Entanglement Renyi Entropy in CFT's

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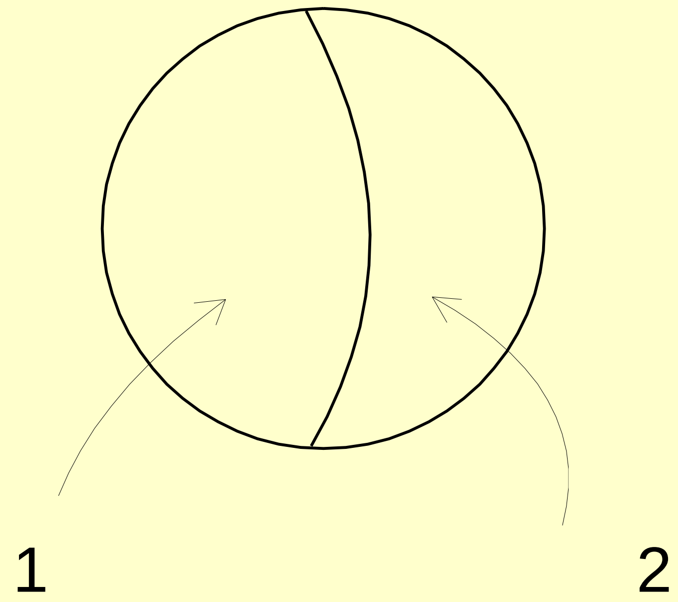
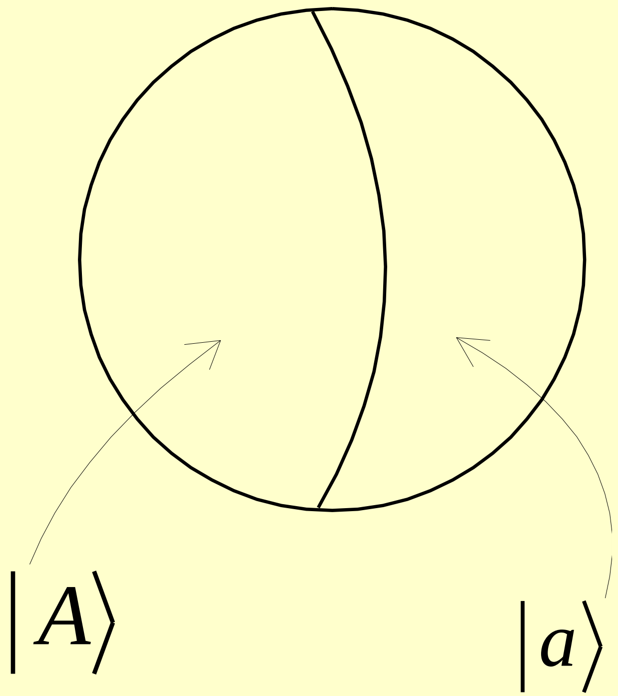
Quantum entanglement

<u>quantum mechanics</u>:
states of subsystems may not be described independently
= states are entangled



<u>importance:</u>
studying correlations of different systems (especially at strong couplings), critical phenomena and etc

reduced density matrix



$$\rho(A,a\,|\,B,b)$$

$$\rho_1(A\,|\,B) = \sum_a \rho(A,a\,|\,B,a),$$

$$\rho_2(a\,|\,b) = \sum_A \rho(A,a\,|\,A,b),$$

$$\rho_1 = Tr_2\,\rho, \quad \rho_2 = Tr_1\,\rho,$$

Entropy as a measure of entanglement

$\rho_1 = \mathrm{Tr}_2\, \rho$ – reduced density matrix

$S_1^{(\alpha)} = \dfrac{\ln \mathrm{Tr}_1\, \rho_1{}^{\alpha}}{1-\alpha}$ – entanglement Renyi entropy

