Unfolding Mixed-Symmetry Tensor Fields in AdS

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May 2009, 4th International Sakharov Conference

Based on `0812.3615[hep-th]` and `0812.4438[hep-th]`
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Plan

1. Introduction, statement of some results
2. Linear equations in flat and AdS backgrounds
3. Some generalities about unfolding
4. Foliations and general strategy

Summary of the main results

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- Complete system of unfolded equations in all the *non-unitary* [Metsaev] massless cases as well, however without identifying the ASV-like potential.

The gauge principle [H. Weyl, 1929]

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- Manifest diffeomorphism invariance without any explicit reference to a metric
- Manifest Cartan integrability, hence *gauge invariance* under infinite-dimensional HS algebra
- Formulation in terms of two infinite-dimensional unitarizable modules of $\mathfrak{so}(2, D-1)$: The *adjoint* and *twisted-adjoint* representations $\rightsquigarrow$ master 1-form and master zero-form, resp.

Unfolded equations and FDA

A free differential algebra $\mathfrak{R}$ is sets $\{X^\alpha\}$ of *a priori* independent variables that are differential forms obeying first-order equations of motion whereby dX^α are equated on-shell to algebraic functions of all the variables expressed entirely using the exterior algebra, *viz.*

$$R^\alpha := dX^\alpha + Q^\alpha(X) \approx 0\,, \quad Q^\alpha(X) := \sum_n f^\alpha_{\beta_1\ldots\beta_n} X^{\beta_1} \cdots X^{\beta_n}\ .$$

The nilpotency of d and the integrability condition $dR^\alpha \approx 0$ require

$$Q^\beta\, \frac{\partial^L Q^\alpha}{\partial X^\beta} \equiv 0 \quad .$$

For $X^\alpha_{[p_\alpha]}$ with $p_\alpha > 0$, gauge transformation preserving $R^\alpha \approx 0$:

$$\delta_\epsilon X^\alpha := d\epsilon^\alpha - \epsilon^\beta \frac{\partial^L}{\partial X^\beta}\, Q^\alpha \quad .$$

The principle of unfolding [Vasiliev, 1988 –]

- The concepts of spacetime, dynamics and observables are *derived* from infinite-dimensional FDA'a [more on this in the talk by Per Sundell].
- Unfolded dynamics is an inclusion of local d.o.f. into field theories described *on-shell* by flatness conditions on generalized curvatures, and generically *infinitely many* local zero-form observables in the presence of a cosmological constant.
- Spin-2 couplings arise (albeit together with exotic higher-derivative couplings) in the limit in which the $\mathfrak{so}(2, D-1)$-valued part of the higher-spin connection one-form is treated exactly while its remaining spin $s > 2$ components become weak fields together with all curvature zero-forms

From totally symmetric to arbitrary shapes ?

- Although a set of fully nonlinear unfolded equations for nonabelian *totally symmetric* gauge fields is now achieved, its extension to nonabelian *mixed-symmetry* gauge fields is presently unknown.

- Such massless gauge fields start being propagated in flat spacetime as soon as $D \geqslant 5$ and in constantly curved spacetime as soon as $D \geqslant 4$. [Unitary massless mixed-symmetry "hook-like" tensor fields in AdS_4 decompose in the flat limit into topological dittos plus one symmetric massless field in $\mathbb{R}^{1,3}$ [Brink–Metsaev–Vasiliev (2000)].]

Free field equations in metric-like formalism

- In flat spacetime, field equations for arbitrary mixed-symmetry fields were *proposed* in [J. M. F. Labastida, 1987 – 1989], then later rederived from generalized Bargmann–Wigner equations (BW), thereby *proving* the correctness of the p.d.o.f. [X. Bekaert, N.B., 2002 – 2006].

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- In AdS_D background, Metsaev [R. Metsaev, 1995 – 1997] gave gauge-fixed equations for arbitrary mixed-symmetry gauge fields $\rightsquigarrow$ unitary (as well as non-unitary) shortened irreps of $\mathfrak{o}(2, D-1)$.

In conformity with the principles of Gauge Invariance and Unfolding $\hookrightarrow$ necessity to obtain the *generalized Bargmann–Wigner equations* for arbitrary mixed-symmetry fields in AdS_D .

Free field equations in frame-like formalism

An important step was achieved by Skvortsov [E. D. Skvortsov, 2008] with the identification and explicit construction of the correct finite-dimensional $\mathfrak{iso}(1, D-1)$ p-form modules to be glued to the corresponding generalized Weyl tensors used in [X. Bekaert, N.B., 2002] for the construction of the generalized BW equations.

$\hookrightarrow$ *Complete unfolded equations* for arbitrary mixed-symmetry free massless fields in flat spacetime [E. D. Skvortsov, 2008] : Our starting point for the derivation of the unfolded equation in AdS_D spacetimes, by the well-known radial reduction technique.

