On the correlation numbers in Minimal Gravity and Matrix Models

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Two approaches to 2D quantum geometry

One is the continuous approach, in which the theory is defined through the functional integral over the Riemannian metric $g_{\mu\nu}(X)$, with appropriate gauge fixing. The choice of the conformal gauge leads to quantum Liouville theory (coupled to matter fields), and for that reason this approach is often called the Liouville Gravity.

The other is the discrete approach, based on the idea of approximating the fluctuating $2D$ geometry by an ensemble of planar graphs, so that the continuous theory is recovered in the scaling limit where the planar graphs of very large size dominate.

The discrete approach is usually referred to as the Matrix Models, since technically the ensemble of the graphs is usually generated by the perturbative expansion of the integral over $N \times N$ matrices, with N sent to infinity to guarantee the planarity of the graphs .

Continuous approach	Discret approach
$\downarrow$	$\downarrow$
"Liouville Gravity"	"Matrix Models"

Impressive body of evidence that the two describe the same reality:

- Operators O_k^{LG} and O_k^{MM} have identical scale dimensions
- Some correlation numbers coincide:

$$\langle\, O_1^{LG}...O_n^{LG}\,\rangle = \langle\, O_1^{MM}...O_n^{MM}\,\rangle$$

But with "naive" identification many correlation numbers are not in agreement.

Resolution: **Resonance relations**:

$$[O_k] \ = [O_{k_1}] + [O_{k_2}]$$

In many cases the disagreement can be fixed by adjusting the parameters in the (nonlinear)relations between the operators O_k^{LG} and O_k^{MM} .

• This work: Trying to find exact map for special class of models:

$$\text{"Minimal Gravity"}\ \mathcal{MG}_{2/2p+1} \leftrightarrow \begin{array}{c}\text{"p − criticality" in}\\ \text{One − Matrix Model}\end{array}$$

◦ The problem is rather "rigid" (more constraints then the parameters).
◦ Nonetheless, the map exists up to the level of four point corr. numbers.
◦ The resulting 1-, 2-, 3-, and 4-point correlation numbers are in perfect agreement.

1. Minimal Gravity

1.1. Quantum Geometry

$$\sum_{\text{topologies}} \int D[g]\, D[\phi]\, e^{-S[g,\phi]}$$

$g(x)$ - Riemannian metric , ϕ - "matter" fields

$$\langle\, \tilde{O}_{k_1} ... \tilde{O}_{k_N}\, \rangle = \int \tilde{O}_{k_1} ... \tilde{O}_{k_N}\, e^{-S[g,\phi]}\, D[g,\phi]$$

$$\tilde{O}_k = \int_{\mathbb{M}} O_k(x)\, d\mu_g(x)$$

$O_k(x)$ - local fields (built from ϕ and g).
Generating function: $\{\lambda\} = \{\lambda_1, ..., \lambda_n\}$

$$Z(\{\lambda\}) = \int D[g,\phi]\, e^{-S_\lambda[g,\phi]}\,,$$

$$S_\lambda[g,\phi] = S_0[g,\phi] + \sum_k \lambda_k\, \tilde{O}_k$$

$$\langle\, \tilde{O}_{k_1} ... \tilde{O}_{k_N}\, \rangle = \frac{\partial^N Z(\{\lambda\})}{\partial\lambda_{k_1} ... \partial\lambda_{k_N}}\Bigg|_{\lambda=0}$$

The parameters $\{\lambda\}$ are the coordinates in the"theory space" Σ.

1.2. Conformal Matter, and Liouville Gravity

$$g^{\mu\nu}\, T^{\text{matter}}_{\mu\nu} = -\frac{c}{12}\, R$$

Conformal Gauge $g_{\mu\nu} = e^{2b\varphi}\, \widehat{g}_{\mu\nu}$: $\quad \Rightarrow \quad$ Decoupling

$$S[g,\phi] \;\to\; S_{\text{L}}[\varphi] + S_{\text{Ghost}}[B,C] + S_{\text{Matter}}[\phi]$$

with

$$S_L[\phi] = \frac{1}{4\pi} \int \sqrt{\widehat{g}}\, \left[\widehat{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + Q\,\widehat{R}\,\varphi + 4\pi\mu\, e^{2b\,\varphi}\right] d^2x\,,$$

$$S_{\text{Ghost}}[B,C] = \frac{1}{2\pi} \int \sqrt{\widehat{g}}\; B_{\mu\nu}\, \nabla^{\mu} C^{\nu}\, d^2x\,,$$

$\Big(B_{\mu\nu} = B_{\nu\mu}\,, \quad \widehat{g}^{\mu\nu} B_{\mu\nu} = 0\Big)$,

$$26 - c = 1 + 6\,Q^2 \qquad\qquad Q = b + 1/b\,.$$

($S_{\text{Matter}}[\phi]$ is conformally invariant, with the central charge c).

