\Omega}{dR}$ , $\frac{dp}{dz}$

iv) Induced Transport

v) Convective Spirals

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# Spectrum of Plasma Modes and Relevant Transport Processes in Astrophysical Disks

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**ABSTRACT**

A simple plasma disk structure imbedded in a magnetic field and in the (prevalent) gravity of a central object is shown to be subject to the excitation of significant axisymmetric and tridimensional modes. The key factors involved in the relevant instability are the plasma pressure vertical gradient and the rotation frequency (around the central object) radial gradient. A modestly peaked vertical profile of the plasma temperature is shown to drive a "thermo-rotational instability" with considerable growth rates. Unstable modes are found as well for "flat" temperature profiles. The tri-dimensional tightly wound spirals that are found have properties that, unlike the familiar galactic spirals, depend on the vertical profiles of their amplitudes. Both radially standing and convective (quasi-modes) spirals are identified. Within the considered spectrum, unstable modes are shown to produce opposing fluxes of particles and thermal energies in the vertical direction. Thus disks with relatively flat vertical temperature profiles can expel particles (winds) from the equatorial plane while transporting thermal energy inward. An effective "diffusion" coefficient, for energy and angular momentum, is derived from the structure of radially convective spiral modes and shown to be consistent with significant radial transport rates.

**Key words.** Accretion, accretion disks; Black hole physics; Magnetohydrodynamics (MHD); Instabilities; Magnetic fields; Gravitation

## 1. Introduction

Identifying the plasma collective modes that can be excited in plasma disk structures (Pringle 1981, Blandford 1976, Lovelace 1976, Coppi & Rousseau 2006) surrounding compact objects such as black holes can be important to explain experimental observations associated with objects of this kind. The geometry of these disk structures (Coppi & Coppi 2001) and their physical parameters, that include in particular the radial gradient of the rotation frequency, the vertical gradients of the particle density and temperature and the effects of the magnetic field in which they are imbedded (Coppi 2008), determine the characteristics and the spectrum of the modes that can be excited. Novel transport processes of particles and thermal energy in the vertical direction as well as radial transport of angular momentum can be produced by the identified modes. In fact, these processes and the tri-dimensional structure of the spiral modes (Coppi 2008b) that are found can be correlated with important observations connected with black holes (Coppi & Rebusco 2008). We point out that the properties of the tightly wound spirals that are introduced depend on the vertical profiles of their amplitudes, a feature that is not shared with the better known theory of galactic spirals (Bertin 2000). We note also that the theory presented here for both axisymmetric and tri-dimensional modes include, among others, the physical ingredients of basic modes such as the so called MRI instability (Velikhov 1959, Chandrasekhar 1960, Balbus & Hawley 1991), that is appropriate for different geometries such as cylinders, or the modes that can be excited in well confined plasmas (Coppi & Spight 1978) and can explain the experimentally observed inward transport of particles in connection with the outward transport of plasma thermal energy.

In Section 2 the characteristic parameters of the simplest plasma disk structure, imbedded in a vertical magnetic field and surrounding a compact object, are defined in view of the analyses presented in the next sections. In Section 3 the basic equations are given for the spectrum of axisymmetric and tri-dimensional normal modes that can be excited in a thin plasma disk. In Section 4 the realistic conditions on the density and temperature vertical profiles are examined under which nearly isobaric, weakly compressible modes are found. In Section 5 the theory of axisymmetric modes is given and the conditions for their marginal stability are derived. The radial gradient of the rotation frequency and the vertical gradient of the plasma pressure are recognized as the key factors for the excitation of these modes. In particular, when the ratio of the relative gradient of the plasma temperature exceeds the relative gradient of the density by 2/3 a new form of instability (the "thermo-rotational instability") emerges (Coppi 2008). This is in addition to that of the ballooning modes analyzed by Coppi and Keyes (2003) that are found when the ratio of the gradients mentioned earlier is 2/3 (adiabatic profile). In Section 6 the transport produced in the vertical direction by the unstable modes described in Section 5 is evaluated by the relevant quasi-linear theory. Then the suggestion is made that the outward particle transport (away from the equatorial plane) occurring in the presence of relatively flat temperature profiles be the seed for the observed winds emanating from disks structures surrounding black holes (Elvis 2000). In this case an inward (toward the equatorial) plane) flux of thermal energy is produced. The mode growth rates of the unstable modes on which the relevant transport coefficients depend are evaluated by completing the analysis given in Section 5. In Section 7 the characteristics of the weakly damped oscillatory modes, can be found when the temperature vertical profile is relatively flat, are examined. In Section 8 the theory of tri-dimensional spiral modes that are localized both in the vertical and the radial direction is given. These modes co-rotate with the plasma at the radius around which they are localized. The excitation of modes

# Thermo-Rotational Instability

(T.R.I.)

Axisymmetric kind represented by

$$\hat{\mathrm{v}}_\phi = \tilde{\mathrm{v}}_\phi(z)\exp\left[\gamma_0 t + ik_R(R-R_0)\right]$$

where

$$\frac{d\tilde{\mathrm{v}}_\phi(z)}{dz} \neq 0, \quad k_R^2 \simeq k_0^2 = \frac{3\Omega_k^2}{\mathrm{v}_A^2}, \quad \mathrm{v}_A^2 = \frac{B_z^2}{4\pi\rho_0}$$

The VERTICAL STRUCTURE is important

and the “thermal” parameter

$$\eta_T = \frac{1}{T}\frac{dT}{dz}\Big/\left(\frac{1}{n}\frac{dn}{dz}\right)$$

has a key role, like the parameter

$$\eta_i = \frac{1}{T_i}\frac{dT_i}{dr}\Big/\left(\frac{1}{n}\frac{dn}{dr}\right)$$

in the theory of Ion Temperature Gradient (ITG) driven modes for magnetically confined plasmas.

# Lowest Eigenfunction

$$\tilde{\xi}_z(z) = \tilde{\xi}_z^0 \exp\left(-\frac{z^2}{2\Delta_z^2}\right)$$

$$\gamma_0^2 \simeq \frac{6\sqrt{3}}{35}\frac{\mathrm{v}_A}{H_0}\Omega_k\left(\eta_T - \frac{2}{3}\right)$$

$\hat{\xi}_z =$ vertical plasma displacement

# Higher Eigenfunctions

Unstable even for $\eta_T < \dfrac{2}{3}$,

e.g.

$$\eta_T < \frac{2}{3} - \frac{7}{2(k_0 H_0)}$$

# 3D Standing Spirals (tightly wound + trailing)

* Do not shear apart

* Radial width of localization

$$\Delta_R \simeq \left(\frac{\gamma_0}{m_\phi \Omega_k}\right)^{1/2} \left(\frac{R_0}{k_0}\right) \sim \frac{\left(\gamma_0 \mathrm{v}_A R_0\right)^{1/2}}{\Omega_k}$$

($\gamma_0 =$ growth rate)

* Height (vertical)

$$\Delta_z \sim \frac{\left(c_s \mathrm{v}_A\right)^{1/2}}{\Omega_k}$$

$$\frac{\Delta_R}{\Delta_z} \sim \left(\frac{\gamma_0 R_0}{c_s}\right)^{1/2} \sim \left(\frac{\gamma_0}{\Omega_k}\frac{R_0}{H_0}\right)^{1/2}$$

# Analytical Expression

$$\hat{\xi}_z \simeq \tilde{\xi}_z\left(R - R_0, z\right)\exp\left[\gamma_0 t - i m_\phi\left(\Omega_0 t - \phi\right) + i k_R\left(R - R_0\right)\right]$$

$$\Omega_0 = \Omega_k\left(R_0\right) \qquad\qquad k_R \simeq k_0 = \sqrt{3}\,\frac{\Omega_k}{\mathrm{v}_A}$$

$$\tilde{\xi}_z\left(R-R_0\,,z\right)\simeq\tilde{\xi}_z^0\exp\left\{-\left[\frac{\left(R-R_0\right)^2}{\Delta_R^2}+\frac{z^2}{2\Delta_z^2}\right]\right\}G_0^0\left(z\right)$$

$$m_\phi k_R\,\frac{d\Omega_k}{dR}<0$$

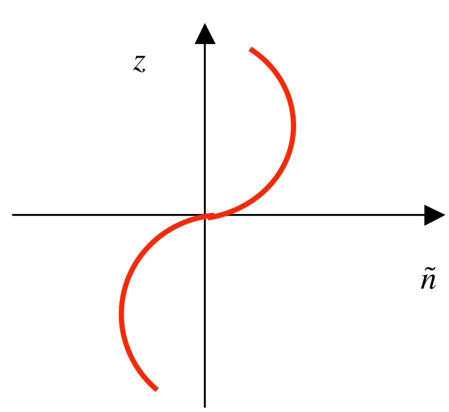
$$\Delta_R^2\simeq-\frac{\gamma_0}{m_\phi k_R\,d\Omega_k/dR}$$