Linearization

- In expansions around maximally symmetric backgrounds with isometry algebras $\mathfrak{g}$, the Weyl zero-form module $\mathfrak{C}^0$ [for one irreducible field, say] is a $\mathfrak{g}$-irrep that is *infinite-dimensional* for generic masses (including critically massless cases in backgrounds with non-vanishing Λ) in which case we refer to it as twisted-adjoint $\mathfrak{g}$-module.

- The twisted-adjoint zero-forms consist of a primary Weyl tensor – such as a scalar field ϕ, Faraday tensor F_{ab} or spin-2 Weyl tensor $C_{ab,cd}$ – and secondary, or descendant, Weyl tensors given on-shell by derivatives of the primary Weyl tensor.

One of our results : BW equations in AdS_D

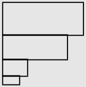
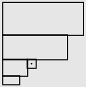
The generalized BW equations for the primary Weyl tensor :

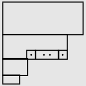
$$\nabla_{(i)} C \approx 0 \,, \quad (\nabla^2 - \overline{M}^2) C \approx 0 \,, \quad \mathbb{B}(\overline{\nabla}) C \equiv 0 \,, \quad C := X^{\mathbf{0}}(\overline{\Theta}^*) \,.$$

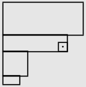
where

- C is the primary Weyl tensor, the Lorentz-tensor with smallest shape $\overline{\Theta}^*$ among all the Lorentz-tensors in the zero-form module $\mathfrak{C}^{\mathbf{0}}$ of the anti-de Sitter algebra ;
- The differential operator $\nabla_{(i)}$ acts by taking a Lorentz-covariant divergence in the ith row of C, projecting afterwards.
- $(\nabla^2 - \overline{M}^2) C \approx 0$ is the wave equation for C and $\mathbb{B}(\overline{\nabla})$ takes some ∇ -curls of C on some of its columns. See examples !

Integration of the 0-form constraints

$\overline{\Theta}_2^* :=$  , $\qquad \mathbb{B}_{2,1}(\cdot)\;:$  $\equiv\; 0 \qquad \Rightarrow$

$C(\overline{\Theta}_2^*) \;=\; (\overline{\nabla}^{(p_2+1)})^{\overline{s}_{23}}\varphi_2(\Theta^*) \;=$  ,

$\delta\varphi_2(\Theta^*) \;=\; \overline{\nabla}^{(p_2)}\epsilon_2(\Theta^{*\prime}) \;=$ 

By means of the integration lemma, the primary Weyl tensor $C(\overline{\Theta}_2^*)$ with Bianchi identity $\mathbb{B}_{2,1}(\overline{\Theta}_2^*) \equiv 0$ is shown to correspond to a massless gauge field $\varphi_2(\Theta^*)$ whose shape is obtained from $\overline{\Theta}_2^*$ by cutting off one row from its second block and by adding one row to its third block. It possesses a one-derivative gauge symmetry with parameter $\epsilon_2(\Theta^{*\prime})$ whose shape is obtained from Θ^* by deleting 1 cell in the 2nd block.

Gluing $\mathfrak{C}^{\mathbf{0}}$ to p-form modules

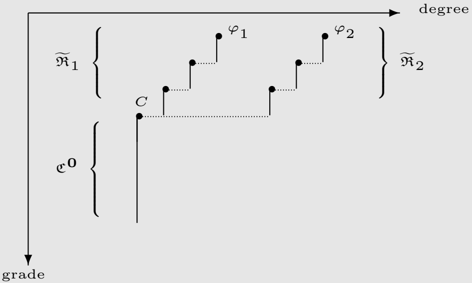


Fig.: An unfolded module of the form $\mathfrak{R} = \mathfrak{R}' \oplus \widetilde{\mathfrak{R}}_2$ where (i) $\mathfrak{R}' = \mathfrak{C}^{\mathbf{0}} \oplus \widetilde{\mathfrak{R}}_1$ is a submodule consisting of a Weyl zero-form module $\mathfrak{C}^{\mathbf{0}}$ with primary Weyl tensor C and dual subcycle $\widetilde{\mathfrak{R}}_1$ ("potential module") with dynamical field φ_1 ; and (ii) $\widetilde{\mathfrak{R}}_2$ is a dual cycle ("dual potential module") with dynamical field φ_2 ("dual potential'). The dashed lines indicate "gluings" by non-trivial generators in σ_0^- (whose existence conditions depend on the nature of the underlying symmetry Lie algebra $\mathfrak{g}$.

Massive-spin 1 field in flat spacetime. A.