Correlation numbers $\langle \tilde{O}_{k_1} ... \tilde{O}_{k_N} \rangle$ with

$$\tilde{O}_k = \int V_k(x)\, \Phi_k(x)\, d^2x$$

$\Phi_k(x)$ - (spinless) primary fields of the matter CFT, with the conformal dimensions (Δ_k, Δ_k) $V_k(x)$ - "gravitational dressings",

$$V_k(x) = e^{2a_k\,\varphi(x)}\,, \qquad a_k(Q - a_k) + \Delta_k = 1$$

Gravitational dimensions of $\tilde{O}_k$ control the scale dependence of the corr. functions:

$$\tilde{O}_k \ \sim \ \mu^{\delta_k}\,, \qquad \delta_k = -\frac{a_k}{b}$$

1.3. Correlation numbers

$$\langle \tilde{O}_{k_1} ... \tilde{O}_{k_n} \rangle = |(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)|^2 \times$$

$$\int d^2x_4 ... d^2x_n \underbrace{\langle O_{k_1}(x_1) O_{k_2}(x_2) O_{k_3}(x_3) O_{k_4}(x_4) ... O_{k_n}(x_n) \rangle}$$

$$\downarrow$$

$$\langle V_{k_1}(x_1) ... V_{k_n}(x_n) \rangle_{\text{Liouville}} \ \langle \Phi_{k_1}(x_1) ... \Phi_{k_n}(x_n) \rangle_{\text{Matter}}$$

1.4. Matter CFT: "Minimal Models"

$$\mathcal{M}_{p/q} \qquad c = 1 - 6\,\frac{(p-q)^2}{pq}$$

Finite number of primary fields

$$\Phi_{(n,m)} \quad (n = 1, ..., p-1\,, \quad m = 1, ..., q-1\,, \quad n \leq m)\,,$$

with (in principle) computable correlation functions, e.g.

$$\langle\, \Phi_{(n_1,m_1)}(x_1)...\Phi_{(n_4,m_4)}(x_4)\,\rangle_{MM} =$$

$$\sum_{(n,m)} \mathbb{C}^{(n,m)}_{(n_1,m_1)(n_2,m_2)}\,\mathbb{C}^{(n,m)}_{(n_3,m_3)(n_4,m_4)} |\mathcal{F}_{(n,m)}(\Delta_i|x)|^2$$

Fusion rules:

$$\Phi_{(n_1,m_1)}\Phi_{(n_2,m_2)} = \sum_{n=|n_1-n_2|+1}^{N}\ \sum_{m=|m_1-m_2|+1}^{M} [\Phi_{(n,m)}]\,,$$

with

$$N = \min(n_1 + n_2 - 1, 2p - n_1 - n_2 - 1)\,,$$

$$M = \min(m_1 + m_2 - 1, 2q - m_1 - m_2 - 1)$$

1.5. "Yang-Lee series" of the Minimal Models $\mathcal{M}_{2/2p+1}$

- $\mathcal{M}_{2/2p+1}$ has p primary fields

$$\Phi_k \equiv \Phi_{(1,k+1)}\,, \qquad k = 0, 1, ..., p-1 \quad (p, p+1, ..., 2p-1)$$

