Finite-Frequency Counting Statistics in Quantum Point Contacts

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Priority Program:
Semiconductor Spintronics

Finite-Frequency Current Statistics in Quantum Contacts

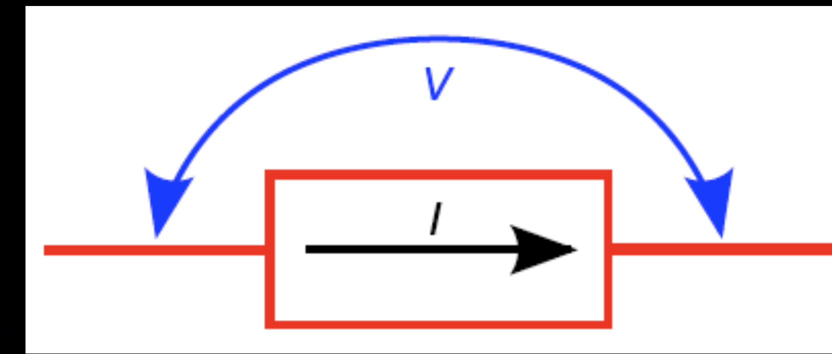
- Full counting statistics: the general view on quantum transport
- Time-resolved measurement and the projection postulate
- Positive operator-valued measure formulation of time-resolved current statistics
- Conclusion

Collaborators:

Adam Bednorz

Theoretical Quantum Transport Group @ University of Konstanz

Full counting statistics



- Mesoscopic electron transport:
  resistor described by coherent scattering matrix
- The transport process is a problem of quantum mechanical time evolution
- At t=0 the electrons are in initial state |0>, then evolves according to the Hamiltonian H:

$$|t_0\rangle = \exp\left(-i\int_0^{t_0} H dt\right)|0\rangle$$

- After time t_0 the state is $|t_0>$ and the probability that N charges are transfered is measured:

$$P_{t_0}(N) = |\langle t_0|N\rangle|^2$$

  this correspond to one **projective** measurement

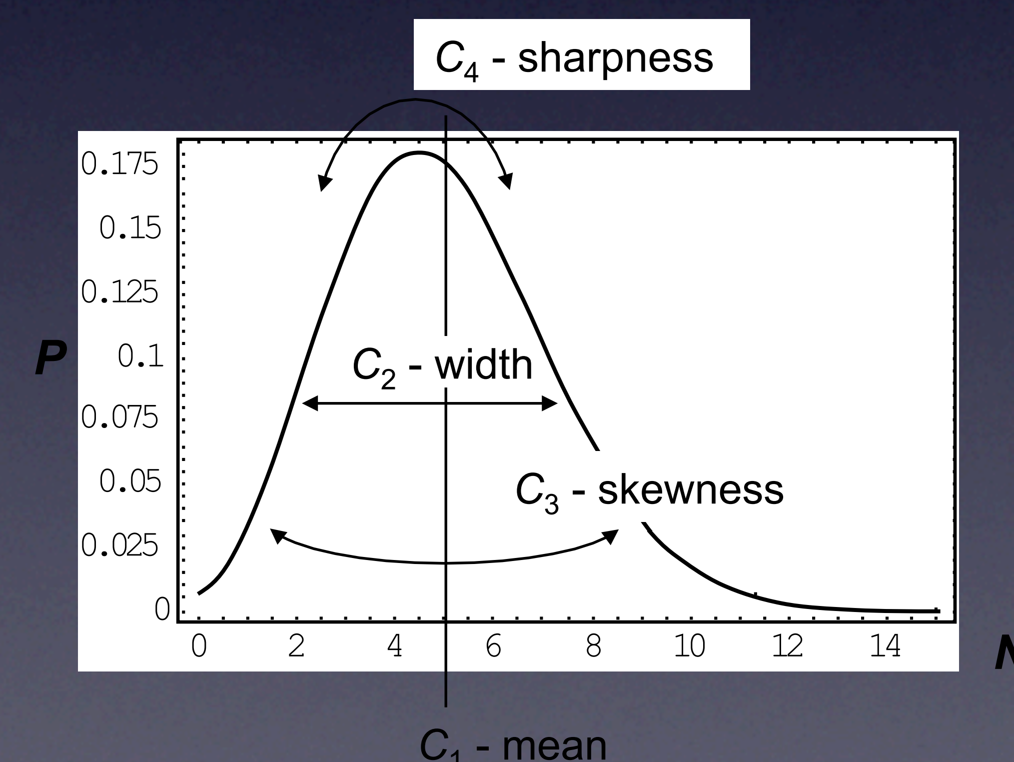
The probability distribution is characterized by cumulants

$$C_1 = \langle N\rangle_P = \sum_N NP(N)$$

$$C_2 = \left\langle (N - C_1)^2 \right\rangle_P$$

$$C_3 = \left\langle (N - C_1)^3 \right\rangle_P$$

$$C_4 = \left\langle (N - C_1)^4 \right\rangle_P - 3(C_2)^2$$



$$e^{S_{t_0}(\chi)} = \langle e^{i\chi N}\rangle_{t_0} = \sum_N P_{t_0}(N) e^{i\chi N}$$

How to calculate the CGF quantum mechanically?
Quantum mechanical current detection has to account for non-commuting current operators!