TheWeyl zero-form module $\mathfrak{C}^{\mathbf{0}}$ is spanned by Lorentz-tensors with shapes $\overline{\Theta}^* = \square$, $\Theta^*_{1_1} = \square\square$, $\Theta^*_{1_2} = \begin{smallmatrix}\square\\\square\end{smallmatrix}$, $\Theta^*_{2_1} = \square\square\square$, $\Theta^*_{2_2} = \begin{smallmatrix}\square\square\\\square\end{smallmatrix}$, *etc.*

The first two levels of the Weyl zero-form constraint read

$$
\begin{aligned}
\nabla C_a + e^b \Phi_{ab} + \frac{\overline{M}}{2}\, e^b \Phi_{a,b} &\approx 0 \qquad (\alpha = 0) \quad ,\\
\nabla \Phi_{ab} + e^c \Phi_{abc} + \frac{\overline{M}}{4}\, e^c \Phi_{ab,c} - \frac{\overline{M}^2}{(D-1)}\, e_{(a} C_{b)} &\approx 0 \qquad (\alpha = 1_1) \quad ,\\
\nabla \Phi_{a,b} + e^c \Phi_{c[a,b]} + \frac{2\overline{M}}{D-1} e_{[a} C_{b]} &\approx 0 \qquad (\alpha = 1_2) \quad .
\end{aligned}
$$

There are no primary Bianchi identities (*i.e.* the primary Weyl tensor C_a is unconstrained), while there is a secondary one at the first level, *viz.* $\nabla_{[a}\Phi_{b,c]} \approx 0$. Its integration yields $dA + \frac{1}{2} e^a e^b \Phi_{a,b} \approx 0$.

Massive spin-1 field in flat space. B.

Revisiting the zeroth level ($\alpha = 0$), its totally anti-symmetric part reads $\nabla_{[a}C_{b]} + \overline{M}\,\nabla_{[a}A_{b]} \approx 0$, which can be integrated using a 0-form χ, yielding the sytem of constraints

$$dA + \frac{1}{2}e^a e^b \Phi_{a,b} \approx 0 \quad , \qquad d\chi + \overline{M}A + e^a C_a \;\approx\; 0\ .$$

For $\overline{M} > 0$ we have a contractible cycle $\mathfrak{S} = \{\chi, Z\}$

$$d\chi + Z \;\approx\; 0\ , \qquad dZ \;\approx\; 0\ , \qquad Z \;:=\; \overline{M}A + e^a C_a\ ,$$

which manifests the massive Stückelberg shift symmetry that can be used to fix the gauge

$$\chi \;\stackrel{!}{=}\; 0 \qquad \Rightarrow \qquad A \;=\; -\frac{1}{\overline{M}}e^a C_a\ .$$

Foliation 1.

Take a FDA $\widehat{\mathfrak{R}}$ with curvature constraints

$$\widehat{T}^{\widehat{\alpha}} := d\,\widehat{W}^{\widehat{\alpha}} + \widehat{Q}^{\widehat{\alpha}}(\widehat{W}) \approx 0$$

over a base $\widehat{\mathscr{M}}_{D+1}$ with a smooth foliation $i : \widehat{\mathscr{M}} \times \mathbb{R} \to \widehat{\mathscr{M}_i} \subseteq \widehat{\mathscr{M}}$ where $\widehat{\mathscr{M}_i}$ is a region of $\widehat{\mathscr{M}}$ foliated with leaves $\mathscr{M}_L := i(\widehat{\mathscr{M}}, L)$ of codimension 1 and a non-vanishing normal 1-form $N = d\phi$, where $\phi : \widehat{\mathscr{M}_i} \to \mathbb{R}$ is defined by $\phi(\mathscr{M}_L) = L \rightsquigarrow (A)dS_D$-radius. Introduce the vector field ξ parallel to N and such that $i_\xi N = 1$. One has $(n \geqslant 0)$

$$(\mathscr{L}_\xi)^n \widehat{W}^{\widehat{\alpha}} = \widehat{U}^{\widehat{\alpha}}_n + N\,\widehat{V}^{\widehat{\alpha}}_n\,, \qquad i_\xi \widehat{U}^{\widehat{\alpha}}_n = 0 = i_\xi \widehat{V}^{\widehat{\alpha}}_n\,,$$

$$\widehat{X}^{\widehat{\alpha}} := \widehat{U}^{\widehat{\alpha}}_{0}\,, \quad \widehat{Y}^{\widehat{\alpha}} := \widehat{V}^{\widehat{\alpha}}_{0}\,, \quad \widehat{U}^{\widehat{\alpha}} := \widehat{U}^{\widehat{\alpha}}_{1}\,, \quad \widehat{V}^{\widehat{\alpha}} := \widehat{V}^{\widehat{\alpha}}_{1}$$

(where $\widehat{V}^{\widehat{\alpha}}_n \equiv 0$ if $p_{\widehat{\alpha}} = 0$) and $\widehat{U}^{\widehat{\alpha}}_n \equiv (\mathscr{L}_\xi)^n \widehat{X}^{\widehat{\alpha}}\,, \quad \widehat{V}^{\widehat{\alpha}}_n \equiv (\mathscr{L}_\xi)^n \widehat{Y}^{\widehat{\alpha}}\,.$

Foliation 2.

