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Lagrange anchor, symmetries and conservation laws of free massless fields

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The BW equations for free massless fields of spin $s \geq 1/2$:

$$T^{\dot\alpha}_{\alpha_1\cdots\alpha_{2s-1}} := \partial^{\alpha\dot\alpha}\varphi_{\alpha\alpha_1\ldots\alpha_{2s-1}} = 0$$

$\varphi_{\alpha_1\ldots\alpha_{2s}}(x)$ is a symmetric, complex-valued spin-tensor on $\mathbb{R}^{3,1}$.

- The equations are non-Lagrangian unless $s = 1/2$.
- For $s > 1/2$ they satisfy the Noether identities

$$\partial^{\alpha_1}_{\dot\alpha} T^{\dot\alpha}_{\alpha_1\cdots\alpha_{2s-1}} \equiv 0,$$

  though there are no gauge symmetries.
- The equations enjoy infinite sets of global symmetries and conservation laws.

D. Lipkin, J. Math. Phys. (1964) .

S. Anco, J. Pohjanpelto, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.(2003) ; SIGMA (2008) .

M.A. Vasiliev, O.A. Gelfond, E.D. Skvortsov, Theor. Math. Phys. (2008) .

SYMMETRIES=(elementary)+(non-elementary)

CONSERVED CURRENTS =(linear) +(quadratic)

The space of Killing spinors $\mathrm{Kil}(k,l)$:

$$\partial^{(\alpha}_{(\dot\alpha}\zeta^{\alpha_1\dots\alpha_k)}_{\dot\alpha_1\dots\dot\alpha_l)}=0.$$

$$\delta_\varepsilon \varphi_{\alpha_1\cdots\alpha_{2s}} = \varepsilon Z_{\alpha_1\cdots\alpha_{2s}}$$

Symmetries of order $p+2s-1$:

$$\begin{aligned} Z_{\alpha_1\cdots\alpha_{2s}} = \xi^{\beta_1\cdots\beta_{p+2s-1}\dot\beta_1\cdots\dot\beta_{p+2s-1}} \partial_{\beta_1\dot\beta_1}\cdots\partial_{\beta_{p+2s-1}\dot\beta_{p+2s-1}} \varphi_{\alpha_1\cdots\alpha_{2s}} \\ + \Upsilon^{\beta_1\cdots\beta_{p-1}\dot\beta_1\cdots\dot\beta_{4s+p-1}} \partial_{\beta_1\dot\beta_1}\cdots\partial_{\beta_{p-1}\dot\beta_{p-1}} \\ \times \partial_{\alpha_1\dot\beta_p}\cdots\partial_{\alpha_{2s}\dot\beta_{2s+p-1}} \bar\varphi_{\dot\beta_{2s+p}\cdots\dot\beta_{4s+p-1}} + \quad \text{(lower orders)} \end{aligned}$$

$$\xi \in \mathrm{Kil}(p+2s-1, p+2s-1) \qquad \Upsilon \in \mathrm{Kil}(p-1, 4s+p-1)$$

$$j_{\alpha_1\dot\alpha_1}=\frac{1}{2}\bar\varphi_{\dot\alpha_1\dot\alpha_2\ldots\dot\alpha_{2s}}Q_{\alpha_1}^{\dot\alpha_2\ldots\dot\alpha_{2s}}+c.c.$$

Characteristics of order p:

$$\begin{aligned}Q_{\alpha_1}^{\dot\alpha_2\cdots\dot\alpha_{2s}}=&\xi^{\beta_1\cdots\beta_{p+2s-1}\dot\beta_1\cdots\dot\beta_p\dot\alpha_2\cdots\dot\alpha_{2s}}\partial_{\beta_1\dot\beta_1}\cdots\partial_{\beta_p\dot\beta_p}\varphi_{\alpha_1\beta_{p+1}\ldots\beta_{2s+p-1}}\\&+\Upsilon^{\beta_1\cdots\beta_{p-1}\dot\beta_1\cdots\dot\beta_{2s+p}\dot\alpha_2\cdots\dot\alpha_{2s}}\partial_{\beta_1\dot\beta_1}\cdots\partial_{\beta_{p-1}\dot\beta_{p-1}}\\&\times\partial_{\alpha_1\dot\beta_p}\bar\varphi_{\dot\beta_{p+1}\cdots\dot\beta_{2s+p}}+\quad\text{(lower orders)}\end{aligned}$$

$$\xi\in i^p\mathrm{Kil}_{\mathbb{R}}(p+2s-1,p+2s-1),$$

$$\Upsilon\in\mathrm{Kil}(p-1,4s+p-1)\quad\left(\begin{array}{ll}s\in\mathbb{N}, & p\in2\mathbb{N}\\ s\in\mathbb{N}-\frac{1}{2}, & p\in2\mathbb{N}-1\end{array}\right).$$

Consider a set of fields ϕ^i subject to a system of PDEs

$$T_a(\phi)=0.$$

Geometrically, one can think of T's as a section of some (infinite-dimensional) vector bundle $\mathcal{E}\to M$ over the configuration space of fields M.

