Towards the exact spectrum of the $\mathrm{AdS}_5 \times S^5$ superstring. I

Gleb Arutyunov

Institute for Theoretical Physics, Utrecht University

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Outline

1. Infinite volume spectrum
2. Spectrum in a large but finite volume
3. Towards the exact spectrum
   - Mirror theory
   - Mirror Bethe-Yang equations
   - String hypothesis

$AdS_5 \times S^5$ superstring in the light-cone gauge

- String sigma model is on a cylinder of circumference $P_+ = J$, where J is an angular momentum of string around the equator of S^5
- When $J \to \infty$ the cylinder decompactifies into a plane. Integrability implies factorized scattering
- In the limit $J \to \infty$ the symmetry algebra of the light-cone model is

$$\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2) \in \mathfrak{psu}(2,2|4)$$

extended by two central charges depending on the world-sheet momentum P

- The world-sheet S-matrix factorises

$$\mathbb{S}(p_1,p_2) = S_0 \cdot S(p_1,p_2) \otimes S(p_1,p_2)$$

each 16×16-matrix S is $\mathfrak{psu}(2|2)_{c.e.}$-invariant Beisert '05

Frolov, Zamaklar and G.A. '06

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Dispersion relation and rapidity torus

- The dispersion relation implied by the symmetry algebra

$$H^2 = 1 + 4g^2 \sin^2 \frac{p}{2}$$

can be uniformized on a torus as

$$p = 2\,\mathrm{am}\, z\,, \qquad \sin\frac{p}{2} = \mathrm{sn}\,(z,k)\,, \qquad H = \mathrm{dn}\,(z,k)$$

where the elliptic modulus is $k = -4g^2$ and the torus the real and imaginary periods equal to $2\omega_1(k)$ and $2\omega_2(k)$.

Janik '06

- Constrained parameters $x^\pm$

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i}{g}\,, \qquad \frac{x^+}{x^-} = e^{ip}$$

On the z-torus $x^\pm$ are meromorphic

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S-matrix for fundamental particles

$$
\begin{aligned}
S(p_1, p_2) = {} & \frac{x_2^- - x_1^+}{x_2^+ - x_1^-} \frac{\eta_1 \eta_2}{\tilde{\eta}_1 \tilde{\eta}_2} \left(E_1^1 \otimes E_1^1 + E_2^2 \otimes E_2^2 + E_1^1 \otimes E_2^2 + E_2^2 \otimes E_1^1\right) \\
& + \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)} \frac{\eta_1 \eta_2}{\tilde{\eta}_1 \tilde{\eta}_2} \left(E_1^1 \otimes E_1^1 + E_2^2 \otimes E_2^2 + E_1^1 \otimes E_2^2 + E_2^2 \otimes E_1^1\right) \\
& - \left(E_1^1 \otimes E_1^1 + E_2^2 \otimes E_2^2 + E_1^1 \otimes E_2^2 + E_2^2 \otimes E_1^1\right) \\
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\end{aligned}
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$\eta_1 = \eta(p_1)\exp(\frac{i}{2}p_2)$, $\eta_2 = \eta(p_2)$, $\tilde{\eta}_1 = \eta(p_1)$, $\tilde{\eta}_2 = \eta(p_2)\exp(\frac{i}{2}p_1)$, $\eta(p) = \exp(\frac{i}{4}p)\sqrt{ix^- - ix^+}$

Spectrum on a large circle

- Bethe-Yang equations

Beisert,Staudacher '04

$$\text{“}e^{ip_k J}\prod_{k\neq i}\mathcal{S}(p_i,p_k)=1\text{”}$$

(+ additional equations with auxiliary roots encoding non-diagonal structure of $\mathcal{S}$)

- Given $\{p_i\}_{i=1}^{M}$, the energy (dimension) is given by

$$E=\sum_{i=1}^{M}\sqrt{1+4g^2\sin^2\frac{p_i}{2}}=E(g,J)$$

- This is NOT the correct answer for finite J!

Wrapping interactions (distinguished Feynman graphs), finite-size corrections to classical string energies, BFKL analysis, all points to this...

