Lagrange anchor and BRST complex for general gauge dynamics

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The talk is based on the series of articles:

`P.O.Kazinski, S.L.Lyakhovich, A.A.Sharapov (2005);`

`S.L.Lyakhovich and A.A.Sharapov (2004-2009);`

`D.S.Kaparulin, S.L.Lyakhovich,A.A.Sharapov (2010-2012)`

The problem setting

Given:

Classical gauge theory defined by ***field equations***.

Problems:

- Construct the classical BRST complex
- Quantize the dynamics
- Connect symmetries with characteristics (conservation laws).

The motivation is two-fold:

- It is a matter of principle to identify a general structure (more general than the variational) which is sufficient to construct the classical BRST complex, to relate symmetries and conservation laws, and to quantize the theory;
- Many important field equations are not variational.

Examples of non-Lagrangian field equations.

The list of the best known examples includes:
Interacting massless higher spin field equations, (anti-)self-dual Yang-Mills and Donaldson-Ulenbeck-Yau equations, Hitchin systems, 5-branes.

Some other classical field equations, being quite reasonable as such, but non-variational, are sided away, because no perspectives are seen to quantise, and/or to apply Noether theorems.
Examples of this type: the gravity equations involving only irreducible components of the curvature tensor

$$R=\Lambda, \qquad \textbf{or} \qquad \tilde{R}_{\mu\nu}=0, \qquad \tilde{R}_{\mu\nu}\equiv R_{\mu\nu}-\frac{1}{d}g_{\mu\nu}R, \quad g^{\mu\nu}\tilde{R}_{\mu\nu}\equiv 0$$

The eq. $R=\Lambda$ probably defines topological theory in $d=4$.
The eqs $\tilde{R}_{\mu\nu}=0$ comprise all Einstein's solutions, with all the possible cosmological constants - noticed by Einstein.

General gauge dynamics: condensed notation.

General classical dynamics are defined by two principal constituents:

- A set of fields ϕ^i;
- A set of field equations $T_a(\phi)=0$.

The "condensed" indices i,a include the space-time point x^μ, and all the discrete indices labeling components of fields, or equations. The field equations $T(\phi)=0$ are PDE's in x^μ.
Functions of fields, $F(\phi)$ are understood as the local functionals, the derivatives ∂_i by fields ϕ^i are variational.
In Lagrangian theory, i and a coincide, in general they don't.
Lagrangian field equations read $T_i(\phi)\equiv\partial_i S(\phi)=0$

The Dynamics Bundle and the Shell

The set of all the field configurations $\mathcal{M} \ni \phi$ is considered a manifold, and the solutions to the field equations form a sub-manifold $\Sigma \subset \mathcal{M}$, called the ***shell***.

$$\Sigma = \{\phi \in \mathcal{M} \,|\, T(\phi) = 0\}.$$

A vector bundle $\mathcal{E} \mapsto \mathcal{M}$ is assumed to exist such that the l.h.s. of the field equations $T_a(\phi)$ are the components of the certain section of this bundle

$$T = T_a(\phi) e^a \in \Gamma(\mathcal{E}).$$

We term $\mathcal{E}$ as the ***dynamics bundle***.

In Lagrangian theory $\mathcal{E}$ is identified with T^*M, and the field equations are just components of an exact one-form:

$$T \equiv dS(\phi) = \partial_i S(\phi) d\phi^i \in \Lambda^1(\mathcal{M}).$$

Regularity conditions in general dynamics.

Consider the Jacobi matrix $J_{ia} \equiv \partial_i T_a(\phi)$
The regularity implies that

$$\mathrm{rank} J|_{U_\Sigma} = const$$

The map defined by J,

$$\Gamma(T\mathcal{M}) \xrightarrow{J} \Gamma(\mathcal{E})$$

in general, is neither surjective, nor is it injective, and the same is true for the dual map defined by the transposed Jacobi matrix J^*
In Lagrangian theory, where $\mathcal{E} = T^*\mathcal{M}$, J is the symmetric Van Wleck matrix: $\partial_i T_j = \partial^2_{ij} S(\phi)$ whose on-shell kernel defines the gauge symmetry, and simultaneously, Noether identities.

Gauge algebra and Noether identities in general dynamics.

The rectangular Jacobi matrix $J_{ia}{=}\partial_i T_a(\phi)$ has different left and right on-shell kernels spanned by basis elements $R^i_\alpha(\phi)$ and $Z^a_A(\phi)$:

$$R^i_\alpha J_{ia}\big|_\Sigma{=}0, \qquad J_{ia}Z^a_A\big|_\Sigma{=}0.$$

Basis elements $R^i_\alpha(\phi)$ of the left kernel are understood as gauge symmetry generators. The right kernel basis elements $Z^a_A(\phi)$ are understood as generators of Noether identities.