Classical definition

$$e^{S_{cl}(\chi)} = \left\langle e^{i\chi N} \right\rangle_P$$

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Quantum generalization I:
no time-ordering

$$e^{S_{q1}(\chi)} = \left\langle e^{i\chi \hat{N}} \right\rangle_{\hat{\rho}}$$

Quantum average

$$\langle \cdots \rangle_{\hat{\rho}} = Tr\left[\hat{\rho} \cdots\right]$$

$$\hat{N} = \int_0^{t_0} \hat{I}(t) dt$$

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Quantum generalization II:
simple time-ordering

Levitov, Lesovik, JETPL 92

$$e^{S_{q2}(\chi)} = \left\langle \mathrm{T} e^{i\chi\hat{N}} \right\rangle_{\hat{\rho}}$$

$$\left[\hat{I}(t), \hat{I}(t')\right] \neq 0$$

$$\mathrm{T}\left(\tilde{\mathrm{T}}\right) = \text{(anti-)time-ordering operator}$$

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Quantum generalization III:
symmetric time ordering

Lee, Levitov, Lesovik, JMP 96

Nazarov, 99
Belzig, Nazarov, PRL 01
Nazarov, Kindermann, EPJB 03

$$e^{S(\chi)} = \left\langle \tilde{\mathrm{T}} e^{i\chi\hat{N}/2} \mathrm{T} e^{i\chi\hat{N}/2} \right\rangle_{\hat{\rho}}$$

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Average current: $$C_1 = t_0 \left\langle \hat{I}(t) \right\rangle$$

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$$C_2 = \int_0^{t_0} dt_1 \int_0^{t_0} dt_2 \left\langle \delta\hat{I}(t_1)\delta\hat{I}(t_2) \right\rangle = \frac{t_0}{2} \int_{-\infty}^{\infty} dt \left\langle \left\{ \delta\hat{I}(t), \delta\hat{I} \right\} \right\rangle$$

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Zero frequency third cumulant:

$$C_3 = \frac{6}{4} \int_0^{t_0} dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \left\langle \left\{ \delta\hat{I}(t_1), \left\{ \delta\hat{I}(t_2), \delta\hat{I}(t_3) \right\} \right\} \right\rangle$$

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Compare to classical definition:
$$C_3^{cl} = \int_0^{t_0} dt_1 \int_0^{t_0} dt_2 \int_0^{t_0} dt_3 \left\langle \delta\hat{I}(t_1)\delta\hat{I}(t_2)\delta\hat{I}(t_3) \right\rangle$$

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E.g. tunnel junction at zero temperature:

$$C_1 = C_2 = C_3$$

Poisson process

$$C_1^{cl} = C_2^{cl} = C_1$$

$$C_3^{cl} = 0$$

Gaussian process

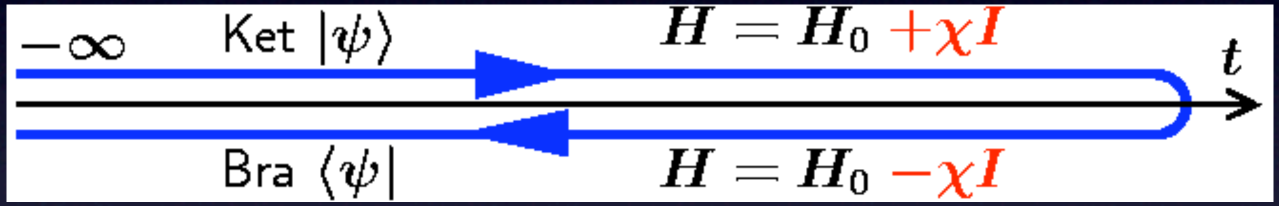
Microscopic justification: time evolution of **ideal current detector** and **projective measurement** at the end

Lee, Levitov, Lesovik, 1996
Kindermann, Nazarov 2003

$$e^{S(\chi)} = \mathrm{Tr}\left[\hat{\rho}\tilde{\mathcal{T}} e^{i\frac{\chi}{2}\int_0^{t_0} \hat{I}(t)dt} \mathcal{T} e^{i\frac{\chi}{2}\int_0^{t_0} \hat{I}(t)dt}\right]$$

influence of the dynamics of measured system on the time evolution of the detector

General result for the CGF of a **quantum point contact** can be obtained using the extended Keldysh technique (viz. the QPC in the presence of an external quantum field)



$$S(\chi) = Tr\ Ln\left[\breve{1} + \frac{T}{2}\left(\frac{\{\breve{G}_1, \breve{G}_2\}}{2} - \breve{1}\right)\right]$$

