Group-Theoretical Derivation of Path Integrals for Particles and "History-Strings"

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Motivation

- **Path Integral** (including a form of **classical action**) is **derived** from **Path Group**
- **Why Path Group?**
  - **Path-dependent functions** (gauge-invariant, Mandelstam 1962)
  - **Path Group** (group-theoretical version, Mensky 1983)
    - Non-Abelian Stokes theorem
    - Quantum equivalent principle
- **What does this teach us?**
  - Looking for **adequate form of non-locality**
  - It may be **geometry of paths** instead of **geometry of points**

Dynamics from a group

- **Propagator** derived **from a group**
  - **Two representations** (“local” and “elementary”)
  - **Intertwining** of these representations resulting in **quantum theory**
- Technics of **induced representations**
- **Projective representations** determines the form of the **classical action**

Group-theoretical derivation of quantum dynamics

$$\boxed{S_1 \in [U_{elem}, U_{loc}],\ S_2 \in [U_{loc}, U_{elem}]}$$

$$S_1 U_{elem}(g) = U_{loc}(g) S_1, \quad S_2 U_{loc}(g) = U_{elem}(g) S_2$$

$$\Pi = S_1 S_2$$

- No classical theory and subsequent “quantization”
- Quantum features are provided by intertwining two reps

Induced representations

$$U_\kappa(G) = \kappa(K) \uparrow G$$

$$\boxed{\begin{array}{l}(U_\kappa(g)\varphi)(g') = \varphi(g'g) \\ \varphi(kg) = \kappa(k)\varphi(g), \quad \forall g \in G,\, k \in K\end{array}}$$

Functions on the homogeneous space may be used

$$\mathcal{X} = K\backslash G \qquad\qquad x = Kg \in \mathcal{X}$$

“Local” representation should be induced

Projective representations

$$U(g)U(g') = (g, g')U(gg')$$

For the operation being associative

$$U(g)\Big[U(g')U(g'')\Big] = \Big[U(g)U(g')\Big]U(g'')$$

multipliers have to satisfy the following conditions

$$\boxed{(g, g')(gg', g'') = (g, g'g'')((g', g'')}$$

Instead, one may take usual (vector) reps but for an **extended group**

1. Galilei group (for the non-relativistic theory)

$$g = (\mathbf{a}, t)_T r \mathbf{v}_G$$

$$(\mathbf{a}, t)_T (\mathbf{a}', t')_T = (\mathbf{a} + \mathbf{a}', t + t')_T, \quad \mathbf{v}_G \mathbf{v}'_G = (\mathbf{v} + \mathbf{v}')_G$$
$$r(\mathbf{a}, t)_T r^{-1} = (r\mathbf{a}, t)_T, \quad r\mathbf{v}_G r^{-1} = (r\mathbf{v})_G$$
$$\mathbf{v}_G (\mathbf{a}, t)_T = (\mathbf{a} + t\mathbf{v}, t)_T \mathbf{v}_G$$

Projective representation $U(g)U(g') = (g, g')U(gg')$

$$\left((\mathbf{a}, t)_T r \mathbf{v}_G, (\mathbf{a}', t')_T r' \mathbf{v}'_G\right) = \exp\left[im\left(\mathbf{v}\mathbf{a}' + \frac{1}{2}\mathbf{v}^2 t'\right)\right]$$

The extended group $g = \xi_C (\mathbf{a}, t)_T r \mathbf{v}_G$ $\quad \xi_C \xi'_C = (\xi + \xi')_C$

$$\mathbf{v}_G (\mathbf{a}, t)_T = \xi_C (\mathbf{a} + t\mathbf{v}, t)_T \mathbf{v}_G, \quad \xi = -\left(\mathbf{a}\mathbf{v} + \frac{1}{2}t\mathbf{v}^2\right)$$