$G_0^0\left(z\right)=$ eigenfunction is vertical modulation

# Density Perturbations

$$\frac{\hat{n}}{n}\simeq-\frac{z}{H_0^2}\frac{6}{5}\left(\eta_T-\frac{2}{3}\right)\hat{\xi}_z$$

$$\eta_T\equiv\frac{1}{T}\frac{dT}{dz}\Big/\left(\frac{1}{n}\frac{dn}{dz}\right)$$



## Quasi-linear Vertical particle Transport

$$\Gamma_p = \langle\langle \hat{n}\hat{\mathrm{v}}_z \rangle\rangle \simeq -\frac{4}{5}\gamma_0 \left\langle\left\langle \left|\hat{\xi}_z\right|^2 \right\rangle\right\rangle \left(1 - \frac{3}{2}\eta_T\right)\frac{\partial n}{\partial z}$$

* Inflow or outflow depending on $\eta_T$

* Observed WINDS from AGN cores (disks) may be connected to this process

## "Isobaric" Modes and Influence on the Electron Runaway Field $E_R^c$

$$\frac{\hat{T}}{T} \simeq -\frac{\hat{n}}{n}$$

$$E_R^c \propto \frac{n}{T} \simeq \frac{n_0}{T_0}\frac{1 - \Delta T/T_0}{1 + \Delta T/T_0}$$

Therefore, the production of non-thermal electron distributions can be strongly affected by the considered modes when densities are relatively low.

# Convective Spirals

$$\hat{\xi}_z \simeq \tilde{\xi}_z^0 \exp\left(-\frac{z^2}{2\Delta_z^2}\right)\cos\left[\frac{\sigma_R}{2}\left(R-R_0\right)^2-\left(\delta\omega_0\right)t\right]$$

$$\exp\left[im_\phi\left(\phi-\Omega_0 t\right)+ik_R\left(R-R_0\right)\right]$$

$$\sigma_R=\frac{7}{3}m_\phi k_R\frac{d\Omega}{dR}\frac{1}{\delta\omega_0}\sim m_\phi\frac{k_0}{R}\frac{\Omega}{\delta\omega_0}$$

$$D_{eff}=\frac{\partial}{\partial\sigma_R}\delta\omega_0\simeq-\frac{3}{7}\frac{\left(\delta\omega_0\right)^2}{m_\phi k_R\left(d\Omega/dR\right)}>0$$

for

$$m_\phi k_R\frac{d\Omega}{dR}<0$$

$$D_{eff}\sim\frac{\mathrm{v}_A R_0}{m_\phi}\left(\frac{\mathrm{v}_A}{c_s}\right)$$

# III. "Crystal" Structures

i) Axisymmetric

ii) Current Carrying

iii) Non-accreting

## 3-D Configurations

$$\mathbf{B} = \frac{1}{R}\left[\nabla\psi \times \mathbf{e}_\phi + I(\psi)\,\mathbf{e}_\phi\right] \qquad \text{(Axisymmetric)}$$

$$\mathbf{B}\cdot\nabla\psi = 0$$

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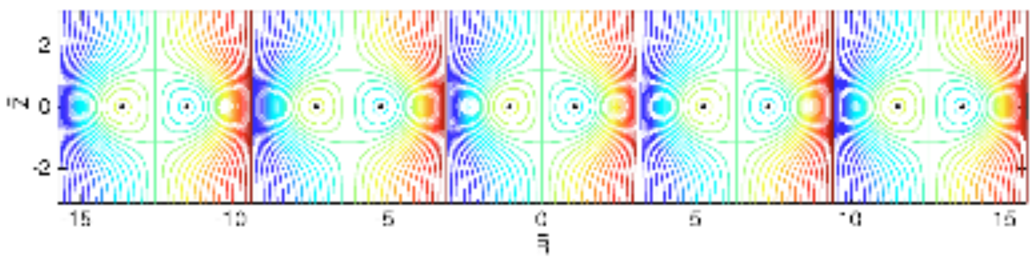


FIG. 1. "Crystal" structure of the magnetic field in a thin plasma accretion disk. Representation of the magnetic surface function ,= $\bar R-2$ $n^0$ $-Y_0^0$ sin $\bar R$ $F_0$ $\bar z^2$ for $Y_0^0=2$ see Eq. 5.3 . Here $n^0=0,1,2,\ldots$

Representative magnetic surface

$$\psi = \left(R - R_0 - 2\pi n^0 R_0\right)\frac{d\psi_0}{dR} + \psi_1 \exp\left(-\frac{z^2}{2\Delta_z^2}\right)\sin\left[k_0\left(R-R_0\right)\right]$$

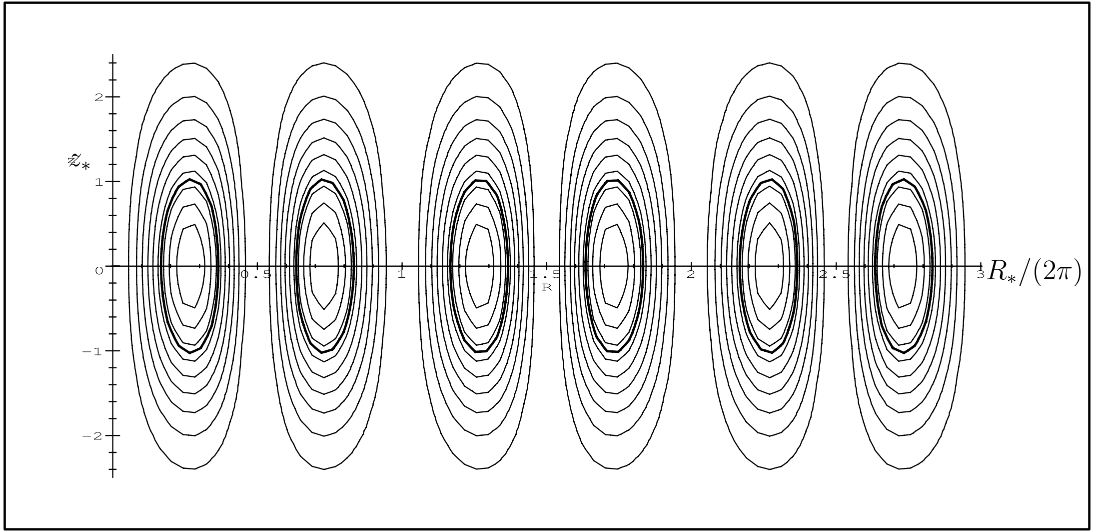


FIG. 4.—Density level lines as a function of $R_*/(2\pi)$ and $z_*$. The heavy line corresponds to the surface represented in Fig. 5, corresponding to $D_*(R_*, z_*) = 0.5$. The function $D_*$ is defined by eqs. (103) and (109), and the variables $R_*$ and $z_*$ are introduced in § 9.

III-4

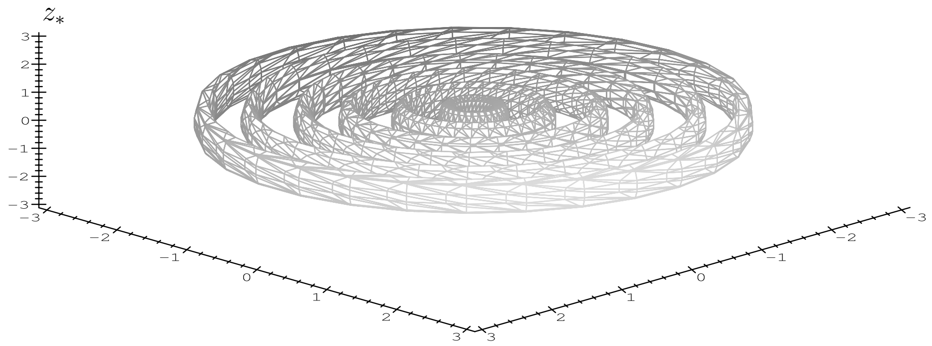


FIG. 5.—Plasma density rings represented by the level lines $D_*(R_*, z_*) = 0.5$. The function $D_*$ is defined by eqs. (103) and (109), and the variables are introduced in § 9.