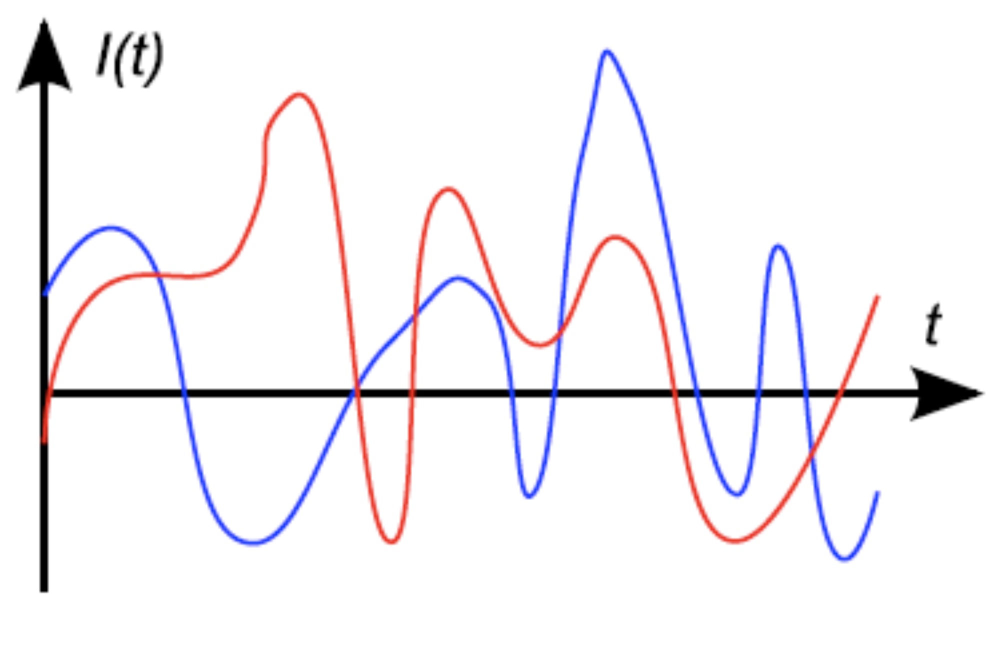
Belzig, Nazarov, 2001

The interpretation problem:

Quantum CGF predicts results of measurement of transfered charge
But: what does the CGF tell us about the transport process?
Is it possible to identify an equivalent classical stochastic process?
What about time-resolved (continuous) measurement?

Gathering information with high time resolution

Detector output should reflect the
time-dependent current
(different for each measurement)



Described by

$$\rho[I(t)]$$

Probability density functional

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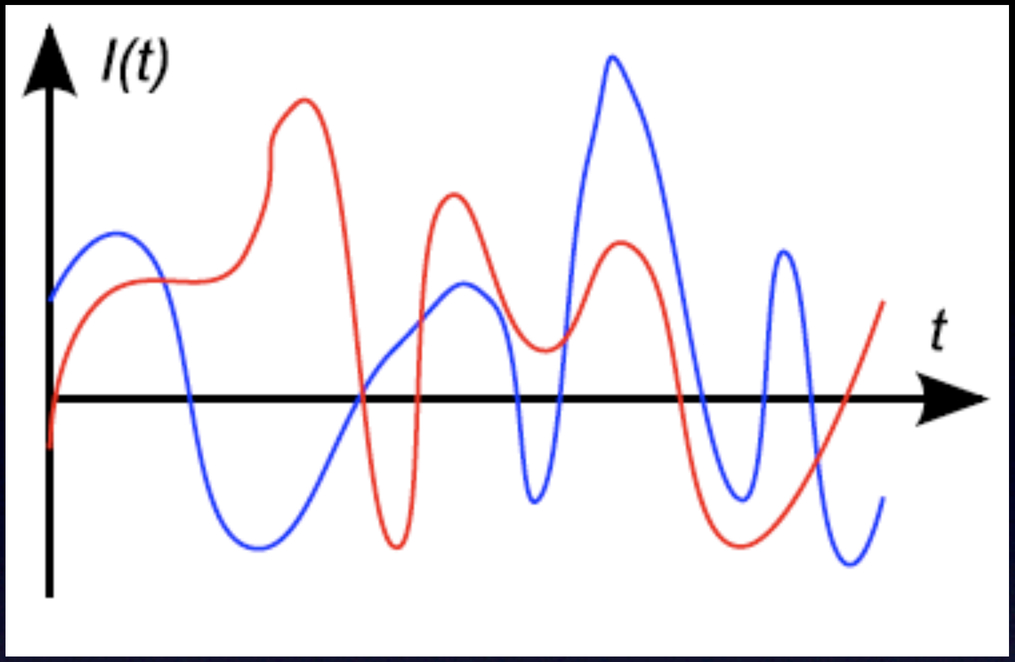

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Probability density functional

- But current operators I(t) at <u>different</u> times do not commute and cannot be measured accurately simultaneously!
- Problem: any intermediate projective measurement will <u>project</u> the state and change the subsequent time-evolution!
- Can we define a useful probability density functional?

Generalization of low-frequency quantum FCS:

$$e^{S(\chi)} = \left\langle \tilde{\mathrm{T}} e^{i\chi\hat{N}/2} \mathrm{T} e^{i\chi\hat{N}/2} \right\rangle_{\hat{\rho}}$$

$$e^{S[\chi]} = \left\langle \tilde{\mathrm{T}} \exp\left(\frac{i}{2e} \int_0^{t_0} \chi(t) \hat{I}(t) dt \right) \mathrm{T} \exp\left(\frac{i}{2e} \int_0^{t_0} \chi(t) \hat{I}(t) dt \right) \right\rangle_{\hat{\rho}}$$

see also: Kindermann and Nazarov (2003), Galaktionov, Golubev, Zaikin (2003)

Let us define a candidate for a probability density through

$$\rho_c[I] = \int D\chi e^{-\frac{i}{e}\int_0^{t_0} \chi(t) I(t)\, dt + S[\chi]}$$

Is this a proper probability density?

The problem: definition does not guarantee a positive definite probability density

Linear and quadratic averages are OK!

