Reconstruction in quantum field theory with a fundamental length

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Motivation and historical remarks

- Higher-energy bound imposed by localizability (**Meyman, Jaffe**): The growth of expectation values of local fields in momentum space is less than exponential
- Ultraviolet finite models of the electroweak and strong interactions, (**Efimov, Moffat, Clayton, Kleppe, Woodard, Cornish,...**)

$$\frac{1}{p^2+m^2} \longrightarrow \frac{f(p^2)}{p^2+m^2}, \qquad f(p^2) \to 0, \quad p^2 \to \infty.$$

$$|f(z)| \leq C \exp\{\ell |z|^\rho\}, \quad z \in \mathbb{C}, \quad \rho \geq 1/2, \quad \ell > 0.$$

- The interplay with string theory and noncommutative field theory (**Fainberg, Kapustin, Moffat,...**)
- Generalization of the CPT and spin-statistics theorems (**Fainberg, Iofa, Lücke, M.S....**)
- A gap in the theory: the absence of Wightman-type reconstruction theorem (i.e. of the operator realization)

An instructive example

$\varphi(x) =: \exp(g\phi^2) : (x)$, where ϕ is a free hermitian scalar field

$$W(x_1 - x_2) = \langle 0|\varphi(x_1)\varphi(x_2)|0\rangle = \sum_{r=0}^{\infty} \frac{(2r)!}{(r!)^2}(g\Delta_+(x_1 - x_2))^{2r}.$$

$$\Delta_+(x_1 - x_2) = \langle 0|\phi(x_1)\phi(x_2)|0\rangle$$

$$(2r)!/(r!)^2 \leq 2^{2r}, \qquad |\Delta_+(\zeta)| \leq \frac{1}{4\pi^2} \cdot \frac{1}{(\mathrm{Im}\,\zeta)^2}, \qquad \mathrm{Im}\,\zeta \in \mathbb{V}^-,$$

$$\Downarrow$$

$$|W(\zeta)| \leq \sum_r \left(\frac{2g}{4\pi^2} \cdot \frac{1}{(\mathrm{Im}\,\zeta)^2}\right)^r, \qquad \mathrm{Im}\,\zeta \in \mathbb{V}^-,$$

$$W(\zeta) = \frac{1}{\sqrt{1 - 4g^2\Delta_+(\zeta)^2}}, \qquad \mathrm{Im}\,\zeta \in \mathbb{V}^-_, \quad (\mathrm{Im}\,\zeta)^2 > g/2\pi^2.$$

Analytic properties of the n-point functions

$$\langle 0|\varphi(x_1)\dots\varphi(x_n)|0\rangle = \sum_R C_R w^R,\ \ w^R :\ = \prod_{i<j} w_{ij}^{r_{ij}}, \quad w_{ij} = \Delta_+(x_i - x_j)$$

$$\mathrm{Im}(z_j - z_{j+1}) \in \mathbb{V}_\ell^- = \{y \in \mathbb{V}^- : y^2 > \ell^2\} \quad \text{for all} \quad j = 1, \dots, n-1$$

$$\Downarrow$$

$$|C_R w^R|^2 \leq (2g)^{2|R|} \prod_{j=1}^{n} \left(\sum_{\substack{i=1 \\ i\neq j}}^{n} |w_{ij}| \right)^{\sum\limits_{i=1}^{n} r_{ij}} \leq \left(\frac{g}{\pi^2 \ell^2} \sum_{k=1}^{n-1} \frac{1}{k^2} \right)^{|R|},$$

where $|R| = \sum_{i<j} r_{ij}$, $\quad \sum_{k=1}^{\infty} (1/k^2) = \pi^2/6$

Theorem 1. The n-point Wightman function W_n of the field $:\exp(g\phi^2):$, considered as a function of the difference variables $\zeta_j = z_j - z_{j+1}$, is defined and analytic in the domain $(\mathbb{R}^4 + V_\ell^-)^{n-1}$, where ℓ is less than and sufficiently close to $\ell_0 = \sqrt{g/6}$.