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# "Crystal" magnetic structure in axisymmetric plasma accretion disks[a]

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A general class of stationary magnetic configurations which can exist in (thin) plasma accretion disks is identified by solving analytically the coupled nonlinear equations that describe the radial and the vertical equilibrium conditions of the disk. These configurations are characterized by a "crystal" structure consisting of a sequence of toroidal current filaments that can involve null points of the poloidal magnetic field. The obtained solutions are valid in the limit where the magnetic energy density is smaller than the thermal energy density ($\beta > 1$). In view of studying magnetic disk configurations from which jets can emerge, and for which the limit where $\beta \sim 1$ is important, the relevant equilibrium equations are derived and their symmetries are pointed out. © *2005 American Institute of Physics*. [DOI: 10.1063/1.1883667]

## I. INTRODUCTION

The basic equilibrium conditions of an axisymmetric thin gaseous plasma disk are well known.[1] The problem of formulating the equivalent conditions for a plasma disk where the electric and magnetic field configurations play an important role is, instead, considerably more complex. In particular, a characteristic set of nonlinear equations has to be identified and solved in physically relevant limits.

In the case of a gaseous disk, the vertical equilibrium condition can be separated from the radial condition, in the thin disk limit, and the vertical pressure profile is determined once the temperature profile is prescribed. By contrast, in the case of plasma disks, the vertical and the radial equilibrium conditions are intrinsically coupled when the toroidal current densities in the disk are significant. In addition, the frozen-in-law condition (corresponding to considering plasmas with relatively large electrical conductivities) leads one to find that the relevant magnetic surfaces can exhibit a "crystal structure" as exemplified by Fig. 1. As a consequence of this, the plasma pressure is modulated radially with a period of the order of the height of the disk. The theory that is presented in this paper is carried out in the limit where the plasma pressure exceeds the magnetic field pressure ($\beta > 1$) as this lends itself to a complete analytic treatment. In order to approach the case where $\beta \sim 1$ that is of special importance in view of dealing with disks from which jets can emerge, the appropriate equations are derived and their symmetries are pointed out.

The main results of the equilibrium analysis are discussed in Sec. VII and are related to the theory of the axisymmetric ballooning modes[2,3] which can fit within thin accretion disks and are marginally stable.

## II. ELECTRIC FIELD CONFIGURATION

Clearly, in the equilibrium state, the electric field is associated with a potential $\Phi$, such that $\mathbf{E} = -\nabla\Phi(R,z)$, and we find it convenient to separate it into the following components:

$$\mathbf{E} = \mathbf{E}_{\text{cor}} + \mathbf{E}_{\Delta}, \tag{2.1}$$

where

$$\mathbf{E}_{\text{cor}} + \Omega(R,z)\frac{R}{c}\mathbf{e}_{\phi} \times \mathbf{B}_p = 0 \tag{2.2}$$

and $\Omega(R,z)$ indicates the local rotation frequency of the plasma (i.e., the ions). The poloidal magnetic field in terms of the magnetic surface function $\Psi(R,z)$ is

$$\mathbf{B}_p = \frac{1}{R}\nabla\Psi \times \mathbf{e}_{\phi} \tag{2.3}$$

and

$$\mathbf{E}_{\text{cor}} = -\frac{\Omega(R,z)}{c}\nabla\Psi. \tag{2.4}$$

As implied by Eq. (2.2), $\mathbf{E}_{\text{cor}} \cdot \mathbf{B}_p = 0$ and $\nabla \times \mathbf{E}_{\text{cor}} = -\nabla \times \mathbf{E}_{\Delta}$, where

$$\nabla \times \mathbf{E}_{\Delta} = \frac{1}{c}\nabla\Omega \times \nabla\Psi. \tag{2.5}$$

The vertical equilibrium equation for the electrons is

$$-enE_{\Delta z} - \frac{\partial p_e}{\partial z} - \frac{en}{c}(u_{eR}B_{\phi} - u_{e\phi}B_R) = 0, \tag{2.6}$$

where $n$ is the particle density, $u_{e\phi}$ the flow velocity of the electrons relative to the (rotating) ions, and we have neglected the rate of collisional momentum transfer between electrons and ions. Standard notations are adopted for the other quantities entering Eq. (2.6). The corresponding equation for the ions (e.g., protons) is

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[b] Invited speaker.

 

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# PLASMA DISKS AND RINGS WITH "HIGH" MAGNETIC ENERGY DENSITIES

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## ABSTRACT

The nonlinear theory of rotating axisymmetric thin structures in which the magnetic field energy density is comparable with the thermal plasma energy density is formulated. The only flow velocity included in the theory is the velocity of rotation around a central object whose gravity is dominant. The periodic sequence, in the radial direction, of pairs of opposite current channels that can form is shown to lead to relatively large plasma density and pressure modulations, while the relevant magnetic surfaces can acquire a "crystal structure." A new class of equilibria consisting of a series of plasma rings is identified, in the regimes where the plasma pressure is comparable to the magnetic pressure associated with the fields produced by the internal currents. The possible relevance of this result to the formation of dusty plasma rings is pointed out.

*Subject headings:* accretion, accretion disks — MHD — planets: rings — plasmas — stars: neutron — stars: rotation

## 1. INTRODUCTION

In the course of the search for the plasma collective modes that can provide the needed rate of angular momentum transport (Shakura & Sunyaev 1973) in accretion disks, we have realized that an elementary theory of the equilibrium configurations of differentially rotating axisymmetric plasma thin structures remained to be formulated. Thus, we have considered (Coppi 2005) the case where the rotation velocity is the only flow velocity present, the gravity of the central object is prevalent, and the magnetic field within the structure (ring sequence or disk) has two components: an "external" one in which the structure is embedded and an "internal" one due to currents flowing within the structure. No torque is associated with the resulting field configuration.

We note that the existing literature on rotating, perfectly conducting objects is extensive but addresses different issues. In particular, Ferraro (1937) dealt with the nonuniform rotation of the Sun and its magnetic field, Mestel (1961) with a rotating star that has a pure poloidal field initially, and Weber & Davis (1967) with the deceleration of the Sun through the action of magnetic stresses. In this context, more recently Lovelace (1976) and Blandford (1976) proposed accretion disk models in which the energy is extracted by electromagnetic torques, and Blandford & Payne (1982) treated the MHD flows from accretion disks and the associated production of radio jets.

Our elementary theory includes the role of marginally stable ballooning modes associated with both the differential rotation and the presence of an external field (Coppi & Coppi 2001; Coppi & Keyes 2003). We have started with the linear and nonlinear analysis (Coppi 2005) of disks in which the vertical confinement is due primarily to the vertical component of the gravitational force produced by the central object, as in the case of "gaseous" disks (Lynden-Bell 1969; Pringle & Rees 1972; Shakura & Sunyaev 1973), where the energy density of both the magnetic field in which they are immersed and that created by the currents within the disk is smaller than the thermal energy density ($\beta > 1$). A main result of this analysis is that the relevant poloidal magnetic field configuration can be characterized by a periodic sequence of pairs of toroidal current channels in which the currents have opposite directions and by the formation of a "crystal structure" of the magnetic surfaces. In fact, the vertical cross section of this structure can be visualized as a string of "field reverse configurations" (FRCs), the periodic reversal of the vertical field relative to the external field component being due to the currents within the structure.

An interesting question that arose from the same analysis (Coppi 2005) is whether a disk configuration would continue to exist in the case in which the magnetic field produced by the currents inside the plasma structure exceeds considerably the external magnetic field component, and the plasma pressure is contained (vertically) by the magnetic field pressure rather than by gravity. In fact, the results described in this paper show that the plasma density becomes strongly modulated by the current channels, and a sequence of plasma rings rather than a disk is formed. The theory is also given for the intermediate case, where the external field pressure is comparable with the plasma pressure and to the internally produced field. In this case a disk with a strongly corrugated (in the radial direction), vertical density profile is found.

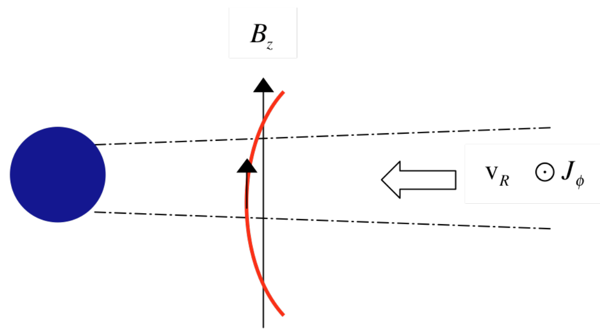
Clearly this calls for a parallel analysis of self-gravitating plasma disks, which has been undertaken already. Moreover, other interesting questions arise. One is whether the tridimensional plasma modes that can arise in these configurations can produce sufficient rates of outward angular momentum transport and consequent rates of mass accretion that are consistent with the values of the effective diffusion coefficients (Shakura & Sunyaev 1973) employed for relevant models of accreting objects. A further question is whether an axisymmetric structure of the type identified in our analysis will evolve into a toroidally modulated coherent structure or a turbulent structure.