Both sets of generators are defined modulo on-shell vanishing terms.

From the regularity of J_{ia} follows that the left kernel distribution is integrable on shell, and the right kernel is generated by Noether identity generators

$$R^j_\alpha\partial_j R^i_\beta{-}R^j_\beta\partial_j R^i_\alpha{=}U(\phi)^\gamma_{\alpha\beta}R^i_\gamma{+}W^{ia}_{\alpha\beta}T_a, \qquad Z^a_A T_a{\equiv}0,$$

In Lagrangian theory J is symmetric, and R^i_α and Z^a_A coincide.

In general, they don't.

The condensed indices α,A labeling symmetries and identities can run the different sets.

1. Maxwell electrodynamics in the strength tensor formalism
Consider anti-symmetric rank 2 tensor subject to free Maxwell equations

$$T^{\nu} \equiv \partial_{\mu} F^{\mu\nu} = 0, \quad T_{\mu\nu\lambda} \equiv \partial_{[\mu} F_{\nu\lambda]} = 0. \tag{1}$$

There are no gauge symmetry for F, but the identities exist:

$$\partial_{\nu} T^{\nu} \equiv 0, \quad \partial_{[\rho} T_{\mu\nu\lambda]} \equiv 0. \tag{2}$$

2. Self-dual Yang-Mills fields
The (anti-)self-duality equations are invariant with respect to the usual gauge transformations of the Yang-Mills field A_{μ}. These equations are independent, however - no gauge identities at all.
A similar phenomenon is observed with DUY equations.

Reducibility of gauge symmetries and identities in general dynamics.

Both gauge symmetry and Noether identity generators can be reducible, i.e. the "null-vectors" $\overset{(1)}{R}, \overset{(1)}{Z}$ exist such that

$$\overset{(1)}{R}{}^{\alpha}_{\alpha_1}(\phi) R^i_{\alpha}(\phi)|_{\Sigma} = 0, \quad \overset{(1)}{Z}{}^{A}_{A_1}(\phi) Z^a_A(\phi)|_{\Sigma} = 0$$

The reducibility generators $\overset{(1)}{R}, \overset{(1)}{Z}$ can be reducible in their own turn, so we have a sequence of the "null-vectors" $\overset{(k)}{R}, \overset{(l)}{Z}, [k] = m, [l] = n$. In Lagrangian theory $m = n$, and the reducibility generators coincide for Noether identities and gauge symmetries. In general, these are different.

Reducibility of (m,n)-type.

The reducibility generators are supposed to define morphism of certain bundles, such that

$$0 \longleftarrow \Gamma(\mathcal{F}^*_m) \overset{\overset{(m-1)}{R}{}^*}{\longleftarrow} \cdots \Gamma(\mathcal{F}^*_1) \overset{R^*}{\longleftarrow} \Gamma(T^*\mathcal{M}) \overset{J^*}{\longleftarrow} \Gamma(\mathcal{E}^*) \overset{Z}{\longleftarrow} \Gamma(\mathcal{G}_1) \overset{\overset{(1)}{Z}}{\longleftarrow} \cdots \Gamma(\mathcal{G}_n) \longleftarrow 0$$

This sequence is on-shell exact as

$$\mathrm{Im}\overset{(k)}{R} = \mathrm{Ker}\overset{(k-1)}{R}, \mathrm{Im}R = \mathrm{Ker}J, \quad \mathrm{Im}\overset{(k)}{Z} = \mathrm{Ker}\overset{(k-1)}{Z}, \quad \mathrm{Im}Z = \mathrm{Ker}J^*$$

In Lagrangian theory $m=n$, $\mathcal{F}_k = \mathcal{G}_k, \forall k$, $\mathcal{E} = T^*M$, $J = J^*, R = Z$,and the "wings"outside the central segment, defined by J^*, can be identified just by taking dual and transposed map.

In general, none of these coincidences occurs, and the theory is termed (m,n)-reducible.

