Near–horizon black holes in diverse dimensions and integrable models

Anton Galajinsky

Tomsk Polytechnic University

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Plan

1. Motivations
2. Near–horizon extremal Kerr geometry in $d = 4$
3. Near–horizon Killing tensor
4. Near–horizon geometry of extremal rotating black hole in arbitrary dimension
5. Conformal mechanics related to near–horizon black hole
6. Near–horizon black holes and integrable models

A.G., O. Lechtenfeld, A. Nersessian, work in progress

Motivations

- In arbitrary dimension, black hole may rotate in various orthogonal spatial 2–planes, the isometry group being $U(1)^n$ for n independent rotation parameters. $U(1)^n \to U(n)$ if all the rotation parameters are equal.

- The metric is invariant under time translations. Near the horizon $U(1) \to SO(2,1)$ (the Kerr/CFT correspondence).

- The model of a massive relativistic particle on such a background inherits $SO(2,1) \times U(n)$ symmetry. The angular sector of the latter gives rise to a reduced (super)integrable model, which accommodates $U(n)$.

- The dynamics of the integrable reduction is governed by the near horizon Killing tensor of the second rank.

Kerr metric in Boyer–Lindquist coordinates

$$ds^2 = dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \frac{2Mr}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 - (r^2 + a^2)\sin^2\theta d\phi^2$$

$$\Delta = r^2 + a^2 - 2Mr, \qquad \rho^2 = r^2 + a^2\cos^2\theta$$

M is the mass and $J = aM$ is the angular momentum.

Extremal solution $\left(\Delta(r_0) = 0,\ \Delta'(r_0) = 0\right)$

$$r_0^2 = M^2 = a^2$$

where r_0 is the horizon radius.

Isometries $\left(U(1) \times U(1)\right)$

$$t' = t + \alpha, \qquad \phi' = \phi + \beta$$

Near–horizon extremal Kerr geometry in four dimensions

A natural definition of the near horizon limit

$$r \quad \rightarrow \quad r_0 + \epsilon r_0 r \; ; \qquad \epsilon \rightarrow 0$$

yields a degenerate metric.

A way out:

- Observation

$$\frac{\rho^2}{\Delta} dr^2 \quad \rightarrow \quad r_0^2 (1+\cos^2\theta) \boxed{\tfrac{dr^2}{r^2}} \, , \qquad\qquad ds^2_{AdS_2} = r^2 dt^2 - \boxed{\tfrac{dr^2}{r^2}}$$

- Rewrite the metric

$$ds^2 = \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - \frac{\sin^2\theta}{\rho^2}\left(adt - (r^2+a^2)d\phi\right)^2$$

- Extend the natural prescription

$$t \quad \rightarrow \quad \frac{2r_0 t}{\epsilon}, \qquad \phi \quad \rightarrow \quad \phi + \frac{t}{\epsilon}$$

Near–horizon extremal Kerr metric (J. Bardeen, G. Horowitz, 1999)

$$ds^2 = r_0^2(1+\cos^2\theta)\left(\boxed{r^2dt^2 - \tfrac{dr^2}{r^2}} - d\theta^2\right) - \frac{4r_0^2\sin^2\theta}{(1+\cos^2\theta)}(rdt+d\phi)^2$$

is a vacuum solution of the Einstein equations.

Extra isometries $(SO(2,1)\times U(1))$

$$t' = t + \gamma t, \qquad r' = r - \gamma r \quad \boxed{\text{dilatation}}$$

$$t' = t + (t^2 + \frac{1}{r^2})\sigma, \quad r' = r - 2tr\sigma, \quad \phi' = \phi - \frac{2}{r}\sigma \quad \boxed{\text{special conf.transf.}}$$

Similar relations hold for Kerr–Newman and Kerr–Newman–AdS black holes.

$SO(2,1)\times U(1) \quad \rightarrow \quad$ Kerr/CFT correspondence (2009)

The second rank Killing tensor in Kerr geometry $(x^m = (t, r, \theta, \phi))$

$$L_{mn} = Q_{mn} + r^2 g_{mn}, \qquad \nabla_{(n} L_{mp)} = 0$$

where

$$Q_{mn} = \begin{pmatrix} -\Delta & 0 & 0 & a\Delta \sin^2\theta \\ 0 & \frac{\rho^4}{\Delta} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a\Delta \sin^2\theta & 0 & 0 & -a^2\Delta \sin^4\theta \end{pmatrix}$$

The near horizon Killing tensor

$$L_{nm} dx^n dx^m = (1+\cos^2\theta)^2 \left[r^2 dt^2 - \frac{1}{r^2} dr^2 \right] = \boxed{p_\theta^2 + \left(\frac{1+\cos^2\theta}{2\sin\theta}\right)^2 p_\phi^2 + m^2 \cos^2\theta}$$

is reducible $((\xi_n^{(1)}, \xi_n^{(2)}, \xi_n^{(3)}, \xi_n^{(4)})$ – Killing vectors)

$$L_{nm} = \frac{1}{2}\left(\xi_n^{(1)}\xi_m^{(3)} + \xi_n^{(3)}\xi_m^{(1)}\right) - \xi_n^{(2)}\xi_m^{(2)} + \xi_n^{(4)}\xi_m^{(4)}$$