Fusion rules

$$[\Phi_{k_1}][\Phi_{k_2}] = \sum_{k=|k_1-k_2|\,:\,2}^{k_1+k_2} [\Phi_k]\,, \qquad [\Phi_k] = [\Phi_{2p-k-1}]$$

$$\Phi_k = \Phi_{2p-k-1}$$

- Correlation functions:

$$\langle\, \Phi_k \,\rangle = \delta_{k,0}, \qquad \langle\, \Phi_k \Phi_{k'} \,\rangle \;\sim\; \delta_{k,k'}$$

$$\langle\, \Phi_{k_1}\Phi_{k_2}\Phi_{k_3} \,\rangle = 0$$

$$\text{if} \quad \begin{cases} k_1 + k_2 < k_3, \ \text{etc}, & \text{for} \quad k_1 + k_2 + k_3 \ \ \text{even} \\ k_1 + k_2 + k_3 < 2p - 1 & \text{for} \quad k_1 + k_2 + k_3 \ \ \text{odd} \end{cases}$$

$$\langle\, \Phi_{k_1} ... \Phi_{k_n} \,\rangle = 0$$

$$\text{if} \quad \begin{cases} k_1 + ... + k_{n-1} < k_n, & \text{for} \quad k_1 + ... + k_n \ \ \text{even} \\ k_1 + ... + k_n < 2p - 1 & \text{for} \quad k_1 + ... + k_n \ \ \text{odd} \end{cases}$$

• Generating function: $\{\lambda\} = \{\lambda_1, \lambda_2, ..., \lambda_{p-1}\}$

$$Z_{\mathcal{MG}}(\mu, \{\lambda\}) = \left\langle \; \mathrm{exp}\left\{ -\sum_{i=1}^{p-1} \lambda_i \, \tilde{O}_i \right\} \right\rangle_{\mathcal{MG}_{2/2p+1}}$$

The cosmological constant μ may be treated as $\mu = \lambda_0$

$$S[\mathcal{MG}] = ... + \mu \underbrace{\int e^{2b\varphi(x)} \, d^2x}_{} + ...$$

$$\tilde{O}_0 = \int V_0(x) \, \Phi_0(x) \, d^2x \,, \quad \Phi_0 = I$$

Dimensions:

$$\lambda_k \;\sim\; \mu^{\frac{k+2}{2}} \,, \quad k = 0, 1, ..., p-1$$

By the definition

$$\langle \, \tilde{O}_{k_1} ... \tilde{O}_{k_n} \, \rangle = \frac{\partial^n Z_{\mathcal{MG}}(\mu, \{\lambda_i\})}{\partial \tau_{k_1} ... \partial \lambda_{k_n}} \bigg|_{\{\lambda_i\}=0} \,, \qquad \{\lambda_i\} = \{\lambda_1, ..., \lambda_n\}$$

2. Matrix Models

Continuous limit of the ensemble of planar graphs =Quantum Geometry

2.1.One-matrix Model The planar graphs = Feynmann diagrams associated with the perturbative evaluation of the matrix integral

$$Z = \log \int dM \; e^{-N \, \mathrm{tr} \left(\frac{1}{2} M^2 - \sum_{n=3} \frac{\alpha_n}{n!} M^n \right)}$$

M- Hermitian $N \times N$ matrix, N being the device for sorting out the topologies

$$Z = N^2 Z_0 + Z_1 + ... + N^{2-2g} Z_g + ...$$

Each term Z_g generates discretized surfaces, of the topology g, made of triangles and higher polygons, with the weights determined by α_i.

• We concentrate on $g = 0$ (sphere) Σ -space of the "potentials" $V(M) = \sum_{n=3} \frac{\alpha_n}{n!} M^n$.

The one-Matrix Model exhibits an infinite set of multi-critical points, labelled by the integer $p = 1, 2, 3, ...$.

In the scaling limit the partition function is expressed through the solution of the "string equation"

$$\mathcal{P}(u) = 0\,, \tag{1}$$

where $\mathcal{P}(u)$ is the $p+1$-degree polynomial

$$\mathcal{P}(u) = u^{p+1} + t_0\, u^{p-1} + \sum_{k=1}^{p-1} t_k\, u^{p-k-1}\,, \tag{2}$$

with the parameters t_k describing the relevant deviations from the p-critical point . The singular part of the Matrix Model partition function $Z(t_0, t_1, ..., t_{p-1})$ is expressed through $\mathcal{P}(u)$ as follows

$$Z = \frac{1}{2}\int_0^{u_*} \mathcal{P}^2(u)\, du\,, \tag{3}$$

where $u_* = u_*(t_0, t_1, ..., t_{p-1})$ is the suitably chosen root of the polynomial , i.e. $\mathcal{P}(u_*) = 0$.
It is important to remember that Z really gives only the singular part of the Matrix Model partition function.

