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New aspects of duality in electromagnetism, linearized gravity and higher spin gauge fields

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Motivation

Special type of symmetries in modern theoretical physics

Simplest context : electromagnetism with electric and magnetic sources

Application to thermodynamics of black holes dyons

New formalism making EM duality manifest

Black hole dyons

First law for Reissner-Nordstrøm BH

RN dyon

$$ds^2 = -N^2dt^2 + N^{-2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$
$$N = \sqrt{1 - \frac{2M}{r} + \frac{Q^2+P^2}{r^2}},$$
$$A = -\frac{Q}{r}dt + P(1-\cos\theta)d\phi,$$

infer thermodynamics for parameter variations from purely electric case using duality

$$\delta M = \frac{\kappa}{8\pi}\delta\mathcal{A} + \phi_H \delta Q + \psi_H \delta P$$

$$\Delta = M^2 - (Q^2+P^2), \quad r_\pm = M \pm \sqrt{\Delta}$$
$$\kappa = \frac{r_+ - r_-}{2r_+^2}, \quad \mathcal{A} = 8\pi\big[M^2 - \frac{Q^2+P^2}{2} + M\sqrt{\Delta}\big],$$
$$\phi_H = \frac{Q}{r_+}, \quad \psi_H = \frac{P}{r_+}$$

Problem: excluded in action based derivations of 1st law and Euclidean approaches because of string singularity and absence of magnetic potential

EM duality — No sources

equations of motion

$$\begin{cases} dF = 0 \\ d^*F = 0 \end{cases}$$

field strength

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$$

$$F_{0i} = -E_i$$

$$F_{ij} = \epsilon_{ijk} B^i$$

invariance under

$$\begin{pmatrix} F \\ {}^*F \end{pmatrix} \to M \begin{pmatrix} F \\ {}^*F \end{pmatrix} \qquad \det M \neq 0$$

action principle

$$dF = 0 \Longrightarrow F = dA$$

$$S[A] = \frac{1}{2} \int F \wedge {}^*F$$

gauge invariance

$$\delta_\Lambda A = d\Lambda, \quad \delta_\Lambda S = 0$$

EM duality   No sources

Hamiltonian action principle

$$S_H[A_\mu, E^i] = \int d^4x(-\dot{E}^i A_i - \mathcal{H} - A_0 \partial_i E^i)$$

canonical structure

$$\{A_i(t,x), -E^j(t,y)\} = \delta_i^j \delta^3(x-y)$$

Hamiltonian density

$$\mathcal{H} = \frac{1}{2}(E^i E_i + B^i B_i) \qquad B^i = \epsilon^{ijk}\partial_j A_k$$

solve Gauss constraint

$$\partial_i E^i = 0 \Longrightarrow E^i = \epsilon^{ijk}\partial_j Z_k$$

reduced action principle

$$S_R[Z^i_T, A^T_i] = \int d^4x(-\epsilon^{ijk}\partial_j Z_k \dot{A}_i - \mathcal{H})$$

invariant under duality rotations

$$\begin{cases} \delta_D A_i = Z_i, \\ \delta_D Z_i = -A_i \end{cases}, \qquad \delta_D S_R = 0$$

Deser & Teitelboim Phys. Rev. D 13 (1976) 1592

EM duality — No sources

doublet notation

$$A_i^a = \begin{pmatrix} A_i \\ Z_i \end{pmatrix}$$

$$B^{ai} = \epsilon^{ijk}\partial_j A_k^a = \begin{pmatrix} B^i \\ E^i \end{pmatrix}$$

$$\delta_D A_{ia} = \epsilon_{ab} A_i^b$$

manifestly invariant action

$$S_{SS}[A_\mu^a] = \frac{1}{2}\int d^4x \left[\epsilon_{ab} B^{ai}(\partial_0 A_i^b - \partial_i A_0^b) - \delta_{ab} B^{ai} B_i^b\right]$$

A_0^a spurious

Schwarz & Sen Nucl. Phys. B411 (1994) 35

Point-particle sources

duality requires both electric and magnetic sources

magnetic pole $q_n^a = (g_n, 0)$ electron $q_n^a = (0, e_n)$ dyon $q_n^a = (g_n, e_n)$

current $$j^{a\mu}(x) = \sum_n q_n^a \int_{\Gamma_n} \delta^{(4)}(x - z_n) dz_n^\mu$$