Defining $\widehat{R}_n^{\widehat{\alpha}} := (1 - N i_\xi)(\mathscr{L}_\xi)^n \widehat{T}^{\widehat{\alpha}}$ and $\widehat{S}_n^{\widehat{\alpha}} := -i_\xi (\mathscr{L}_\xi)^n \widehat{T}^{\widehat{\alpha}}$,

the original constraints $\widehat{T}^{\widehat{\alpha}} \approx 0$ become

$$\widehat{R}_n^{\widehat{\alpha}} = (d - N\mathscr{L}_\xi)\widehat{U}_n^{\widehat{\alpha}} + \widehat{f}_n^{\widehat{\alpha}}(\{\widehat{U}_m\}_{m=0}^n) \approx 0 ,$$

$$\widehat{S}_n^{\widehat{\alpha}} = (d - N\mathscr{L}_\xi)\widehat{V}_n^{\widehat{\alpha}} + \widehat{g}_n^{\widehat{\alpha}}(\{\widehat{U}_m, \widehat{V}_m\}_{m=0}^n) - \widehat{U}_{n+1}^{\widehat{\alpha}} \approx 0 \qquad \text{for} \quad p_{\widehat{\alpha}} \geqslant 1 ,$$

where the structure functions are given by

$$\widehat{f}_n^{\widehat{\alpha}} := (1 - N i_\xi)(\mathscr{L}_\xi)^n \widehat{Q}^{\widehat{\alpha}}(\widehat{X} + N\widehat{Y}) = (\mathscr{L}_\xi)^n \widehat{Q}^{\widehat{\alpha}}(\widehat{X}) ,$$

$$\widehat{g}_n^{\widehat{\alpha}} := -i_\xi (\mathscr{L}_\xi)^n \widehat{Q}^{\widehat{\alpha}}(\widehat{X} + N\widehat{Y}) = -(\mathscr{L}_\xi)^n \left(\widehat{Y}^{\widehat{\beta}} \partial_{\widehat{\beta}} \widehat{Q}^{\widehat{\alpha}}(\widehat{X})\right) \qquad \text{for} \quad p_{\widehat{\alpha}} \geqslant 1 .$$

Foliation 3.

In terms of the pull-back $(U_n^{\widehat{\alpha}}, V_n^{\widehat{\alpha}}; R_n^{\widehat{\alpha}}, S_n^{\widehat{\alpha}}) := i_L^*(\widehat{U}_n^{\widehat{\alpha}}, \widehat{V}_n^{\widehat{\alpha}}; \widehat{R}_n^{\widehat{\alpha}}, \widehat{S}_n^{\widehat{\alpha}})$, one gets

$$R_n^{\widehat{\alpha}} \;=\; dU_n^{\widehat{\alpha}} + \widehat{f}_n^{\widehat{\alpha}}(\{U_m\}_{m=0}^n) \;\approx\; 0 \,,$$

$$S_n^{\widehat{\alpha}} \;=\; dV_n^{\widehat{\alpha}} - U_{n+1}^{\widehat{\alpha}} + \widehat{g}_n^{\widehat{\alpha}}(\{U_m, V_m\}_{m=0}^n) \;\approx\; 0 \qquad \text{for} \quad p_{\widehat{\alpha}} \geqslant 1 \,.$$

Define $f^{\widehat{\alpha}}(X) := \widehat{Q}^{\widehat{\alpha}}(X)$ and $g^{\widehat{\alpha}}(X,Y) := -Y^{\widehat{\beta}}\partial_{\widehat{\beta}} f^{\widehat{\alpha}}(X)$. The closed subsystem

$$R^{\widehat{\alpha}} \;:=\; dX^{\widehat{\alpha}} + f^{\widehat{\alpha}}(X) \;\approx\; 0 \,,$$

$$S^{\widehat{\alpha}} \;:=\; dY^{\widehat{\alpha}} + g^{\widehat{\alpha}}(X,Y) - U^{\widehat{\alpha}} \;\approx\; 0 \qquad \text{for} \quad p_{\widehat{\alpha}} \geqslant 1 \,,$$

$$P^{\widehat{\alpha}} \;:=\; dU^{\widehat{\alpha}} - g^{\widehat{\alpha}}(X,U) \;\approx\; 0 \,,$$

contains three sets of zero-forms, namely

$$\{\Phi^{\widehat{\alpha}^{\mathbf{0}}}\}\,, \quad \{U^{\widehat{\alpha}^{\mathbf{0}}}\} = \{i_L^* \mathscr{L}_\xi \widehat{\Phi}^{\widehat{\alpha}^{\mathbf{0}}}\} \quad \text{and} \quad \{Y^{\widehat{\alpha}^{\mathbf{0}}}\} = \{i_L^* i_\xi \widehat{A}^{\widehat{\alpha}^{\mathbf{1}}}\}\,.$$

Foliation 4.

An irreducible model [one field] may arise from subsidiary constraints on :

I) the normal Lie derivatives

$$i_L^* \mathscr{L}_\xi \widehat{X}^{\widehat{\alpha}} \equiv U^{\widehat{\alpha}} \approx -\Delta^{\widehat{\alpha}}(X,Y) ,$$

where the functions $\Delta^{\widehat{\alpha}}(X,Y)$ thus assign scaling weights to the fields under rescalings in L ; and

II) zero-forms

$$\Xi^{R^{\mathbf{0}}}(X^{\widehat{\alpha}^{\mathbf{0}}}, Y^{\widehat{\alpha}^{\mathbf{0}}}) \approx 0 ,$$

where $\Xi^{R^{\mathbf{0}}}$ denotes a set of functions.

