Holography, Unfolding and Higher-Spin Theories

M.A.Vasiliev

Lebedev Institute, Moscow

Ginzburg conference on physics

Moscow, June 02, 2012

HS theory

Higher derivatives in interactions

A.Bengtsson, I.Bengtsson, Brink (1983), Berends, Burgers, van Dam (1984)

$$S = S^2 + S^3 + \dots , \qquad S^3 = \sum_{p,q,r} (D^p\varphi)(D^q\varphi)(D^r\varphi)\rho^{p+q+r+\frac{1}{2}d-3}$$

HS Gauge Theories $(m = 0)$: Fradkin, M.V. (1987)

$$AdS_d : \qquad [D_n, D_m] \sim \rho^{-2} = \lambda^2$$

AdS/CFT:

$(3d, m = 0) \otimes (3d, m = 0) = \sum_{s=0}^{\infty} (4d, m = 0)$ Flato, Fronsdal (1978);

Sundborg (2001), Sezgin, Sundell (2002,2003), Klebanov, Polyakov (2002),

Giombi, Yin (2009)...

Results

CFT_3 **dual of** AdS_4 **HS theory: 3d conformal HS theory**

Holography: Unfolding

Plan

I **Unfolded dynamics and holographic duality**

II **Free massless HS fields in** AdS_4

III **Conserved currents and massless equations**

IV AdS_4 **HS theory as** $3d$ **conformal HS theory**

V **Conclusion**

Unfolded dynamics

First-order form of differential equations

$$\dot{q}^i(t) = \varphi^i(q(t)) \qquad \textbf{initial values:} \quad q^i(t_0)$$

Unfolded dynamics: multidimensional covariant generalization

$$\frac{\partial}{\partial t} \to d\,, \qquad q^i(t) \to W^\Omega(x) = dx^{n_1} \wedge \ldots \wedge dx^{n_p}$$

$$\mathbf{dW^\Omega(x) = G^\Omega(W(x))\,, \qquad d = dx^n \partial_n}$$

$G^\Omega(W)$: **function of "supercoordinates"** W^Φ

$$G^\Omega(W) = \sum_{n=1}^{\infty} f^\Omega{}_{\Phi_1 \ldots \Phi_n} W^{\Phi_1} \wedge \ldots \wedge W^{\Phi_n}$$

$d > 1$: **Nontrivial compatibility conditions**

$$G^\Phi(W) \wedge \frac{\partial G^\Omega(W)}{\partial W^\Phi} \equiv 0$$

Any solution: FDA Sullivan (1968); D'Auria and Fre (1982)

The unfolded equation is invariant under the gauge transformation

$$\delta W^\Omega(x) = d\varepsilon^\Omega(x) + \varepsilon^\Phi(x) \frac{\partial G^\Omega(W(x))}{\partial W^\Phi(x)}\,,$$

Vacuum geometry

a Lie algebra. $\omega = \omega^\alpha T_\alpha$: h valued 1-form.

$$G(\omega) = -\omega \wedge \omega \equiv -\frac{1}{2}\omega^\alpha \wedge \omega^\beta [T_\alpha, T_\beta]$$

the unfolded equation with $W = \omega$ has the zero-curvature form

$$d\omega + \omega \wedge \omega = 0\,.$$

Compatibility condition: Jacobi identity for h.

FDA: usual gauge transformation of the connection ω.

Zero-curvature equations: background geometry in a coordinate independent way.

If h is Poincare or anti-de Sitter algebra it describes Minkowski or AdS_d space-time

Properties

- **General applicability**
- **Manifest (HS) gauge invariance**
- **Invariance under diffeomorphisms**

  Exterior algebra formalism
- **Interactions: nonlinear deformation of** $G^{\Omega}(W)$
- **Local degrees of freedom are in 0-forms** $C^{i}(x_0)$ **at any** $x = x_0$ **(as** $q(t_0)$**)**

  infinite dimensional module dual to the space of single-particle states
- **Independence of ambient space-time**

  Geometry is encoded by $G^{\Omega}(W)$

Unfolding and holographic duality

Unfolded formulation unifies various dual versions of the same system.

Duality in the same space-time:

ambiguity in what is chosen to be dynamical or auxiliary fields.

Holographic duality between theories in different dimensions:

universal unfolded system admits different space-time interpretations.