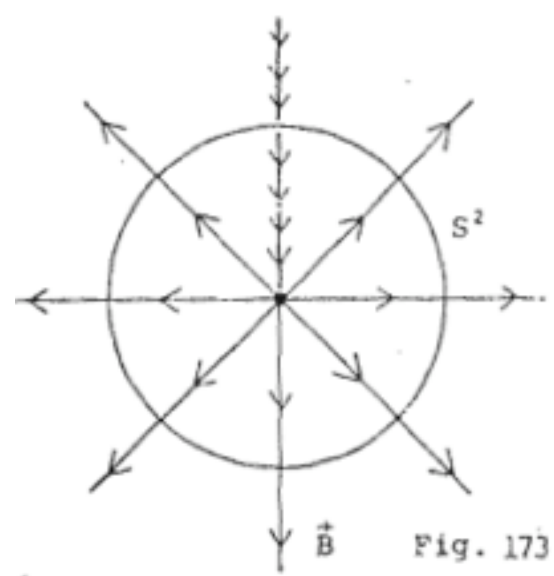
pole at origin $$j_M^\mu(x) = g\delta_0^\mu \delta^{(3)}(x) \qquad B^i = \frac{g}{4\pi}\frac{x^i}{r^3}$$

singular potential $$A = -\frac{g}{4\pi}(\cos\theta + 1)d\varphi \qquad A_i = \frac{g}{4\pi}\begin{pmatrix} \frac{y}{r(r-z)} \\ -\frac{x}{r(r-z)} \\ 0 \end{pmatrix}$$

regularize $$A_i \to A_i^\epsilon = \frac{g}{4\pi}\begin{pmatrix} \frac{y}{R(R-z)} \\ -\frac{x}{R(R-z)} \\ 0 \end{pmatrix}$$

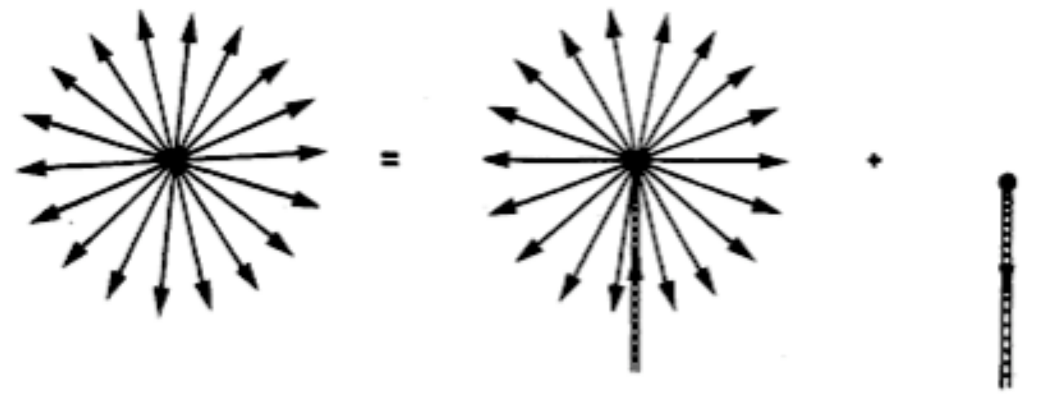
$$R = \sqrt{r^2 + \epsilon^2}$$

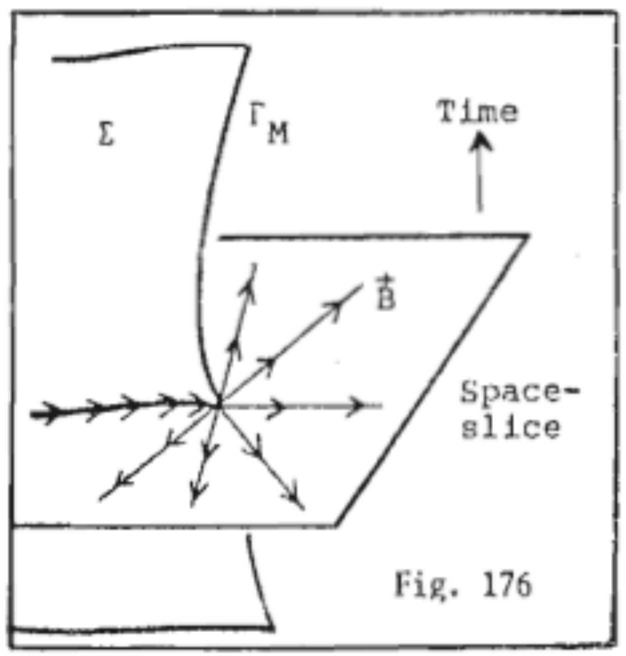
magnetic field: semi-infinite solenoid



EM duality Dirac strings

add dynamical string





$$G^{\mu\nu}_{\Sigma}(x) = g \int_{\Sigma} \delta^{(4)}(x-y) dy^{\mu} \wedge dy^{\nu}$$

$$\partial\Sigma = \Gamma_M \qquad y^{\mu}(\tau,\sigma), \; y^{\mu}(\tau,0) = z^{\mu}_M(\tau)$$

property $$d^*G_{\Sigma} = {}^*G_{\partial\Sigma} = {}^*j_M$$

modified field strength $$F^D = dA + {}^*G_{\Sigma} \qquad dF^D = {}^*j_M$$

action principle

$$S^D[A_{\mu}(x), y^{\mu}(\tau,\sigma), z_e(\tau)] = \frac{1}{2}\int F^D \wedge {}^*F^D + e\int_{\Gamma_e} A$$
$$-m_e \int_{\Gamma_e} \sqrt{-dz^{\mu}_e dz_{\mu e}} - m_M \int_{\Gamma_M} \sqrt{-dz^{\mu}_M dz_{\mu M}}$$