Consider the quadratic observable (functional of the current):

$$X = \int_0^{t_0} dt dt' I(t) I(t') [e^{-(t-t')^2/s^2} - 2e^{-(t-t')^2/9s^2}]$$

The average of the variance gives (for T<<1)

$$\left\langle \delta X^2 \right\rangle_{\rho_c} = \int DI \rho_c [I] \delta X^2 [I] = -\frac{T t_0 e^4}{3 s \pi^{3/2}} < 0$$

Hence ρ_c cannot be interpreted as probability

Bednorz and Belzig, PRL **101**, 206803 (2008)

Weak measurement of non-commuting variables: Positive Operator Valued Measure (POVM)

Orthogonal, projective:

Set of states $|A\rangle$

$$\{\hat{P}_A = |A\rangle\langle A|\}$$

$$\sum_A \hat{P}_A = \hat{1}$$

$$\hat{P}_A \hat{P}_B = \hat{P}_A \delta_{A,B}$$

Probability to find A

$$p_A = \mathrm{Tr}\hat{\rho}\hat{P}_A$$

State after measurement

$$\hat{\rho}_A = \hat{P}_A \hat{\rho} \hat{P}_A / p_A$$

Non-projective, non-orthogonal:

Kraus operators $\left\{\hat{K}_A\right\}$

$$\hat{F}_A = \hat{K}_A^\dagger \hat{K}_A$$

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Valued

Measure

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Neumarks theorem: Every POVM corresponds to a projective measurement in an extended Hilbert space (including the detector)

Kraus operator for current measurement (defines POVM)

$$\hat{K}[I] = \int D\phi \,\mathrm{T}\exp\left(i\int_0^{t_0} \phi(t)\left(I(t) - \hat{I}(t)\right)dt - \int_0^{t_0} \phi^2(t)\,dt \,/\, \tau \right)$$

Phenomenological parameter τ interpolates smoothly between

- full projection $\tau \longrightarrow \infty$
  (accurate measurement, but strong backaction)
- weak measurement $\tau \longrightarrow 0$
  (large detector noise, small signal)

Positive definite probability density functional

$$\rho[I] = \left\langle K^\dagger [I] K[I] \right\rangle_{\hat{\rho}}$$

Final expression for CGF

$$\phi\left(\phi^{\dagger}\right)=\varphi\pm\chi/2$$

$$e^{S[\chi,\varphi]}=\left\langle \tilde{\mathrm{T}}\exp\left(\frac{i}{2e}\int_0^{t_0}(\chi(t)-2\varphi(t))\hat{I}(t)dt\right)\mathrm{T}\exp\left(\frac{i}{2e}\int_0^{t_0}(\chi(t)+2\varphi(t))\hat{I}(t)dt\right)\right\rangle_{\hat{\rho}}$$

$$e^{S[\chi]}=\int D\varphi e^{S[\chi,\varphi]-\int_0^{t_0}\left(\frac{1}{2}\chi^2(t)+2\varphi^2(t)\right)dt/\tau}$$

White noise of the detector

Backaction of the detector

Limiting cases

$$\boxed{\tau\to 0}$$

weak measurement (large detector noise, small signal)

$$\rho[I]=\int DI'\rho_c[I']\rho_\tau^g[I-I']$$

With large white Gaussian offset noise

$$\langle I(t)I(t')\rangle_{\rho_\tau^g}=\frac{e^2}{\tau}\delta(t-t')$$

$$\boxed{\tau\to\infty}$$

strong backaction + small offset noise

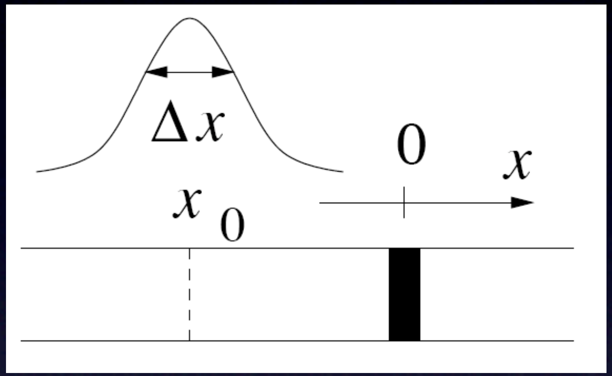
$$\rho[I]=\int DI'\rho_\tau[I']\rho_\tau^g[I-I']$$

$$\rho_\tau[I]\neq\rho_c[I]$$

is positive but mainly caused by backaction

Details of the noise depend on the detector model

Sensitivity function



Detector integrates current over finite region

Interaction time with detector

$$\tau_\Delta = \Delta x / v_F$$

Time of flight to the detector

$$\tau_0 = x_0 / v_F$$

Parameters introduced to model

- time of flight takes into account response time of the detector (e.g. some RC-time)
- interaction time allows for finite measurement uncertainty
- the relation to experimental detectors still needs to be established.