Myers–Perry solution $\left(\sum_{i=1}^{n}\mu_i^2=1,\ a_n=0\right)$

$$ds^2=dt^2-\frac{U}{\Delta}dr^2-\frac{2M}{U}\left(dt-\sum_{i=1}^{n-1}a_i\mu_i^2 d\phi_i\right)^2-\sum_{i=1}^{n}(r^2+a_i^2)d\mu_i^2-$$

$$-\sum_{i=1}^{n-1}(r^2+a_i^2)\mu_i^2 d\phi_i^2,$$

$$\Delta=\frac{1}{r}\prod_{i=1}^{n-1}(r^2+a_i^2)-2M,\qquad U=r\sum_{i=1}^{n}\frac{\mu_i^2}{r^2+a_i^2}\prod_{j=1}^{n-1}(r^2+a_j^2)$$

Near–horizon extremal solution $\left(a_i=a,\ \rho_0^2=\frac{1+(2n-3)\mu_n^2}{2n-3}\right)$

$$ds^2=\boxed{\rho_0^2\left(r^2dt^2-\frac{dr^2}{r^2}\right)}-\frac{4}{(2n-3)^2\rho_0^2}\sum_{i=1}^{n-1}\mu_i^2(rdt+d\phi_i)^2-d\mu_n^2-$$

$$-2(n-1)\sum_{i=1}^{n-1}d\mu_i^2+\frac{2}{(n-1)(2n-3)\rho_0^2}\sum_{i<j}^{n-1}\mu_i^2\mu_j^2(d\phi_i-d\phi_j)^2$$

Near–horizon extremal solution in $d = 2n + 1$

Myers–Perry solution

$$ds^2 = dt^2 - \frac{U}{\Delta} dr^2 - \frac{2M}{U}\left(dt - \sum_{i=1}^{n} a_i \mu_i^2 d\phi_i\right)^2 - \sum_{i=1}^{n} (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2)$$

$$\Delta = \frac{1}{r^2}\prod_{i=1}^{n}(r^2 + a_i^2) - 2M, \quad U = \sum_{i=1}^{n} \frac{\mu_i^2}{r^2 + a_i^2}\prod_{j=1}^{n}(r^2 + a_j^2), \quad \sum_{i=1}^{n}\mu_i^2 = 1$$

There are n rotation parameters corresponding to n azimuthal coordinates ϕ_i.

Near–horizon extremal solution (maximally symmetric configuration)

$$ds^2 = \boxed{r^2 dt^2 - \tfrac{dr^2}{r^2}} - 2n(n-1)\sum_{i=1}^{n} d\mu_i^2 - 2\sum_{i=1}^{n}\mu_i^2 (r dt + d\phi_i)^2 + \\ + \frac{2(n-1)}{n}\sum_{i<j}^{n}\mu_i^2\mu_j^2 (d\phi_i - d\phi_j)^2$$

Consider a massive relativistic particle moving near the horizon of a rotating black hole and solve the mass shell condition $g^{nm}p_n p_m = m^2$ for p_0

$$p_0 = H = r\left(\sqrt{(rp_r)^2 + L} - \sum_i p_{\phi_i}\right)$$

Here $L = L(\mu_i, p_{\mu_i}, \phi_i, p_{\phi_i})$ is the near horizon Killing tensor, which is quadratic in momenta.

Conformal generators

$$D = tH + rp_r, \quad K = \frac{1}{r}\left(\sqrt{(rp_r)^2 + L} + \sum_i p_{\phi_i}\right) + t^2 H + 2trp_r$$

Casimir element of $so(2,1)$

$$HK - D^2 = L - \left(\sum_i p_{\phi_i}\right)^2$$

Conformal mechanics related to near–horizon black hole

Canonical transformation to conventional conformal mechanics
(S. Bellucci, A.G., E. Ivanov, S. Krivonos, 2003; A.G., A. Nersessian, 2011)

$$r,\ p_r \quad \rightarrow \quad X=\sqrt{2K_0},\ P=-\frac{2D_0}{\sqrt{2K_0}}$$

$$H \quad \rightarrow \quad \boxed{H=\tfrac{1}{2}P^2+\tfrac{2g^2}{X^2}}$$

where $D_0 = D|_{t=0}$, $K_0 = K|_{t=0}$ and

$$g^2 = L - \left(\sum_i p_{\phi_i}\right)^2$$

The angular sector can be viewed as a reduced Hamiltonian system governed by

$$\tilde{H} = g^2$$

Hamiltonian (m^2 is a coupling constant)

