New uncertainty relations found due to probability representation of quantum states and recent experiments confirming these relations

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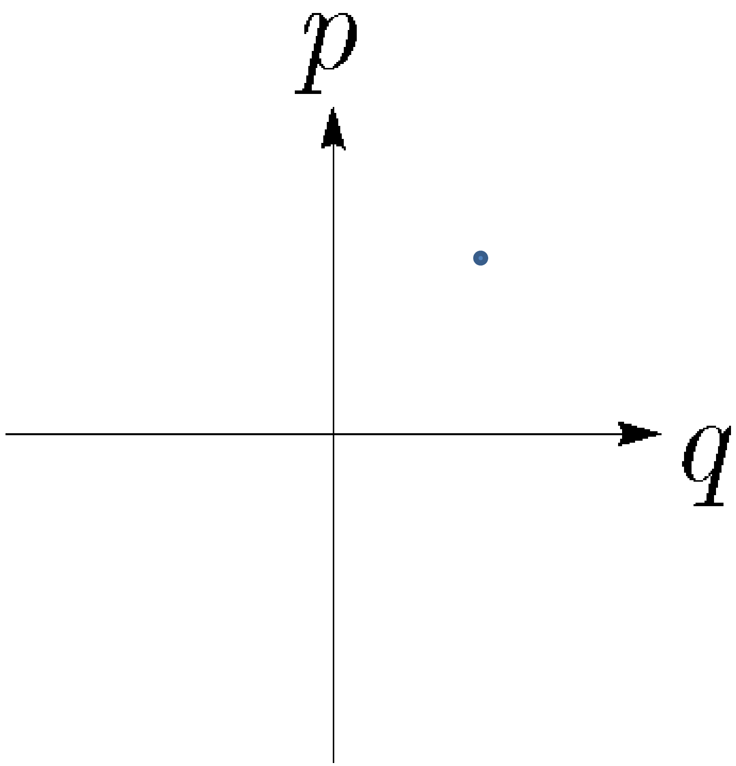
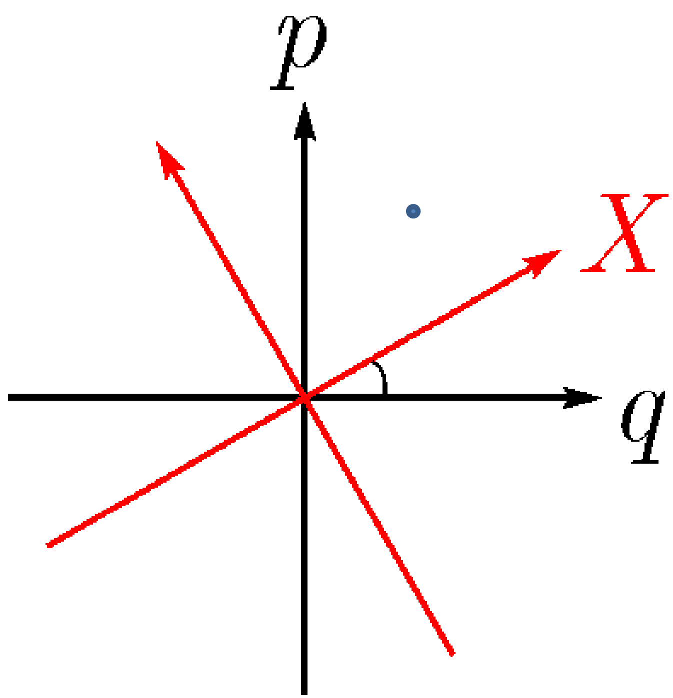
Classical Picture



