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Conformal frame independence in cosmology

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1. Conformal frames / - why bother? -

In cosmology, we encounter various frames of the metric which are conformally equivalent.

Einstein frame, Jordan frame, string frame, ...

They are mathematically equivalent, so one can work in any frame as long as mathematical manipulations are concerned.

But it is often said that there exists a unique physical frame on which we should consider actual ‘physics.’

Is it really so?

Two typical frames in scalar-tensor theory

$(\phi + \boldsymbol{g})$

- Jordan(-Brans-Dicke) frame

  "gravitational" part : $F(\phi)R+L(\phi)$

  matter part: $L(\psi, A,...)$ ~ minimal coupling with g

  (matter assumed to be universally coupled with g
  ··· for baryons, experimentally consistent)

- Einstein frame

  "gravitational" part : $R+L(\phi)$ ~ minimal coupling between g and ϕ

  matter part: $G(\phi)L(\psi, A,...)$ ψ : fermion, A : vector, ...

  (if non-universal coupling:

  $$\Rightarrow \sum_A G_A(\phi) L_A(Q_A); \;\; Q_A = \psi, A, \cdots.$$)

Conformal transformation

A few basics:

- metric and scalar curvature

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$

$$R \to \tilde{R} = \Omega^{-2}\left[R - (D-1)\left(2\frac{\Box\Omega}{\Omega} - (D-4)g^{\mu\nu}\frac{\partial_\mu\Omega\partial_\nu\Omega}{\Omega^2} \right)\right]$$

- matter fields (for $D = 4$)

$$\phi \;\to\; \tilde{\phi} = \Omega^{-(D-2)/2}\phi \quad \left(= \Omega^{-2}\phi \right) \qquad \text{scalar}$$

$$A_\mu \to \tilde{A}_\mu = \Omega^{-(D-4)/2} A_\mu \quad \left(= A_\mu \right) \qquad \text{vector}$$

$$\psi \;\to \tilde{\psi} = \Omega^{-(D-1)/2}\psi \quad \left(= \Omega^{-3/2}\psi \right) \qquad \text{fermion}$$

2. Standard ("baryonic") action in 4D

(for Universe at T $\lesssim$ GeV)

'Jordan' frame (= matter minimally coupled to gravity)

$$S=\int d^4x\sqrt{-g}\left[-i\bar{\psi}_X\gamma^\mu\left(\overleftrightarrow{D}_\mu-ie_XA_\mu\right)\psi_X-m_X\bar{\psi}_X\psi_X-\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}+\cdots\right]$$

$$\bar{\psi}\gamma^\mu\overleftrightarrow{D}_\mu\psi=\frac{1}{2}\left[\bar{\psi}\gamma^\mu D_\mu\psi-(D_\mu\bar{\psi})\gamma^\mu\psi\right],$$

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\,,\quad D_\mu=\partial_\mu-\frac{1}{4}\omega_{ab\mu}\Sigma^{ab}\,,$$

$$\Sigma^{ab}=\frac{1}{2}\left[\gamma^a,\gamma^b\right],\quad \omega_{ab\mu}=e_{a\nu}\nabla_\mu e_b^\nu\,.$$

ψ_X : X = electron/proton/...

A : electromagnetic 4-potential

For the moment, ignore/freeze dilatonic degrees of freedom.

(scalar gravitational)

Effect of conformal transformation

For $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$

$$S = \int d^4x \sqrt{-\tilde{g}} \left[i \bar{\tilde{\psi}} \tilde{\gamma}^\mu \left(\overleftrightarrow{\tilde{D}}_\mu - ieA_\mu \right) \tilde{\psi} - \tilde{m} \bar{\tilde{\psi}} \tilde{\psi} - \frac{1}{4} \tilde{g}^{\mu\alpha} \tilde{g}^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} + \cdots \right]$$

where $\tilde{\gamma}^\mu = \Omega^{-1} \gamma^\mu, \quad \tilde{\psi} = \Omega^{-3/2} \psi, \quad \tilde{m} = \Omega^{-1} m$.

(A_μ is conformal invariant in 4 dim)

Conformal transformation from \`Jordan frame' to any other frame results in spacetime-dependent mass.

And this is the only effect, provided dynamics of Ω (at short distances) can be neglected.