The appropriate test functions

Definition. The space $A_\ell(\mathbb{R}^d)$ consists of functions analytic in the tubular domain $\{x+iy \in \mathbb{C}^d \colon |y| < \ell\}$ and such that

$$\|f\|_{\ell',N} = \sup_{|y|\le\ell'} \sup_{x\in\mathbb{R}^d} (1+|x|)^N |f(x+iy)| < \infty \quad \text{for any } \ell' < \ell \text{ and } N = 0, 1, \ldots$$

$$W_n(f) = \int\limits_{\mathbb{R}^{4(n-1)}} W_n(\xi+i\eta)\, f(\xi+i\eta)\, \mathrm{d}\xi,\ \ f \in A_{\ell_0}(\mathbb{R}^{4(n-1)}),\ \eta \in (\mathbb{V}_\ell^-)^{n-1},\ |\eta| < \ell_0, \ell \lesssim \ell_0$$

$$W_n \in A'_{\ell_0}(\mathbb{R}^{4(n-1)})$$

Th. 2. **The n-point vacuum expectation value of $:\exp g\phi^2:$, viewed as a generalized function of the difference variables $\xi_j = x_j - x_{j+1}$, is well defined on the space $A_{\ell_0}(\mathbb{R}^{4(n-1)})$, where $\ell_0 = \sqrt{g/6}$.**

The Fourier transformed space $\widehat{A}_\ell(\mathbb{R}^d)$ consists of functions $\hat{f}$ satisfying

$$|\partial^\kappa \hat{f}(q)| \le C_{\kappa,\ell'} \exp(-\ell' \sum_j |q_j|), \qquad \text{for any multiindex } \kappa \text{ and any } \ell' < \ell.$$

Carriers instead of supports

Def. 2. Let $O \subset \mathbb{R}^d$ and let $\widetilde{O}_\ell$ be its complex ℓ-neighborhood of the form $\widetilde{O}_\ell = \{z \in \mathbb{C}^d \colon |z - x| < \ell, \exists x \in O\}$, where $|z| = \max_{1 \le j \le d} |z_j|$. The space $A_\ell(O)$ consists of functions analytic in $\widetilde{O}_l$ and satisfying the condition

$$\|f\|_{O,\ell',N} = \sup_{x \in \widetilde{O}_{\ell'}} (1+|z|)^N |f(z)| < \infty, \qquad \text{for all } \ell' < \ell \text{ and } N = 0, 1, 2, \dots$$

This is a presheaf of vector spaces over $\mathbb{R}^d$,

$$O_1 \subset O_2 \Rightarrow A_\ell(O_2) \subset A_\ell(O_1), \qquad A_\ell(\mathbb{R}^d) \subset A_\ell(O) \quad \forall\, O \subset \mathbb{R}^d.$$

If a functional $v \in A'_\ell(\mathbb{R}^d)$ has a continuous extension to the space $A_\ell(O)$, then O can be regarded as a **carrier** of v.

- Elements of $A'_\ell(\mathbb{R}^d)$ have no smallest carrier
- If $v \in A'_\ell(O_1)$, $v \in A'_\ell(O_2)$, and $\mathrm{dist}(O_1, O_2) > \ell$, then $v \equiv 0$.