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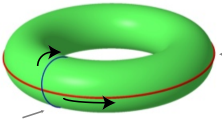
TBA and mirror theory

Follow the TBA approach for relativistic models (Zamolodchikov '90)

Frolov and G.A. '07



- One Euclidean theory – two Minkowski theories. One is related to the other by the double Wick rotation:

$$\tilde{\sigma} = -i\tau\,, \qquad \tilde{\tau} = i\sigma$$

The Hamiltonian $\tilde{\mathrm{H}}$ w.r.t. $\tilde{\tau}$ defines the *mirror theory*.

- Ground state energy ($R \to \infty$) is related to the free energy of its mirror

$$E(L) = Lf(L)$$

Free energy f can be found from the Bethe ansatz for the mirror model because $R \to \infty$

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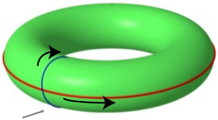

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Mirror dispersion relation

- The mirror momentum ($\mathrm{H} \to i\tilde{p}$) in terms of z: $\tilde{p} = -i\,\mathrm{dn}\,z$
- Shift z by $\omega_2/2$ gives

$$\tilde{p} = -i\,\mathrm{dn}\left(z + \frac{\omega_2}{2}\right) \equiv \sqrt{1+4g^2}\,\frac{\mathrm{sn}\,z}{\mathrm{cn}\,z}$$

  for real z the corresponding values of $\tilde{p}$ are real

- The double-Wick rotation is the shift by $2\omega_2/4$
- No periodicity in $\tilde{p}$ because $\mathrm{cn}\,z$ has zeroes at $z = \pm\frac{1}{2}\omega_1$
- The mirror energy is

$$\widetilde{\mathrm{H}} = 2\,\mathrm{arcsinh}\frac{1}{2g}\sqrt{1+\tilde{p}^2}\,.$$

Ambjorn,Janik and Kristjansen '05

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Mirror S-matrix and boundary conditions for fermions

- The S-matrix of the mirror model:

$$\widetilde{S}(\tilde{p}_1, \tilde{p}_2) = S(p_1(\tilde{p}_1), p_2(\tilde{p}_2))$$

or, equivalently, on the z-torus

$$\widetilde{S}(z_1, z_2) = S\left(z_1 + \frac{\omega_2}{2}, z_2 + \frac{\omega_2}{2}\right)$$

Mirror Bethe-Yang equations are straightforward

Periodicity of fermions

- *Fermions of the string model: periodic or anti-periodic in the spacial direction, anti-periodic in time*
- *Fermions of the mirror model: anti-periodic in the special direction, periodic or anti-periodic in time*

Ground state energy for periodic fermions is related to Witten's index of the mirror theory: $\mathrm{Tr}((-1)^{\mathrm{F}} \mathrm{e}^{-\beta \tilde{\mathrm{H}}})$

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Bethe-Yang equations for the mirror model

$$1 = e^{i\widetilde{p}_k R} \prod_{\substack{l=1\\ l\neq k}}^{K^{\rm I}} S^{11}_{\mathfrak{sl}(2)}(x_k, x_l) \prod_{\alpha=1}^{2} \prod_{l=1}^{K^{\rm II}_{(\alpha)}} \frac{x_k^- - y_l^{(\alpha)}}{x_k^+ - y_l^{(\alpha)}} \sqrt{\frac{x_k^+}{x_k^-}}$$

$$-1 = \prod_{l=1}^{K^{\rm I}} \frac{y_k^{(\alpha)} - x_l^-}{y_k^{(\alpha)} - x_l^+} \sqrt{\frac{x_l^+}{x_l^-}} \prod_{l=1}^{K^{\rm III}_{(\alpha)}} \frac{v_k^{(\alpha)} - w_l^{(\alpha)} - \frac{i}{g}}{v_k^{(\alpha)} - w_l^{(\alpha)} + \frac{i}{g}}$$