2. Group of Agashi-Roman-Santilli (relativistic analogue of Galilei group)

$$g = (a,\tau)_T \lambda v_G$$

$$(a,\tau)_T(a',\tau')_T = (a+a',\tau+\tau')_T, \quad v_G v'_G = (v+v')_G$$
$$\lambda(a,\tau)_T\lambda^{-1} = (\lambda a,\tau)_T, \quad \lambda v_G \lambda^{-1} = (\lambda v)_G$$
$$v_G(a,\tau)_T = (a+\tau v,\tau)_T\, v_G$$

Projective representation $\quad U(g)U(g') = (g,g')U(gg')$

$$\left((a,\tau)_T\lambda v_G, (a',\tau')_T\lambda' v'_G\right) = \exp\left[im\left((v,a') + \frac{1}{2}(v,v)\tau'\right)\right]$$

The extended group $\quad g = \xi_C(a,\tau)_T\lambda v_G \qquad \xi_C\xi'_C = (\xi+\xi')_C$

$$v_G(a,\tau)_T = \xi_C(a+\tau v,\tau)_T v_G, \quad \xi = -\left((a,v) + \frac{1}{2}\tau(v,v)\right)$$

“Local” representation of ARS group

ARS space: homogeneous space of the group *G* in respect to the subgroup *H*

$$\boxed{(x,\tau) = H\,(x,\tau)_T \in H\backslash G} \qquad h = \xi_C \lambda v_G$$

➡ “local” rep is induced from the subgroup *H*

$$\boxed{U_{loc} = U_\chi = \chi(H) \uparrow G} \qquad \chi(h) = \exp(im\xi)\Delta(\lambda)$$

acts in the space of functions $\psi(x,\tau)$

normalized in respect to $d^4x\ dt$

“Elementary” representation of ARS group

$$U_{elem} = U_\kappa = \kappa(K) \uparrow G$$

$$\boxed{k = \xi_C \lambda(a,\tau)_T \quad \kappa(k) = \exp(im\xi)\Delta(\lambda)}$$

acts in the space of functions $\varphi(v)$

Intertwining the representations and constructing the propagator

$$\boxed{S_1 \in [U_{elem}, U_{loc}],\ S_2 \in [U_{loc}, U_{elem}]}$$

$$(S_1\varphi)(x,\tau) = \int d^4v\ \exp\left[-im\left((x,v) + \frac{1}{2}\tau(v,v)\right)\right]\varphi(v),$$

$$(S_2\psi)(v) = \int d\tau \int d^4x\ \exp\left[im\left((x,v) + \frac{1}{2}\tau(v,v)\right)\right]\psi(x,\tau)$$

$$\boxed{\Pi = S_1 S_2}$$

$$(\Pi\psi)(x,\tau) = \int d\tau' \int d^4x'\, \Pi(x,\tau|x',\tau')\,\psi(x',\tau'),$$

$$\boxed{\Pi(x,\tau|x',\tau') = \left(\frac{m}{2\pi i(\tau-\tau')}\right)^2 \exp\left(\frac{im(x-x')^2}{2(\tau-\tau')}\right)}$$

Causal propagator

Relativistic causality: propagation into the future of τ [Stueckelberg 1942]

$$\Pi^c(x,\tau|x',\tau') = \theta(\tau-\tau')\left(\frac{m}{2\pi i(\tau-\tau')}\right)^2 \exp\left(\frac{im(x-x')^2}{2(\tau-\tau')}\right)$$

then integrating over τ- τ’

$$\boxed{\Pi^c(x,x') = \int_0^\infty d\tau \left(\frac{m}{2\pi i\tau}\right)^2 \exp\left(\frac{im(x-x')^2}{2\tau}\right)}$$

3. Paths instead of translations

- **Paths** (classes of curves equivalent under reparametrization)
- **Trajectories**, or **parametrized paths** (narrower classes of curves not equivalent under reparametrization)