Result for the measured noise (flight time very small)

$$S_2(V,\omega) = S_q(V,\omega) + S_{off} + S_\Delta(V)$$

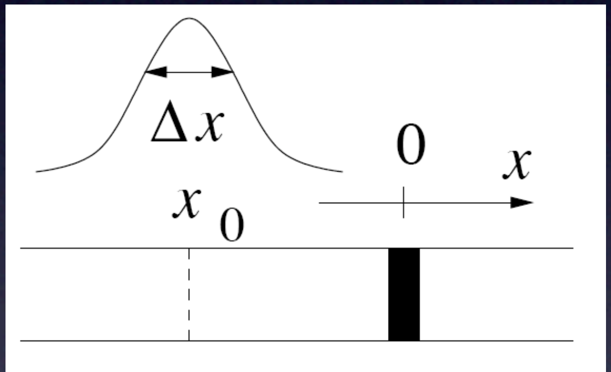
Standard symmetrized quantum noise

$$w(\omega) = \omega \coth\left(\frac{\hbar\omega}{2k_B T}\right)$$

$$S_q(V,\omega) = \frac{e^2}{h}\left[2T^2 w(\omega) + RT\left(w(\omega+eV) + w(\omega-eV)\right)\right]$$

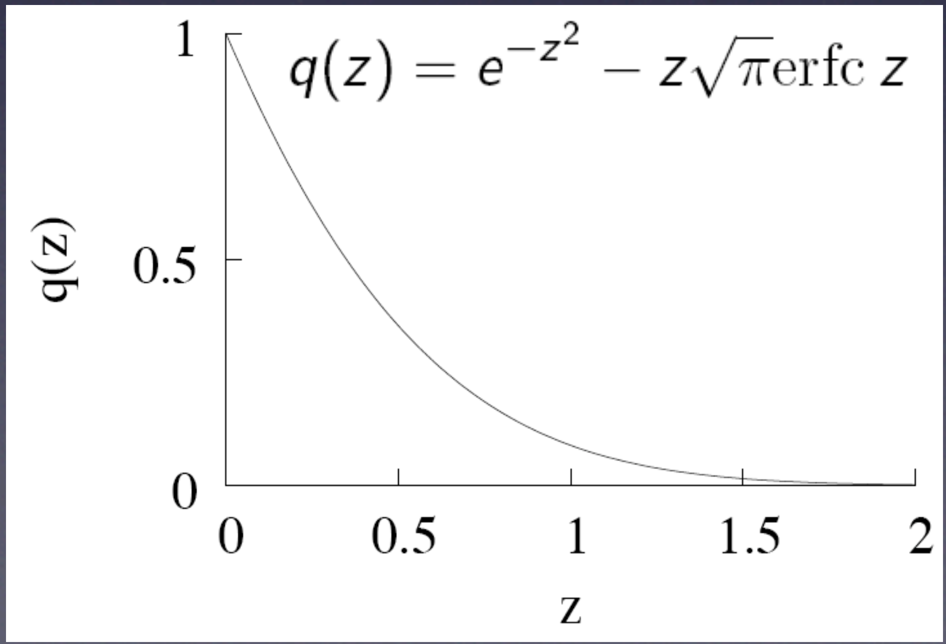
White Gaussian off-set noise

$$S_{off} = \frac{e^2}{\tau}$$ internal detector noise



Backaction noise $k_B T_e \ll \hbar\omega \ll \hbar/\tau_\Delta$

$$S_\Delta(V) = \frac{e^2\tau}{8\pi^2\tau_\Delta^2} RTq(\tau_\Delta eV/\hbar)$$



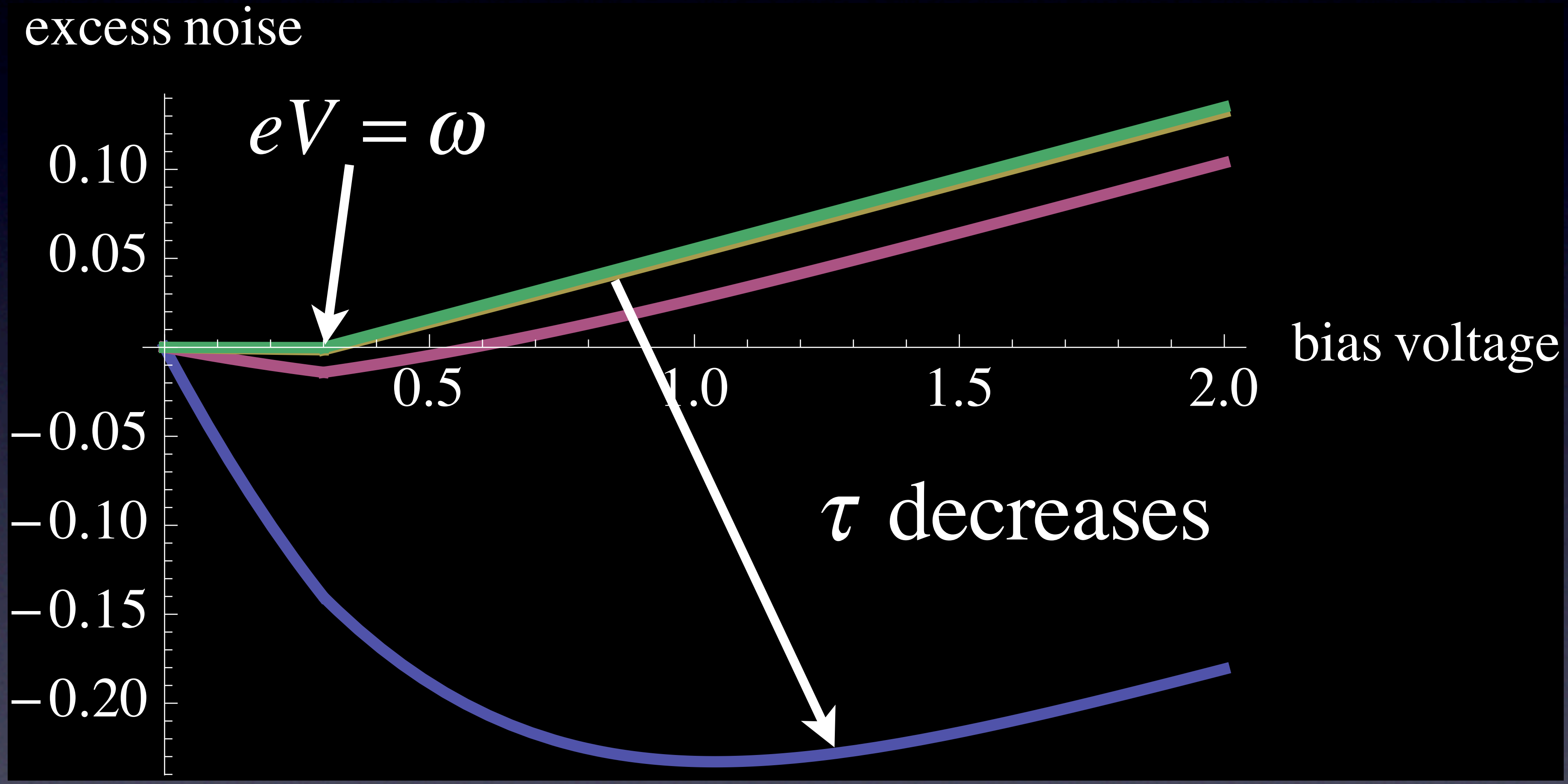
depends on the interaction time with the detector τ_Δ compared to the duration of the wave packet $\hbar/eV$