In general, $\alpha > 0,$ and $\alpha \neq 1$

Next we consider integer values $\alpha = n = 2, 3, 4, \ldots.$

Basic properties

I. $S_1^{(\alpha)} \geq 0, \quad (S_1^{(\alpha)} = 0,$ if and only if ρ_1 is pure state)

Different limits:

$$S_1^{(\alpha)} \to S_1 \quad , \quad \alpha \to 1$$

$$S_1 \equiv -\mathrm{Tr}_1\, \rho_1 \ln \rho_1$$ – entanglement entropy

II. "Symmetry" in a pure state

$$S_1^{(\alpha)} = S_2^{(\alpha)}$$

entanglement has to do with quantum gravity:

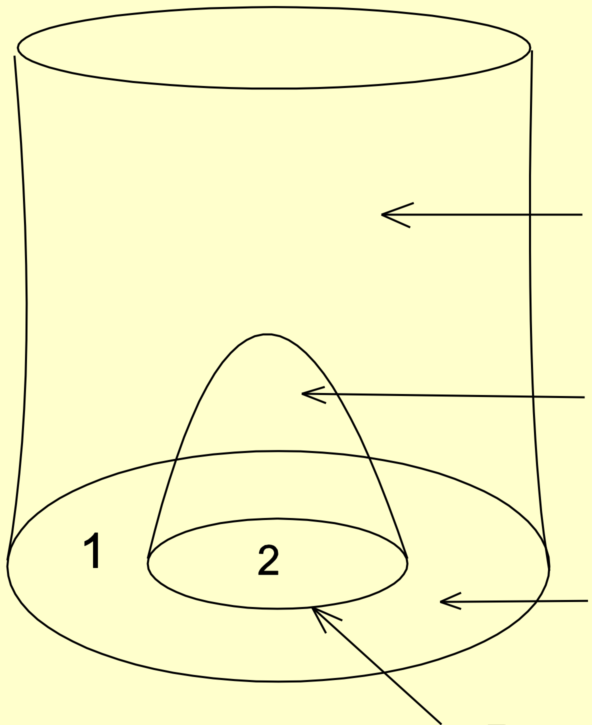
- possible source of the entropy of a black hole (states inside and outside the horizon);

- d=4 supersymmetric BH's are equivalent to 2, 3,… qubit systems

- entanglement entropy allows a *holographic interpretation* for CFT's with AdS duals

Holographic Formula for Entanglement Entropy (n=1)

Ryu and Takayanagi,
hep-th/0603001, 0605073



AdS_{d+1} (bulk space)

$\tilde{B}$ minimal (least area) surface in the bulk

4d space-time manifold (asymptotic boundary of AdS)

B separating surface

entropy of entanglement

$$S = \frac{\tilde{A}}{4G^{(d+1)}}$$

is measured in terms of the area of $\tilde{B}$

$G^{(d+1)}$ is the gravity coupling in AdS

Holographic formula enables one to compute entanglement entropy in strongly correlated systems with the help of geometrical methods (the Plateau problem);

Ryu-Takayanagi formula passes several non-trivial tests:

- in 2D and 4D CFT’s (at weak coupling);
-for different quantum states;

H_1

-for different shapes and topologies of the separating surface in boundary CFT

Is it possible to find a holographic description of entanglement Renyi entropy?

Plan:

- new result: Renyi entropies for 4D CFT's at weak coupling;

$$S^{(n)} = \sum_{p=2}^{d-2} \Lambda^{d-p} \frac{s^{(n)}{}_p}{d-p} + s^{(n)}{}_d \ln(\Lambda/\mu) + \ldots$$

$s^{(n)}{}_2 \sim A(B) -$ area of the separating surface B

$s^{(n)}{}_4 \sim a(\gamma_n) F_a + c(\gamma_n) F_c + b(\gamma_n) F_b$

$F_a, F_c, F_b -$ integrals of curvatures on B, $a(\gamma_n), c(\gamma_n), -$ rational functions of γ_n

- Possible holographic representation of Renyi entropy (analog of RT formula);