Semigroup of trajectories

$$\boxed{\{x\}_{\tau'}^{\tau} = \{x(s) \in \mathcal{M} | \tau' \leq s \leq \tau\}}$$

$$\{x''\}_{\tau''}^{\tau} = \{x\}_{\tau'}^{\tau} \cdot \{x'\}_{\tau''}^{\tau'}$$

$\{x\}_{\tau'}^{\tau} \sim \{x'\}_{\tau'+\Delta\tau}^{\tau+\Delta\tau}$, если $x'(s) = x(s - \Delta\tau) + a$

$$\boxed{\{x\}_{\tau''}^{\tau'} \leftrightarrow [u]_\tau} \qquad \boxed{u(\sigma) = \dot{x}(\sigma - \tau'')}$$

Действие полугруппы траекторий на пространстве-времени

$$\boxed{(x', \tau')[u]_\tau = (x'', \tau'')}$$

если $\{x\}_{\tau''}^{\tau'}$, $x(\tau') = x'$, $x(\tau'') = x''$

Generalized semigroup ARS

$$g = [u]_\tau \lambda [v]$$

$$[v] = \{v(\sigma) \in \mathbb{R}^4 | \infty \leq \sigma \leq \infty\}$$

- generalized proper Galilei transformation

$$[v][v'] = [v + v'], \quad \lambda[v]\lambda^{-1} = [\lambda v] \quad \lambda[u]_\tau \lambda^{-1} = [\lambda u]_\tau$$

$$[v][u]_\tau = [u + \tilde{v}]_\tau [\tilde{v}]$$ where $$\tilde{v}(\sigma) = v(\sigma - \tau)$$

$$\Big([u]_\tau \lambda[v], [u']_{\tau'} \lambda'[v']\Big) = \exp\left[im \int_0^{\tau'} d\sigma \left((u, v') + \frac{1}{2}(v', v') \right) \right]$$

Extended semigroup $g = \xi_C [u]_\tau \lambda [v]$ $\quad \xi_C \xi'_C = (\xi + \xi')_C$

$$[v][u]_\tau = \xi_C [u + \tilde{v}]_\tau [\tilde{v}], \quad \xi = -\int_0^\tau d\sigma \left((u, \tilde{v}) + \frac{1}{2}(\tilde{v}, \tilde{v}) \right)$$

Representations of the generalized semigroup ARS

$$U_{elem} = U_\kappa = \kappa(K) \uparrow G$$

$$\boxed{k = \xi_C \lambda [u]_\tau \qquad \kappa(k) = \exp(im\xi)\Delta(\lambda)}$$

$$U_{loc} = U_\chi = \chi(H) \uparrow G$$

$$\boxed{h = \xi_C \lambda [v] \qquad \chi(h) = \exp(im\xi)\Delta(\lambda)}$$

$$(\Pi^c \psi)[u]_\tau = \int_0^\tau d\tau' \int d[v] \, \exp\left[-im \int_0^\tau d\sigma \left((u, v) + \frac{1}{2}(v, v) \right) \right] \psi[u]_{\tau'}$$

where $[u]_{\tau'}$ is an initial part of $[u]_\tau$

Dynamics of non-local objects is trivial !

Propagator in space-time

- $\psi[u]_\tau$ is an amplitude of propagation along $[u]_\tau$
- Fixing an "anchor point" (x',τ'), one may map each ψ onto the corresponding distribution over space-time points

$$\Psi(x',\tau') = \int d[u]_{\tau'}\psi[u]_{\tau'}, \quad (x',\tau')[u]_\tau = (x'',0)$$

- Fixing instead an initial space-time distribution $\Psi_0(\mathbf{x}',t')$, one has

$$\boxed{\Psi(x',\tau') = \int d\{x\}_0^{\tau'}\,\alpha\{x\}_0^{\tau'}\,\psi[u]_{\tau'}\Psi_0(x'',0)}$$

$\alpha\{x\}_{\tau''}^{\tau'}$ is a **representation** of the **semigroupoid of trajectories:**

$$\alpha\{x\}_{\tau'}^{\tau}\cdot\alpha\{x\}_{\tau''}^{\tau'} = \alpha\{x\}_{\tau''}^{\tau}$$