Dirac veto: electron cannot touch string

quantization condition $$eg = 2\pi N \hbar$$

Dirac Phys. Rev. 74 (1948) 817

Remarks on Dirac strings

(i) asymmetric treatment of both types of sources, manifest Poincaré invariance but no duality

(ii) non trivial U(1) fiber bundles, no strings

Wu & Yang Phys. Rev. D12 (1975) 3845, D14 (1976)437

(iii) Dirac strings related to de Rham currents and Poincaré duality

De Rham, Variétés différentiables, Hermann, 1960

Duality

"complicated" proof of duality between electric and magnetic BH

partition function through semi-classical evaluation of Euclidean path integral

$Z(\beta, P)$ vs $Z(\beta, \phi_H)$

Hawking & Ross Phys. Rev. D52 (1995) 5685

additional Legendre transformation needed to compare

Dyons and duality — Extended formulation in flat space

New construction — dynamical longitudinal fields and non spurious scalar potentials

$$A^a_\mu \equiv (A_\mu, Z_\mu) \qquad C^a \equiv (C, Y) \qquad \vec{B}^a \equiv (\vec{B}, \vec{E}) \qquad \vec{B}^a = \vec{\nabla} \times \vec{A}^a + \vec{\nabla} C^a$$

external sources $\qquad \partial_\mu j^{a\mu} = 0$

action principle

$$I[A^a_\mu, C^a] = I_M[A^a_\mu, C^a] + I_I[A^a_\mu; j^{a\mu}], \qquad I_I[A^a_\mu; j^{a\mu}] = \int d^4x\, \epsilon_{ab} A^a_\mu j^{b\mu}.$$

$$I_M[A^a_\mu, C^a] = \frac{1}{2} \int d^4x \left[\epsilon_{ab}(\vec{B}^a + \vec{\nabla} C^a) \cdot (\partial_0 \vec{A}^b - \vec{\nabla} A^b_0) - \vec{B}^a \cdot \vec{B}_a\right],$$

Maxwell's equations

$$\begin{cases} A^a_0 : \quad \vec{\nabla} \cdot \vec{B}^a \equiv \nabla^2 C^a = j^{0a} \qquad C^a : \ \nabla^2 C_a = \epsilon_{ab}(\vec{\nabla} \cdot \partial_0 \vec{A}^b - \nabla^2 A^b_0) \\ \vec{A}^a : \ -\epsilon_{ab} \partial_0 \vec{B}^b + \vec{\nabla} \times \vec{B}_a = \epsilon_{ab} \vec{j}^b. \end{cases}$$

no strings, duality manifest

G.B. & Gomberoff Phys. Rev. D78 (2008) 025025

Dyons and duality — Extended formulation in flat space

fixed point particle dyon at origin $j^{a\mu}(x) = 4\pi Q^a \delta_0^\mu \delta^3(x)$

Coulomb-type solution $A^a = -\frac{\epsilon^{ab} Q_b}{r} dt, \; C^a = -\frac{Q^a}{r}$

resolution of string singularity

gauge invariance $\delta_\epsilon A^a_\mu = \partial_\mu \epsilon^a, \qquad \delta_\epsilon C^a = 0$

magnetic charge is surface integral, no longer a topologically charge

canonical pairs $(\vec{A}^T, -\vec{\nabla} \times \vec{Z}^T), \; (\vec{A}^L, -\vec{\nabla} Y) \; (\vec{Z}^L, \vec{\nabla} C)$

gauge fixing the magnetic Gauss constraint gives back standard EM

Extended formulation with dynamical dyons

dynamical point particle dyons need strings for Lorentz force law

$$G^{a\mu\nu} = \sum_n q_n^a \int_{\Sigma_n} \delta^{(4)}(x - y_n) dy_n^\mu \wedge dy_n^\nu \qquad y_n^\mu(\sigma_n, \tau_n), \quad y_n^\mu(0, \tau_n) = z_n^\mu(\tau_n)$$

total action $\quad I_M + I_I + I_k + \frac{1}{2} \int d^4x \, \epsilon_{ab} \left[2\partial^i C^a \alpha_i^b - \beta^{ai} \alpha_i^b + \beta_T^{ai} \partial_0 \gamma_i^b) \right] .$