- Strong coupling results

Entanglement Renyi Entropy in CFT’s at weak coupling

1st step: representation in terms of a 'partition function'

if $\rho = e^{-H/T} / \mathrm{Tr}\, e^{-H/T}$ – is a thermal density matrix

$$S^{(\alpha)} = \frac{1}{1-\alpha}\left(\ln Z(T/\alpha) - \alpha \ln Z(T)\right), \quad Z(T) = \mathrm{Tr}\, e^{-H/T}$$

in general $\rho \sim \mathrm{Tr}_2\, e^{-H/T}$, an analog of this relation:

$$S_1^{(n)}(T) = \frac{\ln Z(n,T) - n \ln Z(T)}{1-n}$$

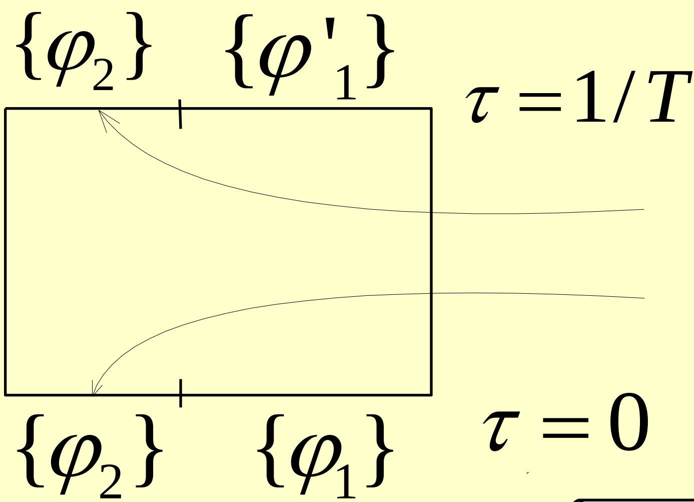
where

$$Z(n,T) \equiv \mathrm{Tr}_1\, (\mathrm{Tr}_2\, e^{-H/T})^n$$ – a "partition function", $Z(T) = Z(1,T)$

$\beta = 2\pi n$ – "inverse temperature"

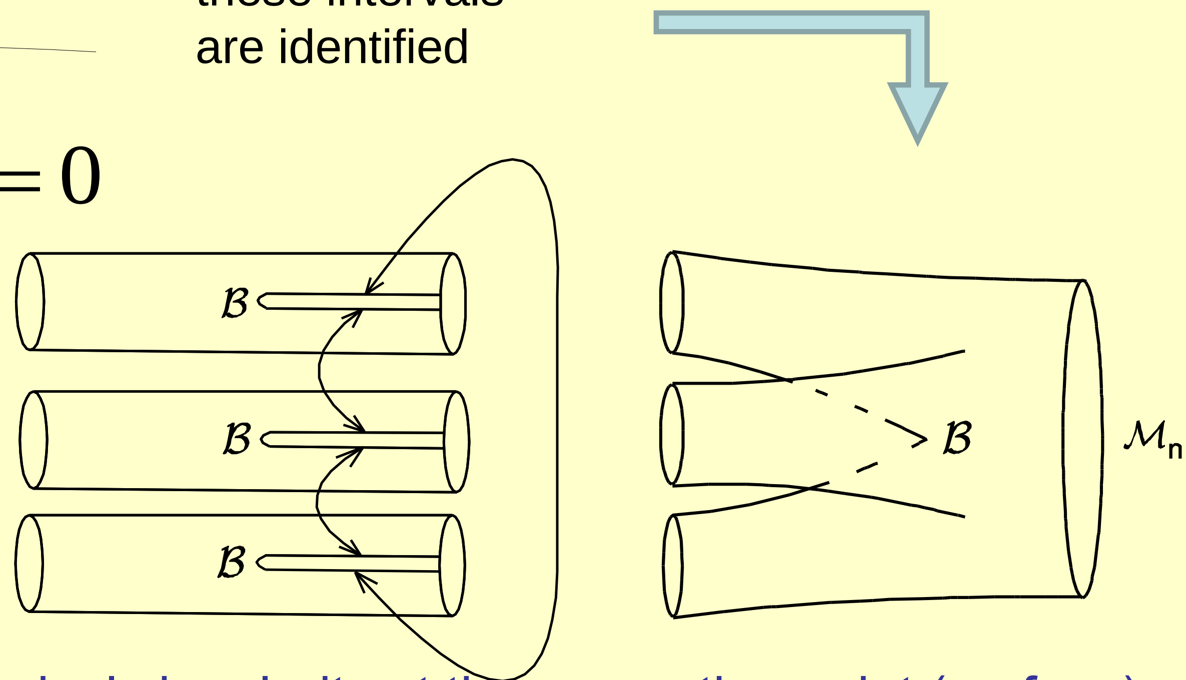
2d step: relation of a 'partition function' to an effective action on a 'curved space'

