Ginzburg conference

Infrared divergences in cosmological perturbation theory

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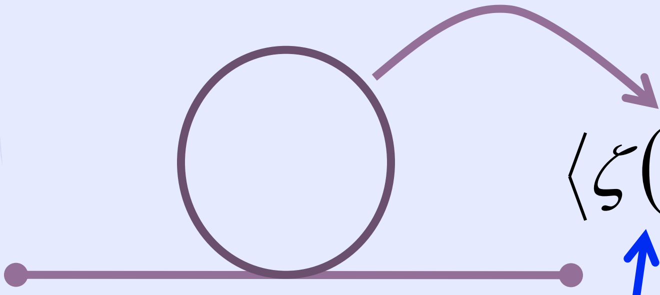
arXiv:1206.XXXX
PTP125:1067 arXiv:1009.2947,
Phys.Rev.D82:121301 arXiv:1007.0468
PTP122: 779 arXiv:0902.3209
PTP122:1207 arXiv:0904.4415

§ IR divergence in single field inflation

Setup: 4D Einstein gravity + minimally coupled scalar field



Factor coming from this loop:

$$\langle \zeta(y)\zeta(y)\rangle \approx \int d^3k\ \underline{P(k)} \approx \log\left(aH / k_{\min}\right)$$

$$\boxed{\propto 1/k^3}$$

curvature perturbation in co-moving gauge.

scale invariant spectrum
- no typical mass scale

$$\begin{cases} \gamma_{ij} = e^{2\rho+2\zeta} \exp(h)_{ij} \\ \delta\phi = 0 \end{cases}$$

Transverse traceless

Special property of single field inflation

Yuko Urakawa and T.T., PTP122: 779 arXiv:0902.3209

- In conventional cosmological perturbation theory, gauge is not completely fixed.

Time slicing can be uniquely specified: $\delta\phi = 0$ OK!

but spatial coordinates are not.

$$h_j^{\ j} = 0 = h_{i\ ,j}^{\ j}$$

Residual gauge d.o.f.

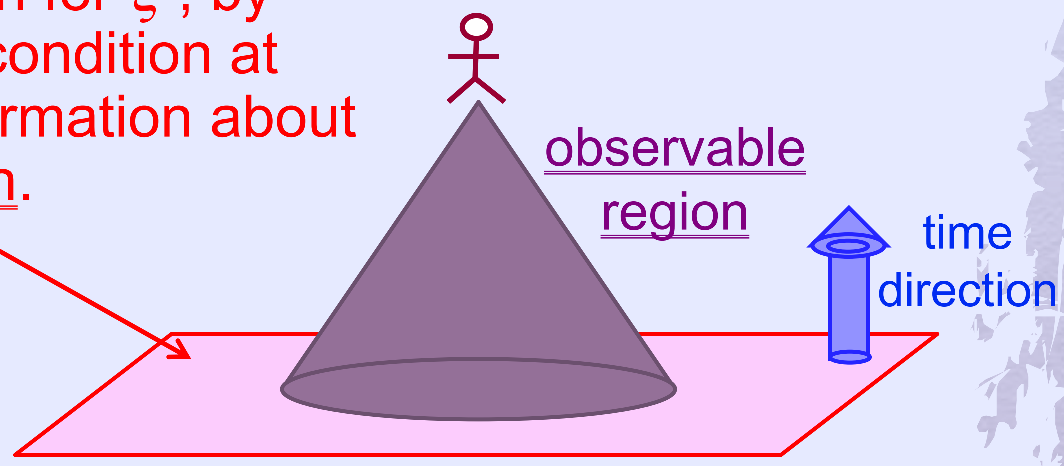
$$\delta_g h_{ij} = \xi_{i,j} + \xi_{j,i}$$

Elliptic-type differential equation for ξ^i.

$$\Delta\xi^i = \cdots$$

Not unique locally!

- To solve the equation for ξ^i, by imposing boundary condition at infinity, we need information about un-observable region.



Basic idea of the proof of IR finiteness in single field inflation

- The local spatial average of ζ can be set to 0 identically by an appropriate (but non-standard) gauge choice.