Consequences of the group-theoretical approach

$$\Pi^c(x',\tau'|x'',\tau'') = \int d\{x\}_{\tau''}^{\tau'} \cdot \exp\left[\frac{1}{2} im \int_{\tau''}^{\tau'} d\sigma\,(\dot{x},\dot{x})\right] \alpha\{x\}_{\tau''}^{\tau'}$$

$$\alpha\{x\}_{\tau''}^{\tau'} = P\ \exp\left\{ i \int_{\tau''}^{\tau'} d\sigma \left[V(x(\sigma),\sigma) + \mathbf{A}(x(\sigma),\sigma)\dot{x} \right] \right\}$$

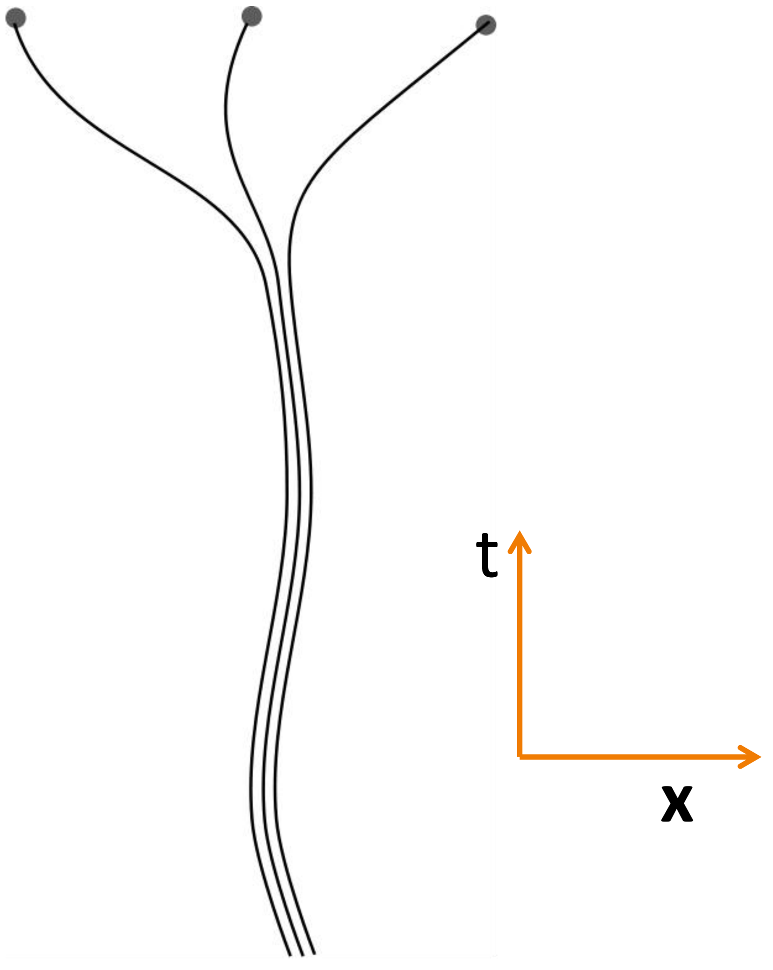
This is a causal propagator in an **external gauge field**
(may be generalized to get gravitational field)

Therefore the following **conclusions** are derived **from Path Group:**

- The **path integral form of dynamics** derived
- Form of **classical action** is derived (if comparing with Feynman postulate)
- Only fields of geometrical nature are predicted **(gauge and gravitational)**
- Interpretation of **non-local objects** ("history-strings") more natural

Quarks as "history-strings" and confinement of color

- "History-strings" of **complementary colors** as a model of a baryon
- With high probability the strings have the **same positions** in most part of the length
- The parts of the strings having the same position and complementary colors are **unobservable**