The present paper is organized as follows. In § 2 we discuss the basic equations for the vertical and radial momentum density balance in a differentially rotating plasma thin structure. The only velocity present is in the toroidal direction, and the magnetic field configuration is connected to it by the "isorotation condition" (Ferraro 1937) that is appropriate for relatively large electrical

# IV. Accretion Scenarios in the Presence of Magnetic Fields

$$E_\phi = 0$$

i) Simplest Disk Structure




$$\mathrm{v}_R \simeq D_m \frac{1}{B_z}\left(\frac{\partial}{\partial R} B_z - \frac{\partial}{\partial z} B_R\right) \qquad (1)$$

* The effects of a significant anomalous resistivity $\eta = 4\pi D_m / c^2$ would have to be important over the entire disk and a process to produce this resistivity would have to be identified

* A large scale violation of the hyperconductivity condition

$$\mathbf{v} = \alpha_{\mathrm{v}} \mathbf{B} + \Omega(\psi) R \mathbf{e}_\phi \qquad (2)$$

such as that represented by Eq. (1), would prevent the consideration of important plasma modes (e.g. driven by the differential rotation and the vertical plasma pressure) needed to produce outward angular momentum transport, “winds”, etc. and relying on the validity of Eq. (2).

## ii. Nearly-Axisymmetric Magnetic Crystal Structures

The presence of a coherent magnetic crystal structure allows the plasma to move radially in steps along channels following the magnetic separatrices, requiring the effects of resistive modes only in restricted regions, between adjacent separatrices (see Figs. 1 and 2) and at the center of each module from which the plasma can invade neighboring magnetic surfaces. The resulting form of accretion would be an intermittent process related to the onset and decay of resistive ballooning modes driven by the local "residual" gravitational acceleration.

## iii. Spiral Structures

Their existence and properties remain to be investigated.

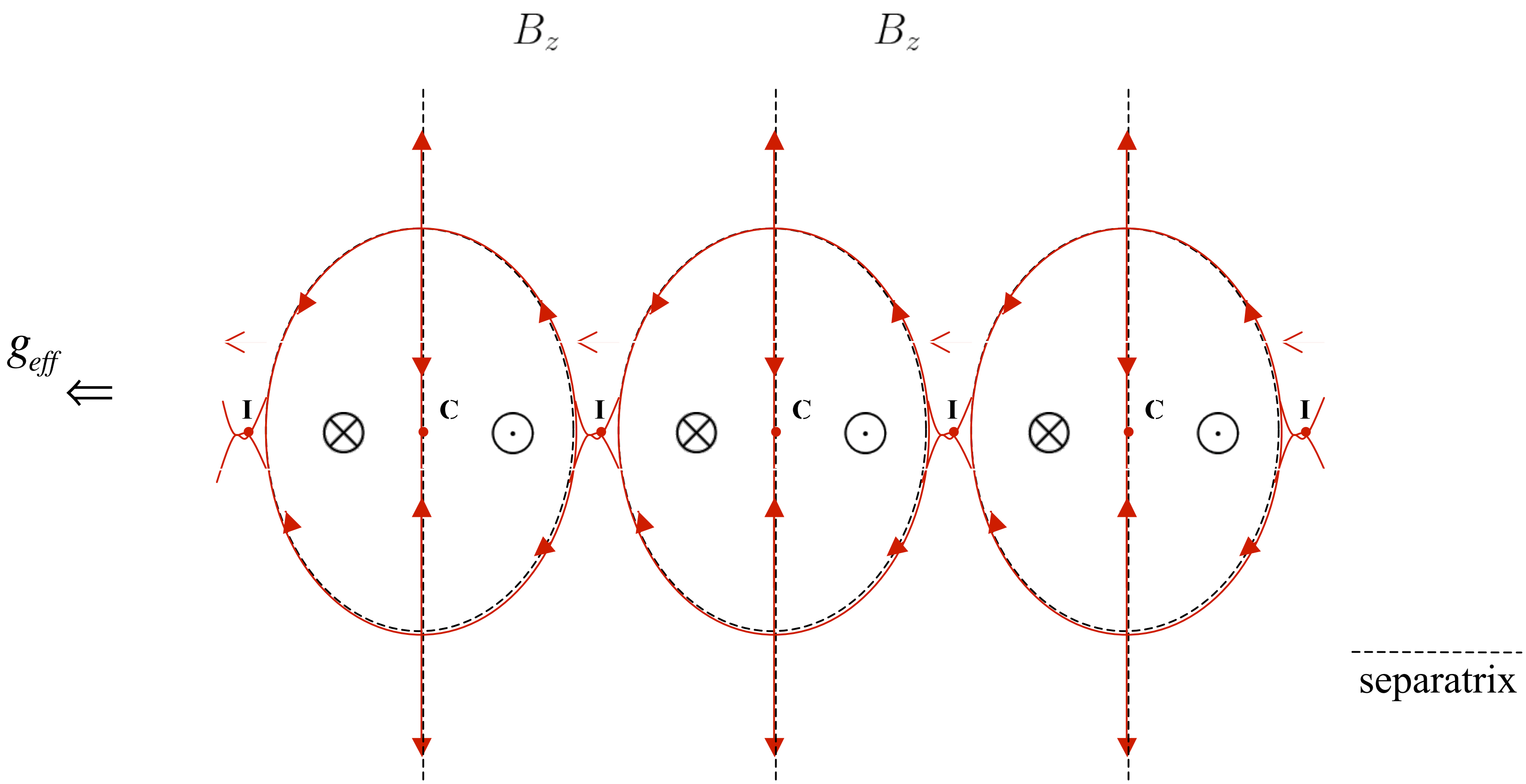



Fig. 1

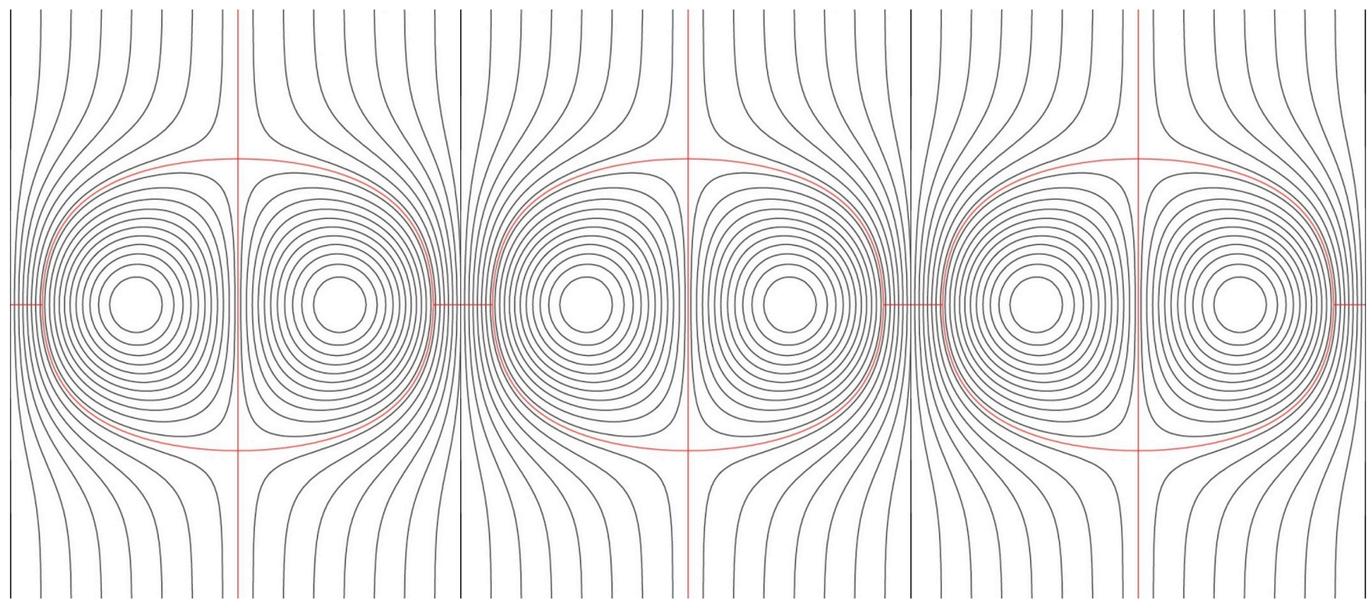
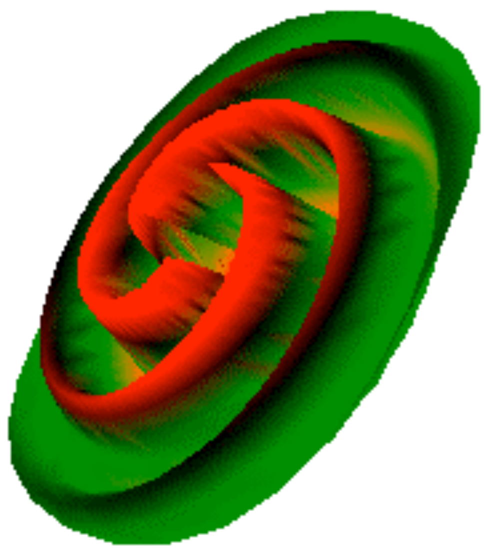