$$\int f(q,p)\,dq\,dp = 1$$



$$X = q\cos\theta + p\sin\theta$$

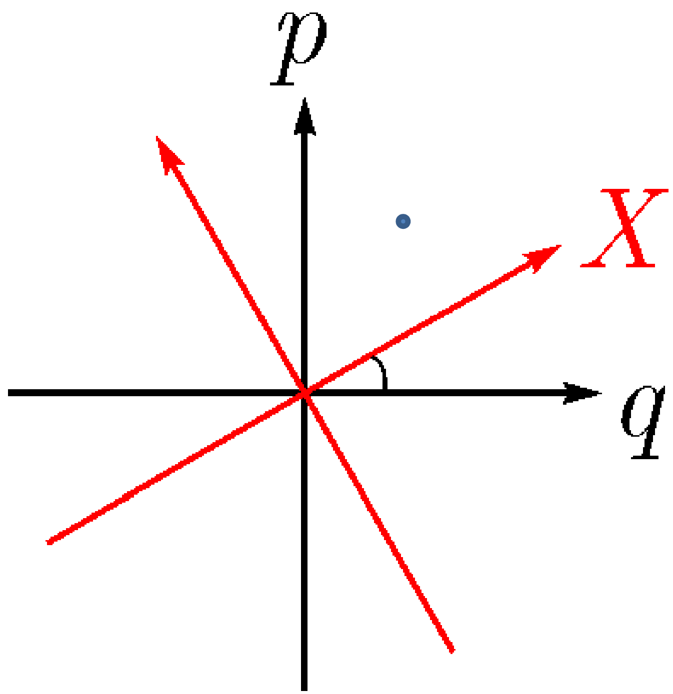
$$w(X,\theta) = \langle \delta\left(X - q\cos\theta - p\sin\theta\right)\rangle$$
$$= \int f(q,p)\delta\left(X - q\cos\theta - p\sin\theta\right)\, dq\, dp$$

$$
\begin{aligned}
M(X,\mu,\nu) &= \langle \delta\left(X-\mu q-\nu p\right)\rangle \\
&= \int f(q,p)\delta\left(X-\mu q-\nu p\right)\, dq\, dp
\end{aligned}
$$

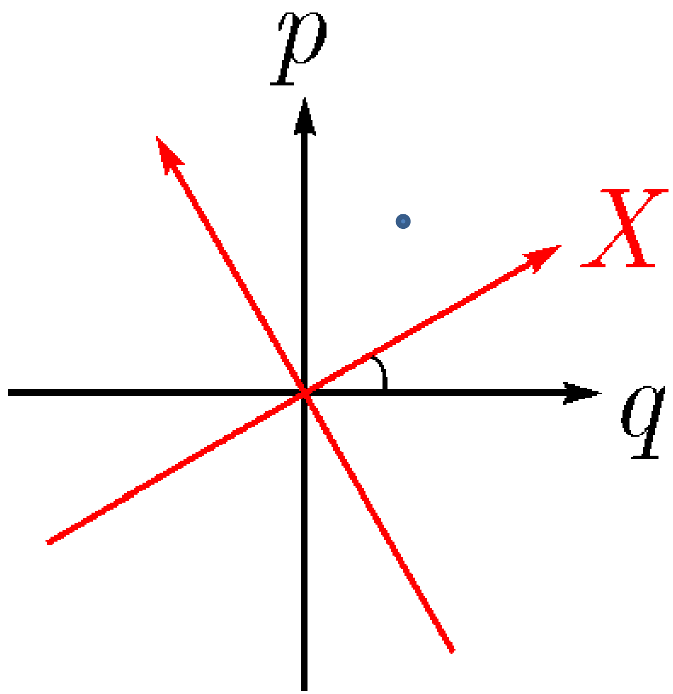
$$
\begin{aligned}
\mu &= s\cos\theta \\
\nu &= s^{-1}\sin\theta
\end{aligned}
$$

Man'ko, O. V., Man'ko, V. I., J. Russ. Laser Res. 18, 407--444 (1997)