$$W(\beta,T) = -\ln Z(\beta,T) - \text{effective action}$$



these intervals are identified

$$\mathrm{Tr}_1 \rho_1^{\,3}$$



one glues n (n=3) copies

a 'curved space' with conical singularity at the separating point (surface)

3[d] step: use results of spectral geometry

$$W=\frac{1}{2}\sum_k \eta_k \ln\det L_k \quad , \quad \eta_k=\pm 1$$

$L_k -$ Laplace operators of different spin fields on M_n

$$W=\sum_{p=0}^{d-1}\Lambda^{d-p}\frac{A_p}{p-d}-A_d \ln(\Lambda/\mu)+\ldots$$ for dimension d even,

$$A_p=\sum_k \eta_k A_{k,p}$$, where $A_{k,p}$: $\mathrm{Tr}\, e^{-tL_k}=\sum_{p=0}^{\infty} t^{\frac{p-d}{2}} A_{k,p}+\ldots;$

$\Lambda -$ is a UV cutoff; μ is a physical scale (mass, inverse syze etc)

an example: a scalar Laplacian $L_0=-\nabla^2$:

$$A_0=O(n)\,,\ A_2=\frac{1}{24\pi}\int_{M_n} R+\frac{1}{12\gamma_n}(\gamma_n{}^2-1)\int_B \quad , \quad \gamma_n=n^{-1}$$

There are non-trivial contributions from conical singularities located at the 'separating' surface B

computations

$$S=\sum_{p=2}^{d-2}\Lambda^{d-p}\frac{s_p}{d-p}+s_d\ln(\Lambda/\mu)+\ldots,$$

$$S^{(n)}=\sum_{p=2}^{d-2}\Lambda^{d-p}\frac{s^{(n)}{}_p}{d-p}+s^{(n)}{}_d\ln(\Lambda/\mu)+\ldots,-\text{ Renyi entropies}$$

$$s_p\equiv-\lim_{n\to1}(n\partial_n-1)A_p(n)\quad,\quad s^{(n)}{}_p\equiv\frac{nA_p(1)-A_p(n)}{n-1}$$

$$s_0=s^{(n)}{}_0=0\quad,\ s_{2k+1}=s^{(n)}{}_{2k+1}=0-\text{(if boundaries are absent)}$$

4D N=4 super SU(N) Yang-Mills theory at weak coup.

6 scalar multiplets, 4 multiplets of Weyl spinors, 1 multiplet of gluon fields

$$S^{(n)} = \frac{1}{2}\Lambda^2 s^{(n)}{}_2 + s^{(n)}{}_4 \ln(\Lambda/\mu) + ...$$

$s^{(n)}{}_2 = \dfrac{d(N)}{4\pi}\gamma_n A(B)$ – area of the separating surface B

Conformal invariance

$$s^{(n)}{}_4 = d(N)(a(\gamma_n)F_a + c(\gamma_n)F_c + b(\gamma_n)F_b)$$

$$F_a = -\frac{1}{2\pi}\int_B \sqrt{\sigma} d^2x \, R(B) \quad ,$$

$$F_c = \frac{1}{2\pi}\int_B \sqrt{\sigma} d^2x \, C_{\mu\nu\lambda\rho} n_i^\mu n_j^\nu n_i^\lambda n_j^\rho \quad ,$$

$$F_b = \frac{1}{2\pi}\int_B \sqrt{\sigma} d^2x \left(\frac{1}{2}\mathrm{Tr}(k_i)\mathrm{Tr}(k_i) - \mathrm{Tr}(k_i k_i) \right) ,$$

$R(B)$ – scalar curvature of B, n_i^μ – a pair of unit orthogonal normals to B,

$C_{\mu\nu\lambda\rho}$ – Weyl tensor of M at B, $(k_i)_{\mu\nu}$ – extrinsic curvatures of B