Result for the measured noise

$$S_{excess}(V,\omega) = S_2(V,\omega) - S_2(0,\omega)$$

frequency unit: $1/\tau_\Delta$

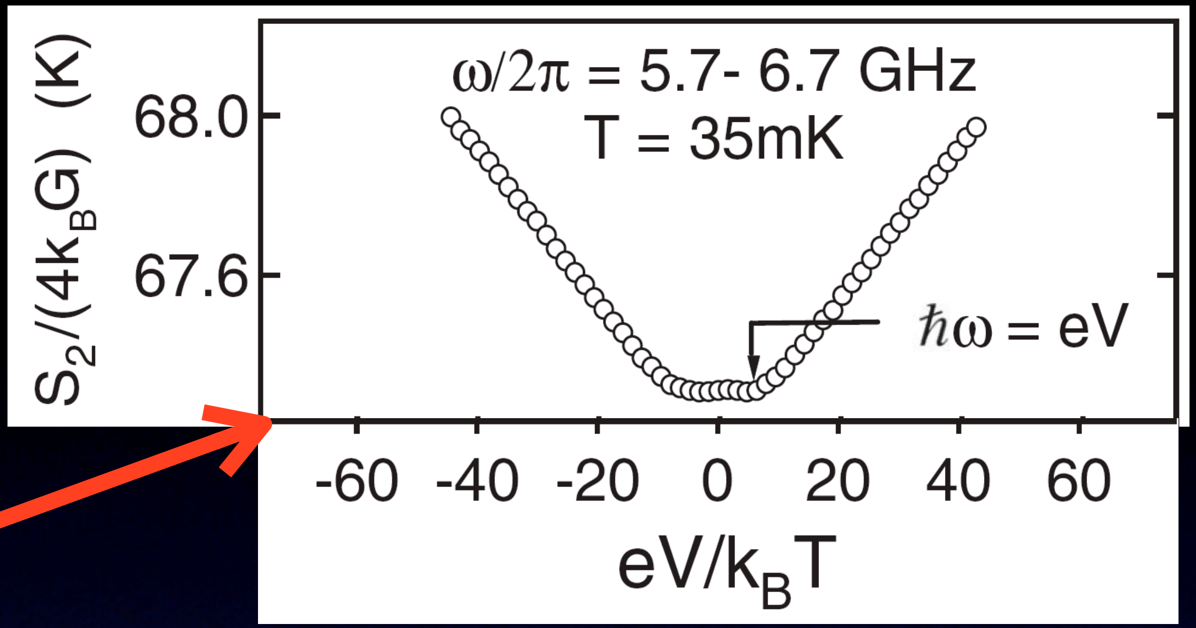
$\omega\tau_\Delta = 0.3$



- Signature of quantum noise is gradually masked by backaction
- A voltage-independent backgound noise is substracted

Experimental observation?