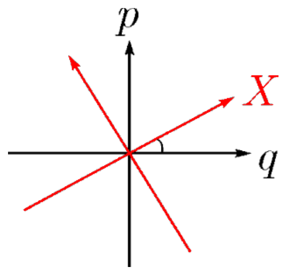
$$
\begin{aligned}
w(X,\theta) &= M(X,\cos\theta,\sin\theta) \\
M(X,\mu,\nu) &= \frac{1}{\sqrt{\mu^2+\nu^2}}\, w\left(\frac{X}{\sqrt{\mu^2+\nu^2}}, \tan^{-1}\frac{\nu}{\mu}\right)
\end{aligned}
$$



$$f(q,p) = \frac{1}{4\pi^2} \int M(X,\mu,\nu) e^{i(X-\mu q-\nu p)} dX\, d\mu\, d\nu \geq 0$$



$$\langle q^n \rangle = \int M(X,1,0)\, X^n \, dX,$$
$$\langle p^n \rangle = \int M(X,0,1)\, X^n \, dX$$



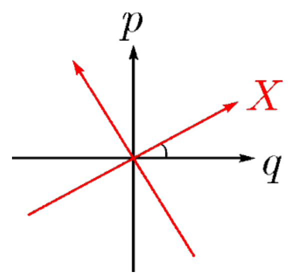
Quantum Picture

$$w(X,\theta) = \langle \delta\left(X - \widehat{q}\,\mathrm{cos}\,\theta - \widehat{p}\,\mathrm{sin}\,\theta\right)\rangle$$

$$M(X,\mu,\nu) = \langle \delta\left(X - \mu\widehat{q} - \nu\widehat{p}\right)\rangle$$

$$\int M(X,\mu,\nu)\,dX = 1$$

Mancini, S., Man'ko, V. I., Tombesi, P, Found. Phys. 27, 801-824 (1997).