Extension of space-time without changing dynamics by letting the differential d and differential forms W to live in a larger space

$$d = dX^n \frac{\partial}{\partial X^n} \to \tilde{d} = dX^n \frac{\partial}{\partial X^n} + d\widehat{X}^{\widehat{n}} \frac{\partial}{\partial \widehat{X}^{\widehat{n}}}\,, \qquad dX^n W_n \to dX^n W_n + d\widehat{X}^{\widehat{n}} \widehat{W}_{\widehat{n}}\,,$$

$\hat{X}^{\widehat{n}}$ are additional coordinates

$$\tilde{d} W^{\Omega}(X, \hat{X}) = G^{\Omega}(W(X, \hat{X}))$$

Particular space-time interpretation of a universal unfolded system, e.g, whether a system is on-shell or off-shell, depends not only on $G^{\Omega}(W)$ but, in the first place, on space-time M^d and chosen vacuum solution $W_0(X)$.

Two unfolded systems in different space-times are equivalent (dual) if they have the same unfolded form.

Direct way to establish holographic duality between two theories: unfold both to see whether their unfolded formulations coincide.

Given unfolded system generates a class of holographically dual theories in different dimensions.

Free massless fields in AdS_4

Infinite set of spins $s = 0, 1/2, 1, 3/2, 2 \ldots$

Fermions require doubling of fields

$$\omega^{ii}(y,\bar{y}\mid x)\,,\qquad C^{i1-i}(y,\bar{y}\mid x)\,,\qquad i=0,1\,,$$

$$\bar{\omega}^{ii}(y,\bar{y}\mid x)=\omega^{ii}(\bar{y},y\mid x)\,,\qquad \bar{C}^{i\,1-i}(y,\bar{y}\mid x)=C^{1-i\,i}(\bar{y},y\mid x)\,.$$

$$A(y,\bar{y}\mid x)=i\sum_{n,m=0}^{\infty}\frac{1}{n!m!}y_{\alpha_1}\ldots y_{\alpha_n}\bar{y}_{\dot\beta_1}\ldots\bar{y}_{\dot\beta_m}A^{\alpha_1\ldots\alpha_n}{}_{,}{}^{\dot\beta_1\ldots\dot\beta_m}(x)$$

The unfolded system for free massless fields is

- $\star$ $$R_1^{ii}(y,\bar{y}\mid x)=\eta\,\overline{H}^{\dot\alpha\dot\beta}\frac{\partial^2}{\partial\bar{y}^{\dot\alpha}\partial\bar{y}^{\dot\beta}}\;C^{1-i\,i}(0,\bar{y}\mid x)+\bar\eta\,H^{\alpha\beta}\frac{\partial^2}{\partial y^{\alpha}\partial y^{\beta}}\;C^{i\,1-i}(y,0\mid x)$$
- $\star$ $$\tilde{D}_0C^{i\,1-i}(y,\bar{y}\mid x)=0$$

$$R_1(y,\bar{y}\mid x)=D_0^{ad}\omega(y,\bar{y}\mid x)\qquad H^{\alpha\beta}=e^{\alpha}{}_{\dot\alpha}\wedge e^{\beta\dot\alpha}\,,\quad \overline{H}^{\dot\alpha\dot\beta}=e_{\alpha}{}^{\dot\alpha}\wedge e^{\alpha\dot\beta}\,,$$

$$D_0^{ad}\omega=D^L-\lambda e^{\alpha\dot\beta}\left(y_\alpha\frac{\partial}{\partial\bar{y}^{\dot\beta}}+\frac{\partial}{\partial y^\alpha}\bar{y}_{\dot\beta}\right)\,,\qquad \tilde{D}_0=D^L+\lambda e^{\alpha\dot\beta}\left(y_\alpha\bar{y}_{\dot\beta}+\frac{\partial^2}{\partial y^\alpha\partial\bar{y}^{\dot\beta}}\right)\,,$$

$$D^L=d_x-\left(\omega^{\alpha\beta}y_\alpha\frac{\partial}{\partial y^\beta}+\bar\omega^{\dot\alpha\dot\beta}\bar{y}_{\dot\alpha}\frac{\partial}{\partial\bar{y}^{\dot\beta}}\right)\,.$$

Non-Abelian HS algebra

Star product

$$(f * g)(Y) = \int dSdT f(Y+S)g(Y+T)\,\mathrm{exp}\, -iS_A T^A$$

$$[Y_A, Y_B]_* = 2iC_{AB}\,, \qquad C_{\alpha\beta} = \epsilon_{\alpha\beta}\,, \qquad C_{\dot\alpha\dot\beta} = \epsilon_{\dot\alpha\dot\beta}$$

Non-Abelian HS curvature

$$R_1(y,\bar y|x) \to R(y,\bar y|x) = d\omega(y,\bar y|x) + \omega(y,\bar y|x) * \omega(y,\bar y|x)$$

$$\tilde D_0 C(y,\bar y|x) \to \tilde D C(y,\bar y|x) = dC(y,\bar y|x) + \omega(y,\bar y|x) * C(y,\bar y|x) - C(y,\bar y|x) * \omega(y,-\bar y|x)$$