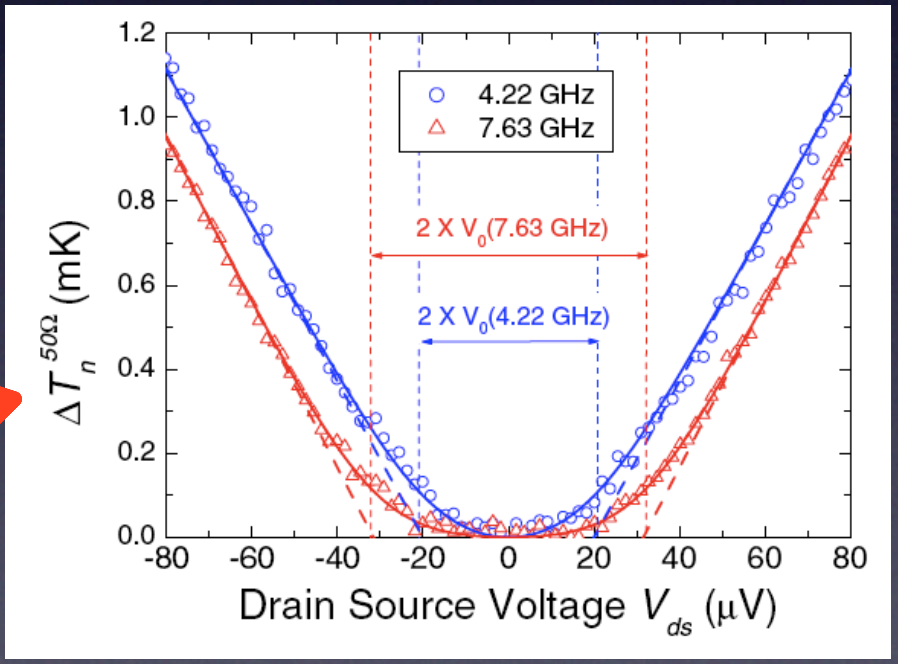
J. Gabelli and B. Reulet, PRL 08



Voltage- and frequency-independent offset noise

E. Zakka-Bajjani et al., PRL 99, 236803 (2007)

Offset noise substracted

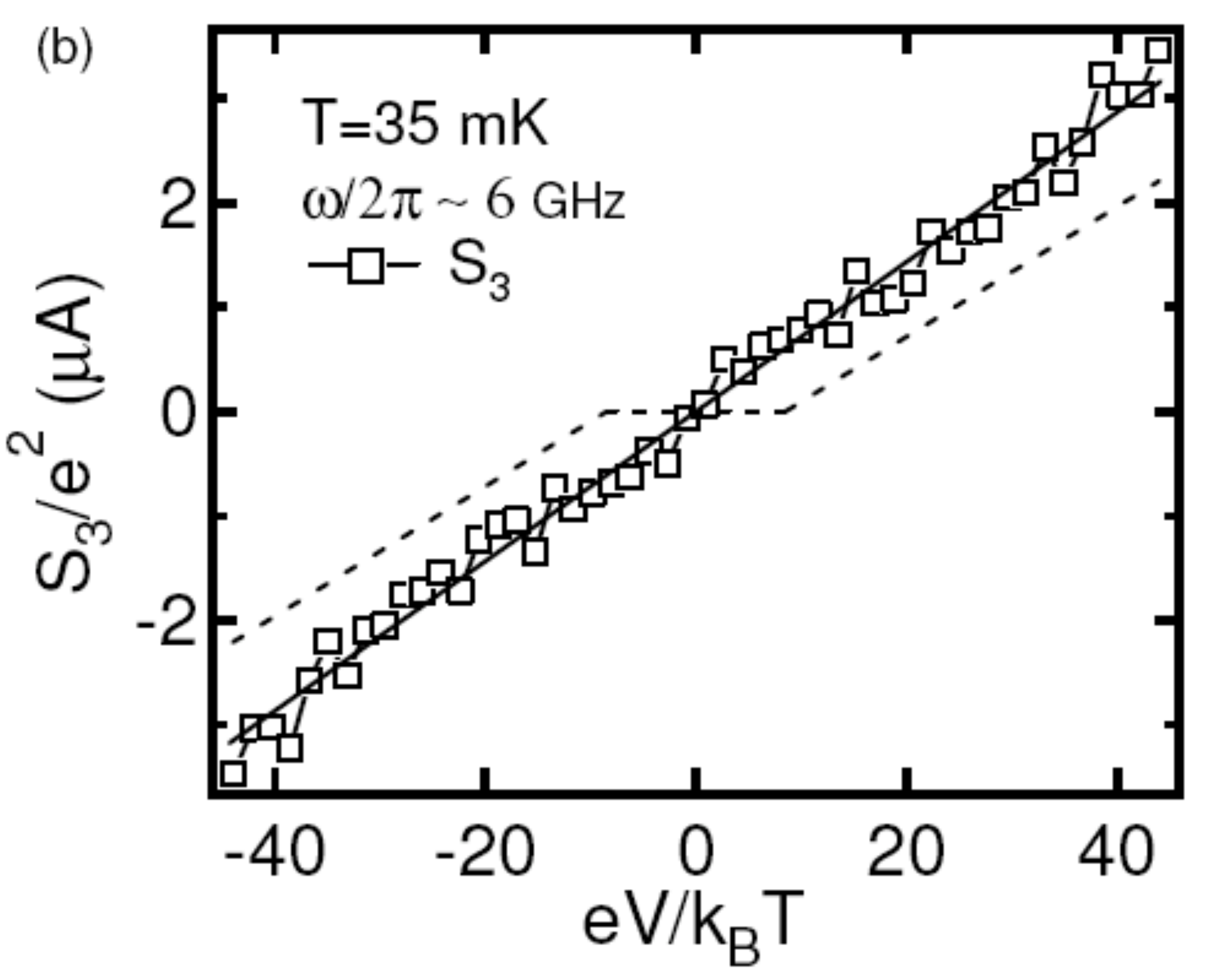


Only offset noise is seen so far.

Third cumulant of a tunnel junction at high frequencies

$$S_3(\omega_1,\omega_2) = \langle \delta I(\omega_1)\delta I(\omega_2)\delta I(-\omega_1-\omega_2)\rangle$$

Measurement by
Gabelli+Reulet, J. Stat. Mech. (2009)



Expectation is the observation of quantum features for $\hbar\omega = eV$ (dashed line) due to the zero point fluctuations

Experiment finds

$$S_3(\omega_1,\omega_2) = e^2 I$$

in agreement with theory!