Fig. 2

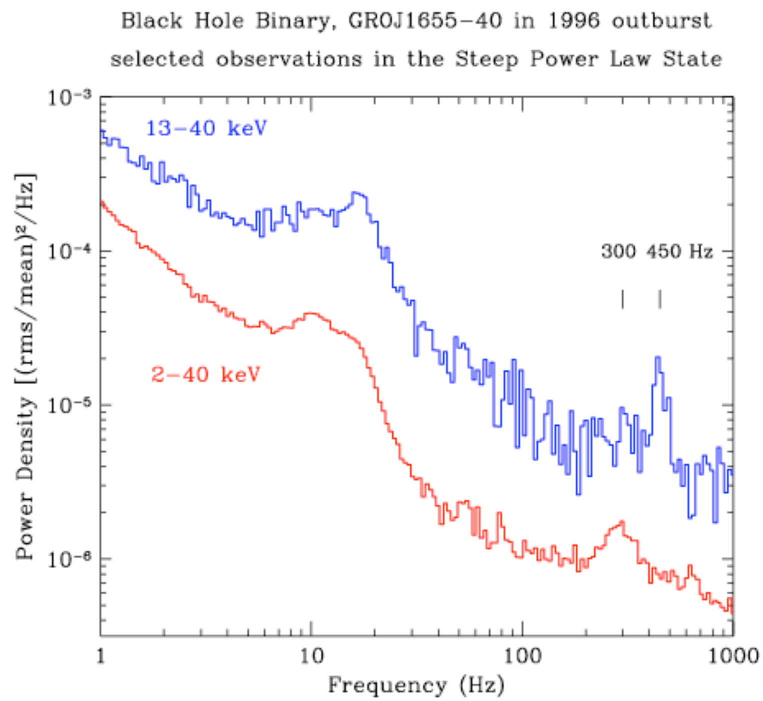
# V) Plasmas in the Vicinity of Black Holes, 3D-Spirals and QPOs (with P. Rebusco and M. Bursa)



# HFQPOs

- Highly Coherent Peaks
  in the X-ray power spectra
- 0.1-1200 Hz
  HF-> few hundred Hz
- Show up alone OR in pairs OR mor
- In **Black Holes**:
  stable 3:2

Courtesy of R.Remillard




The experimental observations show that HFQPOs have frequencies connected to those of orbits related to the MS orbit (e.g., Remillard & McClintock 2006). Moreover HFQPOs occur at the same fixed frequencies for the same black hole candidate (the maximum of the relevant frequency shifts is about 15%) and these scale as $f_0 \propto 1/M_*$ for different sources, $M_*$ being the black hole mass.

We associate the observation of QPO's with the excitation of the lowest harmonics, with $m_\phi = 3$ and $m_\phi = 2$, of the 3D spirals discussed earlier. In particular, we envision that the plasma structure in the vicinity of a black hole is characterized by a "Structured Region" preceded by a "Relaxation Region", as indicated in Fig.1. The inner edge of the "Relaxation Region" is marked by the radius of the marginally stable orbit where the plasma density vanishes. In this region the plasma is considered to be in a strongly turbulent state while in the "Structure Region" the conditions necessary for the onset of coherent spiral modes of the kind discussed previously are assumed to exist.

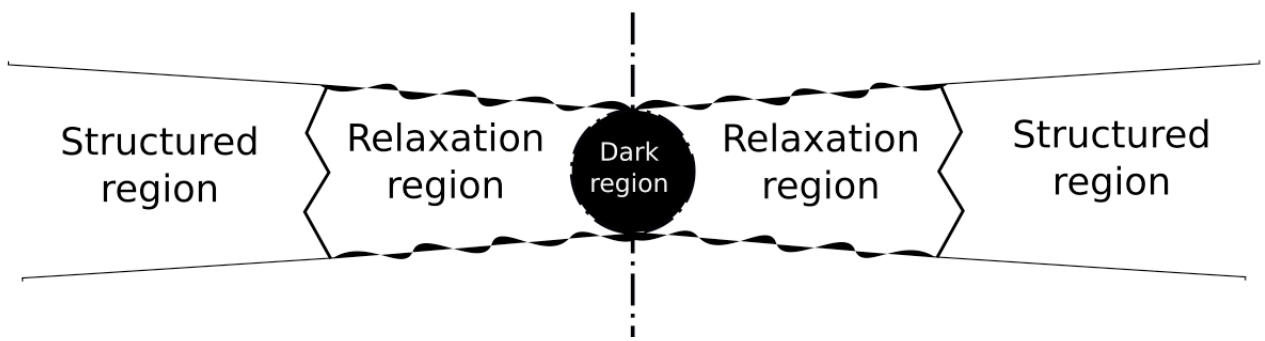



Fig. 1

The prevalent modes will be localized around a radius $R = R_0$ where their growth rate is highest, close to the inner edge of the Structured Region.

# Three Population Model

We assume that in the region around $R = R_0$ the plasma is characterized by 3 components

* The electrons
* A "cold" ion population with the same temperature $T_c^i$ as that of the electrons $T_i^c \simeq T_e$ and about the same (high) density
* A "hot" nuclei population with a considerably lower density, and a temperature $T_i^h \simeq \bar{T}_h m_e c^2$ where $\bar{T}_h$ is the relevant uncertainty parameter

Given the very high electron density considered it is reasonable to regard bremsstrahlung radiation emission as the main form of energy loss. For simplicity, we consider the nuclei to be protons (Z=1).

Thus we may write the approximate thermal energy balance equation as

$$3\nu_{ei}\frac{m_e}{m_i}n_i^h T_i^h \simeq \gamma_B n_e^2 T_e^{1/2} \qquad \text{(V-1)}$$

and, numerically,

$$3\times 2.9\times 10^{-6}\times\bar{\Lambda}\times 1.5\times 10\times\frac{10^{-3}}{1.8}\times\frac{n_i^h}{n_e}T_i^h\left(J\right)\simeq\gamma_B\bar{T}_e^2\times 10^8,$$

where

$$T_i^h\left(J\right)\simeq 0.8\bar{T}_h\times 10^{-13}J\,,$$

$$\gamma_B\simeq 1.7\times 10^{-32}\ \text{W/cm}^3\ ,$$

$$\bar{T}_e=\frac{T_e}{20\text{keV}},\qquad\qquad \bar{\Lambda}\equiv\frac{\ln\Lambda}{15}\ .$$

Consequently, we obtain

$$\frac{n_i^h}{n_e}\simeq\frac{\bar{T}_e^2}{\bar{T}_h\bar{\Lambda}}\times 10^{-3}\ . \qquad \text{(V-2)}$$

We observe that the electron thermal velocity is

$$\mathrm{v}_{the} \simeq 4.2\times 10^9 \bar{T}_e \text{ cm/sec}$$

and, when $\bar{T}_e \simeq 1$, this is below the average rotation velocity $\mathrm{V}_\phi(R_0)$ that we shall consider. For a comparison, the thermal velocity of the hot nuclei is

$$\mathrm{v}_{thi}^h \simeq \left(2\bar{T}_h \frac{m_e}{m_i}\right)^{1/2} c \simeq \left(\bar{T}_h \frac{m_p}{m_i}\right)^{1/2} \times 10^9 \text{ cm/sec},$$

where $m_p$ is the proton mass, and $\mathrm{v}_{thi}^c \simeq \left(\bar{T}_e/\bar{T}_h\right)^{1/2} \times 0.2\mathrm{v}_{thi}^h$ .

Thus we may write the approximate thermal energy balance equation as

$$3\nu_{ei}\frac{m_e}{m_i}n_i^h T_i^h \simeq \gamma_B n_e^2 T_e^{1/2} \tag{V-1}$$

and, numerically,

$$3\times 2.9\times 10^{-6}\times\bar{\Lambda}\times 1.5\times 10\times\frac{10^{-3}}{1.8}\times\frac{n_i^h}{n_e}T_i^h(J)\simeq\gamma_B\bar{T}_e^2\times 10^8,$$

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where $m_p$ is the proton mass, and $\mathrm{v}_{thi}^{c} \simeq \left(\bar{T}_e/\bar{T}_h\right)^{1/2} \times 0.2\mathrm{v}_{thi}^{h}$ .