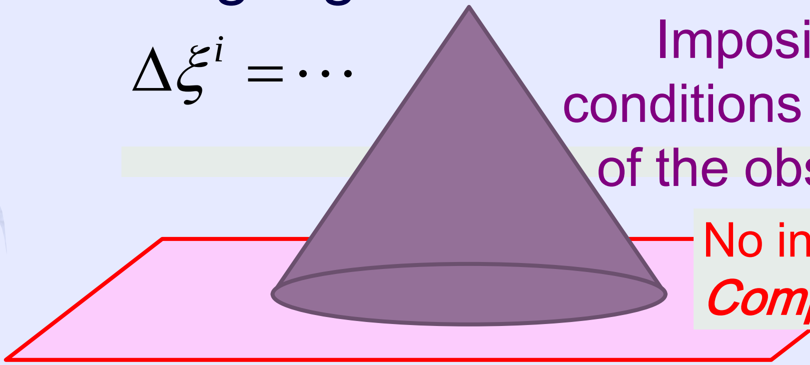
- Even if we choose such a local gauge, the evolution equation for ζ formally does not change, and it is hyperbolic. So, the interaction vertices are localized inside the past light cone.

- Therefore, IR divergence does not appear as long as we compute ζ in this local gauge. But here we assumed that the initial quantum state is free from IR divergence.

Complete gauge fixing vs. Genuine gauge-invariant quantities

- Local gauge conditions.

$\Delta \xi^i = \cdots$



Imposing boundary conditions on the boundary of the observable region

No influence from outside

Complete gauge fixing ☺

But unsatisfactory?

The results depend on the choice of boundary conditions.

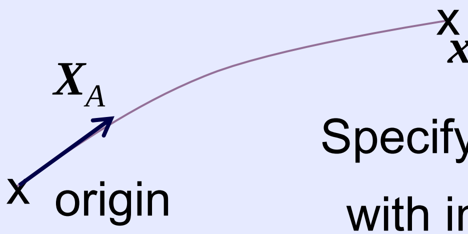
Translation invariance Is lost.

- ***Genuine*** gauge-invariant quantities.

Correlation functions for 3-d scalar curvature on ϕ=constant slice.

$\langle R(\boldsymbol{x}_1)\, R(\boldsymbol{x}_2)\rangle$ Coordinates do not have gauge invariant meaning.

(Giddings & Sloth 1005.1056)
(Byrnes et al. 1005.33307)



$\boldsymbol{x}(\boldsymbol{X}_A, \lambda=1) = \boldsymbol{X}_A + \delta \boldsymbol{x}_A$

Specify the position by solving geodesic eq. $D^2 x^i / d\lambda^2 = 0$

with initial condition $Dx^i / d\lambda \big|_{\lambda=0} = X^i$

${}^gR(\boldsymbol{X}_A) := R(\boldsymbol{x}(\boldsymbol{X}_A, \lambda=1)) = R(\boldsymbol{X}_A) + \delta\boldsymbol{x}_A \nabla R(\boldsymbol{X}_A) + \ldots$

$\langle {}^gR(\boldsymbol{X}_1)\, {}^gR(\boldsymbol{X}_2)\rangle$ should be genuine gauge invariant.

Translation invariance of the vacuum state takes care of the ambiguity in the choice of the origin.

Extra requirement for IR regularity

In $\delta\phi$ =0 gauge, EOM is very simple

$$\left[\partial_t^2+\left(3+\varepsilon_2\right)\dot{\rho}\,\partial_t-e^{-2(\rho+\zeta)}\Delta\right]\zeta\approx 0$$

Only relevant terms in the IR limit were kept.

Non-linearity is concentrated on this term.