$$I_k[z_n^\mu] = -\sum_n m_n \int_{\Gamma_n} \sqrt{-dz_n^\mu dz_{n\mu}} \qquad \beta^{ai} = G^{a\ 0i} \qquad \alpha_i^a = \frac{1}{2} \epsilon_{ijk}\, G^{a\ jk} \qquad \beta_T^{ai} = \epsilon^{ijk} \partial_j \gamma_k^a$$

variation with respect to z_n^μ gives Lorentz force law

veto: string attached to dyon n cannot cross any other dyon

leads to standard quantization condition for dyons $\quad \epsilon_{ab} q_n^a q_m^b = 2\pi N \hbar$

Extended formulation in curved space

need to generalize first order ADM formulation of Einstein-Maxwell

no non singular longitudinal magnetic field $\mathcal{B}^i_{ADM} = \epsilon^{ijk}\partial_j A_k$

curved space $\mathcal{B}^{ai} = \epsilon^{ijk}\partial_j A^a_k + \sqrt{g}\partial^i C^a$

matter action
$$I_M[A^a_\mu, C^a, g_{ij}, N, N^i] = \frac{1}{8\pi}\int d^4x \Big[(\mathcal{B}^{ai} + \sqrt{g}\partial^i C^a)\epsilon_{ab}(\partial_0 A^b_i - \partial_i A^b_0) - \frac{N}{\sqrt{g}}\mathcal{B}^i_a\mathcal{B}^a_i - \epsilon_{ab}\epsilon_{ijk}N^i\mathcal{B}^{aj}\mathcal{B}^{bk}\Big]$$

total action
$$I[z,u] = \int d^4x\,[a_A(z)\partial_0 z^A - u^\alpha\gamma_\alpha]$$

kinetic term
$$a_A(z)\partial_0 z^A = \frac{\pi^{ij}}{16\pi}\partial_0 g_{ij} - \frac{\mathcal{E}^i}{4\pi}\partial_0 A_i + \frac{\sqrt{g}\partial^i C}{4\pi}\partial_0 Z_i$$

Lagrange multipliers and constraints $u^\alpha \equiv (N, N^i, A^a_0)$ $\gamma_\alpha \equiv (\mathcal{H}_\perp, \mathcal{H}_i, \mathcal{G}_a)$

$$\mathcal{H}_\perp = \frac{1}{16\pi}(\mathcal{H}^{ADM}_\perp + \mathcal{H}^{mat}_\perp), \quad \mathcal{H}_i = \frac{1}{16\pi}(\mathcal{H}^{ADM}_i + \mathcal{H}^{mat}_i), \quad \mathcal{G}_a = \frac{1}{4\pi}\epsilon_{ab}\partial_i\mathcal{B}^{bi}$$

gauge parameters $\epsilon^\alpha \equiv (\xi^\perp, \xi^i, \lambda^a)$

smeared constraints $\Gamma[\epsilon] = \int d^3x\,\gamma_\alpha\epsilon^\alpha$

Dyons and duality   Extended formulation in curved space

first class gauge algebra
$$\{\Gamma[\epsilon_1],\Gamma[\epsilon_2]\} = \Gamma[[\epsilon_1,\epsilon_2]] \qquad \{H[\xi],H[\eta]\} = H[[\xi,\eta]_{SD}] + G[[\xi,\eta]_B]$$

surface deformation algebra
$$[\xi,\eta]^{\perp}_{SD} = \xi^i\partial_i\eta^{\perp} - \eta^i\partial_i\xi^{\perp}, \qquad [\xi,\eta]^a_B = \mathcal{B}^{ai}\epsilon_{ijk}\xi^j\eta^k - \frac{\epsilon^{ac}\mathcal{B}_{ci}}{\sqrt{g}}(\xi^{\perp}\eta^i - \eta^{\perp}\xi^i)$$
$$[\xi,\eta]^i_{SD} = g^{ij}(\xi^{\perp}\partial_j\eta^{\perp} - \eta^{\perp}\partial_j\xi^{\perp}) + \xi^j\partial_j\eta^i - \eta^j\partial_j\xi^i$$

diffeomorphisms $\quad \xi^{\perp} = N\eta^0 \quad \xi_i = g_{i\mu}\eta^{\mu} \qquad \mathcal{L}_\eta g_{\mu\nu} \approx \delta_\xi g_{\mu\nu}$

sources $\quad I_{ADM} + I_M + I_J$

$$I_J[A^a_\mu, C^a, y^\mu] = \frac{1}{4\pi}\int d^4x\,\epsilon_{ab}\big[A^a_\mu j^{b\mu} + \sqrt{g}\partial^i C^a\alpha^b_i - \frac{1}{2}\beta^{ai}\alpha^b_i - \frac{1}{2}\beta^{ai}_T\partial_0\gamma^b_i\big]$$
$$\beta^{aTi} = \epsilon^{ijk}\partial_j\gamma^a_k$$