Quasilocality

Lemma. **For any $l > 0$, the function $\Delta_+(\zeta)$ is analytic in the domain $\{\zeta = \xi + i\eta \in \mathbb{C}^4 \colon \xi^2 < -l^2 < \eta^2\}$ and everywhere in this domain the following estimate holds:**

$$|\Delta_+(\xi + i\eta)| \leq \frac{1}{2\pi^2} \cdot \frac{1}{\eta^2 - \xi^2}$$

$$v(x_1 - x_2) \stackrel{\text{def}}{=} \langle 0|\,[\varphi(x_1), \varphi(x_2)]\,|0\rangle, \qquad \varphi(x) =: e^{g\phi^2} : (x)$$

$$v(f) = \sum_{r=0}^{\infty} \frac{(2r)!g^{2r}}{(r!)^2} \int\limits_{\mathbb{R}^4} \left(\Delta_+^{2r}(\xi + i\eta)\, f(\xi + i\eta) - \Delta_+^{2r}(\xi - i\eta)\, f(\xi - i\eta)\right) \mathrm{d}\xi$$

$$f \in A_{\ell_0}(\mathbb{R}^4), \quad \eta \in \mathbb{V}_\ell^-,\ |\eta| < \ell_0,\ \ell \lesssim \ell_0$$

$$v(f) = \sum_{r=0}^{\infty} \frac{(2r)!g^{2r}}{(r!)^2} \int\limits_{\xi^2 > -\ell_0^2} (\Delta_+^{2r}(\xi + i\eta)\, f(\xi + i\eta) - \Delta_+^{2r}(\xi - i\eta)\, f(\xi - i\eta))\, \mathrm{d}\xi + \int\limits_{\sigma} \Delta_+^{2r}(\zeta)\, f(\zeta)\, \mathrm{d}\zeta$$

$$\sigma = \{\zeta \in \mathbb{C}^4 \colon (\mathrm{Re}\,\zeta)^2 = -\ell_0^2, \quad \mathrm{Im}\,\zeta = t\eta,\ -1 \leq t \leq 1\} \subset \widetilde{\mathbb{V}}_{2\ell_0}.$$

$$\sup_{\zeta \in \sigma} |\Delta_+(\zeta)| \leq \frac{1}{2\pi^2 \ell_0^2} \quad \Longrightarrow \quad v \in A'_{2\ell_0}(\mathbb{V}).$$

The general case of n-point functions

$$\tau_{k,k+1}\colon (x_1,\dots,x_k,x_{k+1},\dots,x_n) \longrightarrow (x_1,\dots,x_{k+1},x_k,\dots,x_n)$$

$$(\tau_{k,k+1}W_n)(\xi_1,\dots,\xi_{k-1},\xi_k,\xi_{k+1},\dots,\xi_{n-1}) = \\ W_n(\xi_1,\dots,\xi_{k-1}+\xi_k,-\xi_k,\xi_k+\xi_{k+1},\dots,\xi_{n-1}).$$

Th. 3. Let W_n be the n-point vacuum expectation value of $:\exp(g\phi^2):$, considered as a generalized function of the difference variables. For any transposition $\tau_{k,k+1}$, the functional $W_n - \tau_{k,k+1}W_n$ has a continuous extension to the space $A_{2\ell_0}(\mathbb{V}_{(k)})$, where

$$\ell_0 = \sqrt{g/6} \quad \textbf{and} \quad \mathbb{V}_{(k)} = \{\xi \in \mathbb{R}^{4(n-1)}\colon \xi_k^2 > 0\}$$

Cluster decomposition property

If $a^2 < 0$, then

$$\mathcal{W}_{k+m}(x_1, \dots, x_k, y_1 + \lambda a, \dots, y_m + \lambda a) \quad \longrightarrow$$
$$\longrightarrow \mathcal{W}_k(x_1, \dots, x_k)\mathcal{W}_m(y_1, \dots, y_m) \quad \text{as } \lambda \longrightarrow \infty.$$

$$\mathcal{W}_n(x_1 - i\eta, x_2 - 2i\eta, \dots, x_n - ni\eta) = W_n(\xi_1 + i\eta, \dots, \xi_{n-1} + i\eta), \; \xi_j = x_j - x_{j+1}$$