Cartan integrability requires that

$$d\Delta^{\widehat{\alpha}} - g^{\widehat{\alpha}}(X,\Delta) \equiv (R^{\widehat{\beta}} \partial^{(X)}_{\widehat{\beta}} + S^{\widehat{\beta}} \partial^{(Y)}_{\widehat{\beta}}) \Delta^{\widehat{\alpha}} ,$$

$$d\Xi^{R^{\mathbf{0}}} \equiv (R^{\widehat{\alpha}^{\mathbf{0}}} \partial^{(X)}_{\widehat{\alpha}^{\mathbf{0}}} + S^{\widehat{\alpha}^{\mathbf{0}}} \partial^{(Y)}_{\widehat{\alpha}^{\mathbf{0}}}) \Xi^{R^{\mathbf{0}}} .$$

Foliation 5.

The former condition ensures the integrability of the constrained curvature equations

$$S^{\widehat{\alpha}}|_{U=-\Delta} \;=\; dY^{\widehat{\alpha}} + \Delta^{\widehat{\alpha}}(X,Y) + g^{\widehat{\alpha}}(X,Y) \;\approx\; 0\;,$$

since the U-dependent terms in $dS^{\widehat{\alpha}}$ cancel separately prior to imposing $U^{\widehat{\alpha}} \approx -\Delta^{\widehat{\alpha}}(X,Y)$. The subsidiary constraints can equivalently be imposed directly on $\widehat{\mathscr{M}}_{D+1}$ as

$$\left(\widehat{U}^{\widehat{\alpha}}, \widehat{V}^{\widehat{\alpha}}\right) \;\approx\; \left(\Delta^{\widehat{\alpha}}(\widehat{X},\widehat{Y}), \Upsilon^{\widehat{\alpha}}(\widehat{X},\widehat{Y})\right)\;, \quad \Xi^{R^{\mathbf{0}}}(\widehat{X}^{\widehat{\alpha}^{\mathbf{0}}}, \widehat{Y}^{\widehat{\alpha}_1}) \;\approx\; 0\;,$$

where the functions $\Upsilon^{\widehat{\alpha}}$ can be determined from $\Delta^{\widehat{\alpha}}$ using Cartan integrability. This is the approach we actually used to reduce Skvotsov's system from flat $(D+1)$ spacetime to AdS_D .

Brink–Metsaev–Vasiliev spectrum

- As found by Metsaev (1995), a given $\mathfrak{so}(D-1)$-spin of shape Θ consisting of B blocks yields B inequivalent massless lowest-weight spaces $\mathfrak{D}(e^I_0;\Theta)$ of $\mathfrak{so}(2,D-1)$, each having a single singular vector associated with the Ith block of Θ ($I=1,\dots,B$).

- The partially massive nature of the cases with $B>1$ later led Brink, Metsaev and Vasiliev (2000) to conclude that upon adding Stückelberg fields $\{\chi(\Lambda;\Theta')\}_{\Theta'\in\Sigma^I_{\rm BMV}(\Theta)}$ (associated with all blocks except the Ith one) the resulting extended system must have a smooth flat limit in the sense of counting local degrees of freedom.

- Since only $\mathfrak{D}(e^1_0;\Theta)$ is unitary, BMV conjectured that the fully gauge invariant action $S^\Lambda_I := S[\varphi(\Lambda;M^2_I;\Theta),\{\chi(\Lambda;\Theta')\}]$ should have the flat-space limit

$$\text{BMV conjecture} \quad : \quad S^\Lambda_I \quad \xrightarrow{\lambda\to 0} \sum_{\Theta'\in\Sigma^I_{\rm BMV}(\Theta)} (-1)^{\epsilon_I(\Theta')} S^{\Lambda=0}[\varphi(\Lambda=0,\Theta')] \ ,$$

$$\Sigma^I_{\rm BMV}(\Theta) \ = \ \Theta|_{\mathfrak{so}(D-2)} \setminus \Sigma_{I^{\rm th}\ {\rm block}}(\Theta) \ ,$$

where :

(I) $\Sigma_{I^{\rm th}\ {\rm block}}(\Theta)$ is the subset of $\Theta|_{\mathfrak{so}(D-2)}$ obtained by deleting at least one cell in the $I^{\rm th}$ block ; and

(II) the phase factors $(-1)^{\epsilon_I(\Theta')}$ are all positive iff $I=1$.