1-1 correspondence of solutions to those of covariant equations

resolved RN dyon
$$ds^2 = -N^2dt^2 + N^{-2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$
$$N = \sqrt{1 - \frac{2M}{r} + \frac{Q^2+P^2}{r^2}},$$
$$A^a = -\epsilon^{ab}Q_b(\frac{1}{r} - \frac{1}{r_+})dt$$
$$C^a = -Q^a\int_r^\infty \frac{dr'}{r'^2N(r')}$$

Dyons and duality   1st law of black hole mechanics

surface integrals $\delta_z(\gamma_\alpha \epsilon^\alpha) = \delta z^A \frac{\delta(\gamma_\alpha \epsilon^\alpha)}{\delta z^A} - \partial_i k^i_\epsilon$ Regge & Teitelboim Ann. Phys. 88 (1974) 286

solution of constraints z^A_s solution of linearized constraints δz^A_s

time independent solution z^a_s, u^α_s $\partial_0 z^A_s = 0$

$$\frac{\delta a_B}{\delta z^A} \partial_0 z^A - \partial_0 a_A = \frac{\delta(\gamma_\alpha u^\alpha)}{\delta z^A}$$

implies

$$\partial_i k^i_{u_s}[z^A_s, \delta z^A_s] = 0$$

Stokes theorem

$$\oint_{S_{r_1}} d^3x_i\ k^i_{u_s}[z^A_s, \delta z^A_s] = \oint_{S_{r_2}} d^3x_i\ k^i_{u_s}[z^A_s, \delta z^A_s].$$

standard gravitational part

$$k^i_\epsilon[z^A; \delta z^A] = k^{grav,i}_\epsilon[g_{ij}, \pi^{ij}; \delta g_{ij}, \delta\pi^{ij}] + k^{mat,i}_\epsilon[z^A; \delta z^A$$

$$k^{grav,i}_\epsilon = \frac{1}{16\pi}\Big[G^{ljki}(\xi^\perp \nabla_k \delta g_{lj} - \partial_k \xi^\perp \delta g_{lj}) + 2\xi_k \delta\pi^{ki} + (2\xi^k \pi^{ji} - \xi^i \pi^{jk})\delta g_{jk}\Big],$$

$$G^{ljki} = \frac{1}{2}\sqrt{g}(g^{lk}g^{ji} + g^{il}g^{jk} - 2g^{lj}g^{ki})$$

new matter part

$$k^{mat,i}_\epsilon = \frac{1}{4\pi}\Big(\frac{\xi^\perp}{\sqrt{g}}\epsilon^{ijk}\mathcal{B}^a_j \delta A_{ak} - \xi^\perp \mathcal{B}^{ai}\delta C_a + \epsilon_{ab}(\xi^k \mathcal{B}^{ai} - \xi^i \mathcal{B}^{ak})\delta A^b_k$$

includes electric and magnetic contributions

$$-\epsilon_{ab}\sqrt{g}g^{il}\epsilon_{ljk}\xi^j \mathcal{B}^{ak}\delta C^b + \epsilon_{ab}(\sqrt{g}\partial^i \lambda^a \delta C^b - \lambda^a \delta \mathcal{B}^{bLi})\Big)$$

for RN dyon

$$\oint_{S_r} d^{n-1}x_i\ k^{mat,i}_u = -\frac{1}{4\pi}\int_0^\pi d\theta \int_0^{2\pi} d\phi\ \epsilon_{ab} A^a_0 \delta \mathcal{B}^{bLi}$$

1st law of black hole mechanics

first law

$$\oint_{S^\infty} d^3x_i\, k_u^i[z,\delta z] = \oint_{S_{r_+}} d^3x_i\, k_u^i[z,\delta z]$$

at infinity

$$\oint_{S^\infty} d^3x_i\, k_u^i[z,\delta z] = \delta_z\mathcal{M} - \phi_H\delta_z\mathcal{Q} - \psi_H\delta_z\mathcal{P}$$

at horizon

$$\oint_{S_{r_+}} d^{n-1}x_i\, k_u^{grav,i} = \frac{\kappa}{8\pi}\delta_z\mathcal{A}$$

geometric derivation of first law for RN for arbitrary perturbations

$$\delta_z\mathcal{M} = \frac{\kappa}{8\pi}\delta_z\mathcal{A} + \phi_H\delta_z\mathcal{Q} + \psi_H\delta_z\mathcal{P}$$