$3d$ conformal equations

Conformal invariant massless equations in $d=3$

$$dx^{\alpha\beta}\Big(\frac{\partial}{\partial x^{\alpha\beta}} \pm \frac{\partial^2}{\partial y^\alpha \partial y^\beta}\Big)C(y|x) = 0\,, \qquad \alpha,\beta = 1,2$$

Shaynkman, MV (2001)

Rank r unfolded equations: tensoring of Fock modules Gelfond, MV (2003)

$$dx^{\alpha\beta}\Big(\frac{\partial}{\partial x^{\alpha\beta}} + \eta_{ij}\frac{\partial^2}{\partial y_i^\alpha \partial y_j^\beta}\Big)C(y|x) = 0\,, \qquad i,j = 1,\ldots r\,.$$

For diagonal η^{ij} higher-rank equations are satisfied by

$$C(y_i|x) = C_1(y_1|x)C_2(y_2|x)\ldots C_r(y_r|x)\,.$$

Rank-two equations: conserved currents

$$\left\{\frac{\partial}{\partial x^{\alpha\beta}} - \frac{\partial^2}{\partial y^{(\alpha}\partial u^{\beta)}}\right\}T(u,\,y|x) = 0$$

$T(u,\,y|x)$: **generalized stress tensor. Rank-two equation is obeyed by**

$$T(u,\,y\,|x) = \sum_{i=1}^{N} C_{+\,i}(y-u|x)\,C_{-\,i}(u+y|x)$$

Rank-two fields: bilocal fields in the twistor space.

Dynamical currents (primaries)

$$J(u|x) = T(u,0\,|x)\,, \qquad \tilde{J}(y|x) = T(0,y|x)$$

Gelfond, MV (2003)

$$J^{asym}(u,y|x) = u_\alpha y^\alpha \Big(\frac{\partial^2}{\partial u^\beta \partial y_\beta} T(u,y|x)\Big|_{u=y=0}\Big)$$

$J(u|x)$ **generates** $3d$ **currents of all integer and half-integer spins**

$$J(u|x) = \sum_{2s=0}^{\infty} u^{\alpha_1}\dots u^{\alpha_{2s}} J_{\alpha_1\dots\alpha_{2s}}(x)\,, \qquad \tilde{J}(u|x) = \sum_{2s=0}^{\infty} u^{\alpha_1}\dots u^{\alpha_{2s}} \tilde{J}_{\alpha_1\dots\alpha_{2s}}(x)\,.$$

$$J^{asym}(u,y|x) = u_\alpha y^\alpha J^{asym}(x)$$

$$\Delta J_{\alpha_1\dots\alpha_{2s}}(x) = \Delta \tilde{J}_{\alpha_1\dots\alpha_{2s}}(x) = s+1 \qquad \Delta J^{asym}(x) = 2$$

Differential equations: conventional conservation condition

$$\frac{\partial}{\partial x^{\alpha\beta}} \frac{\partial^2}{\partial u_\alpha \partial u_\beta} J(u|x) = 0\,, \qquad \frac{\partial}{\partial x^{\alpha\beta}} \frac{\partial^2}{\partial y_\alpha \partial y_\beta} \tilde{J}(y|x) = 0$$

$3d$ conformal setup in AdS_4 HS theory

For manifest conformal invariance introduce

$$y^+_\alpha = \frac{1}{2}(y_\alpha - i\bar{y}_\alpha)\,, \qquad y^-_\alpha = \frac{1}{2}(\bar{y}_\alpha - iy_\alpha)\,, \qquad [y^-_\alpha\,, y^{+\beta}]_* = \delta^\beta_\alpha$$

$3d$ **conformal realization of the algebra** $sp(4;\mathbb{R}) \sim o(3,2)$

$$L^\alpha{}_\beta = y^{+\alpha} y^-_\beta - \frac{1}{2}\delta^\alpha_\beta y^{+\gamma} y^-_\gamma\,, \qquad D = \frac{1}{2} y^{+\alpha} y^-_\alpha$$

$$P_{\alpha\beta} = i y^-_\alpha y^-_\beta\,, \qquad K^{\alpha\beta} = -i y^{+\alpha} y^{+\beta}$$

Conformal weight of HS gauge fields

$$[D, \omega(y^\pm|X)] = \frac{1}{2}\left(y^{+\alpha}\frac{\partial}{\partial y^{+\alpha}} - y^-_\alpha \frac{\partial}{\partial y^-_\alpha}\right)\omega(y^\pm|X)\,.$$

Pullback $\hat{\omega}(y^\pm|x)$ **of** $\omega(y^\pm|x)$ **to** Σ: $3d$ **conformal HS gauge fields**

Conformal frame

D **in the twisted adjoint representation is realized by the second-order operator**

$$\{D\,,C\}_* = \left(y^{+\alpha} y^-_\alpha - \frac{1}{4} \frac{\partial^2}{\partial y^{+\alpha} \partial y^-_\alpha} \right) C$$

Fields C inherited from AdS_4 theory are not manifestly conformal.