$$\tilde{H} = \frac{1}{(2n-3)(2n-2)} \sum_{i,j=1}^{n-1} ((2n-3)\rho_0^2 \delta_{ij} - \mu_i \mu_j) p_{\mu_i} p_{\mu_j} + \\ + \sum_{i,j=1}^{n-1} \left(\frac{(2n-3)(2n-2)\rho_0^2}{4\mu_i^2} \delta_{ij} - \frac{(2n-3)^2 \rho_0^2}{4} - 1 \right) p_{\phi_i} p_{\phi_j} + m^2 \rho_0^2$$

where

$$\rho_0^2 = \frac{2(n-1)}{2n-3} - \sum_{i=1}^{n-1} \mu_i^2$$

In particular, for $d = 4$ one finds ($\mu_1 = \sin\theta$, $\mu_2 = \cos\theta$)

$$\mathcal{H} = p_\theta^2 + \left(\left[\frac{1 + \cos^2\theta}{2\sin\theta} \right]^2 - 1 \right) p_\phi^2 + m^2 (1 + \cos^2\theta)$$

Because p_ϕ is conserved, this is an integrable model.

Further reduction (set p_{ϕ_i} to be coupling constants)

$$\tilde{H}_{red} = \frac{1}{(2n-3)(2n-2)} \sum_{i,j=1}^{n-1} ((2n-3)\rho_0^2 \delta_{ij} - \mu_i \mu_j) p_{\mu_i} p_{\mu_j} + \sum_{i=1}^{n-1} \frac{g_i^2 \rho_0^2}{\mu_i^2} + \\ + \nu \sum_{i=1}^{n-1} \mu_i^2$$

where ν and g_i are coupling constants and

$$\rho_0^2 = \frac{2(n-1)}{2n-3} - \sum_{i=1}^{n-1} \mu_i^2$$

For $d = 4$ this yields

$$\tilde{\mathcal{H}}_{red} = p_\theta^2 + g^2 \cot^2 \theta + \nu \cos^2 \theta$$

Hamiltonian ($\mu_n^2 = 1 - \sum_{i=1}^{n-1} \mu_i^2$)

$$\tilde{H} = \frac{1}{2n(n-1)} \sum_{i,j=1}^{n-1} (\delta_{ij} - \mu_i \mu_j) p_{\mu_i} p_{\mu_j} + \sum_{i,j=1}^{n} \left(\frac{n}{2\mu_i^2} \delta_{ij} - \frac{(n+1)}{2} \right) p_{\phi_i} p_{\phi_j}$$

In particular, for $d = 5$ one finds $(\mu_1 = \sin\theta,\ \mu_2 = \cos\theta)$

$$\mathcal{H} = \frac{1}{4} p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} + \frac{p_\psi^2}{\cos^2\theta} - \frac{3}{2} (p_\phi + p_\psi)^2 = J_i J_i - \frac{3}{2} J_0^2$$

where J_i and J_0 form $su(2) \oplus u(1)$

$$J_0 = p_\psi + p_\phi, \qquad J_1 = p_\psi - p_\phi$$

$$J_2 = \frac{1}{2} p_\theta \cos\left(\tfrac{1}{2}(\psi - \phi)\right) + (p_\phi \cot\theta + p_\psi \tan\theta) \sin\left(\tfrac{1}{2}(\psi - \phi)\right)$$

$$J_3 = \frac{1}{2} p_\theta \sin\left(\tfrac{1}{2}(\psi - \phi)\right) - (p_\phi \cot\theta + p_\psi \tan\theta) \cos\left(\tfrac{1}{2}(\psi - \phi)\right)$$

The Hamiltonian $\mathcal{H}$ describes a minimally superintegrable model.

Rotating black holes in $d = 2n + 1$ and integrable models

Further reduction (set p_{ϕ_i} to be coupling constants)

$$\tilde{H}_{red} = \frac{1}{2n(n-1)} \sum_{i,j=1}^{n-1} (\delta_{ij} - \mu_i \mu_j) p_{\mu_i} p_{\mu_j} + \sum_{i=1}^{n} \frac{g_i^2}{\mu_i^2}$$

In $d = 5$ one reveals a dihedral systems on a circle

$$\tilde{\mathcal{H}}_{red} = \frac{1}{4} p_\theta^2 + \frac{\nu_1^2}{\sin^2\theta} + \frac{\nu_2^2}{\cos^2\theta}$$

where ν_1 and ν_2 are coupling constants.

Work to be done

- Explicit realization of symmetries, which underlie the (super)integrable mechanics related to the near horizon extremal black hole in arbitrary dimension.
- Construction of mechanics models related to less symmetric configurations (not all rotation parameters a_i are equal)
- Generalization of the analysis to the case of a rotating black hole on AdS background.