$$1 = \prod_{l=1}^{K^{\rm II}_{(\alpha)}} \frac{w_k^{(\alpha)} - v_l^{(\alpha)} + \frac{i}{g}}{w_k^{(\alpha)} - v_l^{(\alpha)} - \frac{i}{g}} \prod_{\substack{l=1\\ l\neq k}}^{K^{\rm III}_{(\alpha)}} \frac{w_k^{(\alpha)} - w_l^{(\alpha)} - \frac{2i}{g}}{w_k^{(\alpha)} - w_l^{(\alpha)} + \frac{2i}{g}}$$

where the S-matrix of the $\mathfrak{sl}(2)$-sector enters

$$S^{11}_{\mathfrak{sl}(2)}(x_1, x_2) = \frac{x_1^+ - x_2^-}{x_1^- - x_2^+} \frac{1 - \frac{1}{x_1^- x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}} \sigma_{12}^{-2}, \qquad v = y + \frac{1}{y}$$

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Bound states of the mirror model

The $\mathfrak{sl}(2)$ S-matrix

$$S^{11}_{\mathfrak{sl}(2)}(x_1, x_2) = \frac{x_1^+ - x_2^-}{x_1^- - x_2^+} \frac{1 - \frac{1}{x_1^- x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}} \sigma_{12}^{-2}$$

exhibits a pole for complex values of momenta

$$\tilde{p}_1 = \frac{p}{2} + iq\,, \qquad \tilde{p}_2 = \frac{p}{2} - iq\,, \qquad \mathrm{Re}\, q > 0$$

for which $x^-(\tilde{p}_1) - x^+(\tilde{p}_2) = 0 \quad \Longrightarrow \quad q = q(p)$

This pole leads to the existence of a Q-particle bound state

$$x_1^- = x_2^+\,, \quad x_2^- = x_3^+\,, \quad \ldots\,, \quad x_{Q-1}^- = x_Q^+$$

The mirror asymptotic spectrum contains fundamental particles and their bound states. Mirror bound states transform in the atypical anti-symmetric irreps of $\mathfrak{su}(2|2)_{c.e.}$

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Bethe-Yang for mirror particles and their bound states

The Bethe-Yang equations for bound states are obtained by fusing the equations for the constituent fundamental particles:

$$
\begin{aligned}
1 &= e^{i\widetilde{p}_k R} \prod_{\substack{l=1 \\ l\neq k}}^{K^{\mathrm{I}}} S_{\mathfrak{sl}(2)}^{Q_k Q_l}(x_k, x_l) \prod_{\alpha=1}^{2} \prod_{l=1}^{K^{\mathrm{II}}_{(\alpha)}} \frac{x_k^- - y_l^{(\alpha)}}{x_k^+ - y_l^{(\alpha)}} \sqrt{\frac{x_k^+}{x_k^-}} \\
-1 &= \prod_{l=1}^{K^{\mathrm{I}}} \frac{y_k^{(\alpha)} - x_l^-}{y_k^{(\alpha)} - x_l^+} \sqrt{\frac{x_l^+}{x_l^-}} \prod_{l=1}^{K^{\mathrm{III}}_{(\alpha)}} \frac{v_k^{(\alpha)} - w_l^{(\alpha)} - \frac{i}{g}}{v_k^{(\alpha)} - w_l^{(\alpha)} + \frac{i}{g}} \\
1 &= \prod_{l=1}^{K^{\mathrm{II}}_{(\alpha)}} \frac{w_k^{(\alpha)} - v_l^{(\alpha)} + \frac{i}{g}}{w_k^{(\alpha)} - v_l^{(\alpha)} - \frac{i}{g}} \prod_{\substack{l=1 \\ l\neq k}}^{K^{\mathrm{III}}_{(\alpha)}} \frac{w_k^{(\alpha)} - w_l^{(\alpha)} - \frac{2i}{g}}{w_k^{(\alpha)} - w_l^{(\alpha)} + \frac{2i}{g}} .
\end{aligned}
$$

$S_{\mathfrak{sl}(2)}^{Q_k Q_l}$ is obtained by fusing the fundamental constituents $S_{\mathfrak{sl}(2)}^{11}$