Take

$$t_0 = \mu \quad -\text{"cosmological constant"}$$

Then

$$[u] = [\mu^{\frac{1}{2}}], \quad [t_k] = [\mu^{\frac{k+2}{2}}], \qquad [Z] = [\mu^{\frac{2p+3}{2}}],$$

exactly the gravitational dimensions of $\mathcal{MG}_{2/2p+1}$,

$$t_k \; \sim \; \lambda_k, \quad k = 0, 1, 2, ..., p-1.$$

Convenient to separate $t_0 = \mu$ and $\{t_i\} = \{t_1, t_2, ..., t_{p-1}\}$

Matrix Model correlation numbers:

$$\langle \mathcal{O}_{k_1} ... \mathcal{O}_{k_n} \rangle_{MM} \equiv \left. \frac{\partial^n Z_{MM}(\mu, \{t_i\})}{\partial t_{k_1} ... \partial_{k_n}} \right|_{\{t_i\}=0}, \qquad \{t_i\} = \{t_1, ..., t_n\}$$

With the (naive) identification $t_k \;\sim\; \lambda_k$ one would expect

$$\langle\, \mathcal{O}_{k_1} ... \mathcal{O}_{k_n} \,\rangle_{MM} = \langle\, \tilde{O}_{k_1} ... \tilde{O}_{k_n} \,\rangle_{\mathcal{MG}} \;\times \text{[Leg factors]}$$

This expectation fails.

Since

$$\mathcal{P}(u) = u^{p+1} + \mu\, u^{p-1} + \sum_{k=1}^{p-1} t_k\, u^{p-k-1}\,, \qquad\qquad Z = \frac{1}{2} \int_0^{u_*} \mathcal{P}^2(u)\, du$$

we have $u_*(\mu, 0, ..., 0) = \sqrt{\mu}$, and

$$\frac{\partial Z}{\partial t_k}\bigg|_{\{t=0\}} = \int_0^{u_*} \mathcal{P}(u)\, \frac{\partial \mathcal{P}(u)}{\partial t_k}\, du \bigg|_{\{t=0\}} = -\frac{2\, \mu^{\frac{2p-k+1}{2}}}{(2p-k-1)(2p-k+1)}$$

$$\frac{\partial^2 Z}{\partial t_k \partial t_{k'}}\bigg|_{\{t=0\}} = \int_0^{u_*} \frac{\partial \mathcal{P}(u)}{\partial t_k}\, \frac{\partial \mathcal{P}(u)}{\partial t_{k'}}\, du \bigg|_{\{t=0\}} = \frac{\mu^{\frac{2p-k-k'-1}{2}}}{2p-k-k'-1}$$

etc

in sharp contrast with

$$\langle\, \tilde{O}_k \,\rangle_{\mathcal{MG}} \;=\; 0\,, \quad k = 1, 2, ..., p-1 \quad (\text{since } \;\langle\, \Phi_k \,\rangle_{CFT} = 0)$$

$$\langle\, \tilde{O}_k \tilde{O}_{k'} \,\rangle_{\mathcal{MG}} \;\sim\; \delta_{kk'}\,, \qquad\qquad (\text{since } \langle\, \Phi_k \Phi_{k'} \,\rangle_{CFT} \sim \delta_{kk'})$$

2.3. Resonance transformations

$$[t_k] = [\mu^{\frac{k+2}{2}}]\,, \qquad [\lambda_k] = [\mu^{\frac{k+2}{2}}]$$

It is possible to have, e.g.

$$[t_k] = [\lambda_{k_1}][\lambda_{k_2}] \qquad (k = k_1 + k_2 + 2 \geq 2)$$

$(k = 0, 1, 2, ..., p-1)$. I.e.

$$t_k \;=\; \lambda_k \;+\; \sum_{\substack{k_1, k_2 = 0 \\ k_1 + k_2 = k+2}}^{p-1} c_k^{k_1 k_2}\, \lambda_{k_1}\lambda_{k_2} \;+\; \text{higher order terms}$$