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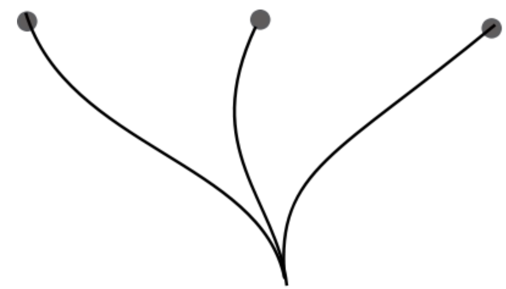

- **This configuration** is now argued in the **CHD theory to imply confinement**
- This hints that **quarks are “history-strings”** in the fundamental theory

Concluding remarks

- It is not space-time but **trajectory space** that is **fundamental**
- The resulting theory may be interpreted as theory of **non-local objects, "history-strings"**
- **Conventional dynamics** (including **classical action**) is derived if **trajectories (strings) are not observable**
- Generally **effects of non-locality** (observability of trajectories, or strings) are predicted
- They may explain **confinement of quarks**

Literature

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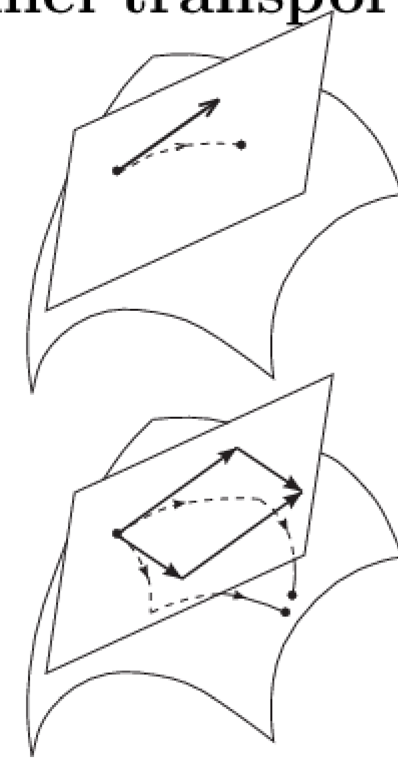
Non-commutative parallel transports

- A short translation in a tangent space corresponds to a **parallel transport along the geodesic**

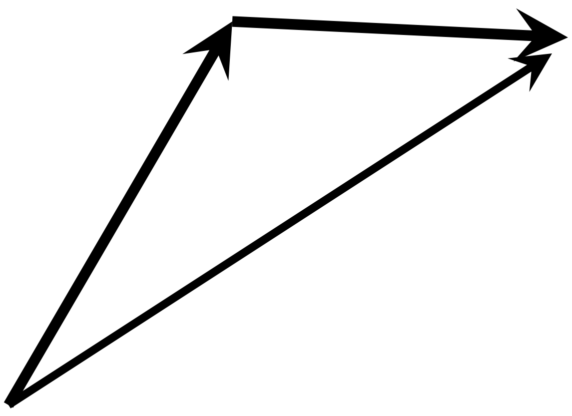
- Two short translations correspond to **non-commutative parallel transports**



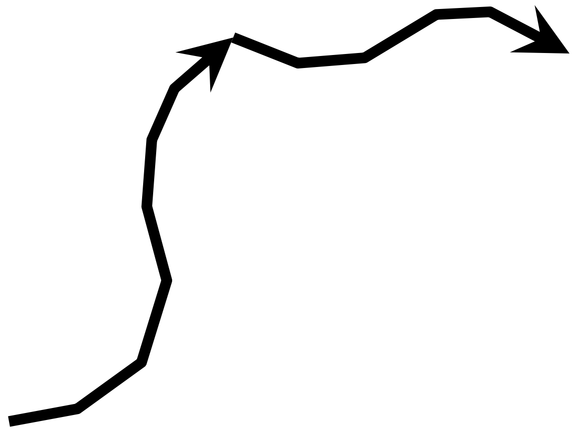
- A series of short translations (in a flat space!) is a **'path'**