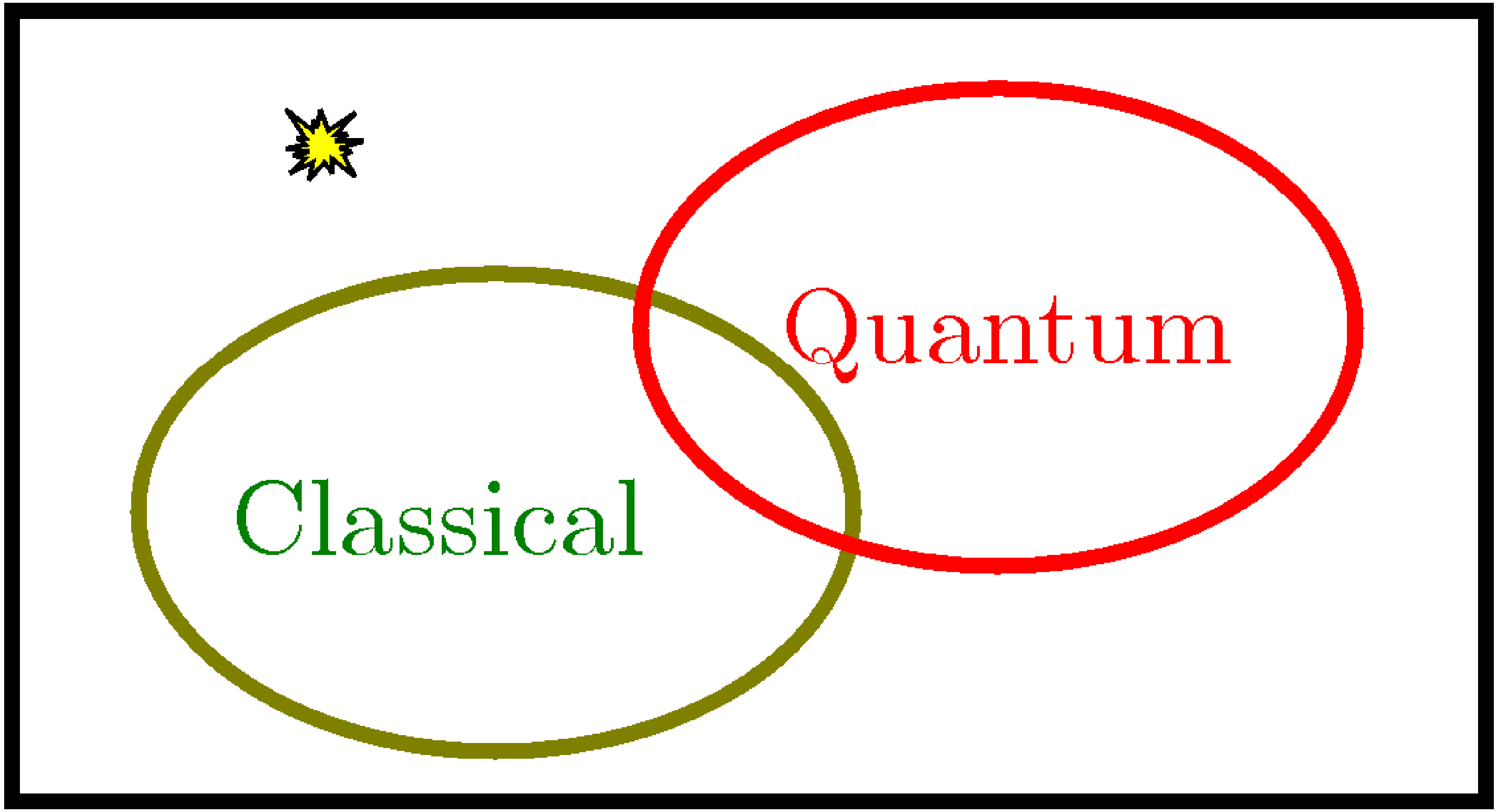
$$\psi(y)$$

$$M(X,\mu,\nu) = \frac{1}{2\pi|\nu|}\left|\int \psi(y)\exp\left[i\left(\frac{\mu}{2\nu}y^2 - \frac{Xy}{\nu}\right)\right]\,dy\right|^2$$

Man'ko V. I., Mendes, R. V, Phys. Lett. A 263, 53--56 (1999) [ArXiv Physica/9712022]

$$\widehat{\rho} = \frac{1}{2\pi} \int M(X, \mu, \nu) e^{i(X - \mu\widehat{q} - \nu\widehat{p})} \, dX \, d\mu \, d\nu$$

See also review:
Ibort, A., Man'ko, V. I., Marmo, G., Simoni, A., Ventriglia F., Phys. Scr. 79, 065013 (2009)



Review “An introduction to the tomographic picture of quantum mechanics”
Ibort, A., Man'ko, V. I., Marmo, G., Simoni, A., Ventriglia F., Phys. Scr. 79, 065013 (2009)

$$\langle X^n \rangle (\mu, \nu) = \int X^n M(X, \mu, \nu)\, dX, \qquad n = 1, 2, \ldots$$

$$\sigma_{PP}\sigma_{QQ} = \left(\int X^2 M(X, 0, 1)\, dX - \left[\int X M(X, 0, 1)\, dX \right]^2 \right)$$
$$\times \left(\int X^2 M(X, 1, 0)\, dX - \left[\int X M(X, 1, 0)\, dX \right]^2 \right) \geq \frac{1}{4}.$$

$$\sigma_{QQ}\sigma_{PP} - \sigma_{QP}^2 \geq \frac{1}{4}$$

$$\sigma_{XX}(\mu,\nu) = \mu^2\sigma_{QQ} + \nu^2\sigma_{PP} + 2\,\mu\nu\sigma_{QP}$$

$$\boxed{\sigma_{QP} = \sigma_{XX}\left(\theta = \frac{\pi}{4}\right) - \frac{1}{2}(\sigma_{QQ} + \sigma_{PP})}$$