Thus

$$t_0 = \lambda_0 = \mu\,,$$

$$t_1 = \lambda_1\;, \qquad ([t_1] = [\mu^{3/2}])$$

$$t_2 = \lambda_2 + A_2\,\mu^2\;, \qquad ([t_2] = [\mu^2])$$

$$t_3 = \lambda_3 + B_3\,\mu\,\lambda_1\;, \qquad ([t_3] = [\mu][t_1])$$

$$t_4 = \lambda_4 + A_4\,\mu^3 + B_4\,\mu\,\lambda_2 + C_4\,\lambda_1^2$$

etc

generally

$$t_k = \lambda_k + \underbrace{A_k\,\mu^{\frac{k+2}{2}}} + \sum_{n=0}^{n\leq k/2} \underbrace{B_k^{k-2n}\,\mu^n\,\lambda_{k-2n}} +$$

$$\frac{1}{2}\sum_{n=0}\ \sum_{k_1+k_2=k-2-2n} \underbrace{C_k^{k_1,k_2}\,\mu^n\,\lambda_{k_1}\lambda_{k_2}} + ...$$

$$\uparrow$$

$$Z_{MM}(\{t\}) \ \rightarrow\ \tilde{Z}_{MM}(\{\lambda\}) \equiv Z_{MM}(\{t(\lambda)\})$$

The right thing to expect is

$$\frac{\partial^N \tilde{Z}_{MM}(\{\lambda\})}{\partial\lambda_{k_1}...\partial\lambda_{k_N}} = \langle\, \tilde{O}_{k_1}...\tilde{O}_{k_n}\,\rangle_{\mathcal{MG}}$$

under special choice of the "Liouville coordinates" $\{\lambda_1, ..., \lambda_n\}$.

Thus,Problem: Finding the "Liouville coordinates" $\{\lambda\}$, such that

- One-point numbers:

$$\langle \tilde{O}_k \rangle_{MM} = \frac{\partial \tilde{Z}(\mu,\{\lambda\})}{\partial \lambda_k}\Big|_{\{\lambda\}=0} = 0 \qquad \text{for} \quad k = 1,2,...,p-1$$

- Two-point numbers:

$$\langle \tilde{O}_k \tilde{O}_{k'} \rangle_{MM} = \frac{\partial^2 \tilde{Z}(\mu,\{\lambda\})}{\partial \lambda_k \partial \lambda_{k'}}\Big|_{\{\lambda\}=0} \sim \delta_{kk'}$$

- Three-point numbers:

$$\langle \tilde{O}_{k_1} \tilde{O}_{k_2} \tilde{O}_{k_3} \rangle_{MM} = \frac{\partial^3 \tilde{Z}(\mu,\{\lambda\})}{\partial \lambda_{k_1} \partial \lambda_{k_2} \partial \lambda_{k_3}}\Big|_{\{\lambda\}=0} = 0$$

obey the fusion rules.

• Multi-point numbers obey fusion rules, e.g. For even $k_1 + ... + k_n$

$$\langle \tilde{O}_{k_1} \tilde{O}_{k_2} ... \tilde{O}_{k_n} \rangle_{MM} = 0 \qquad \text{if} \quad k_n > k_1 + k_2 + ... + k_{n-1}$$

For odd $k_1 + ... + k_n$

$$\langle \tilde{O}_{k_1} \tilde{O}_{k_2} ... \tilde{O}_{k_n} \rangle_{MM} = 0 \qquad \text{if} \quad k_1 + k_2 + ... + k_n < 2p - 1$$

Building the Liouville coordinates order by order in $\{\lambda\}$:

∘ The resonance transforms do not affect odd parity correlation functions.

∘ Starting from $n = 4$ there are not enough parameters to exterminate the "wrong" correlation numbers:

$$[\lambda_k] = [\mu^{\frac{k+2}{2}}] \;\rightarrow\; [\lambda_{k_1+k_2}] = [\lambda_{k_1}][\lambda_{k_2}][\mu^2]$$

3. Finding the Liouville coordinates

When one plugs $t_k(\lambda)$, the polynomial

$$\mathcal{P}(u) = u^{p+1} + t_0\, u^{p-1} + \sum_{k=1}^{p-1} t_k\, u^{p-k-1}\,, \tag{4}$$

takes the form

$$\mathcal{P}(u) = \mathcal{P}_0(u) + \sum_{k=1}^{p-1} \lambda_k\, \mathcal{P}_k(u) + ... + \sum_{k_i=1}^{p-1} \frac{\lambda_{k_1}...\lambda_{k_n}}{n!}\, \mathcal{P}_{k_1...k_n}(u) + ...$$

where $\mathcal{P}_0(u)$ and $\mathcal{P}_{k_1...k_n}(u)$ are the polynomials of u whose coefficients involve non-negative powers of μ.