F_a, F_b, F_c – invariant with respect to the Weyl transformations $g_{\mu\nu}{}'(x) = e^{2\omega(x)} g_{\mu\nu}(x)$

the problem is to determine $a(\gamma_n)$, $c(\gamma_n)$, $b(\gamma_n)$

Entanglement entropy (n=1)

$$s_4 = \lim_{n\to 1} s^{(n)}{}_4 = cF_c + aF_a + bF_b$$

$$c = \lim_{n\to 1} c(\gamma_n) = \frac{1}{4}, \quad a = \lim_{n\to 1} a(\gamma_n) = \frac{1}{4},$$

$$a = c$$

relation to the trace anomaly in $D = 4$

$$\langle T_\mu^\mu \rangle = -aE_4 - cI_4$$

$$E_4 = \frac{1}{16\pi^2}\left(R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right)$$

$$I_4 = -\frac{1}{16\pi^2}C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho}$$

$$C_{\mu\nu\lambda\rho} = R_{\mu\nu\lambda\rho} + \frac{1}{2}(g_{\mu\rho}R_{\nu\lambda} + g_{\nu\lambda}R_{\mu\rho} - g_{\mu\lambda}R_{\nu\rho} - g_{\nu\rho}R_{\mu\lambda}) + \frac{R}{6}(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda})$$

Cardy's conjecture: "charge" a decreases monotonically along RG flows

Computation of coefficient functions

$$s^{(n)}{}_4 = d(N)(a(\gamma_n)F_a + c(\gamma_n)F_c + b(\gamma_n)F_b)$$

contributions to heat kernel coefficients from conical singularities

$$\bar{A}_4(\Delta^{(i)}) = \bar{a}_i(\gamma_n)F_a + \bar{c}_i(\gamma_n)F_c + \bar{b}_i(\gamma_n)F_b$$

$$a(\gamma_n) = \frac{1}{n-1}(6\bar{a}_0(\gamma_n) - 4\bar{a}_{1/2}(\gamma_n) + \bar{a}_1(\gamma_n)) = \frac{1}{32}(\gamma_n{}^3 + \gamma_n{}^2 + 7\gamma_n + 15)$$

$$c(\gamma_n) = \frac{1}{n-1}(6\bar{c}_0(\gamma_n) - 4\bar{c}_{1/2}(\gamma_n) + \bar{c}_1(\gamma_n)) = \frac{1}{32}(\gamma_n{}^3 + \gamma_n{}^2 + 3\gamma_n + 3)$$

$\lim_{n\to 1} b(\gamma_n) = 1,$ ('holographic' arguments by S.N. Solodukhin, arXiv:0802.3117)

Toward a holographic description of Entanglement Renyi Entropy in CFT’s

Holographic Renyi Entropy (a suggestion)

$$S^{(n)} = \frac{1}{4G_N^{(5)}}\Big(f(\gamma_n)A(\tilde{B}) + 2\pi(\tilde{a}(\gamma_n)\tilde{F}_a + \tilde{c}(\gamma_n)\tilde{F}_c + \tilde{b}(\gamma_n)\tilde{F}_b)\Big) + ...$$

$A(\tilde{B})$ – volume of $\tilde{B}$;

$\tilde{F}_a, \tilde{F}_c, \tilde{F}_b$ – are some local (bulk) invariant functionals set on $\tilde{B}$;

$f(\gamma_n), \tilde{a}(\gamma_n), \tilde{c}(\gamma_n), \tilde{b}(\gamma_n)$ – some coefficient functions;