Paths generalize translations

- Translations are vectors

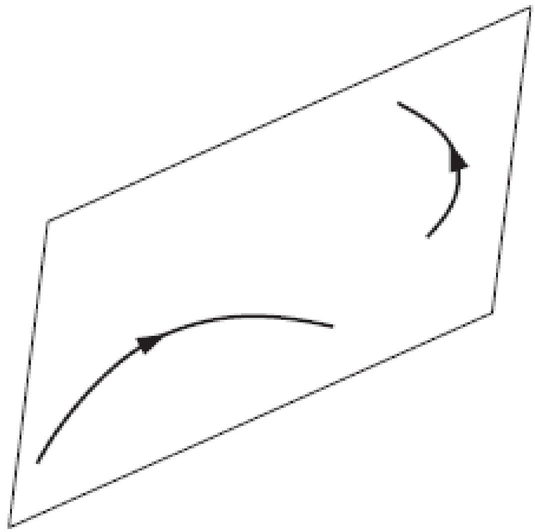


- Paths are (classes of) curves

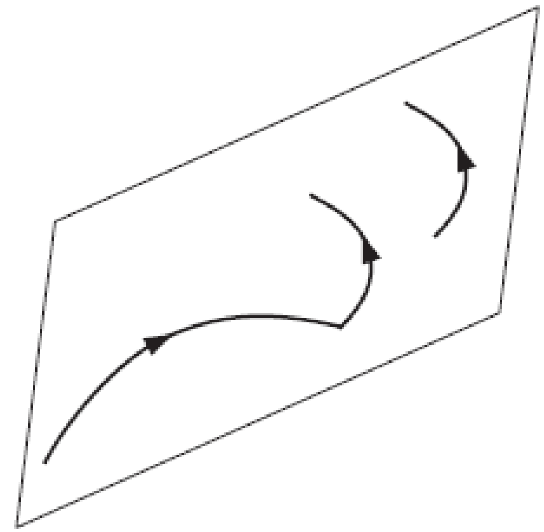


Two arbitrary paths may be multiplied:

$$p'' = p'\,p \in P \quad \text{for any} \quad p, p' \in P$$



Representative curves

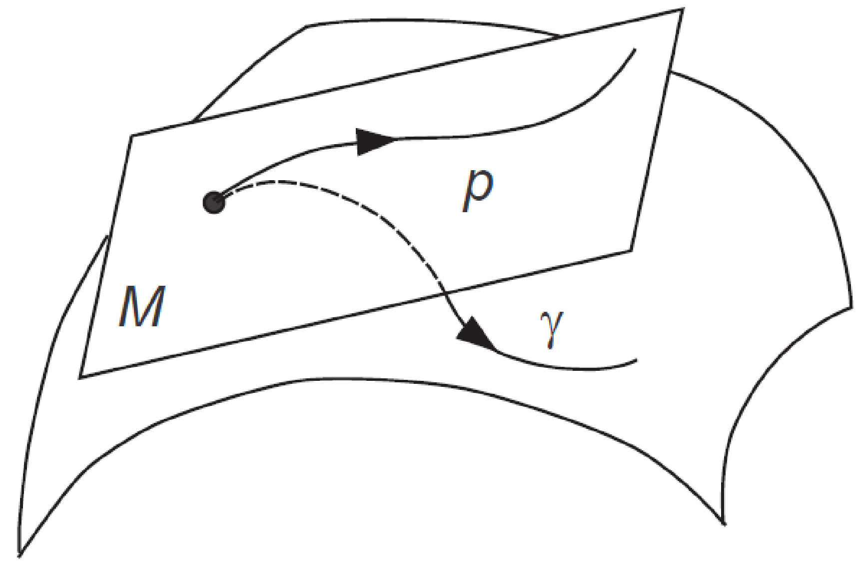


Multiplication of curves

$\Longrightarrow$ Group axioms are satisfied in P

“Flat model” of a curve

- Map of a line in the (flat) tangent space onto the line in the curved space



$$\dot{x}^{\mu}(\tau) = \dot{\xi}^{\alpha}(\tau) b^{\mu}_{\alpha}(\tau)$$

$$\dot{b}^{\lambda}_{\beta}(\tau) = \Gamma_{\mu\nu}(x(\tau))\, \dot{x}^{\mu}(\tau) b^{\nu}_{\beta}(\tau)$$