$$\begin{aligned}
\mathcal{P}_0(u) &= \quad u^{p+1} \quad + C'_0\, \mu\, u^{p-1} \quad + C''_0\, \mu^2\, u^{p-3} \quad + ... \\
\mathcal{P}_k(u) &= C_k\, u^{p-k-1} + C'_k\, \mu\, u^{p-k-3} + C''_k\, \mu^2\, u^{p-k-5} + ... \\
&...
\end{aligned}$$

$C'_k, C''_k, ...$ are dimensionless constants related to the higher-order coefficients in $t_k(\lambda)$, and in general $\mathcal{P}_{k_1...k_n}(u)$ are polynomials of the degree

$$p + 1 - 2n - \sum k_i\,,$$

of similar structure. Of course, only polynomials of non-negative degree appear, so that the sum in $\mathcal{P}(u)$ is finite.

3.1 One- and two-point correlation numbers

. The first order of business is to determine $\mathcal{P}_0(u)$ and $\mathcal{P}_k(u)$. One finds

$$Z_0 = \frac{1}{4}\int_{-u_0}^{u_0} \mathcal{P}_0^2(u)\,du\,,$$
$$Z_k \quad = \frac{1}{2}\int_{-u_0}^{u_0} \mathcal{P}_0(u)\mathcal{P}_k(u)\,du\,,$$
$$Z_{k_1k_2} = \frac{1}{2}\int_{-u_0}^{u_0} \Big[\mathcal{P}_{k_1}(u)\mathcal{P}_{k_2}(u) + \mathcal{P}_0(u)\mathcal{P}_{k_1k_2}(u)\Big]\,du\,.$$

All Z_k vanish. It means that all the polynomials $\mathcal{P}_k(u)$ must be orthogonal to $\mathcal{P}_0(u)$ with the measure 1. Since the second term in the 2-nd eq. may be disregarded, then the diagonal form of the two-point correlation numbers requires that $\mathcal{P}_k(u)$ themselves form an orthogonal set of polynomials . $\mathcal{P}_k(u)$, up to normalization, are the Legendre polynomials,

$$\mathcal{P}_k(u) = C_k\, g_k\, u_0^{p-k-1}\; L_{p-k-1}(u/u_0)\,.$$

Furthermore, since $\mathcal{P}_0(u)$ is $p+1$ degree polynomial, and vanishing at u_0, one finds

$$\mathcal{P}_0(u) = g\, u_0^{p+1} \left[L_{p+1}(u/u_0) - L_{p-1}(u/u_0)\right]$$

$$g = \frac{(p+1)!}{(2p+1)!!}$$

3.2. Three- and four-point correlation numbers

Before proceeding to the higher-order correlation numbers, it is useful to get rid of annoying factors in the eq-s above. We trade λ_k for the dimensionless couplings

$$s_k = \frac{g_k\ u_0^{-k-2}}{g\,(2p+1)}\,\lambda_k\,,$$

and write the polynomial $\mathcal{P}_k(u)$ as

$$\mathcal{P}(u) = g\,(2p+1)\,u_0^{p+1}\ Q(u/u_0)\,,$$

where $Q(x)$ is the polynomial of degree $p+1$; as in (5), we will think of it as the power series in s_k,

$$Q(x) = Q_0(x) + \sum_{k=1}^{p-1} s_k\,Q_k(x) + \sum_{k_1 k_2}^{p-1} \frac{s_{k_1} s_{k_2}}{2}\,Q_{k_1 k_2}(x) + ...$$

Eq's above then tell us that

$$Q_0(x) = \frac{L_{p+1}(x) - L_{p-1}(x)}{2p+1} = \int L_p(x)\,dx$$

and

$$Q_k(x) = L_{p-k-1}(x)\,.$$

3.3. Three point numbers

Evaluation of the coefficients $\mathcal{Z}_{k_1k_2k_3}$ is straightforward:

$$\mathcal{Z}_{k_1k_2k_3} = -1 + \frac{1}{2}\int_{-1}^{1} \Big[Q_{k_1k_2}(x)Q_{k_3}(x) + Q_{k_1k_3}(x)Q_{k_2}(x) + Q_{k_2k_3}(x)Q_{k_1}(x$$