Conformal frame: Wick star product

$$(f_N \star g_N)(y^\pm) = \int \mu(u^\pm) \exp(-u^-_\alpha u^{+\alpha}) f_N(y^+, y^- + u^-) g_N(y^+ + u^+, y^-)$$

$$f_N(y^\pm) = \exp -\frac{1}{2} \epsilon^{\alpha\beta} \frac{\partial^2}{\partial y^{-\alpha} \partial y^{+\beta}} f(y^\pm)$$

$$\{D_N\,, \ldots\}_\star = \frac{1}{2} \left(y^{+\alpha} \frac{\partial}{\partial y^{+\alpha}} + y^{-\alpha} \frac{\partial}{\partial y^{-\alpha}} \right) + y^-_\alpha y^{+\alpha} + 1$$

$$T(y^\pm|x) = \exp -y^-_\alpha y^{+\alpha} C_N(y^\pm|x)$$

$\star$ $$D_N(T(y^\pm)) = \frac{1}{2} \left(y^{+\alpha} \frac{\partial}{\partial y^{+\alpha}} + y^{-\alpha} \frac{\partial}{\partial y^{-\alpha}} + 2 \right) T(y^\pm)$$

Holography at infinity

AdS_4 **foliation:** $x^n = (\mathbf{x}^a, z)$: $\mathbf{x}^a$ **are coordinates of leafs** $(a = 0, 1, 2,)$ z **is a foliation parameter**

Poincaré coordinates

$$W = \frac{i}{z} d\mathbf{x}^{\alpha\beta} y^-_\alpha y^-_\beta - \frac{dz}{2z} y^-_\alpha y^{+\alpha}$$

$$e^{\alpha\dot{\alpha}} = \frac{1}{2z} dx^{\alpha\dot{\alpha}}\,, \qquad \omega^{\alpha\beta} = -\frac{i}{4z} d\mathbf{x}^{\alpha\beta}\,, \qquad \bar{\omega}^{\dot{\alpha}\dot{\beta}} = \frac{i}{4z} d\mathbf{x}^{\dot{\alpha}\dot{\beta}}$$

$$\left[d_{\mathbf{x}} + \frac{i}{z} d\mathbf{x}^{\alpha\beta} \left(y_\alpha \frac{\partial}{\partial y^\beta} - \bar{y}_\alpha \frac{\partial}{\partial \bar{y}^\beta} + y_\alpha \bar{y}_\beta - \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^\beta} \right) \right] C(y, \bar{y} | \mathbf{x}, z) = 0$$

Rescaling y^α and $\bar{y}^{\dot{\alpha}}$ **via**

$$C(y, \bar{y} | \mathbf{x}, z) = z \exp(y_\alpha \bar{y}^\alpha) T(w, \bar{w} | \mathbf{x}, z)\,,$$

$$w^\alpha = z^{1/2} y^\alpha\,, \qquad \bar{w}^\alpha = z^{1/2} \bar{y}^\alpha$$

$T(w, \bar{w} | \mathbf{x}, z)$ **satisfies the** $3d$ **conformal invariant current equation**

$$\left[d_{\mathbf{x}} - i d\mathbf{x}^{\alpha\beta} \frac{\partial^2}{\partial w^\alpha \partial \bar{w}^\beta} \right] T(w, \bar{w} | \mathbf{x}, z) = 0$$

Connections

Setting

$$W^{jj}(y^{\pm}|\mathbf{x},z) = \Omega^{jj}(v^-,w^+|\mathbf{x},z)$$

$$\mathbf{v}^{\pm} = \mathbf{z}^{-1/2}\mathbf{y}^{\pm}\,, \qquad \mathbf{w}^{\pm} = \mathbf{z}^{1/2}\mathbf{y}^{\pm}$$

manifest z–dependence disappears

$$D_{\mathbf{x}}\Omega^{jj}(v^-,w^+|\mathbf{x},z) = \left(d_{\mathbf{x}} + 2id\mathbf{x}^{\alpha\beta}v^-_\alpha\frac{\partial}{\partial w^{+\beta}}\right)\Omega^{jj}(v^-,w^+|\mathbf{x},z)$$