(Ω may be dynamical on cosmological scales)

3. Big Bang Cosmology

Conventional wisdom

$$ds^2 = -dt^2 + a^2(t) d\sigma_{(K)}^2 \ ;$$

$d\sigma_{(K)}^2$: homogeneous and isotropic 3-space $\left(K = \pm 1, 0\right)$

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}$$ ··· expanding universe

➡ cosmological redshift $E_{\text{obs}} = \dfrac{E_{\text{emit}}}{1+z}$ ⇔ Hubble's law

This is how we interpret observational data.

This is regarded as a `proof' of cosmic expansion.

But ....

Conformal transformation:

$$ds^2 \to d\tilde{s}^2 = \Omega^2 ds^2; \quad \Omega = \frac{1}{a}$$

$$\Rightarrow \; d\tilde{s}^2 = -d\eta^2 + d\sigma_{(3)}^2; \quad d\eta = \frac{dt}{a(t)}$$

In this conformal frame, the universe is static.

no Hubble flow.

photons do not redshift...

Is this frame unphysical?

In this static frame,

- electron mass varies in time: $\tilde{m}(\eta) = m\Omega^{-1} = \dfrac{m}{1+z}$

  where "z" is defined by

$$1 + z \equiv \Omega = \frac{1}{a(\eta)} \quad \left(a_0 = a(\eta_0) = 1\right)$$

- Bohr radius $\propto m^{-1}$ ⇔ atomic energy levels $\propto m$:

energy level in 'static' frame

$$\tilde{E}_n = \frac{E_n}{1+z}$$

energy level in 'Jordan' frame

Thus frequency of photons emitted from a level transition $n \to n'$ at time $z = z(\eta)$ is

$$\tilde{E}_{nn'} = \frac{E_{nn'}}{1+z}$$

this is exactly what we observe as Hubble's law!

Gravity in the static frame

Assume canonical Einstein theory with matter minimally coupled to gravity:

Jordan frame = Einstein frame

- Gravity is stronger in the early universe:

$$\frac{1}{G}\sqrt{-g}R = \frac{1}{G\Omega^2}\sqrt{-\tilde{g}}\tilde{R} + \cdots \Rightarrow \tilde{G} = G\Omega^2 = \frac{G}{a^2}$$

- This is what we observe in the original frame:

$$G\frac{m_1 m_2}{r_p^2} = G\frac{m_1 m_2}{a^2 r^2} = \tilde{G}\frac{m_1 m_2}{r^2}$$

proper distance   comoving distance

(gravity is prop to a^{-2} at a fixed comoving distance)

Interpretation of CMB in this frame

- CMB photons have never redshifted.
- The universe was in thermal equilibrium when the electron mass was small by a factor >10^3, ie, at time z >10^3, at fixed temperature T=2.725K.

(we have set the scale $\Omega(z=0)=\frac{1}{a_0}=1$)

Just to check physics...

- Thomson cross section: $\tilde{\sigma}_T \propto \tilde{m}^{-2} \to \tilde{\sigma}_T = \sigma_T (1+z)^2$

  electron density: $\tilde{n}_e = \text{const.} = n_e (1+z)^{-3}$

⇨ rate of scattering/interaction per unit proper time:

$$\tilde{n}_e \tilde{\sigma}_T d\eta = \frac{n_e \sigma_T}{1+z} d\eta = n_e \sigma_T dt$$

$\underbrace{\text{local/non-gravitational}}$

Thus physics is the same. It's only the scale that differs.

4. Cosmological Perturbations

Makino & MS (1991), ...

- tensor-type perturbation

$$ds^2 = -dt^2 + a^2(t)\left(\delta_{ij} + h_{ij}\right)dx^i dx^j$$

$$= a^2(\eta)\left[-d\eta^2 + \left(\delta_{ij} + h_{ij}\right)dx^i dx^j\right]$$

$$\Downarrow$$

$$\boxed{\partial_j h^{ij} = h^j{}_j = 0}$$

$$d\tilde{s}^2 = \Omega^2 ds^2$$

$$= \Omega^2(x^\mu)a^2(\eta)\left[-d\eta^2 + \left(\delta_{ij} + h_{ij}\right)dx^i dx^j\right]$$

Definition of h_{ij} is apparently **Ω-independent**.