# Average Rotation Velocity

As is well known, if a black hole has a significant angular momentum represented by the parameter $a_* = J/(M_* c R_G) < 1$, where $R_G \equiv GM_*/c^2$, the value of the marginally stable orbit radius $R_{MS}$ is given by the equation

$$\bar{R}_{MS}\left(\bar{R}_{MS} - 6\right) \pm 2a_* \sqrt{\bar{R}_{MS}} + 3a_*^2 = 0\,, \qquad \text{(V-3)}$$

where $\bar{R}_{MS} \equiv \bar{R}_{MS}/R_G$.

Thus, when $a_* = 1$ (extreme Kerr hole), $\bar{R}_{MS} = 1$ (direct orbit), or $\bar{R}_{MS} = 9$ (retrograde orbit). Clearly $R_{MS} = 3R_S$ when $a_* = 0$ where $R_S = 2R_G$ is the Schwarzschild radius.

The analysis of the spiral modes that has been carried out in the Newtonian limit can be extended to include relevant general relativity corrections in the case where $a_* = 0$ (Schwartzchild metric) by adopting an effective gravitational potential. A particularly simple one is the Paczynsky-Wiita potential that is $\phi_G \simeq -GM_*/(R - R_s)$.

We shall refer to values of $R_0$ that exceed the larger of the two solutions for $R_{MS}$ of Eq. (V-3). Considering the properties of the electron thermal distribution, we choose, as indicated earlier, values of the electron thermal velocity that do not exceed $\mathrm{V}_\phi\left(R_0\right)$.

In particular we assume that the half thermal thickness of the disk is related to $R_S$ by

$$H_0\left(R_0\right) \simeq \left(\frac{2T_i^h}{m_i}\right)^{1/2} \frac{R_0}{\mathrm{V}_\phi\left(R_0\right)} \simeq \alpha_{MS} R_S \tag{V-4}$$

where $\alpha_{MS}$ is a finite coefficient that depends on the lowest value of $R_{MS}$ for a given angular momentum parameter $a_*$. Then the mean toroidal velocity $\mathrm{V}_\phi\left(R_0\right)$ is given by

$$\mathrm{V}_\phi\left(R_0\right) \simeq \frac{c}{\alpha_*}\left(\frac{m_e}{2m_i}\right)^{1/2} \frac{R_0}{R_G} \simeq c\left(\frac{R_G}{R_0}\right)^{1/2},$$

where $\alpha_* \equiv \alpha_{MS} / \sqrt{\bar{T}_n}$, and

$$R_0 \simeq R_G \left( 2\alpha_*^2 \frac{m_i}{m_e} \right)^{1/3} \simeq 15.45 R_G \left( \alpha_*^2 \frac{m_i}{m_p} \right)^{1/3} .$$

We observe, that given the choice (V-2), both

$$V_\phi \left( R_0 \right) \simeq \frac{c}{\alpha_*^{1/3}} \left( \frac{m_e}{2m_i} \right)^{1/6} \simeq 0.76 \frac{1}{\alpha_*^{1/3}} \left( \frac{m_p}{m_i} \right)^{1/6} \times 10^{10} \text{ cm/sec}$$

and the specific angular momentum

$$R_0 V_\phi \left( R_0 \right) \simeq \alpha_*^{1/3} c R_G \left( \frac{2m_i}{m_e} \right)^{1/6}$$

**have only a weak dependence on** $\alpha_*$.

From this we obtain

$$\Omega_k\left(R_0\right) \simeq \frac{c}{\alpha_* R_G}\left(\frac{m_e}{2m_i}\right)^{1/2}\frac{m_\phi^0}{2\pi} \quad ,$$

where $m_\phi^0 = 3,\ 2$ is the toroidal mode number of the relevant 3D-spirals and, numerically,

$$f_0 \simeq \frac{m_\phi^0}{3}\frac{10M_\odot}{M_*}\frac{1}{\alpha_*}\left(\frac{m_p}{m_i}\right)^{1/2} \times 2.2 \times 10^2\, Hz.$$

# Other Characteristic Parameters

In order to identify a range of plausible plasma parameters in the region around $R = R_0$ and $z = 0$, we note that the electron collision frequency is

$$\nu_e \simeq 1.5 \times 10^9 \left( \frac{n}{10^{20}} \right) \frac{\bar{\Lambda}}{\bar{T}_e^{3/2}} \text{ sec}^{-1}$$

and the mean free path

$$\lambda_e = \frac{\text{v}_{the}}{\nu_e} \simeq 2.7 \bar{T}_e^2 \left( \frac{10^{20}}{n} \right) \frac{1}{\bar{\Lambda}} \text{ cm} .$$

Thus, for the range of temperature and densities that we consider, it is clear that the cold particle populations are well thermalized. The relevant plasma pressure $p \equiv p_e + p_i \simeq 2 n_e T_e$ is of the order of $6 \times 10^{12} \left( n / 10^{20} \right)$erg cm$^{-3}$ for $\bar{T}_e \sim 1$.

Then, following of Eq. (V-2), we have $n_i^h \sim 10^{17}\left[\bar{T}_e^2/\left(\bar{T}_n\bar{\Lambda}\right)\right]\left(n_e/10^{20}\right)$ cm$^{-3}$ and we notice that the mean free path due to hot-hot ion collisions is of the order of $H_0$ for $n_i^h \sim 10^{17}\,\mathrm{cm}^{-3}$.

The magnetic field can be evaluated in terms of the parameter $\beta = 8\pi p/B^2$ as

$$B \simeq \left[\frac{3.2}{\beta}\bar{T}_e\left(\frac{n}{10^{20}}\right)\right]^{1/2} 2\times 10^7\,\mathrm{G}$$

and the Alfvén velocity as

$$\mathrm{v}_A \simeq \frac{4.4}{\sqrt{10}}\left(\frac{4\bar{T}_e}{\beta}\right)^{1/2}\times 10^8\ \mathrm{cm/sec}\ .$$

Finally, we observe that in order to account for the densities that would be compatible with considerable accretion rates such as $\dot{M} \simeq 10^{-4}\,M_\odot/\mathrm{year}$ we have to assume that a higher density and colder plasma region surrounds the structured region.

# VI. Modulated Emission and Relevant Ray Tracing

THE ASTROPHYSICAL JOURNAL, 617:L45–L48, 2004 December 10




# THE UPPER KILOHERTZ QUASI-PERIODIC OSCILLATION: A GRAVITATIONALLY LENSED VERTICAL OSCILLATION

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## ABSTRACT

We show that a luminous torus orbiting a Schwarzschild black hole gives rise to a periodically varying flux of radiation when oscillating along its own axis, even though the source of radiation is steady and perfectly axisymmetric. This implies that the simplest oscillation mode in an accretion flow, axisymmetric up and down motion at the meridional epicyclic frequency, may be directly observable when it occurs in the inner parts of accretion flow around neutron stars and black holes. The high-frequency modulations of the X-ray flux observed in low-mass X-ray binaries at two frequencies (twin kilohertz quasi-periodic oscillations) could then be a signature of strong gravity both because radial and meridional oscillations have different frequencies in non-Newtonian gravity and because strong gravitational deflection of light rays causes the flux of radiation to be modulated at the higher frequency.

*Subject heading:* X-rays: general

## 1. HIGHEST FREQUENCY IN ACCRETING BLACK HOLES

The highest frequencies modulating the X-ray flux observed from accreting neutron stars and black holes continue to attract attention because their values are as high as those of orbital frequencies close to the neutron star surface or to the circular photon orbit around a black hole. The origin of the modulations, known as quasi-periodic oscillations (QPOs) because they are not quite coherent, still remains a major puzzle (see van der Klis 2000 for a review).

Typically these QPOs come in pairs, with the higher frequency larger by about 50% than the lower frequency of the pair, and reaching values as high as 1.2 kHz for neutron stars and 0.5 kHz for black holes. It has been recognized for some time that the high-frequency QPOs may correspond to accretion disk oscillations not present in Newtonian $1/r$ gravity, e.g., modes trapped close to the maximum of the epicyclic frequency[6] (Kato 2001; Wagoner 1999). Nonaxisymmetric modes have been preferred, as it was thought that a considerable degree of nonaxisymmetry is a necessary condition for modulating the X-rays. It has also been suggested that the high-frequency QPO phenomenon is caused by a nonlinear resonance between two modes of oscillation of the accretion disk or torus (Kluźniak & Abramowicz 2001; Abramowicz & Kluźniak 2001, 2004). In a resonance, there should be a rational ratio of frequencies, and indeed the observed pairs of high-frequency QPOs in black holes are in a 3 : 2 ratio (McClintock & Remillard 2004). More recently, it has been specifically suggested that the higher frequency corresponds to vertical oscillations of the accretion disk/torus occurring at the meridional epicyclic frequency (Kluźniak & Abramowicz 2002, 2004; Kluźniak et al. 2004; Lee et al. 2004). However, the mechanism of X-ray modulation remained a puzzle.