The *shell*: $\Sigma=\{\ \phi\in M\ \mid\ T(\phi)=0\ \}$

The operator of *universal linearization*

$$J_{ia}=\partial_i T_a$$

defines the map of vector bundles $J: TM\to\mathcal{E}$.

A vector field $Z = Z^i \partial_i$ on M is called a *symmetry* of the field equations if

$$J(Z)|_\Sigma = 0 \quad \Leftrightarrow \quad Z^i J_{ia} = A_a^b T_b.$$

$\mathrm{Sym}(T)$ is the space of all symmetries.

Let $\mathcal{E}^*$ denote the vector bundle dual to the dynamics bundle $\mathcal{E}$.

A section $P = \{P^a\}$ of $\mathcal{E}^*$ is called an *adjoint symmetry* if

$$J^*(P)|_\Sigma = 0 \quad \Leftrightarrow \quad J_{ia} P^a = B_i^b T_b.$$

$\mathrm{AdSym}(T)$ is the space of all adjoint symmetries.

A section Q of $\mathcal{E}^*$ is said to generate an identity for the equations if

$$Q^a T_a = \int \mathrm{div} j \qquad \Rightarrow \qquad \mathrm{div} j|_\Sigma = 0$$

j is a conserved current.

Each characteristic Q is an adjoint symmetry, but not vice versa.

$\mathrm{Char}(T)$ is the space of all nontrivial characteristics.

$\mathrm{CL}(T)$ is the space of all nontrivial conservation laws.

There is a one-to-one correspondence between the spaces of *nontrivial characteristics and conservation laws:*

$$\mathrm{Char}(T) = \mathrm{CL}(T)$$

A vector bundle homomorphism $V:\mathcal{E}^*\to TM$ is called a *Lagrange anchor*, if the following diagram of maps on-shell commutes:

$$\begin{tikzcd} TM \arrow[r, "J"] & \mathcal{E} \\ \mathcal{E}^* \arrow[u, "V"] \arrow[r, "J^*"] & T^*M \arrow[u, "V^*"'] \end{tikzcd}$$

Explicitly,

$$V_a^i\partial_i T_b - V_b^i\partial_i T_a = C_{ab}^d T_d\,.$$

[P.O. Kazinski , S.L. Lyakhovich, A.A. Sharapov, JHEP (2005)]

Example: For Lagrangian equations

$$\mathcal{E}=T^*M, \qquad T_i=\partial_i S=0, \qquad V=\mathrm{id}:TM\to TM.$$

Any Lagrange anchor defines the map

$$V\colon \mathrm{AdSym}(T) \to \mathrm{Sym}(T)$$

In particular, if $Q \in \mathrm{Char}(T) \subset \mathrm{AdSym}(T)$ is a characteristic, then

$$Z = V(Q)$$

is a symmetry.

[D.S. Kaparulin, S.L. Lyakhovich, A.A. Sharapov, JMP (2010)]

Example: For Lagrangian equations $T_i = \partial_i S$

$$J = J^* \quad \Rightarrow \quad \mathrm{AdSym}(T) = \mathrm{Sym}(T), \qquad V = \mathrm{id}, \qquad \mathrm{Char}(T) = \mathrm{Sym}(S)$$

In general, the map V from the characteristics to symmetries is neither injective nor surjective.

A Lagrange anchor is *strongly integrable* if

$$[V_a, V_b] = C^d_{ab} V_d, \qquad C^d_{ab} C^e_{cd} + V^i_c \partial_i C^e_{ab} + cycle(a,b,c) = 0.$$

The vector fields $V_a = V^i_a \partial_i$ form an integrable distribution on M.

The strongly integrable Lagrange anchor defines the Lie algebroid $V : \mathcal{E}^* \to TM$ over M with Lie bracket

$$[e_a, e_b] = C^d_{ab} e_d$$

e_a being frame sections in $\mathcal{E}^*$.

The symmetries form a Lie algebra acting on the space of characteristics

$$(Z,Q) \quad \mapsto \quad Q'=Z\,Q$$

Given a strongly integrable Lagrange anchor V, one can endow $\mathrm{Char}(T)$ with the structure of Lie algebra:

$$[Q_1,Q_2]_V=V(Q_1)Q_2=-V(Q_2)Q_1.$$

The map

$$V{:}\mathrm{Char}(T)\to\mathrm{Sym}(T)$$

is then a Lie algebra homomorphism,

$$[V(Q_1),V(Q_2)]=V([Q_1,Q_2]_V).$$

Let $Q=Q(\varphi,\partial\varphi,\cdots)$ be a characteristic for the BW equations, i.e.,