- vector-type perturbation

$$ds^2 = a^2\left[-d\eta^2 + 2B_j dx^j d\eta + \left(\delta_{ij} + \partial_i H_j + \partial_j H_i\right) dx^i dx^j\right]$$

$$\boxed{\partial_j B^j = \partial_j H^j = 0}$$

$$\Downarrow$$

$$d\tilde{s}^2 = \Omega^2 ds^2$$

$$= \Omega^2 a^2\left[-d\eta^2 + 2B_j dx^j d\eta + \left(\delta_{ij} + \partial_i H_j + \partial_j H_i\right) dx^i dx^j\right]$$

Definitions of B_j and H_j are aslo **Ω-independent**.

tensor & vector perturbations are
conformal frame-independent

- scalar-type perturbation

$$ds^2 = a^2(\eta)\Big[-(1+2A)d\eta^2 + 2\partial_j B dx^j d\eta$$
$$+\big((1+2\mathcal{R})\delta_{ij} + 2\partial_i\partial_j E\big)dx^i dx^j\Big]$$

$$\Downarrow$$

$$d\tilde{s}^2 = \Omega^2 ds^2$$
$$= \Omega^2 a^2\Big[-(1+2A)d\eta^2 + 2\partial_j B dx^j d\eta$$
$$+\big((1+2\mathcal{R})\delta_{ij} + 2\partial_i\partial_j E\big)dx^i dx^j\Big]$$

Definitions of B and E are Ω-independent.

But A and $\mathcal{R}$ are Ω-dependent!

$$\Omega(t,x^i) = \Omega_0(t)\left[1+\omega(t,x^i)\right]$$

$$\Rightarrow \quad A \to A+\omega\,, \quad \mathcal{R} \to \mathcal{R}+\omega$$

Nevertheless, for $\Omega = \Omega(\phi)$

- The important, curvature perturbation $\mathcal{R}_c$, conserved on superhorizon scales, is defined on comoving hypersurfaces.

uniform ϕ ($\delta\phi = 0$)

$$\mathcal{R}_c \equiv \mathcal{R} - \frac{H}{\dot{\phi}}\delta\phi = \mathcal{R} - \frac{1}{a}\frac{da}{d\phi}\delta\phi$$

frame-independent

- For scalar-tensor theory with

$$L = \frac{1}{2}f(\phi)R + K(X,\phi)\,, \quad X \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

we have $\Omega = \Omega(\phi)$

$\boxed{\mathcal{R}_c = \mathcal{R}_{\delta\phi=0} \text{ is } \Omega\text{-independent!}}$

$\boxed{\mathcal{R}_c \text{ is conformal inv if there is no isocurvature perturbation}}$

generalization to nonlinear perturbation

Gong, Hwang, Park, Song & MS (2011)

- Generalization is straightforward for perturbations on superhorizon scales

δN formalism:

$\mathcal{R}_c(t_f)$ = perturbation in the number of e-folds, δN, between the final comoving surface ($t=t_f$) and an initial flat surface

δN can be O(1)

although the number of e-folds N depends on conformal frames, δN is frame-independent

5. Summary

- A variety of conformal frames appear in cosmology.
- There is no unique *physical* frame;
  - all frames are observationally equivalent.
  - interpretations may differ from frames to frames (can be extremely unconventional in some frames).

frame in which mass is constant gives most intuitive (natural) interpretation

- Curvature perturbation $\mathcal{R}_c$ is frame-dependent
  - but is frame-independent if there is no isocurvature pert.
  - if $\exists$ isocurvature pert., matter coupling is essential in determining which $\mathcal{R}_c$ is directly related to observables.

- Caveat: what if two metrics are related by a singular conformal transformation?
  - eg, can we solve the initial cosmological singularity problem by a singular conformal transformation?

    Probably not, because physics should be the same. But maybe worth studying more carefully.

Regularizing Singularity?

Sch BH: $$ds^2 = -\left(1-\frac{2GM}{c^2 r}\right)c^2dt^2 + \left(1-\frac{2GM}{c^2 r}\right)^{-1}dr^2 + r^2 d\Omega_{(2)}^2$$

conf trans

$$d\tilde{s}^2 = \Omega^2 ds^2 = -c^2dt^2 + \left(1-\frac{2GM}{c^2 r}\right)^{-2}dr^2 + \left(1-\frac{2GM}{c^2 r}\right)^{-1}r^2 d\Omega^2;$$

$$\Omega = \left(1-\frac{2GM}{c^2 r}\right)^{-1/2}$$

If we start from the `tilded' frame, the metric and the scalar field Ω have a real singularity at $r=2GM/c^2$.

But the singularity disappears by the conformal transf.,

$$d\tilde{s}^2 \to ds^2 = \Omega^{-2} d\tilde{s}^2$$

$r=2GM/c^2$ is a perfectly regular sphere.