Formal solution in IR limit can be obtained as

$$\zeta=\zeta_I-2\zeta_I\mathcal{L}^{-1}e^{-2\rho}\Delta\zeta_I+\cdots$$

$$\varepsilon_2=-\frac{d^2}{d\rho^2}\log H$$

with $\mathcal{L}^{-1}$ being the formal inverse of $\mathcal{L}=\partial_t^2+\left(3+\varepsilon_2\right)\dot{\rho}\,\partial_t-e^{-2\rho}\Delta$

$$^{g}R\approx-4e^{-2\rho}\Delta\left[\zeta_I-\zeta_I\left(2\mathcal{L}^{-1}e^{-2\rho}\Delta+\boldsymbol{x}\cdot\partial_{\boldsymbol{x}}\right)\zeta_I+\cdots\right]$$

$$\left\langle {}^{g}R(\boldsymbol{x}_1)\,{}^{g}R(\boldsymbol{x}_2)\right\rangle\ni\underbrace{\left\langle\zeta_I^2\right\rangle}_{\text{IR divergent factor}}\left\langle\Delta\left(2\mathcal{L}^{-1}e^{-2\rho}\Delta+\boldsymbol{x}\cdot\partial_{\boldsymbol{x}}\right)\zeta_I(\boldsymbol{x}_1)\times\Delta\left(2\mathcal{L}^{-1}e^{-2\rho}\Delta+\boldsymbol{x}\cdot\partial_{\boldsymbol{x}}\right)\zeta_I(\boldsymbol{x}_1)\right\rangle$$

IR regularity may require $\left[2\mathcal{L}^{-1}e^{-2\rho}\Delta+\left(\boldsymbol{x}\cdot\nabla\right)\right]\zeta_I=0$

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However, $\mathcal{L}^{-1}$ should be defined for each Fourier component.

$$\mathcal{L}^{-1}f(t,\boldsymbol{x})=\int d^3k\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\mathcal{L}_k^{-1}\tilde{f}_k(t)$$ for arbitrary function $f(t,\boldsymbol{x})$

with $\mathcal{L}_k=\partial_t^2+(3+\varepsilon_2)\dot{\rho}\partial_t+e^{-2\rho}k^2$

Then, $2\mathcal{L}^{-1}e^{-2\rho}\Delta\zeta_I+(\boldsymbol{x}\cdot\nabla)\zeta_I=0$ is impossible,

Because for $\zeta_I\equiv\int d^3k\,(e^{ikx}v_{\boldsymbol{k}}(t)\,a_{\boldsymbol{k}}+h.c.)$,

$\mathcal{L}^{-1}e^{-2\rho}\Delta\zeta_I\propto e^{i\boldsymbol{k}\cdot\boldsymbol{x}}a_{\boldsymbol{k}}$ while $(\boldsymbol{x}\cdot\nabla)\zeta_I\propto i\boldsymbol{k}\cdot\boldsymbol{x}\,e^{i\boldsymbol{k}\cdot\boldsymbol{x}}a_{\boldsymbol{k}}$

Instead, one can impose

$$\left[2\mathcal{L}^{-1}e^{-2\rho}\Delta+(\boldsymbol{x}\cdot\nabla)\right]\zeta_I=\int d^3k\left(a_{\boldsymbol{k}}D_k e^{i\boldsymbol{kx}}v_{\boldsymbol{k}}(t)+h.c.\right)$$

with $D_k\equiv k^{-3/2}e^{-i\phi(\boldsymbol{k})}\dfrac{d}{d\log k}k^{3/2}e^{i\phi(\boldsymbol{k})}$,

which reduces to conditions on the mode functions.

$$-2k^2\mathcal{L}_k^{-1}e^{-2\rho}v_k=D_k v_k$$

- extension to the higher order:

$$\left[\left(2\mathcal{L}^{-1}e^{-2\rho}\Delta\right)^2+\frac{1}{2}(2+\boldsymbol{x}\cdot\nabla)\boldsymbol{x}\cdot\nabla\right]\zeta_I=\int d^3k\left(a_{\boldsymbol{k}}D_k^2 e^{i\boldsymbol{kx}}v_{\boldsymbol{k}}(t)+h.c.\right)$$

With this choice, IR divergence disappears.

$$\left\langle {}^gR(\boldsymbol{X}_1)\,{}^gR(\boldsymbol{X}_2)\right\rangle^{(4)}\propto\underbrace{\left\langle\zeta_I^2\right\rangle}_{\text{IR divergent factor}}\times\underbrace{\int d(\log k)\,\partial_{\log k}^2\left(k^7|v_k|^2e^{i\boldsymbol{k}(\boldsymbol{X}_1-\boldsymbol{X}_2)}\right)}_{\text{total derivative}}$$

Physical meaning of IR regularity condition

In addition to considering ${}^{g}R$, we need additional conditions

$$-2k^2\mathcal{L}_k^{-1}e^{-2\rho}v_k = D_k v_k$$ and its higher order extension.