Interpretation of derivatives

- Derivatives: generate translations

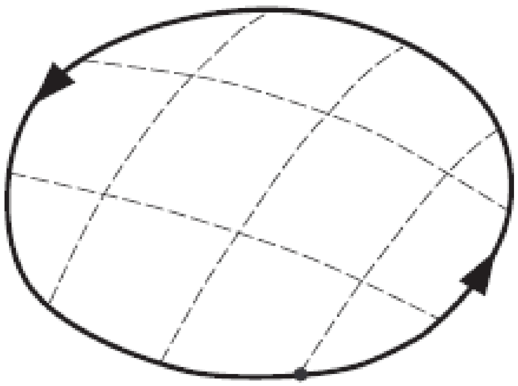
$$e^{\Delta x^\mu \ \partial_\mu} f(x) = f(x + \Delta x)$$

- **Covariant derivatives**: generate the action of paths
  - –gauge th: $U(p) = \mathcal{P} \exp\left\{ - \int_p d\xi^\mu \nabla_\mu \right\}$
  - –Gravity: $U(p) = \mathcal{P} \exp\left(\int_p d\xi^\alpha \, B_\alpha \right)$

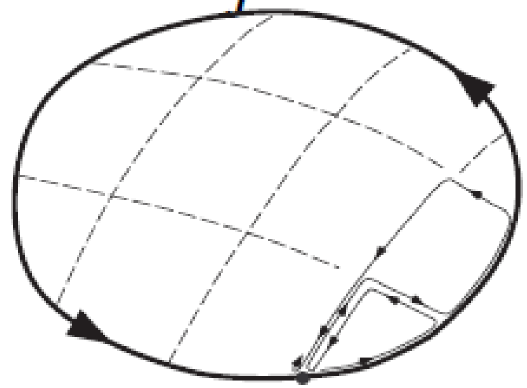
Results on Path Group

- Group-theoretical interpretation of the **path-dependent formalism** (explicitly gauge-invariant)
- **Non-Abelian Stokes theorem**
- **Quantum equivalent principle**
- Explicitly **covariant path integral** (Klein-Gordon euqation without R-potential)
- Quarks as “history-strings” **explaining confinement**

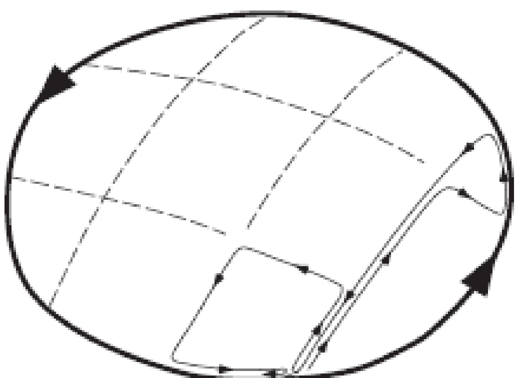
Factorization of a loop: $l = \mathcal{P} \prod_j p_j^{-1} \lambda_j p_j$