The main issue is to understand the structure of solutions to the BY equations in the thermodynamic limit:

$$R \to \infty, \quad K^I/R = \text{fixed}, \quad K^{II}_{(\alpha)}/R = \text{fixed}, \quad K^{III}_{(\alpha)}/R = \text{fixed}$$

This is done by formulating the corresponding

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Root structure

Consider a generic term in the first BY equation

$$1 = e^{i\widetilde{p}_k R} \dots \frac{x_k^- - y_l^{(\alpha)}}{x_k^+ - y_l^{(\alpha)}} \sqrt{\frac{x_k^+}{x_k^-}} \dots$$

For the physical mirror particles $x^{\pm *} = 1/x^{\mp}$, therefore,

$$1 = e^{-i\widetilde{p}_k R} \dots \frac{\frac{1}{x_k^+} - y_l^{(\alpha)*}}{\frac{1}{x_k^-} - y_l^{(\alpha)*}} \sqrt{\frac{x_k^+}{x_k^-}} \dots \implies 1 = e^{i\widetilde{p}_k R} \dots \frac{x_k^- - \frac{1}{y_l^{(\alpha)*}}}{x_k^+ - \frac{1}{y_l^{(\alpha)*}}} \sqrt{\frac{x_k^+}{x_k^-}} \dots$$

- A single y-root must be on the unit circle: $|y| = 1$ and, therefore, $-2 \leq v = y + 1/y \leq 2$
- y-roots which are not on the circle come in pairs $(y_1, y_2 = 1/y_1^*)$ and they lead to the vw-string configurations

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$$1 = e^{-i\widetilde{p}_k R} \ldots \frac{\frac{1}{x_k^+} - y_l^{(\alpha)*}}{\frac{1}{x_k^-} - y_l^{(\alpha)*}} \sqrt{\frac{x_k^+}{x_k^-}} \ldots \quad \Longrightarrow \quad 1 = e^{i\widetilde{p}_k R} \ldots \frac{x_k^- - \frac{1}{y_l^{(\alpha)*}}}{x_k^+ - \frac{1}{y_l^{(\alpha)*}}} \sqrt{\frac{x_k^+}{x_k^-}} \ldots$$

- A single y-root must be on the unit circle: $|y| = 1$ and, therefore, $-2 \leq v = y + 1/y \leq 2$
- y-roots which are not on the circle come in pairs $(y_1, y_2 = 1/y_1^*)$ and they lead to the vw-string configurations

String hypothesis

In the thermodynamic limit $R, K^{\mathrm{I}}, K^{\mathrm{II}}_{(\alpha)}, K^{\mathrm{III}}_{(\alpha)} \to \infty$ with K^{I}/R and so on fixed solutions arrange themselves into four different classes of Bethe strings

1. A single Q-particle with real momentum $\tilde{p}_k$
2. A single $y^{(\alpha)}$-particle corresponding to a root $y^{(\alpha)}$ with $|y^{(\alpha)}| = 1$
3. $2M$ roots $y^{(\alpha)}$ and M roots $w^{(\alpha)}$ combining into a $M|vw^{(\alpha)}$-string

$$v_j^{(\alpha)} = v^{(\alpha)} + (M+2-2j)\frac{i}{g}\,, \quad v_{-j}^{(\alpha)} = v^{(\alpha)} - (M+2-2j)\frac{i}{g}\,,$$

$$w_j^{(\alpha)} = v^{(\alpha)} + (M+1-2j)\frac{i}{g}\,, \quad j = 1, \ldots, M\,, \quad v \in \mathbf{R}\,.$$

4. N roots $w^{(\alpha)}$ combining into a single $N|w^{(\alpha)}$-string

$$w_j^{(\alpha)} = w^{(\alpha)} + \frac{i}{g}(N+1-2j)\,, \quad j = 1, \ldots, N\,, \quad w \in \mathbf{R}$$

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String hypothesis

Bethe strings of type 2,3, and 4 are similar to those in the Hubbard model. Indeed, the level II and III BY equations coincide with that of the inhomogenious Hubbard model.