What is the physical meaning of these conditions?

Background gauge: $\tilde{\boldsymbol{x}} = e^{-s}\boldsymbol{x}$ $\quad \tilde{\zeta}(\tilde{\boldsymbol{x}}) = \zeta(\boldsymbol{x})$

$$ds^2 = -dt^2 + e^{2\rho}d\boldsymbol{x}^2 \Longrightarrow d\tilde{s}^2 = -dt^2 + e^{2\rho+2s}d\tilde{\boldsymbol{x}}^2$$

$$H = H_0[\zeta] + H_{\rm int}[\zeta] \Longrightarrow \underline{\tilde{H} = H_0[\tilde{\zeta}] + H_{\rm int}[\tilde{\zeta} - s]}$$

- Quadratic part in ζ and s is identical to $s = 0$ case.
- Interaction Hamiltonian is obtained just by replacing the argument ζ with $\tilde{\zeta} - s$.

Therefore, one can use

1) common mode functions for ζ_I and $\tilde{\zeta}_I$

$$\zeta_I \equiv \int d^3k\, (e^{i\boldsymbol{k}\boldsymbol{x}}\, v_{\boldsymbol{k}}(t)\, a_{\boldsymbol{k}} + h.c.) \Longrightarrow \tilde{\zeta}_I \equiv \int d^3k\, (e^{i\boldsymbol{k}\boldsymbol{x}}\, v_{\boldsymbol{k}}(t)\, \tilde{a}_{\boldsymbol{k}} + h.c.)$$

2) common iteration scheme.

$$\zeta = \zeta_I + \delta\zeta[\zeta_I] \Longrightarrow \tilde{\zeta} = \tilde{\zeta}_I + \delta\zeta[\tilde{\zeta}_I - s]$$

We may require

$$\langle 0|\zeta(\boldsymbol{x}_1)\zeta(\boldsymbol{x}_2)\cdots\zeta(\boldsymbol{x}_n)|0\rangle = \langle\tilde{0}|\tilde{\zeta}(\tilde{\boldsymbol{x}}_1)\tilde{\zeta}(\tilde{\boldsymbol{x}}_2)\cdots\tilde{\zeta}(\tilde{\boldsymbol{x}}_n)|\tilde{0}\rangle$$

Identification $\zeta(\boldsymbol{x})=\tilde{\zeta}(\tilde{\boldsymbol{x}})$ under the assumption $a_{\boldsymbol{k}}=\tilde{a}_{\boldsymbol{k}}$,

$$\Rightarrow \left[2\mathcal{L}^{-1}e^{-2\rho}\Delta+(\boldsymbol{x}\cdot\nabla)\right]\zeta_I=0$$

condition incompatible with Fourier decomposition

$a_{\boldsymbol{k}} \approx \tilde{a}_{e^s\boldsymbol{k}}$

$$\Rightarrow -2k^2\mathcal{L}_k^{-1}e^{-2\rho}v_k=D_k v_k$$

condition compatible with Fourier decomposition

Retarded integral with $\zeta(\eta_0)=\zeta_I(\eta_0)$ guarantees the commutation relation of ζ

$\Rightarrow$ $D_k v_k(\eta_0)=0$: incompatible with the normalization condition.

It looks quite non-trivial to find consistent IR regular states.

However, the Euclidean vacuum state ($\eta_0\to\pm i\infty$) satisfies this condition. (Proof will be given in our new paper)

Summary

We obtained the conditions for the absence of IR divergences.

"Wave function must be homogeneous in the direction of background scale transformation"

Euclidean vacuum and its excited states satisfy the IR regular condition.

It requires further investigation whether there are other (non-trivial and natural) quantum states compatible with the IR regularity.