$$\sigma_{XX}\left(\theta = \frac{\pi}{4}\right) = \left\langle X^2\right\rangle\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right) - \left[\langle X\rangle\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)\right]^2$$

$$
\begin{aligned}
F(\theta) \;=\; & \left(\int X^2 w(X,\theta) dX - \left[\int X w(X,\theta)\, dX\right]^2\right) \\
& \times \left(\int X^2 w\left(X, \theta + \frac{\pi}{2}\right) dX - \left[\int X w\left(X, \theta + \frac{\pi}{2}\right) dX\right]^2\right) \\
& - \left\{\int X^2 w\left(X, \theta + \frac{\pi}{4}\right) dX - \left[\int X w\left(X, \theta + \frac{\pi}{4}\right) dX\right]^2 \right. \\
& - \frac{1}{2}\left[\int X^2 w(X,\theta) dX - \left[\int X w(X,\theta) dX\right]^2 \right. \\
& \left.\left. + \int X^2 w\left(X, \theta + \frac{\pi}{2}\right) dX - \left[\int X w\left(X, \theta + \frac{\pi}{2}\right) dX\right]^2\right]\right\}^2 - \frac{1}{4}
\end{aligned}
$$

$$F(\theta) \geq 0$$

Man'ko, V. I., Marmo, G., Simoni, A., Ventriglia F., Adv. Sci. Lett. 2, 517-520 (2009)

$$i\frac{\partial\rho(x,x',t)}{\partial t} = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2}\right)\rho(x,x',t) + \left(U(x) - U(x')\right)\rho(x,x',t)$$

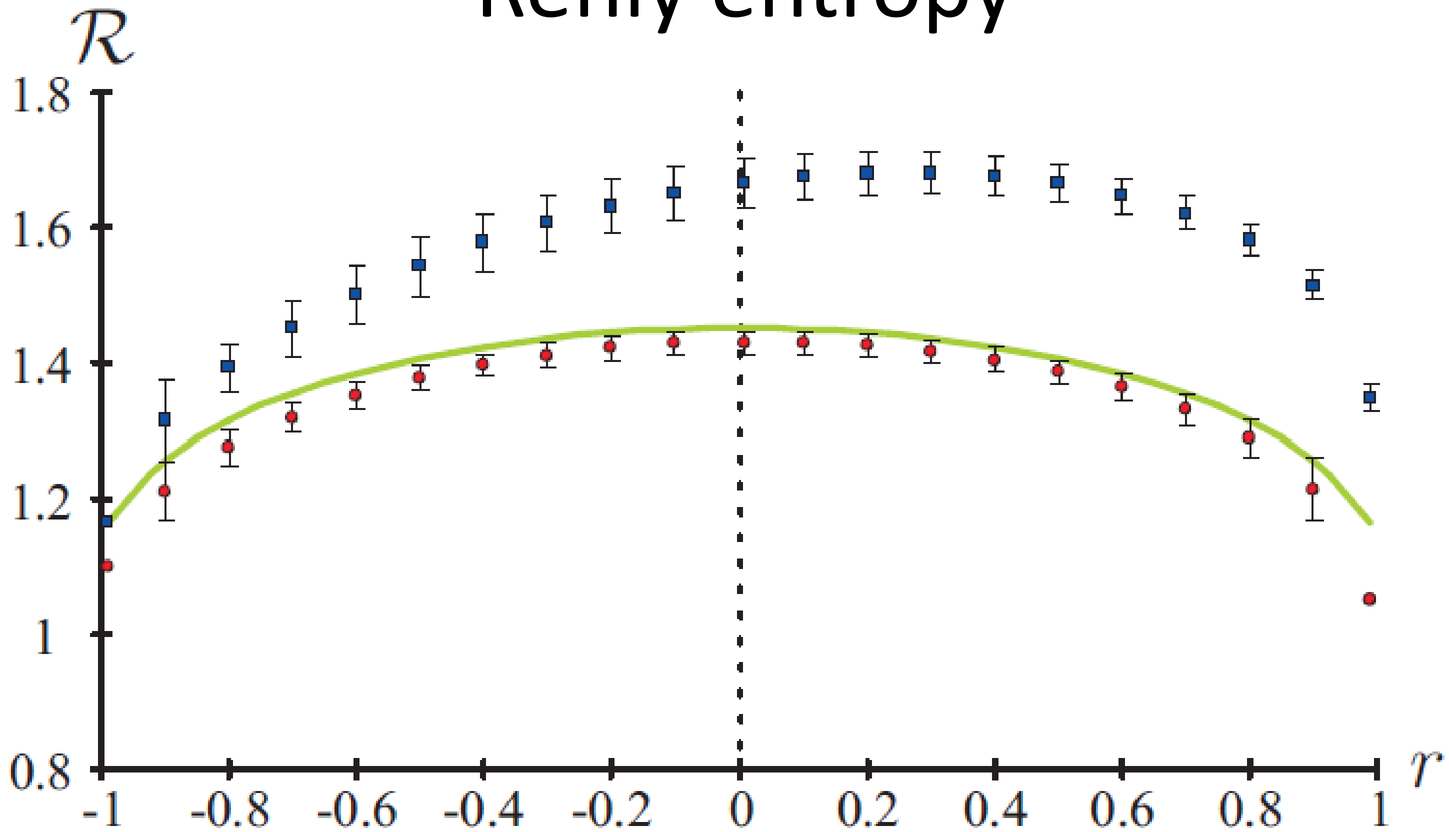
$$\frac{\partial W(q,p,t)}{\partial t} + p\frac{\partial W(q,p,t)}{\partial q} + \frac{1}{i}\left[U\left(q - \frac{i}{2}\frac{\partial}{\partial p}\right) - \text{c.c.}\right]W(q,p,t) = 0.$$