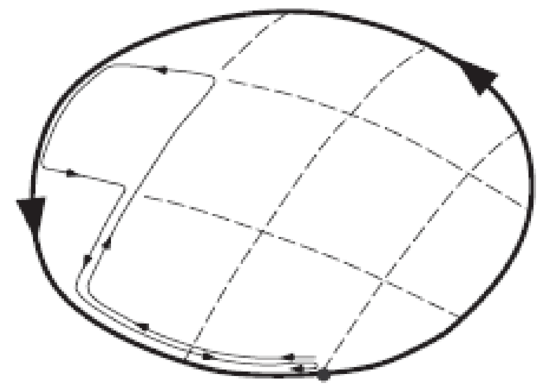
Partition of the area



Starting factorization



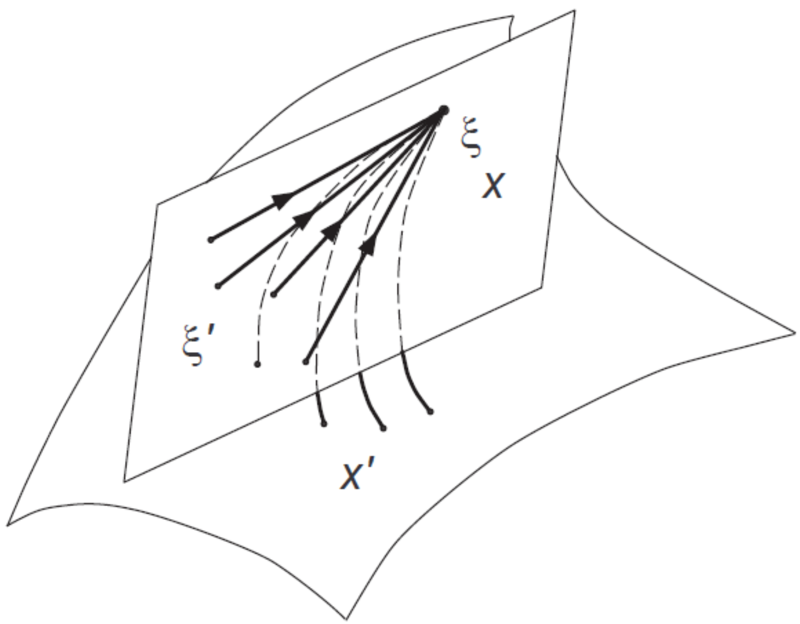
Factorization continued

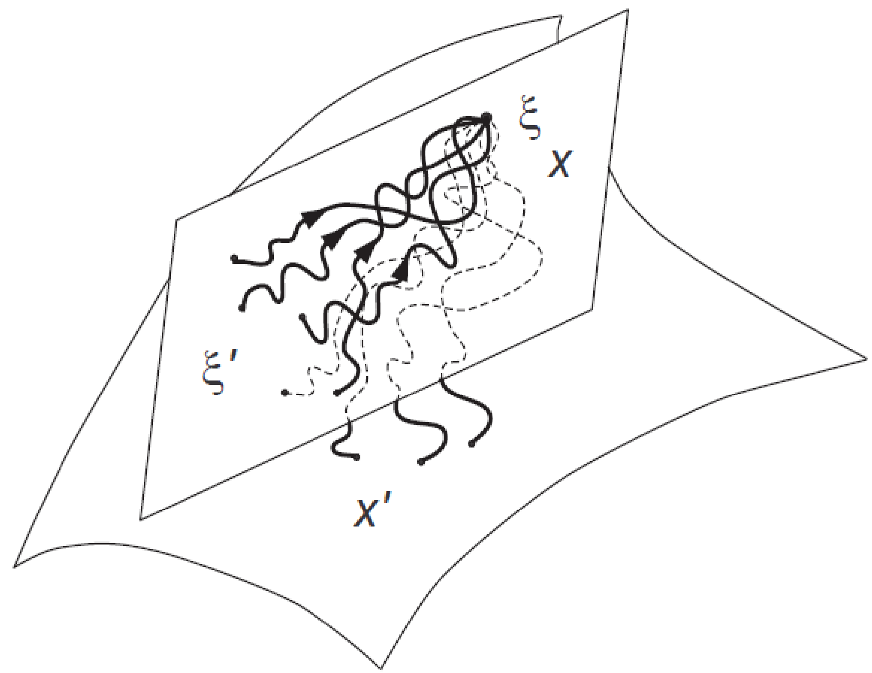


Factorization finished

Classical and quantum EP

- **Classical**: motion along **straight lines**
- Quantum: Feynman path integral





Path integral in a curved space