Tree level 2-point function

- 2-point function of the usual curvature perturbation is divergent even at the tree level.

  $$\langle \zeta(\boldsymbol{X}_1)\,\zeta(\boldsymbol{X}_2)\rangle^{(2)} = \langle \zeta_I(\boldsymbol{X}_1)\,\zeta_I(\boldsymbol{X}_2)\rangle^{(2)} = \int d(\log k)\,k^3 \left[u_{\boldsymbol{k}}(\boldsymbol{X}_1)\, u^*_{\boldsymbol{k}}(\boldsymbol{X}_2) \right] + \text{c.c.}$$

  where $\zeta_I = u_{\boldsymbol{k}}\, a_{\boldsymbol{k}} + u^*_{\boldsymbol{k}}\, a^\dagger_{\boldsymbol{k}}$

  $u_{\boldsymbol{k}} = k^{-3/2}(1\text{-}ik/aH)e^{ik/aH}$ for Bunch Davies vacuum

  $\int d(\log k)\,k^3 \left[u_{\boldsymbol{k}}(\boldsymbol{X}_1)\, u^*_{\boldsymbol{k}}(\boldsymbol{X}_2) \right] \propto \int d(\log k)$ Logarithmically divergent!

- Of course, artificial IR cutoff removes IR divergence

  $\int d(\log k)\,k^3 \left[u_{\boldsymbol{k}}(\boldsymbol{X}_1)\, u^*_{\boldsymbol{k}}(\boldsymbol{X}_2) \right] \propto \int d(\log k) P_{\boldsymbol{k}}$ but very artificial!

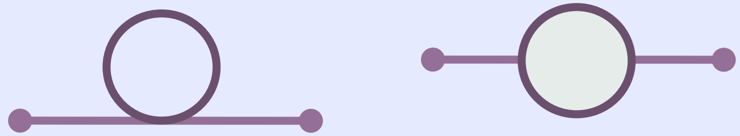
- Why there remains IR divergence even in BD vacuum?

  ζ is not gauge invariant, but ${}^gR(\boldsymbol{X}) \approx R(\boldsymbol{X})$ is.

  $$\langle {}^gR(\boldsymbol{X}_1)\, {}^gR(\boldsymbol{X}_2)\rangle^{(2)} \approx \int d(\log k)\,k^3 \left[\Delta u_{\boldsymbol{k}}(\boldsymbol{X}_1)\, \Delta u^*_{\boldsymbol{k}}(\boldsymbol{X}_2) \right] \propto \int d(\log k)\, k^4$$

Local gauge-invariant quantities do not diverge for the Bunch-Davies vacuum state.

One-loop 2-point function at leading slow-roll exp.



- No interaction term in the evolution equation at $O(\varepsilon^0)$ in flat gauge.

◎flat gauge → synchronous gauge ◎$R(\boldsymbol{X}_A) \sim e^{-2\zeta}\Delta\zeta$ ◎$R \to {}^gR$

$$\langle {}^gR(\boldsymbol{X}_1)\, {}^gR(\boldsymbol{X}_2)\rangle^{(4)} = \langle {}^gR^{(3)}(\boldsymbol{X}_1)\, {}^gR^{(1)}(\boldsymbol{X}_2)\rangle + \langle {}^gR^{(2)}(\boldsymbol{X}_1)\, {}^gR^{(2)}(\boldsymbol{X}_2)\rangle + \langle {}^gR^{(1)}(\boldsymbol{X}_1)\, {}^gR^{(3)}(\boldsymbol{X}_2)\rangle$$

$$\propto \langle \zeta_I{}^2\rangle \int d(\log k) k^3 \Big[\Delta(\boldsymbol{D}^2 u_{\boldsymbol{k}}(\boldsymbol{X}_1))\Delta(u^*_{\boldsymbol{k}}(\boldsymbol{X}_2)) + 2\Delta(\boldsymbol{D}u_{\boldsymbol{k}}(\boldsymbol{X}_1))\Delta(\boldsymbol{D}u^*_{\boldsymbol{k}}(\boldsymbol{X}_2)) + \Delta(u_{\boldsymbol{k}}(\boldsymbol{X}_1))\Delta(\boldsymbol{D}^2u^*_{\boldsymbol{k}}(\boldsymbol{X}_2))\Big] + \text{c.c.}$$

+ (manifestly finite pieces)

where $\zeta_I = u_{\boldsymbol{k}} a_{\boldsymbol{k}} + u^*_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}}$ & $\boldsymbol{D} := \partial_{\log a} - (\boldsymbol{x}\cdot\boldsymbol{\nabla})$

- IR divergence from $\langle \zeta_I{}^2\rangle$, in general.