Hence $\mathcal{W}_n \in A'_{nl_0}(\mathbb{R}^{4n})$

Th. 4. Let a be a space-like vector, $f \in A_{nl_0}(\mathbb{R}^{4n})$, $n = k + m$, and let

$$f_{\lambda a}(x_1, \dots, x_k, y_1, \dots, y_n) = f(x_1, \dots, x_k, y_1 - \lambda a, \dots, y_m - \lambda a), \quad \lambda > 0.$$

The vacuum expectation values of $:\exp(g\phi^2):$ **possess a cluster decomposition property which can be written as**

$$|(\mathcal{W}_{k+m} - \mathcal{W}_k \otimes \mathcal{W}_m, \; f_{\lambda a})| \le C_a \|f\|_{nl_0, 4(n+1)} \frac{1}{1 + \lambda^2}.$$

Functional analytic properties of the spaces $A_\ell(O)$

1. For each $\ell > 0$ and each $O \subset \mathbb{R}^d$, $A_\ell(O)$ is a nuclear Fréchet space and, in particular, is reflexive and separable.

2. If $v \in A'_\ell(\mathbb{R}^d)$ admits a continuous extension to $A_\ell(O_1 \cup O_2)$, then it can be decomposed as $v = v_1 + v_2$, where v_1 and v_2 have continuous extensions to $A_\ell(O_1)$ and $A_\ell(O_2)$, respectively.

3. **Th. 5.** **(Kernel or nuclear theorem) For any $O_1 \subset \mathbb{R}^{d_1}$, $O_2 \subset \mathbb{R}^{d_2}$, and $\ell > 0$, we have the canonical isomorphism**

$$A_\ell(O_1) \,\widehat{\otimes}\, A_\ell(O_2) = A_\ell(O_1 \times O_2).$$

Corollary. Let w be a separately continuous multilinear functional of arguments $f_j \in A_\ell(O_j)$, $j = 1, \ldots, n$. Then there exists a unique continuous linear functional $v \in A'_\ell(O_1 \times \cdots \times O_n)$ such that

$$w(f_1, \ldots, f_n) = (v, f_1 \otimes \cdots \otimes f_n).$$

$A_\infty(\mathbb{R}^d) = \bigcap_{\ell \to \infty} A_\ell(\mathbb{R}^d)$, $A'_\infty(\mathbb{R}^d)$ is the space of tempered ultrahyperfunctions

QFT with a fundamental length in terms of Wightman functions

1. (*Initial functional domain*) $\mathcal{W}_n \in A'_\infty(\mathbb{R}^{4n})$
2. (*Hermiticity*) $\overline{(\mathcal{W}_n, f)} = (\mathcal{W}_n, f^\dagger)$, where $f^\dagger(z_1, \dots, z_n) = \overline{f(\bar{z}_n, \dots, \bar{z}_1)}$
3. (*Positive definiteness*) $\sum_{k,m=0}^{N} (\mathcal{W}_{k+m}, f_k^\dagger \otimes f_m) \geq 0$,
4. (*Poincaré covariance*) $(\mathcal{W}_n, f) = (\mathcal{W}_n, f_{(a,\Lambda)})$, $\forall (a, \Lambda) \in \mathcal{P}_+^\uparrow$
   This is equivalent to the existence of Lorentz invariant functionals $W_n \in A'_\infty(\mathbb{R}^{4(n-1)})$ such that $\mathcal{W}_n(x_1, \dots x_n) = W_n(x_1 - x_2, \dots, x_n - x_{n-1})$
5. (*Spectrum condition*) $\mathrm{supp}\, \hat{W}_n \subset \bar{\mathbb{V}}^+ \times \cdots \times \bar{\mathbb{V}}^+$
6. (*Cluster decomposition property*) If $a^2 < 0$, then $(\mathcal{W}_{k+m}, f \otimes g_{(\lambda a, I)}) \longrightarrow (\mathcal{W}_k, f)(\mathcal{W}_m, g)$ as $\lambda \to \infty$.

7.1. (**Quasilocalizability**) There exists $\ell < \infty$ such that every functional W_n has a continuous extension to the space $A_\ell(\mathbb{R}^{4(n-1)})$.