The first term -1 reproduces MG result, except for the fusion rule factor $N_{k_1k_2k_3}$. The role of the second term is to fix that discrepancy. When $k_1+k_2+k_3$ is odd and $< 2p-1$, the terms with $Z_{k_1k_2k_3}$ are regular . Therefore, we only need to look at the case when $k_1+k_2+k_3$ is even and the second term turns to 1 at all configurations of k_1, k_2, k_3 such that $k_1+k_2 > k_3$, To cancel the first term and to reproduce the fusion rule factor $N_{k_1k_2k_3}$ we need to have

$$\frac{1}{2}\int_{-1}^{1} Q_{k_3}(x)Q_{k_1k_2}(x)\,dx = \begin{cases} 1 \quad \text{if} \quad k_1+k_2 < k_3 \\ \hline 0 \quad \text{if} \quad k_1+k_2 \geq k_3 \end{cases}$$

Since $Q_k(x) = P_{p-k-1}(x)$, this is achieved by taking

$$Q_{k_1k_2}(x) = L'_{p-k_1-k_2-2}(x)\,,$$

where prime denotes the derivative of the Legendre polynomial with respect to x. Thus, we have

$$\mathcal{Z}_{k_1k_2k_3}/\mathcal{Z}_0 = -N_{k_1k_2k_3}\mathcal{N}_p\,,$$

3.4.Four point numbers

Direct calculation yields

$$\mathcal{Z}_{k_1k_2k_3k_4} = \mathcal{Z}^{(0)}_{k_1k_2k_3k_4} + \mathcal{Z}^{(\mathrm{I})}_{k_1k_2k_3k_4}\,,$$

where

$$\mathcal{Z}^{(0)}_{k_1k_2k_3k_4} = \sum_{i=1}^{4} F(k_i-1) - F(-2) - F(k_{(12|34)}) \\ -F(k_{(13|24)}) - F(k_{(14|23)})\,,$$

$$\mathcal{Z}^{(\mathrm{I})}_{k_1k_2k_3k_4} = \frac{1}{2}\int_{-1}^{1}\left[Q_{k_1k_2k_3}Q_{k_4} + Q_{k_4k_1k_2}Q_{k_3} + Q_{k_3k_4k_1}Q_{k_2} + Q_{k_2k_3k_4}Q_{k_1}\right] d$$

In (5)

$$F(k) = L'_{p-k-2}(1) = \frac{1}{2}(p-k-1)(p-k-2)\,,$$

we use the notation

$$k_{(ij|lm)} = \mathrm{min}(k_i+k_j, k_l+k_m)\,.$$

Like in the previous case, the role of the 2-nd term is to enforce the fusion rules, and the polynomials $Q_{k_1k_2k_3}(x)$ are to be determined from this requirement.

The even sector

Assume again that the numbers k_1, k_2, k_3, k_4 are arranged as usual, so that in $\mathcal{Z}^{(0)}_{k_1k_2k_3k_4}$ we always have

$$k_{(12|34)} = k_1 + k_2\,, \qquad k_{(13|24)} = k_1 + k_3\,.$$

The 2-nd term vanishes,if the even sector fusion rules are satisfied. When the fusion rules are violated,we have also

$$k_{(14|23)} = k_2 + k_3 < p - 1$$

The expression $\mathcal{Z}^{(0)}_{k_1k_2k_3k_4}$ evaluates then to

$$-\frac{1}{2}(k_4 - k_1 - k_2 - k_3 - 2)(2p - 3 - k_1 - k_2 - k_3 - k_4)\,.$$

Thus, for $\mathcal{Z}_{k_1k_2k_3k_4}$ to satisfy the fusion rules the integral

$$\int_{-1}^{1} Q_{k_1k_2k_3}(x)Q_{k_4}(x)\,dx$$

which is not equal 0,if $k_{123} < k_4 - 2$ has to return it with the opposite sign. This uniquely determines the polynomials $Q_{k_1k_2k_3}$,

$$Q_{k_1k_2k_3}(x) = L''_{p-\sum k_i-3}(x)$$

It ensures vanishing $\mathcal{Z}_{k_1k_2k_3k_4}$ besides the case when $k_{123} = k_4 - 2$, however in this case the fusion is satisfed authomatically.