However, the integral vanishes for the Bunch-Davies vacuum state.

$\because$ $u_{\boldsymbol{k}} = k^{-3/2}(1 - ik/aH)e^{ik/aH}$ ⟹ $\boldsymbol{D}u_{\boldsymbol{k}} = k^{-3/2}\,\partial_{\log k}(k^{3/2}u_{\boldsymbol{k}})$

⟹ $\langle {}^gR(\boldsymbol{X}_1)\, {}^gR(\boldsymbol{X}_2)\rangle^{(4)} \propto \langle \zeta_I{}^2\rangle \times \int d(\log k)\, \partial^2_{\log k}\, [\Delta(k^{3/2}u_{\boldsymbol{k}}(\boldsymbol{X}_1))\Delta(k^{3/2}u^*_{\boldsymbol{k}}(\boldsymbol{X}_2))\,] + \text{c.c.}$

- To remove IR divergence, the positive frequency function corresponding to the vacuum state is required to satisfy $\boldsymbol{D}u_{\boldsymbol{k}} = k^{-3/2}\,\partial_{\log k}(k^{3/2}u_{\boldsymbol{k}})$.

IR regularity requests scale invariance!

One-loop 2-point function at the next leading order of slow-roll.

(YU and TT, *in preparation*)

$$ {}^{g}R_2 \approx \zeta_I \Delta \left[\left(1 - \frac{\dot{H}}{H^2}\right)\partial_{\log a} + \frac{\ddot{\phi}}{H\dot{\phi}} - \frac{2\dot{H}}{H^2} - x^i\partial_i \right]\zeta_I $$

At the lowest order in ε, $\boldsymbol{D}u_{\boldsymbol{k}} = (\partial_{\log a} - x^i\partial_i)\, u_{\boldsymbol{k}} = k^{-3/2}\,\partial_{\log k}\,(k^{3/2}u_{\boldsymbol{k}})$ was requested.

Some extension of this relation to $O(\varepsilon)$ is necessary:

Natural extension is

$$ \left[\left(1 - \frac{\dot{H}}{H^2}\right)\partial_{\log a} + \frac{\ddot{\phi}}{H\dot{\phi}} - \frac{2\dot{H}}{H^2} - \underline{\underline{x^i\partial_i}} \right] u_{\boldsymbol{k}} = \underline{\underline{k^{-3/2}\partial_{\log k} k^{3/2} u_{\boldsymbol{k}}}} $$

should have the same coefficient

Notice that $u_{\boldsymbol{k}} \propto e^{i\boldsymbol{k}\boldsymbol{x}}$

$$ (u_{\mathbf{k}}, u_{\mathbf{k}'}) = \delta(\mathbf{k} - \mathbf{k}') \Rightarrow u_{\mathbf{k}} = \frac{H^2}{k^{3/2}\dot{\phi}} e^{i\mathbf{kx}} f(k/aH) $$

EOM for f:

$$ \left(\partial^2_{\log aH} + 3\partial_{\log aH} + \frac{k^2}{a^2H^2} - 3\left(\frac{\ddot{\phi}}{H\dot{\phi}} - \frac{2\dot{H}}{H^2}\right) \right) f = 0 $$

consistent with the above requirement !!

$$ \left\langle {}^{g}R(\boldsymbol{X}_1)\, {}^{g}R(\boldsymbol{X}_2) \right\rangle^{(4)} \propto \left\langle \zeta_I^2 \right\rangle \times \int d(\log k) \left(\partial^2_{\log k} + \left(\frac{\ddot{\phi}}{H\dot{\phi}} - 2\frac{\dot{H}}{H^2} \right)\partial_{\log k} \right) \times \left[\Delta\left(k^{3/2}u_{\boldsymbol{k}}(\boldsymbol{X}_1)\right) \Delta\left(k^{3/2}u_{\boldsymbol{k}}(\boldsymbol{X}_2)\right) \right] $$

IR divergence can be removed by an appropriate choice of the initial vacuum even if we consider the next leading order of slow roll.