Euclidean approach in the grand canonical ensemble

improved constraints for suitable fall-off conditions contain surface terms

$$\boldsymbol{H} = \int d^3x\,(\mathcal{H}_\perp N + \mathcal{H}_i N^i) + \mathcal{M} \qquad \boldsymbol{Q} = -\frac{1}{\phi^c}\int d^3x\,(\mathcal{G}_1 A_0) + \mathcal{Q},\ \ \boldsymbol{P} = -\frac{1}{\psi^c}\int d^3x\,(\mathcal{G}_2 Z_0) + \mathcal{P}$$

three commuting observables $\{\boldsymbol{H},\boldsymbol{Q}\} = 0 = \{\boldsymbol{H},\boldsymbol{P}\} = \{\boldsymbol{Q},\boldsymbol{P}\}$

partition function $Z[\beta,\phi^c,\psi^c] = \mathrm{Tr}\, e^{-\beta(\widehat{\boldsymbol{H}} - \phi^c\widehat{\boldsymbol{Q}} - \psi^c\widehat{\boldsymbol{P}})} = e^{\Psi_G}$

Massieu function $\Psi_G(\beta, -\beta\phi^c, -\beta\psi^c)$

path integral

$$Z[\beta, \phi, \psi] = \int \mathcal{D}\Phi e^{I_e^T}$$

Euclidean action

$$I_e^T = \int_0^\beta d\tau \Big(i \int d^3x \, a_A(z) \partial_0 z^A - (\boldsymbol{H} - \phi^c \boldsymbol{Q} - \psi^c \boldsymbol{P}) \Big) + \text{ghost terms}$$

leading contribution: action evaluated at Euclidean Reissner-Nordstrom dyon

$$I_e^{mat}(RND) = \beta \phi_H Q + \beta \psi_H P$$

$$I_e^{grav}(RND) = -\beta M + \frac{1}{4}\mathcal{A}$$

total result

$$\Psi_G = -\beta M + \frac{1}{4}\mathcal{A} + \beta \phi_H Q + \beta \psi_H P$$

Conclusion

construction of an explicitly duality invariant version of electromagnetism

enhanced gauge invariance

static dyon described by Coulomb fields without string singularities

electric and magnetic charges are surface integrals

applications in the context of thermodynamics of BH dyons

Spin 2 Reduced phase space duality

first order formulation

$$S_{PF}[h_{mn}, \pi^{mn}, n_m, n] = \int dt \Big[\int d^3x \left(\pi^{mn} \dot{h}_{mn} - n^m \mathcal{H}_m - n\mathcal{H} \right) - H_{PF} \Big],$$

Hamiltonian

$$H_{PF}[h_{mn}, \pi^{mn}] = \int d^3x \big(\pi^{mn}\pi_{mn} - \frac{1}{2}\pi^2 + \frac{1}{4}\partial^r h^{mn} \partial_r h_{mn} - $$
$$ - \frac{1}{2}\partial_m h^{mn} \partial^r h_{rn} + \frac{1}{2}\partial^m h \partial^n h_{mn} - \frac{1}{4}\partial^m h \partial_m h \big),$$

constraints

$$\mathcal{H}_m = -2\partial^n \pi_{mn}, \quad \mathcal{H}_\perp = \Delta h - \partial^m \partial^n h_{mn}.$$

decomposition

$$\phi_{mn} = \phi^{TT}_{mn} + \phi^T_{mn} + \phi^L_{mn},$$
$$\phi^L_{mn} = \partial_m \psi_n + \partial_n \psi_m,$$
$$\phi^T_{mn} = \frac{1}{2}\left(\delta_{mn}\Delta - \partial_m \partial_n\right) \psi^T.$$

$$\psi_m = \Delta^{-1}\left(\partial^n \phi_{mn} - \frac{1}{2}\Delta^{-1}\partial_m \partial^k \partial^l \phi_{kl} \right),$$
$$\psi^T = \Delta^{-1}\left(\phi - \Delta^{-1}\partial^m \partial^n \phi_{mn} \right),$$

conjugate pairs

$$\big(h^{TT}_{mn}(x),\, \pi^{kl}_{TT}(\vec{y})\big), \quad \big(h^L_{mn}(\vec{x}),\, \pi^{kl}_L(y)\big), \quad \big(h^T_{mn}(x),\, \pi^{kl}_T(y)\big).$$

solving constraints + gauge fixing

$$\mathcal{H}_m = 0 = \mathcal{H} \iff \pi^{kl}_L = 0 = h^T_{mn}$$

$$h^L_{mn} = 0 = \pi^{kl}_T$$

reduced phase space

$$H^R = \int d^3x \left(\pi^{mn}_{TT}\pi^{TT}_{mn} + \frac{1}{4}\partial_r h^{TT}_{mn} \partial^r h^{mn}_{TT} \right).$$