Here we show that gravitational lensing of the photon trajectories in the vicinity of a black hole (in Schwarzschild metric) suffices to appreciably modulate the flux observed at infinity even if the source is symmetric about the axis of a black hole, provided that it moves parallel to the symmetry axis. Previous ray-tracing computations in the context of QPOs were performed for hot spots in the plane of the accretion disk (Karas 1999; Schnittman & Bertschinger 2004; Schnittman 2004). Our three-dimensional computations are performed for an axisymmetric and optically thin source. Specifically, we show that an otherwise steady toroidal source oscillating in an incompressible $m = 0$ mode about the equatorial plane of the black hole gives rise to a periodically modulated flux. A realistic torus will oscillate in a combination of intrinsic modes, rather than at a unique frequency, and in the case of two modes we demonstrate that the observed power ratio exhibits strong dependence on the viewing angle.

## 2. CALCULATION OF TRAJECTORIES AND OBSERVED FLUX

In order to compute the amount of radiation coming from the source, we have developed a new three-dimensional ray-tracing code. Following the method used by Rauch & Blandford (1994), we integrate geodesic and geodesic deviation equations in the Schwarzschild spacetime. Photon trajectories are integrated backward in time from the observer positioned at infinity at some inclination angle $i$ with respect to the $z$-axis. At certain points along the trajectory, the current position, momentum, time delay, and magnification are recorded. This information is then used to reconstruct each photon's path and calculate the total amount of incoming radiation.

The intensity observed at infinity is an integration of the emissivity $f$ over the path length along geodesics, and it can be written down as

$$I_{\rm obs}(t) = \int f(r,\,\theta,\,\phi,\,t-\Delta t)\sqrt{-g_{tt}}\,k^t g^4\,d\lambda. \qquad (1)$$

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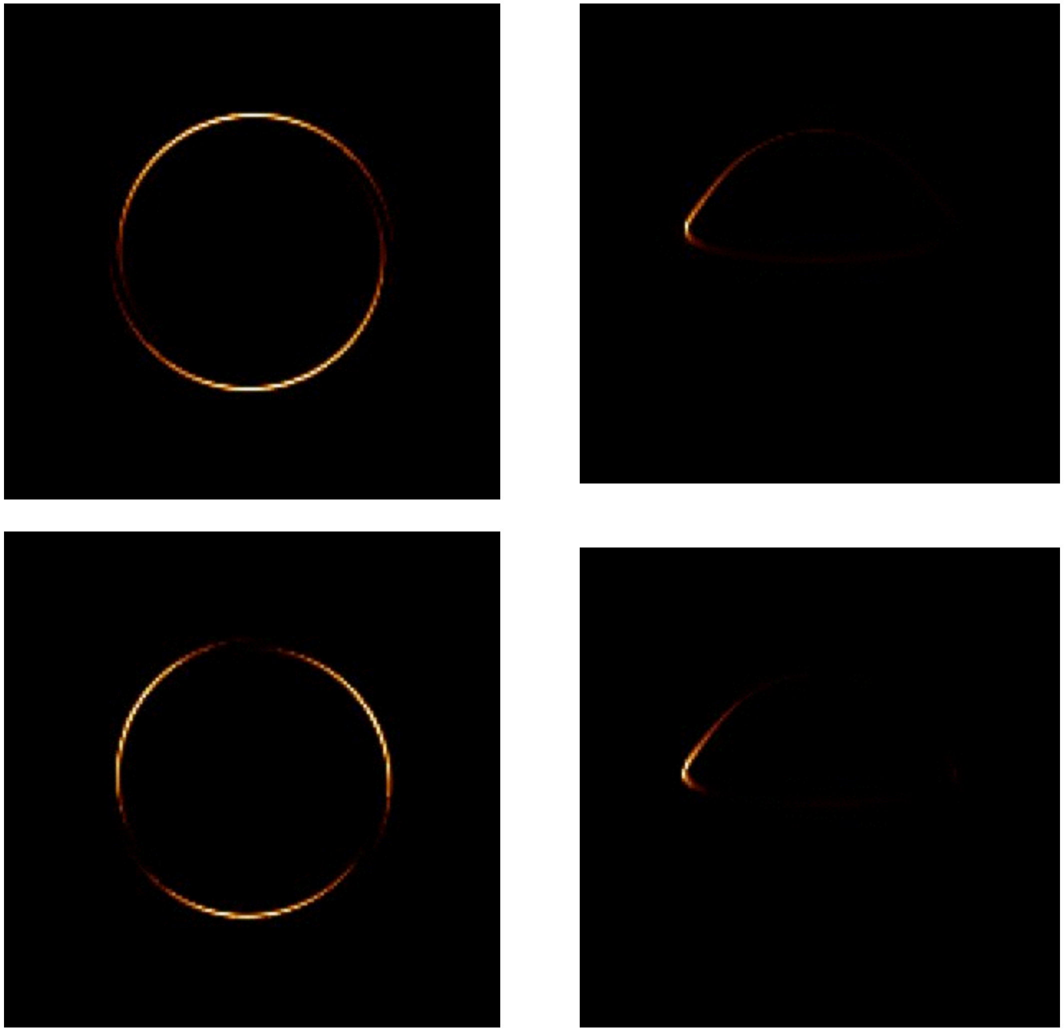
[6] The radial epicyclic frequency may also have a maximum for rapidly rotating Newtonian stars, and it is even possible that an innermost (marginally) stable circular orbit may then exist outside the stellar surface (Amsterdamski et al. 2002; Zdunik & Gourgoulhon 2001).

# Ray-tracing



The Astrophysical Journal, 606:1098–1111, 2004 May 10


# THE HARMONIC STRUCTURE OF HIGH-FREQUENCY QUASI-PERIODIC OSCILLATIONS IN ACCRETING BLACK HOLES

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## ABSTRACT

Observations from the *Rossi X-Ray Timing Explorer* have shown the existence of high-frequency quasi-periodic oscillations (HFQPOs) in the X-ray flux from accreting black hole binary systems. In at least two systems, these HFQPOs come in pairs with a 2 : 3 frequency commensurability. We employ a simple "hot spot" model to explain the position and amplitude of the HFQPO peaks. Using the exact geodesic equations for the Kerr metric, we calculate the trajectories of massive test particles, which are treated as isotropic, monochromatic emitters in their rest frames. Photons are traced from the accretion disk to a distant observer to produce time- and frequency-dependent images of the orbiting hot spot and background disk. The power spectrum of the X-ray light curve consists of multiple peaks at integral combinations of the black hole coordinate frequencies. In particular, if the radial frequency is one-third of the azimuthal frequency (as is the case near the innermost stable circular orbit), beat frequencies appear in the power spectrum at two-thirds and four-thirds of the fundamental azimuthal orbital frequency, in agreement with observations. In addition, we model the effects of shearing the hot spot in the disk, producing an arc of emission that also follows a geodesic orbit, as well as the effects of nonplanar orbits that experience Lens-Thirring precession around the black hole axis. By varying the arc length, we are able to explain the relative amplitudes of the QPOs at either $2\nu$ or $3\nu$ in observations from XTE J1550−564 and GRO J1655−40. In the context of this model, the observed power spectra allow us to infer values for the black hole mass and angular momentum and also constrain the parameters of the model, such as the hot spot size and luminosity.

*Subject headings:* accretion, accretion disks — black hole physics — X-rays: binaries

## 1. INTRODUCTION

In the past decade, observations of X-ray emission from accreting neutron stars and black holes have introduced new possibilities for astrophysical tests of fundamental physics. Recent discoveries made by satellites such as *ASCA*, the *Rossi X-Ray Timing Explorer* (*RXTE*), *BeppoSAX*, *Chandra*, and *XMM-Newton* provide direct evidence for strong-field gravitational effects in compact binary systems and active galactic nuclei (AGNs). These results include Doppler-broadened iron K$\alpha$ fluorescent emission from microquasars (Miller et al. 2002) and millisecond variability of the X-ray flux from black holes in low-mass X-ray binaries (LMXBs; Strohmayer 2001a; Lamb 2003; Remillard et al. 2002; McClintock & Remillard 2004). These measurements give the exciting prospect of determining a black hole's mass and spin, as well as testing general relativity in the strong-field regime.