4. Multi-Point correlation numbers

One thing is known: when the fusion rules are violated, the correlation numbers then vanish as well. This requirement for the n-point numbers imposes strong conditions on the form of the polynomials $Q_{k_1 \ldots k_{n-1}}(x)$, which fix them uniquely.

In fact, the problem seems over-determined. Suppose we have already constructed the expansion up to the order $n-1$, and thus $Q_0, Q_k, \ldots, Q_{k_1 \ldots k_{n-2}}$ are already determined. Then $Q_{k_1 \ldots k_{n-1}}$ enters the expression for the n-th order coefficient $\mathcal{Z}_{k_1 \ldots k_n}$ only through the "counterterm"

$$\frac{1}{2} \int_{-1}^{1} Q_{k_1 \ldots k_{n-1}}(x) Q_{k_n}(x)\, dx$$

The polynomials $Q_{k_1 \ldots k_{n-1}}$ must be chosen in such a way that these terms cancel all other contributions to $\mathcal{Z}_{k_1 \ldots k_n}$ when the even-sector fusion rules are violated, i.e. when $k_1 + \ldots + k_{n-1} > k_n$. But since the degree of the polynomial $Q_{k_1 \ldots k_{n-1}}(x)$ is $p + 3 - 2n - (k_1 + \ldots + k_{n-1})$, the integral actually vanishes at $k_1 + \ldots + k_{n-1} > k_n + 4 - 2n$.

For $n \geq 4$ a window

$$k_n \; > \; \sum_{i=1}^{n-1} k_i \; > \; k_n + 4 - 2n$$

opens in configurations of k_i violating the even-sector fusion rules, where the counterterm seems to be incapable of doing its job of fixing the fusion rules. A similar problem exists in the odd sector. For $n \geq 4$ there is a window

$$2p - 1 \; > \; \sum_{i=1}^{n} k_i \; > \; 2p + 3 - 2n$$

in the configurations of k_i, where the odd-sector fusion rules are violated, but corresponding coefficients $Z_{k_1 \ldots k_n}$ are singular .

We have seen that at $n = 4$ the problem takes care of itself, in both even and odd sectors. We have calculated the five-point correlation numbers $C_{k_1 k_2 k_3 k_4 k_5}$, and indeed they automatically vanish within both even and odd sector windows. As the byproduct of this calculation we have determined the four-index polynomials $Q_{k_1 \ldots k_4}$,

$$Q_{k_1 k_2 k_3 k_4}(x) = L'''_{p - \sum k - 4}(x) \, ,$$

where $\sum k = k_1 + k_2 + k_3 + k_4$.

5.Discussion

Identification of $\mathcal{MG}_{2/2p+1}$ as the world-sheet theory of the p-critical one-Matrix Model suggests that, by choosing suitable resonance terms in the of the relation between the couplings t_k and λ_k, the Matrix Model correlation numbers can be put in agreement with the fusion rules of $\mathcal{MG}_{2/2p+1}$.

Technically, this is done by constructing the polynomial $Q(u)$, order by order in s_k. We have executed this program up to the fifth order. For higher n direct calculations become rather involved. But a quick glance at the above results immediately suggests the general form,

$$Q_{k_1 \ldots k_n}(u) = \left(\frac{d}{du}\right)^{n-1} L_{p-\sum k-n}(u)\,,$$

where again $\sum k = k_1 + \ldots + k_n$.

The conjecture I

The partition function of the one-MM is expressed through $Q(u)$

$$\mathcal{Z} = \frac{1}{2}\int_0^{u_*} Q^2(u)\, du\,,$$

u_* is the solution of the "string equation"

$$Q(u_*) = 0$$

$$Q(u) = \sum_{n=0} \sum_{k_1,\dots k_n=1}^{p-1} \frac{s_{k_1}\dots s_{k_n}}{n!} L^{(n-1)}_{p-\sum k-n}(u)$$

Here we denote

$$L^{(n)}_k(u) = \left(\frac{d}{du}\right)^n L_k(u)$$

The conjecture II

$\mathcal{Z}$ coincides with the generating functions of the correlation numbers in $\mathcal{MG}_{2/2p+1}$

$$\mathcal{Z} = \left\langle \exp\left\{-\sum_{i=1}^{p-1} s_i\,\tilde{O}_i\right\}\right\rangle_{\mathcal{MG}_{2/2p+1}}$$