introduce potentials to make reduced phase space manifestly duality invariant

Henneaux & Teitelboim Phys. Rev. D71 (2005) 024018

Spin 2 — Extended gauge invariance

extended gauge invariance to couple to both electric and magnetic sources

degree of freedom count

$$\#\,\mathrm{dof} = 2 \qquad \#\,\mathrm{fcc} = 8 \qquad \#\,\mathrm{cp} = 10$$

$$2 * (\#\,\mathrm{cp}) = 2 * (\#\,\mathrm{dof}) + 2 * (\#\,\mathrm{fcc})$$

phase space variables

$$z^A = (H^a_{mn}, A^a_m, C^a).$$

change of variables

$$h^a_{mn} = (h_{mn}, h^D_{mn}) \qquad \pi^{mn}_a = (\pi^{mn}_D, \pi^{mn})$$

$$h^a_{mn} = \epsilon_{mpq}\partial^p H^{aq}{}_n + \epsilon_{npq}\partial^p H^{aq}{}_m + \partial_m A^a_n + \partial_n A^a_m + \frac{1}{2}(\delta_{mn}\Delta - \partial_m\partial_n)C^a$$

$$\pi^a_{mn} = -\Delta H^a_{mn}.$$

brackets

$$\left(H^{1TT}_{mn}(x),\, -2\Delta\left(\mathcal{O}H^2_{TT}\right)^{kl}(y)\right),\ \left(C^1(x),\, -\frac{1}{2}\Delta(\Delta H^2_T - \partial_p\partial_q H^{2pq}_T)\,(y)\right),$$

$$\left(A^1_m(x),\, 2\Delta\partial_r H^{2rn}_L(y)\right),$$

$$\left(A^2_m(x),\, -2\Delta\partial_r H^{1rn}_L(y)\right), \left(C^2(x),\, \frac{1}{2}\Delta(\Delta H^1_T - \partial_p\partial_q H^{1pq}_T)(y)\right).$$

$$\{h^a_{mn}(x), \pi^{bkl}(y)\} = \epsilon^{ab}\frac{1}{2}(\delta^k_m\delta^l_n + \delta^k_m\delta^l_n)\delta^{(3)}(x,y).$$

abelian gauge structure

$$\mathcal{H}_{am} = -2\epsilon_{ab}\partial^n\pi^b_{mn} = 2\epsilon_{ab}\Delta\partial^n H^b_{mn},$$

$$\mathcal{H}_{a\perp} = \Delta h_a - \partial_m\partial_n h^{mn}_a = \Delta^2 C_a.$$

Spin 2 Duality + sources

first order action

$$S_G[z^A,u^\alpha]=\int d^4x\,(a_A[z]\dot z^A-u^\alpha\gamma_\alpha[z])-\int dt\,H[z],$$

duality invariant Hamiltonian

$$H=\int d^3x\,\Big(\Delta H^a_{mn}\Delta H^{mn}_a-\frac{1}{2}\Delta H^a\Delta H_a+\frac{1}{8}\Delta C^a\Delta^2C_a\Big).$$

interacting theory

$$S^J=\int d^4x\,\frac{1}{2}h^a_{\mu\nu}T^{\mu\nu}_a,\qquad \partial_\mu T^{\mu\nu}_a=0$$

$$h^a_{0m}=n^a_m=h^a_{m0},\qquad h^a_{00}=-2n^a,$$

$$S_T[z^A,u^\alpha;T^{a\mu\nu}]=\frac{1}{16\pi G}S_G+S^J$$

linearized Taub-NUT

$$T^{\mu\nu}_a(x)=\delta^\mu_0\delta^\nu_0M_a\delta^{(3)}(x^i),\qquad M_a=(M,N).$$

$$\mathcal{H}_{a\perp}=-16\pi GM_a\delta^{(3)}(x).$$

$$C^a=GM^a(2r),\ n^a=GM^a(-\frac{1}{r}),\ A^a_m=GM^a(-\frac{1}{2}\frac{x_m}{r}),\ n^{am}=H^a_{mn}=0,$$

$$h^a_{mn}=GM^a(\frac{2x_mx_n}{r^3}),\quad \pi^{mn}_a=0.$$

singularity resolved by additional pure gauge degrees of freedom

Spin 2 Surface charges + global symmetries

Regge-Teitelboim

$$\gamma_\alpha \varepsilon^\alpha = (\partial^m \xi^{an} + \partial^n \xi^{am})\epsilon_{ab}\pi^b_{mn} + (\delta^{mn}\Delta - \partial^m\partial^n)\xi^{a\perp}h_{amn} - \partial_i \widetilde{k}^i_\varepsilon[z]$$