Using

$$w_\alpha = w^+_\alpha + izv^-_\alpha\,, \qquad \bar{w}_\alpha = iw^+_\alpha + zv^-_\alpha$$

in the limit $z \to 0$ free HS equations take the form

$$\star \quad D_{\mathbf{x}}\Omega^{jj}_{\mathbf{x}}(v^-,w^+|\mathbf{x},0) = d\mathbf{x}_\alpha{}^\gamma d\mathbf{x}_{\beta\gamma}\frac{\partial^2}{\partial w^{+\alpha}\partial w^{+\beta}}\mathcal{T}^{jj}(w^+,0\mid\mathbf{x},0)\,,$$

$$\star \quad \left[d_{\mathbf{x}} - id\mathbf{x}^{\alpha\beta}\frac{\partial^2}{\partial w^{+\alpha}\partial w^{-\beta}}\right]T^{j\,1-j}(w^+,w^-|\mathbf{x},0) = 0\,.$$

$$\mathcal{T}^{jj}(w^+,w^-|\mathbf{x},0) = \eta T^{j\,1-j}(w^+,w^-\mid\mathbf{x},0) - \bar{\eta}T^{1-j\,j}(-iw^-,iw^+\mid\mathbf{x},0)$$

Towards nonlinear $3d$ conformal HS theory

Conformal HS theory is nonlinear since conformal HS curvatures inherited from the AdS_4 HS theory are non-Abelian Fradkin, Linetsky (1990)

$$R_{\mathbf{xx}}(v^-, w^+ \mid \mathbf{x}) = d_{\mathbf{x}}\Omega_{\mathbf{x}}(v^-, w^+ \mid \mathbf{x}) + \Omega_{\mathbf{x}}(v^-, w^+ \mid \mathbf{x}) \star \Omega_{\mathbf{x}}(v^-, w^+ \mid \mathbf{x})$$

It is important

$$[v^-_\alpha, w^{+\beta}]_\star = \delta^\beta_\alpha$$

The equation on 0-forms deforms to nonlinear twisted adjoint representation

$$dT(w^\pm|x) + \Omega(\frac{\partial}{\partial w^{+\beta}}, w^+_\alpha) \circ T(w^\pm|x) - T(w^\pm|x) \circ \Omega(-i\eta\frac{\partial}{\partial w^{-\alpha}}, -i\eta w^-|x) = O(T^2)\,.$$

Matter fields can be added via the Fock module

$$(d + \Omega_0(v^-, w^+|\mathbf{x})) \star C^i(w^+|\mathbf{x}) \star F = 0$$

Doubling of AdS

$z = 0$ is smooth point in rescaled variables

Continuation $z \to -z$: AdS doubling

Parity automorphism

$$P(z) = -z$$

P-even solution: Neumann boundary condition

P-odd solution: Dirichlet boundary condition

Reduction to free CFT_3

The unfolded equation

$$D_{\mathbf{x}}\Omega^{jj}_{\mathbf{x}}(v^-,w^+|\mathbf{x},0) = \mathcal{H}^{\alpha\beta}\frac{\partial^2}{\partial w^{+\alpha}\partial w^{+\beta}}\mathcal{T}^{jj}(w^+,0\mid \mathbf{x},0)$$

remains free if

$$\mathcal{T}^{jj}=0 \qquad \longrightarrow J^{asym}=0 \qquad \text{or} \qquad J^{sym}=0$$

depending on whether A-model or B-model is considered. For these cases the model remains free in accordance with the Klebanov-Polyakov Sezgin-Sundell conjecture.

Free models are equivalent to the reductions of the HS theory with respect to P-involution $y \leftrightarrow \bar{y}$ which is possible for the A and B models.

For HS theory with general phase η parameter such reduction is not possible: no realization as a free conformal theory.

Non-Abelian contribution of superconformal HS connections has to be taken into account.

Conclusions

Holographic duality relates theories that have equivalent unfolded for-mulation: equivalent twistor space description.

Beyond $1/N$

AdS_4 **HS theory is dual to nonlinear** $3d$ **conformal HS theory of** $3d$ **currents**

Both of holographically dual theories are HS theories of gravity

Holography at any surface is nonlocal

Free boundary theories are dual to truncations of HS theories under P **reflection automorphism of** z **in the doubled** AdS_4

AdS doubling

To do

Nonlinear $3d$ conformal HS theory

Actions

Correlators

AdS_3/CFT_2 **and Gaberdiel-Gopakumar conjecture**