Gauge theory of (1,1)-type: introducing ghosts

The bundle $\mathcal{F} \longmapsto M$ that "hosts" the gauge symmetries, is called the Gauge Algebra Bundle.
The bundle $\mathcal{G} \longmapsto M$ that "hosts" the generators of Noether identities, is called the Noether Identity Bundle.
Ghost extension of the configuration space for general (1,1) dynamics
Consider $Z_2 \bigotimes Z$-graded bundle

$$\mathcal{L} \longmapsto M : \mathcal{L} = \Pi(\mathcal{F}[1]) \oplus \Pi(\mathcal{E}[-1]) \oplus (\mathcal{G}[-2])$$

The coordinates are denoted correspondingly:

$$C^{\alpha}, \quad \eta_a, \quad \xi_A, \qquad gh(C) = 1, \quad gh(\eta) = -1, \quad gh(\xi) = -2.$$

In Lagrangian case, η_a would be the anti-field ϕ^*_i to the original field ϕ^i; C^{α} - the gauge ghost, and ξ_A identified as anti-field to C.

Classical BRST embedding of general (1,1) dynamics

The BRST-differential $Q, gh(Q) = 1$ is sought for in the form

$$Q \equiv Q^I(\varphi)\frac{\partial}{\partial\varphi^I} = T_a\frac{\partial}{\partial\eta_a} \quad + \quad \eta_a Z^a_A\frac{\partial}{\partial\xi_A} \quad + \quad C^\alpha R^i_\alpha\frac{\partial}{\partial\phi^i} \quad + \quad \cdots,$$

carrying all the information about the classical system $(\mathcal{E}, T)$ as such. Evaluating the condition $Q^2 = 0$ in the lowest order in r-degree, $|\xi|_r = 2, \quad |\eta|_r = 1$, one immediately comes to the relations $Z^a_A T_a \equiv 0$, $R^i_\alpha \partial_i T_b = U^a_{\alpha b}(\phi) T_a$ characterizing $T_a(x) = 0$ as a set of gauge invariant and linearly dependent equations of motion, with R and Z being the generators of gauge transformations and Noether identities, respectively.

Lagrange anchor - a preliminary observation.

Consider first (0,0) type dynamics with $\mathcal{E}=T^*\mathcal{M}$, so the left hand sides of dynamical equations are the components of one-form:

$$T_i(\phi)d\phi^i=T\in\Lambda T^*\mathcal{M}, \qquad \Sigma=\{\phi\in\mathcal{M}\,|\,T_i(\phi)=0\} \qquad J_{ij}=\partial_i T_j$$

The fact that the dynamics are Lagrangian means that $dT=0$, or that the Jacobi matrix is symmetric, $J^*=J$, i.e. the following diagram commutes:

$$\begin{array}{ccc}
\Gamma(T\mathcal{M}) & \xrightarrow{J} & \Gamma(T^*\mathcal{M}) \\
\uparrow{\scriptstyle id} & & \uparrow{\scriptstyle id} \\
\Gamma(T\mathcal{M}) & \xrightarrow{J^*} & \Gamma(T^*\mathcal{M})
\end{array}$$

Lagrange anchor: a diagrammatic definition.

The Lagrange anchor V defines a bundle homomorphism $V{:}\mathcal{E}^*\longrightarrow TM$ such that the diagram

$$\begin{tikzcd} \Gamma(TM) \arrow[r, "J"] & \Gamma(\mathcal{E}) \\ \Gamma(\mathcal{E}^*) \arrow[u, "V"] \arrow[r, "J^*"] & \Gamma(T^*M) \arrow[u, "V^*"'] \end{tikzcd} \tag{3}$$

commutes on the shell. Off shell this explicitly reads

$$V_a^i\partial_i T_b - V_b^i\partial_i T_a = C_{ab}^c T_c$$

If the anchor was invertible, V^{-1} would be an integrating multiplier for the inverse problem of variational calculus, i.e. $\exists S(\phi){:}\partial S_i=(V^{-1})_i^a T_a$.

Lagrange anchor for general (1,1) dynamics.

Consider now the case of (1,1) dynamics, and denote gauge algebra bundle $\mathcal{F}$, and Noether identity bundle $\mathcal{G}$. Then, the regularity of the (1,1) dynamics is formulated in terms of the following exact sequence of homomorphisms

$$0 \longrightarrow \Gamma(\mathcal{F}) \xrightarrow{R} \Gamma(TM) \xrightarrow{J} \Gamma(\mathcal{E}) \xrightarrow{Z^*} \Gamma(\mathcal{G}^*) \longrightarrow 0$$

Its transpose reads:

$$0 \longleftarrow \Gamma(\mathcal{F}^*) \xleftarrow{R^*} \Gamma(T^*M) \xleftarrow{J^*} \Gamma(\mathcal{E}^*) \xleftarrow{Z} \Gamma(\mathcal{G}) \longleftarrow 0$$

Upon restriction to Σ these sequences make cochain complexes; the properties $Z^*\circ J|_\Sigma=0$ and $J^*\circ Z|_\Sigma=0$ follow from the differential consequences of the identity $Z^a T_a=0$.