$$\widetilde{k}^i_\varepsilon[z] = 2\xi^a_m \epsilon_{ab}\pi^{bmi} - \xi^{a\perp}(\delta^{mn}\partial^i - \delta^{mi}\partial^n)h_{amn} + h_{amn}(\delta^{mn}\partial^i - \delta^{ni}\partial^m)\xi^{a\perp}$$

$$\begin{cases} \partial^m \xi^{an}_s + \partial^n \xi^{am}_s = 0 = \partial_0 \xi^{am}_s - \partial^m \xi^{a\perp}_s, \\ (\delta^{mn}\Delta - \partial^m\partial^n)\xi^{a\perp}_s = 0 = \partial_0 \xi^{a\perp}_s, \end{cases} \iff \xi^a_{\mu s} = -\omega^a_{[\mu\nu]}x^\nu + a^a_\mu,$$

electric and magnetic ADM type conserved surface charges

$$\mathcal{Q}_{\varepsilon_s}[z_s] = \frac{1}{16\pi G}\oint_S d^2x_i\, \widetilde{k}^i_{\varepsilon_s}[z_s] = \frac{1}{2}\omega^a_{\mu\nu}J^{\mu\nu}_a - a^a_\mu P^\mu_a,$$

dyon

$$P^\perp_a = M_a$$

1 copy of global Poincaré transformations acting

$$T'^{\mu\nu}_a(x') = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}_a(x)$$

$$\xi'^\nu_{as}(x') = \Lambda^\mu{}_\alpha \xi^\alpha_{as}(x) = -(\Lambda\omega_a\Lambda^{-1}x')^\nu + (\Lambda\omega_a\Lambda^{-1}b + \Lambda a_a)^\nu,$$

$$\mathcal{Q}_{\varepsilon'_s}[z'_s] = \mathcal{Q}_\varepsilon[z_s].$$

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Spin 1 Duality generator as Chern-Simons term

what about global symmetries ?

spin1 reduced phase space description

$$S^R[A_i^{Ta}] = \int dt \Big[\int d^3x \, \frac{1}{2}\epsilon_{ab}(\mathcal{O}A^{Ta})^i \partial_0 A_i^{Tb} - H_1 \Big],$$

$$H_1 = \frac{1}{2}\int d^3x \, (\mathcal{O}A^{Ta})_i(\mathcal{O}A_a^T)^i = -\frac{1}{2}\int d^3x \, A^{Tai}\Delta A^T_{ai},$$

curl

$$(\mathcal{O}A)^i = \epsilon^{ijk}\partial_j A_k$$

$$\int d^3x \, g_i(\mathcal{O}f)^i = \int d^3x \, (\mathcal{O}g)_i f^i,$$

$$(\mathcal{O}^2 f^T)^i = -\Delta f^{Ti}$$

reduced phase space Poisson brackets

$$\{A_i^{Ta}(x), A^{Tbj}(y)\}_1 = \epsilon^{ab}\Delta^{-1}\epsilon^{jkl}\partial_k^y \delta_{il}^{T(3)}(x-y) = \epsilon^{ab}\Delta^{-1}\big(\mathcal{O}^y\delta_i^{T(3)}(x-y)\big)^j,$$

vacuum Maxwell's equations

$$\partial_0 A^{Tai}(x) = \{A^{Tai}(x), H_1\}_1 = -\epsilon^{ab}(\mathcal{O}A_b^T)^i(x),$$

duality generator

$$H_0 = -\frac{1}{2}\int d^3x \, A_T^{ai}(\mathcal{O}A_a^T)_i, \qquad \{H_0, H_1\}_1 = 0.$$

SO(2) Chern-Simons term

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Higher Spins Bi-Hamiltonian systems

new simplified Poisson bracket $\{A_i^{Ta}(x), A_j^{Tb}(y)\}_0 = \epsilon^{ab}\delta_{ij}^{T(3)}(x-y),$

duality generator as 2nd Hamiltonian $\{A^{Tai}(x), H_1\}_1 = \{A^{Tai}(x), H_0\}_0$

recursion operator $\mathcal{R}^{ai}{}_{bj} = -\delta_b^a(\mathcal{O})_j^i$

hierarchy $K_p^{Tai} = (-)^p\epsilon^{ab}(\mathcal{O}^p A_b^T)^i \qquad \partial_0 A^{Tai}(x) = K_p^{Tai}(x)$

$$\{A^{Tai}(x), H_p\}_1 = \{A^{Tai}(x), H_{p-1}\}_0$$

$$H_{p-1} = \frac{(-)^p}{2}\int d^3x\, A_i^{Ta}(\mathcal{O}^p A^T)_i$$

infinity of Hamiltonians in involution $\{H_n, H_m\}_1 = 0 = \{H_n, H_m\}_0, \quad \forall n, m \geqslant 0.$

generalizes directly to reduced phase space
description of massless HS fields

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Talk based on

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