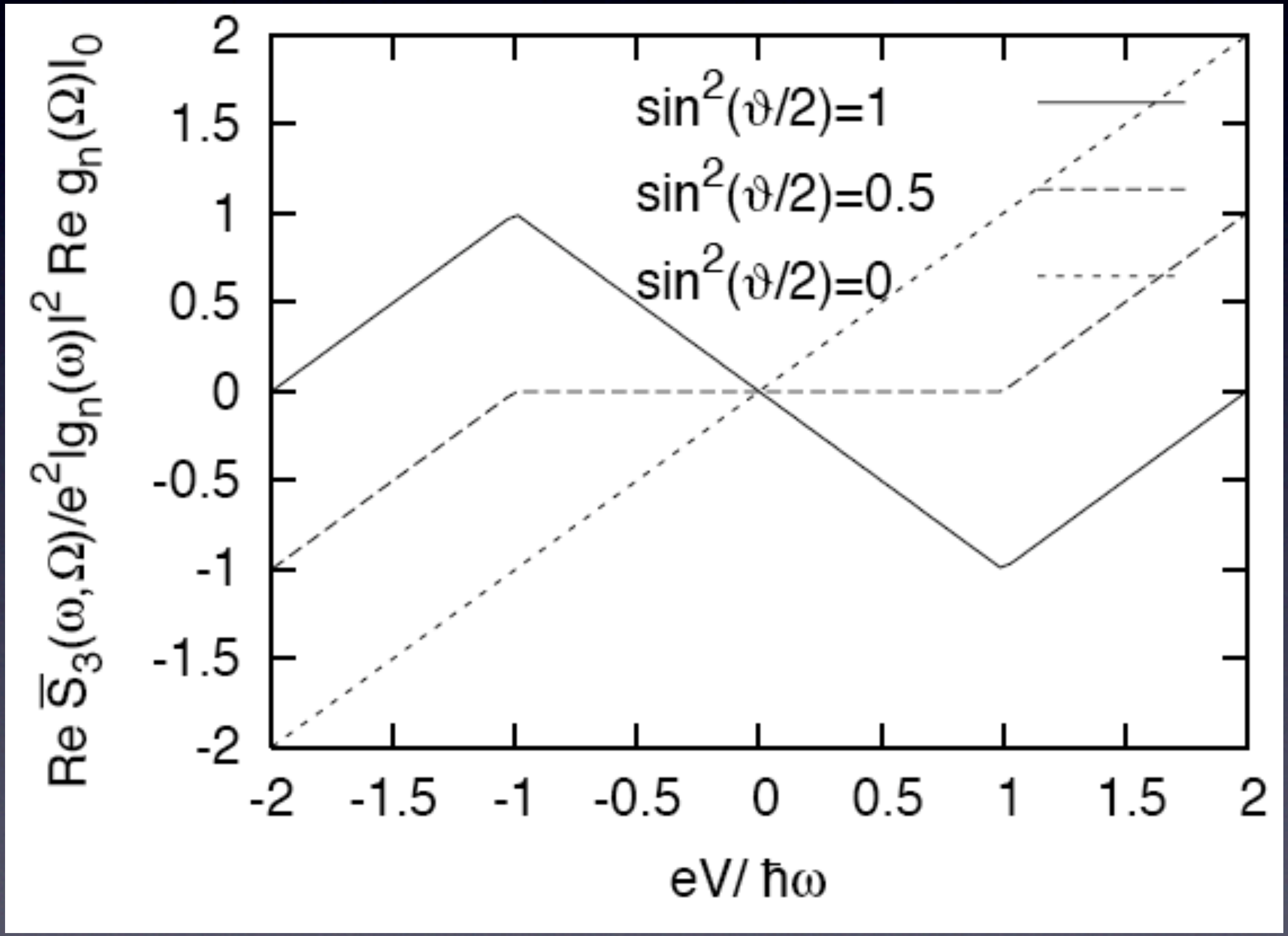
Th: Galaktionov, Golubev, Zaikin, PRB 2003; Golubev, Galaktionov, Zaikin, PRB 2005; Salo, Hekking, Pekkola, PRB 2006

Why is there no signature of zero point fluctuations in S_3?

A different detector may help!

Measured current is coupled to QPC current by a complex conductance

$$I(\omega) = g(\omega) I_{QPC}(\omega) = g(\omega) \frac{2e^2}{h} T V(\omega)$$



The phase of the coupling conductance strongly influences the measured third cumulant in the quantum regime

$$\vartheta = -2\arg\left[g(\omega)\right]$$

Finite phase mixes in the noise susceptibilities

$$\partial S_2(\omega_1,\omega_2)/\partial V(\omega_3)$$

viz. the response of the quantum noise to an ac-exciation

Bednorz, Belzig, unpublished

Conclusion and Summary

Open positions for PhD students

<u>Finite-frequency charge transfer statistics has to account for quantum measurement</u>

- Constructed a POVM of high-frequency current-correlation measurements
- Crossover from large additional white background noise with accurate measurement to backaction-dominated smearing of quantum features
- Consistent with experiments on high-frequency quantum noise and third cumulant

Adam Bednorz and Wolfgang Belzig,
Phys. Rev. Lett. **101**, 206803 (2008)