to reproduce Ryu-Takayanagi formula for entanglement entropy

$$S^{(n=1)} = \frac{1}{4G_N^{(5)}} A(\tilde{B})$$

$$f(1) = 1, \;\; \tilde{a}(1) = \tilde{c}(1) = \tilde{b}(1) = 0$$

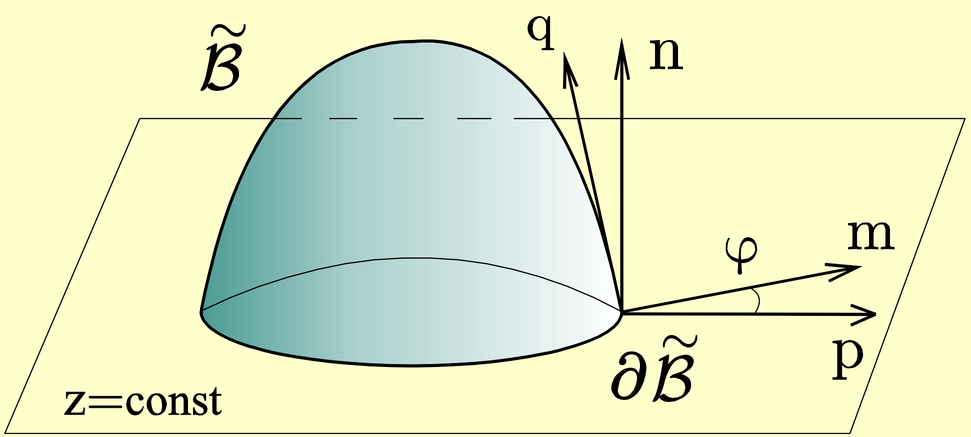
$\tilde{F}_a, \tilde{F}_c, \tilde{F}_b -$ are fixed by conformal invariance (should not depend on the coupling)

$$\tilde{F}_a \to F_a, \ \tilde{F}_c \to F_c, \quad \tilde{F}_b \to F_b,$$

the strategy is to fix $\tilde{F}_a, \tilde{F}_c, \tilde{F}_b$ in the limit of weak couplings

$f(\gamma_n), \tilde{a}(\gamma_n), \tilde{c}(\gamma_n), \tilde{b}(\gamma_n) -$ may depend on the coupling, extra information is required

Asymptotics at AdS



$\tilde{B}-$ is a holographic surface in the bulk;

$\partial\tilde{B}-$ belongs to conformal class of B;

$\tilde{M}$ – asymptotically AdS solution to:

$$\tilde{R}_{MN} - \frac{1}{2}\tilde{R}\,\tilde{g}_{MN} - \frac{3}{l^2}\tilde{g}_{MN} = 0$$

$$ds^2 = \tilde{g}_{MN}dx^M dx^N = z^{-2}(dz^2 + g_{\mu\nu}(z,x)dx^\mu dx^\nu),$$

$$g_{\mu\nu}(z,x) = g_{\mu\nu}(x) - \frac{z^2}{2}\left(R_{\mu\nu} - \frac{1}{6}g_{\mu\nu}R\right) + \ldots$$

$\tilde{B}-$ a minimal codimension 2 hypersurface in $\tilde{M}$, ($\partial\tilde{B}$ conformal to B)

$$ds^2(\tilde{B}) = z^{-2}\left(\frac{dz^2}{\cos^2\varphi} + \sigma_{ab}(z,y)dy^a dy^b\right)$$

Asymptotics at AdS (continued)

tilt angle: $\varphi = \frac{z}{2}k + \ldots,\quad k$ – extrinsic curvature of B

$$\sigma_{ab}(z,y) = \sigma_{ab}(y) - \frac{z^2}{2}\left(kk_{ab} + R_{ab} - \frac{1}{6}\sigma_{ab}R\right) + \ldots,\quad R_{ab} = R_{\mu\nu}x^{\mu}_{,a}x^{\nu}_{,b},$$

volume of $\tilde{B}$: $\quad A(\tilde{B}) = \frac{1}{2z^2}A(B) + \frac{\pi}{2}(F_a + F_c + F_b)\ln\frac{\mu}{z} + \ldots$

z – position of the boundary (a UV cutoff in CFT)

$$2\tilde{R}_{KLMN}(z,y)l^K m^L l^M m^N = -2 + z^2 C_{ijij}(y) + \ldots,$$

$$\tilde{R}_B(z,y) = -6 + z^2\left(C_{ijij} + \frac{k^2}{3} - \mathrm{Tr}(k^2)\right) + \ldots,$$

$$K_{MN}K^{MN} = -z^2\left(\frac{k^2}{3} - \mathrm{Tr}(k^2)\right) + \ldots,$$

$\tilde{R}_{KLMN}$ – Riemann tensor of $\tilde{M}$, $\tilde{\sigma}$ – metric induced on $\tilde{B}$, $\quad l, m$ – normal vectors of $\tilde{B}$,

l – is time-like, $(l \cdot m) = 0$, $\;K_{MN}$ – extrinsic curvature tensor of $\tilde{B}$ for m^N