7.2. (**Quasilocality**) For any $n \geq 2$ and $1 \leq k \leq n-1$, the difference

$$W_n(\xi_1, \dots, \xi_{k-1}, \xi_k, \xi_{k+1}, \dots, \xi_{n-1}) - W_n(\xi_1, \dots, \xi_{k-1} + \xi_k, -\xi_k, \xi_k + \xi_{k+1}, \dots, \xi_{n-1})$$

extends continuously to the space $A_\ell(\mathbb{V}_{(k)})$, where
$\mathbb{V}_{(k)} = \{\xi \in \mathbb{R}^{4(n-1)} \colon \xi_k^2 > 0\}$

Reconstruction. Step 1

The tensor algebra over $A_\infty(\mathbb{R}^4)$ (an analog of the Borchers algebra)

$$T(A_\infty) = \bigoplus_{n=0}^{\infty} T_n, \quad T_0 = \mathbb{C}, \quad T_n = A_\infty(\mathbb{R}^4)^{\hat{\otimes} n} = A_\infty(\mathbb{R}^{4n}) \quad \text{for } n \geq 1.$$

$T(A_\infty)$ consists of all sequences of the form $\mathfrak{f} = (f_0, f_1, \dots)$, where $f_n \in A_\infty(\mathbb{R}^{4n})$ and only a finite number of f_n's are different from zero.

$$(\mathfrak{f} \otimes \mathfrak{g})_n = \sum_{k=0}^{n} f_k \otimes g_{n-k}, \qquad f_n^\dagger(z_1, \dots, z_n) = \overline{f_n(\bar{z}_n, \dots, \bar{z}_1)}$$

$$s(\mathfrak{f}, \mathfrak{g}) = \sum_{k,m \geq 0} \left(\mathcal{W}_{k+m}, f_k^\dagger \otimes g_m \right)$$

$$T(A_\infty) \xrightarrow{q} \overline{T(A_\infty)/\ker s} = \mathcal{H}\colon\ \mathfrak{f} \to \Psi_{\mathfrak{f}}, \quad \mathrm{im}(q) = D.$$

$$\varphi(h)\Psi_{\mathfrak{f}} \stackrel{\text{def}}{=} \Psi_{h\otimes\mathfrak{f}}, \qquad h \otimes \mathfrak{f} = (0, hf_0, h\otimes f_1, h \otimes f_2, \dots).$$

$$U(a,\Lambda)\Psi_{\mathfrak{f}} = \Psi_{\mathfrak{f}_{(a,\Lambda)}}, \qquad |0\rangle = \Psi_{(1,0,0,\dots)}, \quad U(a,\Lambda)\,|0\rangle = |0\rangle$$

Th. 6. Let $\{\mathcal{W}_n\}$, $n = 1, 2, \dots$ be a sequence of generalized functions defined on the spaces $A_\infty(\mathbb{R}^{4n})$. Suppose the $\{\mathcal{W}_n\}$ have the cluster decomposition property and satisfy the conditions of hermiticity, relativistic covariance, positive definiteness, and the spectrum condition. Then there exist a separable Hilbert space $\mathcal{H}$, a continuous unitary representation $U(a, \Lambda)$ of the group $\mathcal{P}_+^\uparrow$ in $\mathcal{H}$, a unique state Ψ_0 invariant under $U(a, \Lambda)$, and a hermitian scalar field φ with an invariant dense domain $D \subset \mathcal{H}$ such that

$$\langle \Psi_0, \varphi(f_1) \dots \varphi(f_n) \Psi_0 \rangle = (\mathcal{W}_n, f_1 \otimes \dots \otimes f_n), \quad f_j \in A_\infty(\mathbb{R}^4), \; j = 1, \dots, n.$$

The field φ satisfies all Wightman's axioms except for locality and with A_∞ instead of the Schwartz space. Any other field theory with the same vacuum expectation values is unitary equivalent to this one.