Procedure. 1

Group-theoretically, the BMV conjecture implies that

$$\mathfrak{D}(e_0^I;\Theta) \quad \xrightarrow{\lambda\to 0} \quad \bigoplus_{\Theta'\in\Sigma^I_{\rm BMV}(\Theta)} (-1)^{\epsilon_I(\Theta')}\mathfrak{D}(\Lambda{=}0;M^2{=}0;\Theta')\ . \tag{1}$$

The dimensional reduction in $\Sigma^I_{\rm BMV}(\Theta)$ and the fact that *the zero-forms carry the local unfolded degrees of freedom* suggests the following procedure :

I) Unfold the tensor gauge field $\widehat{\varphi}(\widehat{\Theta})$ in $\mathbb{R}^{2,D-1}$ and foliate a region of $\mathbb{R}^{2,D-1}$ with AdS_D leaves of inverse radius $\lambda = 1/L$ and with normal vector field ξ obeying $\xi^2 = -1$, the radial vector field ;

II) Set the radial Lie derivative $(\mathscr{L}_\xi + \lambda\Delta)\widehat{X} = 0$, where Δ are scaling dimensions compatible with Cartan integrability ;

PROCEDURE. 2

III) Constrain the shapes $\widehat{\Theta}_{\widehat{\alpha}}$ ($\widehat{\alpha} = 0, 1, \dots$) in the Weyl zero-form module $\widehat{\mathfrak{C}}^{\mathbf{0}}(\Lambda{=}0; \overline{M}^2{=}0; \widehat{\widehat{\Theta}})$ by demanding their $(p_{_I} + 1)$st row to be transverse to ξ where $p_{_I} = \sum_{J=1}^{I} h_{_J}$;

IV) Demonstrate (via harmonic expansion) that the unfolded system in AdS_D carries the massless degree of freedom $\mathfrak{D}(e_{_0}^{I}; \Theta)$ found by Metsaev for massless mixed-symmetry fields in AdS_D ;

V) Take the flat limit without fixing any massive shift symmetries and show that the resulting unfolded system in flat space carries the massless degrees of freedom on the right-hand-side of (1) and contains the corresponding D-dimensional Skvortsov modules.

Oscillator realization and Howe-dual algebra

Take bosonic (+) or fermionic (−) oscillators satisfying

$$[\alpha_{i,a}, \bar{\alpha}^{j,b}] \;:=\; \alpha_{i,a}\bar{\alpha}^{j,b} + (-1)^{\frac{1}{2}(1\pm1)}\bar{\alpha}^{j,b}\alpha_{i,a} \;=\; \delta_i^j\delta_a^b \;,$$

where $a, b = 1, \ldots, D$ transform in the fundamental representation of $\mathfrak{l} \cong (\mathfrak{gl}(D;\mathbb{C}), \mathfrak{so}(D;\mathbb{C}), \mathfrak{sp}(D;\mathbb{C}))$, and $i = 1, 2, \ldots, \nu_\pm$ are auxiliary indices. One has the associated Howe-dual algebras $\widetilde{\mathfrak{l}}^\pm$

$$\mathfrak{l} = \mathfrak{gl}(D;\mathbb{C}) \quad : \quad \widetilde{\mathfrak{l}}^\pm \;=\; \mathfrak{gl}(\nu_\pm) \;,$$

$$\mathfrak{l} = \mathfrak{so}(D;\mathbb{C}) \quad : \quad \widetilde{\mathfrak{l}}^+ \;=\; \mathfrak{sp}(2\nu_+;\mathbb{C}) \;, \qquad \widetilde{\mathfrak{l}}^- \;=\; \mathfrak{so}(2\nu_-;\mathbb{C}) \;,$$

$$\mathfrak{l} = \mathfrak{sp}(D;\mathbb{C}) \quad : \quad \widetilde{\mathfrak{l}}^+ \;=\; \mathfrak{so}(2\nu_+;\mathbb{C}) \;, \qquad \widetilde{\mathfrak{l}}^- \;=\; \mathfrak{sp}(2\nu_-;\mathbb{C}) \;.$$

Generalized Schur Modules

The oscillator realization of the generators of $\widetilde{\mathfrak{l}}^{\pm}$ reads

$$\begin{aligned} N^i_j &= \tfrac{1}{2}\{\bar{\alpha}^{i,a}, \alpha_{j,a}\} := \tfrac{1}{2}(\bar{\alpha}^{i,a}\alpha_{j,a} + \alpha_{j,a}\bar{\alpha}^{i,a}) \;, \\ T_{ij} &= \alpha_{i,a}\alpha_{j,b}J^{ab} \;, \quad \overline{T^{ij}} = \bar{\alpha}^{i,a}\bar{\alpha}^{j,b}J_{ab} \;. \end{aligned}$$

The oscillator algebra can be realized in various oscillator-algebra modules $\mathscr{M}^{\pm}$. For given $\mathscr{M}^{\pm}$, the corresponding *generalized Schur module*

$$\mathscr{S}^{\pm} \equiv \bigoplus_{\widetilde{\lambda}^{\pm}} \mathbb{C} \otimes |\widetilde{\lambda}^{\pm}\rangle \;,$$

where $|\widetilde{\lambda}^{\pm}\rangle$, which we shall refer to as the *Schur states*, are the ground states of $\widetilde{\mathfrak{l}}^{\pm}$ in $\mathscr{M}^{\pm}$ with Howe-dual highest weights $\widetilde{\lambda}^{\pm} = \{\widetilde{\lambda}^{\pm}_i\}^{\nu_{\pm}}_{i=1}$.