Frolov and G.A. '09

cf. Beisert '06 for the string model

For $R \to \infty$ the relevant solutions are

1. N_Q Q-particles, $Q = 1, 2, \ldots, \infty$
2. $N_y^{(\alpha)}$ $y^{(\alpha)}$-particles
3. $N_{M|vw}^{(\alpha)}$ $M|vw^{(\alpha)}$-strings, $\alpha = 1, 2;\ M = 1, 2, \ldots, \infty$
4. $N_{N|w}^{(\alpha)}$ $N|w^{(\alpha)}$-strings, $\alpha = 1, 2;\ N = 1, 2, \ldots, \infty$

BY equations for the real centers of the string complexes as well as for $y^{(\alpha)}$ and Q-particles are obtained by multiplying the constituent BY equations. Taking thermodynamic limit leads to the TBA system for the particle/hole densities

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Introduce a function

$$x(u) = \frac{1}{2}\left(u - i\sqrt{4-u^2}\right) \,, \quad \mathrm{Im}(x(u)) < 0 \ \text{ for any } u \in \mathbb{C} \,,$$

the cuts in the u-plane run from $\pm\infty$ to ± 2 along the real lines.

Thermodynamic limit

Densities $\rho(u)$ of particles, and $\bar{\rho}(u)$ of holes; $u \in \mathbf{R}$, $\alpha = 1, 2$.

1. $\rho_Q(u)$ of Q-particles, $-\infty \leq u \leq \infty$, $Q = 1, \ldots, \infty$
2. $\rho^{(\alpha)}_{y^-}(u)$ of y-particles with $\text{Im}(y) < 0$, $-2 \leq u \leq 2$. The y-coordinate is expressed in terms of u as $y = x(u)$
3. $\rho^{(\alpha)}_{y^+}(u)$ of y-particles with $\text{Im}(y) > 0$, $-2 \leq u \leq 2$. The y-coordinate is expressed in terms of u as $y = \frac{1}{x(u)}$
4. $\rho^{(\alpha)}_{M|vw}(u)$ of $M|vw$-strings, $-\infty \leq u \leq \infty$, $M = 1, \ldots, \infty$
5. $\rho^{(\alpha)}_{N|w}(u)$ of $N|w$-strings, $-\infty \leq u \leq \infty$, $N = 1, \ldots, \infty$,

and the corresponding densities of holes.

Thermodynamic limit

- Integral eqs in the thermodynamic limit

$$\rho_i(u) + \bar{\rho}_i(u) = \frac{R}{2\pi} \frac{d\widetilde{p}_i}{du} + K_{ij} \star \rho_j(u)$$

where $\widetilde{p}_i$ does not vanish only for Q-particles.

- Star operation is defined as

$$K_{ij} \star \rho_j(u) = \int \mathrm{d}u' \, K_{ij}(u, u') \rho_j(u')$$

- Kernels $K's$ are expressed via the corresponding S-matrices as

$$K_{ij}(u, v) = \frac{1}{2\pi i} \frac{d}{du} \log S_{ij}(u, v)$$

- The right action which is defined as

$$\rho_j \star K_{ji}(u) = \int \mathrm{d}u' \, \rho_j(u') K_{ji}(u', u)$$

Free energy and equations for pseudo-energies

To describe both sectors, we consider generalized free energy

$$\mathcal{F}_\gamma(L) = \mathcal{E} - \frac{1}{L} S + \frac{i\gamma}{L}(N_F^{(1)} - N_F^{(2)}) \,,$$

- $\mathcal{E}$ is the energy per unit length carried by Q-particles

$$\mathcal{E} = \int \mathrm{d}u \sum_{Q=1}^{\infty} \widetilde{\mathcal{E}}^Q(u)\rho_Q(u) \,, \quad \widetilde{\mathcal{E}}^Q(u) \text{ is } Q\text{-particle energy}$$

- S is the total entropy
- $i\gamma/L$ plays the role of a chemical potential
- $N_F^{(\alpha)}$ is the fermion number which counts the number of $y^{(\alpha)}$-particles

$$N_F^{(1)} - N_F^{(2)} = \int \mathrm{d}u \, (\rho_{y-}^{(1)}(u) + \rho_{y+}^{(1)}(u) - \rho_{y-}^{(2)}(u) - \rho_{y+}^{(2)}(u))$$

- Minus sign between $N_F^{(1)}$ and $N_F^{(2)}$ is needed for the reality of $\mathcal{F}_\gamma(L)$
- $\gamma = \pi \implies$ Witten's index. $\gamma = 0 \implies$ the usual free energy.