$$
\frac{\partial}{\partial t} w(X,\theta,t) = \left[\cos^2\theta \frac{\partial}{\partial\theta} - \frac{1}{2}\sin 2\theta \left\{1 + X\frac{\partial}{\partial X}\right\}\right] w(X,\theta,t)
$$

$$
+2\left[\mathrm{Im}\ U\left\{\sin\theta \frac{\partial}{\partial\theta}\left[\frac{\partial}{\partial X}\right]^{-1} + X\cos\theta + i\,\frac{\sin\theta}{2}\frac{\partial}{\partial X}\right\}\right] w(X,\theta,t).
$$

Ya.A. Korennoy and V.I. Man'ko, Journal of Russian Laser Research, **32**, 75 (2011)

Réniy entropy



$$\mathcal{R}(r) \equiv \frac{1+r}{r}\ln\left\{\int [w(X,0)]^{(1+r)^{-1}}dX\right\} - \frac{1-r}{r}\ln\left\{\int [w(X,\tfrac{\pi}{2})]^{(1-r)^{-1}}dX\right\} \geq$$

$$\geq \ln(\pi\hbar) + \tfrac{1}{2r}[(1+r)\ln(1+r) - (1-r)\ln(1-r)]$$

M. Bellini, A. S. Coelho, S. N. Filippov, V. I. Man'ko, A. Zavatta. Towards higher precision and operational use of optical homodyne tomograms. ArXiv:1203.2974 [quant-ph], in press Phys. Rev. A

Conclusions

- We reviewed the *probability representation of quantum mechanics*, where the quantum states are described by probability distributions as an alternative to density operators.
- The probability representation is discussed for continuous variables.
- The essence of this work is the *consideration of experiments to check quantum mechanics*, or better to say, to study the accuracy with which one can check experimentally the uncertainty relations for photon quadratures.
- It is worth pointing out that in the classical domain one does not have these bounds on $F(\theta)$.