Reconstruction. Step 2

Th. 7. The quasilocalizability condition implies that the reconstructed field $\varphi(x)$ defined initially on the invariant domain D as an operator-valued generalized function over the space $A_\infty(\mathbb{R}^4)$ can be continuously extended to the space $A_{\ell/2}(\mathbb{R}^4)$. Moreover, this condition is equivalent to the fact that each field monomial $\prod_{j=1}^n \varphi(x_j)$ has a continuous extension to the space of functions g of the form $\mathbf{g}(x_1,\dots,x_n) = g(x_1, x_1-x_2,\dots,x_{n-1}-x_n)$, where

$$g \in A_{\ell/2,\ell}(\mathbb{R}^4\times\mathbb{R}^{4(n-1)}) = A_{\ell/2}(\mathbb{R}^4)\widehat{\otimes}A_\ell(\mathbb{R}^{4(n-1)}).$$

The extension is unique because A_∞ is dense in A_ℓ.

$$\|\varphi(g)\Psi_0\|^2 = \mathcal{W}(g^\dagger\otimes g) = W\left(\int dx\overline{g(x+\xi)}g(x)\right) \le \|W\|_{l,N}\left\|\int dx\overline{g(x+\xi)}g(x)\right\|_{l,N}$$

$$= \|W\|_{l,N}\sup_{|\eta|<l}\sup_\xi(1+|\xi|)^N\left|\int dx\overline{g(x+\xi-i\eta)}g(x)\right|$$

$$\le \|W\|_{l,N}\sup_{|\eta|<l}\sup_\xi(1+|\xi|)^N\int dx\,|g(x+\xi-i\eta/2)g(x-i\eta/2)|$$

$$\le \|W\|_{l,N}\|g\|_{l/2,N}\|g\|_{l/2,N+5}\int\frac{dx}{(1+|x|)^5} \le C\|g\|^2_{l/2,N+5},$$

Corollary. The operator-valued generalized function $f \to \varphi(f)$, $f \in A_{\ell/2}(\mathbb{R}^4)$, has a unique continuous extension to the domain $D_\ell \subset \mathcal{H}$ spanned by all vectors of the form

$$\int dx_1 \dots dx_n \, g(x_1, x_1 - x_2 \dots, x_{n-1} - x_n) \prod_{j=1}^{n} \varphi(x_j)\Psi_0,$$

where $g \in A_{3\ell/2,\ell}(\mathbb{R}^4 \times \mathbb{R}^{4(n-1)})$.

Th. 8. From the quasilocality condition, it follows that for any $\Psi, \Phi \in D_\ell$, the bilinear functional

$$\langle \Phi, [\varphi(f_1), \varphi(f_2)]_- \Psi \rangle$$

on $A_{\ell/2}(\mathbb{R}^4) \times A_{\ell/2}(\mathbb{R}^4)$, which by the nuclear theorem is identified with an element of $A'_{\ell/2}(\mathbb{R}^{4\cdot 2})$, extends continuously to the space $A_{\ell/2}(\mathbb{W})$, where

$$\mathbb{W} = \{(x, x') \in \mathbb{R}^{4\cdot 2} \colon x - x' \in \mathbb{V}\}$$

Conclusions

• The reconstruction theorem has a natural extension to nonlocal QFT with a fundamental length.

• The presheaf of analytic function spaces $A_\ell(O)$ provides a appropriate framework for describing departures from locality in quantum field theories with exponential high-energy behavior and for the formulation of causality.

• An important feature of the reconstruction in nonlocal theory is that the primitive common invariant domain of the reconstructed field operators in the Hilbert space must be suitably extended to implement the quasilocalizability and causality conditions.

• The model $:\exp(g\phi^2):$, and more generally, any Wick-ordered entire function of a free field of order 2 and finite type, satisfies all assumptions of QFT with a fundamental length, which proves the self-consistency of this theory.