The strong gravitational fields near a black hole introduce significant deviations from Newtonian physics, including the existence of an innermost stable circular orbit (ISCO), a feature absent in the classical Kepler problem. Since accreting gas can efficiently lose energy and angular momentum only outside of the ISCO, the hydrodynamic and radiative behavior of the inner accretion disk should be strongly dependent on the structure of the spacetime metric near the ISCO. The famous "no hair" theorem states that the only observable features of an electrically neutral black hole are functions of its mass $M$ and specific angular momentum $a \equiv J/M$. By understanding the behavior of matter near the ISCO, we can determine the mass and angular momentum and thus completely describe the black hole.

Many authors have approached the problem of accretion in compact binaries with a variety of different methods, including early analytic models by Shakura & Sunyaev (1973), Ghosh & Lamb (1978), and Mitsuda et al. (1984). Some have simplified the hydrodynamics in favor of a flat, thin, steady state disk and a more detailed treatment of general relativistic effects (George & Fabian 1991; Laor 1991; Karas, Vokrouhlicky, & Polnarev 1992; Reynolds & Begelman 1997). To include dynamic effects, essential for modeling quasi-periodic oscillations (QPOs), others have included magnetohydrodynamics (MHD) in a pseudo-Newtonian model (Hawley & Krolik 2001; Armitage, Reynolds, & Chiang 2001) or with smoothed particle hydrodynamics (Lanzafame, Molteni, & Chakrabarti 1998; Lee & Ramirez-Ruiz 2002). A family of perturbative models has given rise to the field of diskoseismology (Kato 2001; Wagoner, Silbergleit, & Ortega-Rodriguez 2001), in which different global modes in the disk oscillate with different frequencies. One recent model of an oscillating, pressure-supported torus naturally predicts eigenfrequencies with a 2:3 ratio (Rezzolla et al. 2003a; Rezzolla, Yoshida, & Zanotti 2003b). Wang (2003) proposes a magnetic coupling between the rotating black hole and the accretion disk as a means of producing high-frequency QPOs (HFQPOs), analogous to the Blandford-Znajek process (Blandford & Znajek 1977) used to describe AGNs. However, even with the recent advances in computational power and the ability to do MHD in full general relativity (De Villiers & Hawley 2003), no MHD calculation to date has yet predicted the existence of QPOs at any particular frequency, much less those that have been observed.

Recent observations of commensurate relationships in the HFQPOs of black hole accretion disks (Remillard et al. 2002), as well as the long-standing puzzles of the frequency variability of low-frequency QPO (LFQPO) peaks and their correlations with X-ray flux and energy (Lamb 2003), motivate more detailed study of the QPO phenomenon as a means to determining black hole parameters. We have developed a model that is a combination of many of the above approaches in which additional physics ingredients can be added incrementally to a framework grounded in general relativity. The model does not currently include scattering, radiation pressure, magnetic fields, or hydrodynamic forces, instead treating the disk as a collection of cold test particles radiating isotropically in their respective rest frames. The trajectories of emitted photons are integrated through the metric to a distant observer to construct time-dependent images and spectra of the disk. The dynamic model uses the geodesic trajectory of a massive particle as a guiding center for a small region of excess emission, a "hot spot," that creates a time-varying X-ray signal, in addition to the steady state background flux from the disk.

This hot spot model is motivated by the similarity between the QPO frequencies and the black hole (or neutron star) coordinate frequencies near the ISCO (Stella & Vietri 1998, 1999), as well as the suggestion of a resonance leading to integer commensurabilities between these coordinate frequencies (Abramowicz & Kluźniak 2001, 2003). Stella & Vietri (1999) investigated primarily the QPO frequency pairs found in LMXBs with a neutron star accretor, but their basic methods can be applied to black hole systems as well (Abramowicz & Kluźniak 2001; Abramowicz et al. 2003). Both neutron star and black hole binaries also show strong LFQPOs ($\nu \approx 5$–10 Hz), at frequencies that vary between observations. One critical difference between these systems is the variability of the HFQPOs ($\nu > 50$ Hz) in neutron star systems as opposed to the generally constant frequencies of the black hole HFQPOs (McClintock & Remillard 2004). If anything, it seems more appropriate to apply the geodesic hot spot model to the black hole systems, since they lack the complications of magnetic fields and X-ray emission from the rotating neutron star surface, which confuse the interpretation of coordinate frequencies. In fact, the QPOs from the two different types of binaries may be caused by two completely different physical mechanisms. Unless explicitly stated otherwise, the QPOs we refer to in this paper are the nonvariable HFQPOs that are seen exclusively in black hole LMXBs.

Marković & Lamb (2000) have presented a thorough analysis of this hot spot model for a collection of neutron star binaries for which pairs of QPOs have been observed. Based on a number of observational and theoretical arguments, they conclude that the geodesic hot spot model is not a physically viable explanation for the observed neutron star QPOs. For low- to moderate-eccentricity orbits, the coordinate frequencies simply do not agree with the QPO data. For highly eccentric geodesics, they argue that the relative power in the different frequency modes is qualitatively at odds with the observations. Furthermore, they show that hydrodynamic considerations place strong constraints on the possible size, luminosity, coherency, and trajectories of the hot spots.

Many of these points are addressed in our version of the hot spot model. In addition, by including full three-dimensional ray-tracing, we can quantitatively predict how much QPO power will be produced by a hot spot of a given size and emissivity moving along a geodesic orbit near the ISCO. Along with the special relativistic beaming of the emitted radiation, we find that strong gravitational lensing can cause high-amplitude modulations in the light curves, even for relatively small hot spots. The issues of differential rotation and shearing of the emission region are addressed below when we consider the generalization of the hot spot model to include arcs and nonplanar geometries.

Perhaps the most powerful feature of the hot spot model is the facility with which it can be developed and extended to more general accretion disk geometries. In addition to providing a possible explanation for the commensurate HFQPOs in at least two systems (XTE J1550−564 and GRO J1655−40), the hot spot model with full general relativistic ray-tracing is a useful building block toward any other viable model of a dynamic three-dimensional accretion disk. Within the computational framework of the Kerr metric, we can investigate many different emission models and compare their predicted X-ray spectra and light curves with observations. For example, our ray-tracing code could be used in conjunction with a three-dimensional MHD calculation of an accretion disk to simulate the time-dependent X-ray flux and spectrum from such a disk.

It is with this motivation that we present the initial results of the model. Section 2 describes the relativistic ray-tracing methods and discusses numerical techniques. In § 3 we develop the basic model of a steady state disk with application to relativistically broadened line emission and reproduce published results as a confirmation of the code's accuracy. Section 4 introduces the basic hot spot model for circular geodesics and shows the effect of binary inclination and black hole spin on the QPO power spectrum. In § 5 the model is extended to trajectories with nonzero eccentricity and inclination, as well as to elongated hot spots or arcs, giving a set of model parameters that best fit the QPO data from XTE J1550−564. In § 6 we present our conclusions and a discussion of future work.

## 2. RAY-TRACING IN THE KERR METRIC

We begin by dividing the image plane into regularly spaced "pixels" of equal solid angle in the observer's frame, each corresponding to a single ray. Following the sample rays backward in time, we calculate the original position and direction that a photon emitted from the disk would require in order to arrive at the appropriate position in the detector. The gravitational lensing and magnification of emission from the plane of the accretion disk are performed automatically by the geodesic integration of these evenly spaced photon trajectories, so that high magnification occurs in regions where nearby points in the disk are projected to points with large separation in the image plane. To model the time-varying emission from the disk, each photon path is marked with the time delay along the path from the observer to the emission point in the disk.

To integrate the geodesic trajectories of photons or massive particles, we use a Hamiltonian formalism that takes advantage of certain conserved quantities in the dynamics. The resulting equations of motion do not contain any sign ambiguities from turning points in the orbits, as are introduced by many classical treatments of the geodesic equations in the Kerr metric. We define a Hamiltonian function of eight phase-space variables ($x^\nu$, $p_\mu$) and an integration variable (affine parameter) $\lambda$ along the path length. For a general spacetime metric $g_{\mu\nu}(\mathbf{x})$ with inverse $g^{\mu\nu}(\mathbf{x})$, we can define a Hamiltonian $H_2$ quadratic in the momenta as

$$H_2(x^\mu,\ p_\nu;\ \lambda) = \frac{1}{2} g^{\mu\nu}(\mathbf{x}) p_\mu p_\nu = -\frac{1}{2} m^2, \tag{1}$$