Asymptotics at AdS (continued)

If we put:

$$\tilde{F}_c = \frac{1}{\pi}\int_{\tilde{B}} \sqrt{\tilde{\sigma}}\, d^3y \left[\tilde{R}_{KLMN} l^K m^L l^M m^N + \frac{1}{l^2} \right]$$

$$\tilde{F}_b = -\frac{1}{2\pi}\int_{\tilde{B}} \sqrt{\tilde{\sigma}}\, d^3y\, K_{MN} K^{MN}$$

it follows that:

$$\tilde{F}_{b,c} = F_{b,c}\, l \ln\frac{\mu}{z} + \ldots$$

no other invariants which yield Weyl invariant structures appear:

$\tilde{R}, \tilde{R}_{KL}$ – are constant (gravity eqs.)

the 3d functional, $\tilde{R}_B$, is not independent (Gauss-Codazzi eq.)

Asymptotics at AdS (continued)

$$\tilde{F}_a = F_a \ln\frac{\mu}{z} + ... = -4\ln\frac{\mu}{z}$$

$\tilde{F}_a$ cannot be defined as a local functional (similar to $\tilde{F}_b, \tilde{F}_c$)

one (but not single) option:

$$\tilde{F}_a \equiv -2l\ln\frac{A(\tilde{B})}{l^3}$$

other options are possible;

to match weak coupling calculations one should choose

$$\tilde{a}(\gamma_n) = a(\gamma_n) - \frac{1}{4} = \frac{1}{96}(\gamma_n - 1)(\gamma_n{}^2 + 2\gamma_n - 15),$$

$$\tilde{c}(\gamma_n) = c(\gamma_n) - \frac{1}{4} = \frac{1}{32}(\gamma_n - 1)(\gamma_n{}^2 + 2\gamma_n - 3),,$$

$$\tilde{b}(\gamma_n) = b(\gamma_n) - \frac{1}{4} = ?$$

coefficient functions at strong coupling?

holographic type computation by Hung, Myers, Smolkin, Yale 1110.1084 [hep-th] for spherical entangling surface in Minkowsky spacetyme

(since reduced density is themal $\rightarrow$ CFT $\rightarrow$ dual to some 5D black hole in the AdS bulk)

$$S^{(n)} \simeq N^2 a_{strong}(n)\left[\# \Lambda^2 A - \# \ln(\Lambda^2 A)\right]$$

- some numerical coefficients, $A = 4\pi R^2$

$$a_{strong}(n) = \frac{n}{n-1}(2 - x_n^2(1 + x_n^2)), \quad x_n = \frac{1}{4n}\left(1 + \sqrt{1 + 8n^2}\right)$$

because $F_c = F_b = 0,\ F_a = -4,$ it appears that

$a_{strong}(n)$ should correspond to $a(\gamma_n)$

to compare with our result (weak coupling):

$$S^{(n)} \simeq N^2\left[\frac{\Lambda^2}{8\pi n}A - \frac{1}{48n^3}(15n^3 + 7n^2 + n + 1)\ln(\Lambda^2 A)\right]$$

Summary:

● new result for the entanglement Renyi entropies (ERE) in D=4 CFT's

● ERE is a local invariant functional which have a structure similar to EE -> possibility to find a holographic description of ERE

● a conjecture for holographic ERE: modification of RT formula:

- local and non-local invariant structures in the bulk;
- explicit dependence on dimensionality and the replica parameter of holographic ERE;

● more insights are needed to fix coefficient functions (thus, practical use of the suggested formula for ERE is not obvious)

● if a holographic ERE formula does exist, this may indicate a deeper relation between gravity and quantum phenomena (goes beyond thermodynamic analogies in the presence of black holes)

thank you for attention