Lagrange anchor for general (1,1) dynamics.

Given the Lagrange anchor, V, the previous two diagrams can be combined into the following unified one:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(\mathcal{F}) \arrow[r, "R"] & \Gamma(TM) \arrow[r, "J"] & \Gamma(\mathcal{E}) \arrow[r, "Z^*"] & \Gamma(\mathcal{G}^*) \arrow[r] & 0 \\
0 \arrow[r] & \Gamma(\mathcal{G}) \arrow[r, "Z"] \arrow[u, "W"'] & \Gamma(\mathcal{E}^*) \arrow[r, "J^*"] \arrow[u, "V"'] & \Gamma(T^*M) \arrow[r, "R^*"] \arrow[u, "V^*"'] & \Gamma(\mathcal{F}^*) \arrow[r] \arrow[u, "W^*"'] & 0
\end{tikzcd}$$

We know that the horizontal arrows of this diagram make cochain complexes upon restriction to the shell. Then, the on-shell commutativity of the squares implies that the upward arrows define a co-chain map. It is sufficient to have only V providing commutativity of the central block, then the map W can always be constructed.

A brief preview of BRST quantization algorithm for not necessarily Lagrangian dynamics.

1. The classical BRST differential Q is constructed on the bundle $\mathcal{L}\mapsto M$: $\mathcal{L}=\Pi(\mathcal{F})[1]\oplus\Pi(\mathcal{E})[-1]\oplus\mathcal{G}[-2]$.
2. Given the Lagrange anchor, the classical BRST differential Q is promoted to a BRST charge Ω, being a function(al) on a bundle $T^*\mathcal{L}$. Q defines the first order of Ω in the momenta in $\mathcal{L}$. **The second order is defined by the Lagrange anchor** and the higher orders are sought from the equation $\{\Omega,\Omega\}=0$.
3. Upon quantisation, Ω turns into the quantum BRST operator $\hat{Q}=\Omega(\phi,\bar{\phi}\mapsto i\hbar\partial)=Q_{classical}+i\hbar(\cdots)$, $\hat{Q}^2=0$. The operator $\hat{Q}$ defines the probability amplitude by the equation $\hat{Q}\Psi(x)=0$.

Identification for Lagrangian system:

$a\equiv i,\quad T_i=\partial_i s(x),\quad V_a^j=\delta_a^j,\quad Q=(\cdot,S),\quad \hat{Q}=Q+i\hbar\Delta,\quad \Psi=e^{\frac{i}{\hbar}S}$

Lagrange anchor and Schwinger-Dyson equation.

Consider field equations T_a and Lagrange anchor V_a^i for them

$$V_a^i(\phi)\partial_i T_b(\phi) - V_b^i(\phi) T_a(\phi) = C_{ab}^c(\phi) T_c(\phi) \qquad (4)$$

The quantum probability amplitude $\Psi(\phi)$ is then defined by the following generalization of Schwinger-Dyson equation

$$\hat{T}_a \Psi(\phi) = 0, \qquad \hat{T}_a = i\hbar V_a^j \partial_j - T_a(\phi) \qquad (5)$$

The anchor definition (4) is a compatibility condition for (5). Consider adapted coordinates: $\phi \longmapsto (x, y): V_x = 1, V_y = 0$, Then $T_x = \partial_x s(x), det \partial_y T_y(x, y) \neq 0$, and probability amplitude reads:

$$\Psi(x, y) \sim \delta(T_y(x, y)) e^{\frac{i}{\hbar} S(x)}$$

The transformation to adapted coordinates breaks locality, in general, but for the original fields, the equation (5) is local.

What is the main impact of the Lagrange anchor existence for general dynamics?

The Lagrange anchor, being found for the system of classical field equations, allows one to solve the following problems:

- Covariantly quantize dynamics in three different ways:
  1. Construct the quantum BV master (or Schwinger-Dyson) equation for the amplitude;
  2. Convert not necessarily Lagrangian model in d into an equivalent topological Lagrangian theory in $d+1$ dimensions;
  3. Embed any field theory model into an *augmented* Lagrangian theory that allows to derive the quantum correlators for original fields.
- Connect conservation laws with symmetries;
- Equip the variety of conserved currents with the structure of Poisson algebra (in Lagrangian case that reduces to Gelfand-Dickey algebra).