Cell operators and master fields

The oscillator formalism can be used to define the cell operators [Olver (1983), Metsaev(1995)] $\left\{\beta_{\pm(i),a}, \bar{\beta}^{\pm(i),a}\right\}_{i=1}^{\nu_\pm}$ as a set of operators on the oscillator module $\mathscr{M}^\pm$ that induces a non-trivial action on the corresponding Schur modules $\mathscr{S}^\pm$ and obeying the amputation and generation properties

$$
\begin{aligned}
(N^i_j - \delta^i_j(\widetilde{\lambda}^\pm_i - 1))\beta_{\pm(i),a}|\Delta\rangle &= 0\,, \\
(N^i_j - \delta^i_j(\widetilde{\lambda}^\pm_i + 1))\bar{\beta}^{\pm(i),a}|\Delta\rangle &= 0\,, \quad 1 \leqslant i \leqslant j \leqslant \nu_\pm\,, \qquad |\Delta\rangle \in \mathscr{S}^\pm\,.
\end{aligned}
$$

In terms of these cell operator, we provided a reformulation of Skvortsov's equations using master-fields :

$$
\mathbf{X} \;:=\; \sum_{p=0}^{\infty} \mathbf{X}^{\mathbf{p}} \;\in\; \mathfrak{R} \;=\; \bigoplus_{p\geqslant 0} \mathfrak{R}^{\mathbf{p}}\,, \quad \mathfrak{R}^{\mathbf{p}} \;:=\; \Omega^{\mathbf{p}}(U) \otimes \mathscr{S}^\pm\,,
$$

Master-field reformulation

The Skvortsov equations amount to subjecting $\mathbf{X}$ to : i) curvature constraints ; and ii) irreducibility conditions

$$\mathbf{R} \;:=\; \Big(\nabla + \sigma_0^-\Big)\mathbf{X} \;\approx\; 0\;, \qquad \sigma_{_0}^- \;:=\; \sum_{p\geqslant p'} (\sigma_{_0})^{\mathbf{p+1}}_{\mathbf{p'}}\;,$$

$$(\sigma_{_0})^{\mathbf{p+1}}_{\mathbf{p'}} \;:=\; -ie_{(p'+1)}\cdots e_{(p+1)}\mathbb{P}(p+1,p'+1)\;,$$

where $\nabla := d - \frac{i}{2}\omega^{ab}M_{ab}$, $e_{(i)} := e^a\beta_{(i),a}$ and $\mathbb{P}(p+1,p'+1) : \mathfrak{R} \to \mathfrak{R}^{\mathbf{p}}$ is a projector defined by

$$\mathbb{P}(p+1,p'+1)\mathbf{X} \;:=\; \begin{cases} \delta\left\{N(p'+1,p'+2), N(p'+2,p'+3),\ldots,N(p,p+1)\right\}\mathbf{X}^{\mathbf{p}'} & (p > p \\ \mathbf{X}^{\mathbf{p}} & (p = p \end{cases}$$

where $\delta\{\lambda_1,\ldots,\lambda_k\} := \delta_{\lambda_1,0}\cdots\delta_{\lambda_k,0}$ for $\lambda_i \in \mathbb{Z}$, $i = 1,\ldots,k$.

Step-by-step procedure. 1

1) Skvortsov's equations in $\widehat{\mathscr{M}}_{D+1}$ with signature $(2, D-1)$ read

$$\widehat{\mathbf{T}} \;:=\; \left(\widehat{\nabla} + \widehat{\sigma}_0^-\right)\widehat{\mathbf{W}} \approx 0\;, \quad \widehat{\sigma}_0^- \;:=\; -i\sum_{p\geqslant p'} \widehat{E}_{(p'+1)} \cdots \widehat{E}_{(p+1)}\widehat{\mathbb{P}}(p+1, p'+1)\;,$$

with $\widehat{\nabla} := d - \frac{i}{2}\widehat{\Omega}^{AB}\widehat{M}_{AB}\,,\; \widehat{E}_{(i)} := \widehat{E}^A\widehat{\beta}_{A,(i)}$ and $\widehat{\mathbf{W}} \in \widehat{\mathfrak{R}} = \bigoplus_{p\geqslant 0}\Omega^{\mathbf{p}}(\widehat{U}) \otimes \widehat{\mathscr{S}}_{D+1}\,,$ where $\widehat{U}$ is a region of $\widehat{M}_{D+1}$ that admits a foliation with AdS_D leaves.