Free energy and equations for pseudo-energies

Free energy: $\mathcal{F}_\gamma(L) = \int \mathrm{d}u \sum_k \left[\widetilde{\mathcal{E}}_k \, \rho_k - \frac{i\gamma_k}{L}\, \rho_k - \frac{1}{L}\mathfrak{s}(\rho_k)\right]$

Variations of the densities of particles and holes are subject to

$$\delta\rho_k(u) + \delta\bar{\rho}_k(u) = K_{kj} \star \delta\rho_j \,.$$

Using the extremum condition $\delta\mathcal{F}_\gamma(L) = 0$, one derives the TBA eqs

$$\epsilon_k = L\,\widetilde{\mathcal{E}}_k - \log\left(1 + e^{i\gamma_j - \epsilon_j}\right) \star K_{jk} \,,$$

where the pseudo-energies ϵ_k are $e^{i\gamma_k - \epsilon_k} = \frac{\rho_k}{\bar{\rho}_k}$,

At the extremum $\mathcal{F}_\gamma(L) = -\frac{R}{L} \int \mathrm{d}u \sum_k \frac{1}{2\pi} \frac{d\widetilde{p}_k}{du} \log\left(1 + e^{i\gamma_k - \epsilon_k}\right)$

The energy of the ground state of the l.c. string theory

$$E_\gamma(L) = \lim_{R\to\infty} \frac{L}{R}\mathcal{F}_\gamma(L) = -\int \mathrm{d}u \sum_{Q=1}^{\infty} \frac{1}{2\pi} \frac{d\widetilde{p}^Q}{du} \log\left(1 + e^{-\epsilon_Q}\right)$$

TBA equations for pseudo-energies of mirror particles

- Q-particles
$$\epsilon_Q = L\,\widetilde{\mathcal{E}}_Q - \log\left(1+e^{-\epsilon_{Q'}}\right)\star K^{Q'Q}_{\mathfrak{sl}(2)} - \log\left(1+e^{-\epsilon^{(\alpha)}_{M'|vw}}\right)\star K^{M'Q}_{vwx}$$
$$-\log\left(1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y-}}\right)\star K^{yQ}_{-} - \log\left(1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y+}}\right)\star K^{yQ}_{+}$$

- y-particles
$$\epsilon^{(\alpha)}_{y\pm} = -\log\left(1+e^{-\epsilon_Q}\right)\star K^{Qy}_{\pm} + \log\frac{1+e^{-\epsilon^{(\alpha)}_{M|vw}}}{1+e^{-\epsilon^{(\alpha)}_{M|w}}}\star K_M$$

- $M|vw$-strings
$$\epsilon^{(\alpha)}_{M|vw} = -\log\left(1+e^{-\epsilon_{Q'}}\right)\star K^{Q'M}_{xv}$$
$$+\log\left(1+e^{-\epsilon^{(\alpha)}_{M'|vw}}\right)\star K_{M'M} - \log\frac{1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y+}}}{1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y-}}}\star K_M$$

- $M|w$-strings
$$\epsilon^{(\alpha)}_{M|w} = \log\left(1+e^{-\epsilon^{(\alpha)}_{M'|w}}\right)\star K_{M'M} - \log\frac{1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y+}}}{1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y-}}}\star K_M$$

- The ground state energy
$$E(L) = -\int \mathrm{d}u \sum_{Q=1}^{\infty} \frac{1}{2\pi}\frac{d\widetilde{p}^Q}{du}\log\left(1+e^{-\epsilon_Q}\right)$$

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See also, Bombardelli, Fioravanti, Tateo '09; Gromov, Kazakov,Kozak, Vieira '09