$$Q_{\dot\alpha}^{\alpha_1\cdots\alpha_{2s-1}}\,T^{\dot\alpha}_{\alpha_1\cdots\alpha_{2s-1}}+c.c.=\partial_\mu j^\mu,$$

then

$$Z_{\alpha_1\cdots\alpha_{2s}}=V(Q)_{\alpha_1\cdots\alpha_{2s}}=i^{2s}\partial_{(\alpha_2\dot\alpha_2}\cdots\partial_{\alpha_{2s}\dot\alpha_{2s}}\bar{Q}^{\dot\alpha_2\cdots\dot\alpha_{2s}}_{\alpha_1)}$$

is a generator of global symmetry

$$\delta_\varepsilon\varphi_{\alpha_1\cdots\alpha_{2s}}=\varepsilon Z_{\alpha_1\cdots\alpha_{2s}}.$$

The differential operator V of order $2s-1$ is the Poincaré covariant and strongly integrable **Lagrange anchor**.

The quantum average

$$\langle \mathcal{O} \rangle = \int [D\phi]\mathcal{O}[\phi]\Psi[\phi],$$

$\mathcal{O}$ is a physical observable,

Ψ is a probability amplitude on the configuration space of fields.

For Lagrangian equations

$$\left(\frac{\delta S}{\delta \phi^i} + i\hbar \frac{\delta}{\delta \phi^i} \right) \Psi[\phi] = 0 \quad \Rightarrow \quad \Psi[\phi] = e^{\frac{i}{\hbar} S[\phi]}.$$

$e^{\frac{i}{\hbar} S[\phi]}$ is Feynman's probability amplitude.

The SD equation for a strongly integrable Lagrange anchor:

$$\mathbb{T}_a\Psi[\phi]=\Big(T_a(\phi)+i\hbar V_a^i(\phi)\partial_i\Big)\Psi[\phi]=0.$$

[S.L. Lyakhovich, A.A. Sharapov, JHEP (2006)]
Compatibility conditions:

$$[\mathbb{T}_a,\mathbb{T}_b]=C_{ab}^c(\phi)\mathbb{T}_c.$$

For the Bargmann-Wigner equations we have

$$\left(T^{\dot\alpha}_{\alpha_1\cdots\alpha_{2s-1}}-i^{2s}\hbar\partial_{\alpha_1\dot\alpha_1}\cdots\partial_{\alpha_{2s-1}\dot\alpha_{2s-1}}\frac{\delta}{\delta\bar\varphi_{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{2s-1}}}\right)\Psi[\varphi]=0,$$

$$\left(\bar T^{\alpha}_{\dot\alpha_1\cdots\dot\alpha_{2s-1}}-i^{-2s}\hbar\partial_{\alpha_1\dot\alpha_1}\cdots\partial_{\alpha_{2s-1}\dot\alpha_{2s-1}}\frac{\delta}{\delta\varphi_{\alpha\alpha_1\cdots\alpha_{2s-1}}}\right)\Psi[\varphi]=0.$$

$$\Psi[\varphi]=\int[DY]e^{\frac{i}{\hbar}S_{\mathrm{aug}}[\varphi,Y]}$$

The action of *augmented theory* :

$$S_{\mathrm{aug}}[\varphi,Y]=\int d^4x\Big(Y_{\dot\alpha}^{\alpha_1\dots\alpha_{2s-1}}\,T^{\dot\alpha}_{\alpha_1\cdots\alpha_{2s-1}}(\varphi)$$

$$+\frac{i^{2s}}{2}\partial_{\alpha_1(\dot\alpha_1}\dots\partial_{\alpha_{2s-1}\dot\alpha_{2s-1}}\,Y^{\alpha_1\cdots\alpha_{2s-1}}_{\dot\alpha_{2s})}\,\partial^{\alpha_{2s}\dot\alpha_1}\,\bar Y^{\dot\alpha_2\cdots\dot\alpha_{2s}}_{\alpha_{2s}}+\mathrm{c.c.}\Big).$$

Classically, there is no interaction between φ's and Y's:

$$\delta S_{\mathrm{aug}}=0\quad\Rightarrow\quad\partial^{\alpha\dot\alpha}\varphi_{\alpha\alpha_1\dots\alpha_{2s-1}}=0,\quad\partial^{(\alpha_1\dot\alpha}\,Y^{\alpha_2\cdots\alpha_{2s})}_{\dot\alpha}=0.$$

[S.L. Lyakhovich, A.A. Sharapov, JHEP (2007)]

- The Bargmann-Wigner equations for massless fields of spin $s \geq 1/2$ admit a Poincaré invariant Lagrange anchor. It is given by a differential operator of order $2s-1$.
- The Lagrange anchor takes each conservation law to a symmetry and almost all the symmetries of the Bargmann-Wigner equations comes from the adjoint symmetries via the anchor map.
- The corresponding quantum probability amplitude on the configurations space of fields is given by an essentially nonlocal functional for $s > 1/2$.