2) Decompose the variables and generalized curvatures into components parallel and transverse to the radial vector field

$$\widehat{E}_{(i)} = \widehat{e}_{(i)} + N\widehat{\xi}_{(i)}\,, \quad \widehat{\nabla}\widehat{e}_{(i)} = \lambda N\widehat{e}_{(i)}\;, \quad \widehat{\nabla}\widehat{\xi}_{(i)} = \lambda\widehat{e}_{(i)}\;,$$

$$\widehat{\nabla}\lambda \;=\; -\lambda^2 N\,, \quad \widehat{\mathbf{W}}^{\mathbf{p}} := \widehat{\mathbf{X}}^{\mathbf{p}} + N\,\widehat{\mathbf{Y}}^{\mathbf{p-1}}\;, \quad \widehat{\mathbf{T}} = \widehat{\mathbf{R}} + N\,\widehat{\mathbf{S}}$$

Step-by-step procedure. 2

3) Constrain the radial derivatives in terms of a massive parameter f :

$$(\mathscr{L}_\xi + \lambda\Delta_{[p]})\widehat{\mathbf{X}}^{\mathbf{p}} \approx 0 \;, \qquad (\mathscr{L}_\xi + \lambda\Upsilon_{[p]})\widehat{\mathbf{Y}}^{\mathbf{p-1}} \approx 0 \;,$$

where $\Delta_{[p]} = \Delta_{[p]}(\{\widehat{N}^i_i\}^\nu_{i=1})$ *idem* $\Upsilon_{[p]}$, the reduced curvatures $\widehat{\mathbf{R}}$ and $\widehat{\mathbf{S}}$ form a closed subsystem with variables $\widehat{\mathbf{X}}$ and $\widehat{\mathbf{Y}}$. Cartan integrability (on $\widehat{\mathscr{M}}$: $\widehat{\nabla}\widehat{\mathbf{R}}^{\mathbf{p+1}} \approx 0$) fixes the scaling dimensions

$$\Delta_{[p]} = \Delta^f_{[p]} := \widehat{N}^{p+1}_{p+1} + f_{[p]}(\{\widehat{N}^i_i\}^\nu_{i=1,i\neq p+1}) \;,$$
$$f_{[p]} = -p + f\left(\widehat{N}^1_1 + 1, \ldots, \widehat{N}^p_p + 1, \widehat{N}^{p+2}_{p+2}, \ldots, \widehat{N}^\nu_\nu\right)$$

4) Show that a generic value μ for $C_2[\mathfrak{g}_\lambda]|_{\widehat{\mathscr{S}}(\Lambda;f;\widehat{\widehat{\Theta}})}$ corresponds to two dual values $f^\pm$ s.t. $f^+ + f^- = D-1$ that turn out to be $f^+ = e_0$, the lowest energy of the lowest-weight space $\mathfrak{D}(e_0,\Theta)$: $[\epsilon_0 := \frac{1}{2}(D-3)]$

$$f^\pm_{[0],\mu} := \epsilon_0 + 1 \pm \sqrt{(\widehat{N}^1_1 + \epsilon_0 + 1)^2 + \mu - C_2[\widehat{\mathfrak{m}}]} \equiv f^\pm_\mu(\widehat{N}^2_2, \widehat{N}^3_3, \ldots) \,.$$

4) Examine the critical limit where f^+ approaches Metsaev's massless values $e_0^I = s_I + D - 2 - p_I$, for which f_I^- admits a *projection* of the radially reduced Weyl zero-form : $\widehat{\xi}_{(p_I+1)}\widehat{X}^{\mathbf{0}} \approx 0$. Cartan integrability of the above constraint amounts to

$$\left(\lambda\,\widehat{e}_{(p_I+1)} + i[\widehat{\xi}_{(p_I+1)}, \widehat{e}_{(1)}]\right)\widehat{X}^{\mathbf{0}} \equiv 0 \quad \text{modulo} \left(\lambda\Delta^f_{[0]} + i\widehat{\xi}_{(1)}\right)\widehat{X}^{\mathbf{0}} \approx 0 \,.$$

This equation indeed has the unique solution

$$f^- = f^-_{p_I} := p_I + 1 - \widehat{N}^{p_I+1}_{p_I+1} \,.$$

5) Show the smoothness of the flat limit of the projected massless system, and how the BMV conjecture is realized in an enlarged setting with extra topological fields arising in the flat limit. The latter represent the unfolded “frozen” Stückelberg fields of the Ith block whose Weyl zero form is set to zero in the aforementioned projection of the zero-form.

6) Show the appearance of contractible cycle in the potential sector, except for $I = 1$ where we identify the dynamical h_1-form potential as the Alkalaev–Shaynkman–Vasiliev potential and obtain its full unfolded equations

$$\widehat{R}^{\mathbf{h_1+1}}_{\text{ASV}} \quad := \quad (\widehat{\nabla} - iN\widehat{\xi}_{(h_1+1)})\widehat{U}^{\mathbf{h_1}} - i\,\widehat{e}_{(1)}\cdots\widehat{e}_{(h_1+1)}\widehat{\mathbb{P}}(h_1+1,1)\widehat{\mathbf{X}}^{\mathbf{0}} \;\approx\; 0\;.$$