Form factors in N=4 SYM

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N=4 SYM theory.

- **N=4 SYM is perfect theoretical laboratory for studying properties of D=4 gauge theories.**
- **The correlation functions** $G_n = \langle 0|\mathcal{O}(x_1)\ldots\mathcal{O}(x_n)|0\rangle$ **in this theory can be studied in the weak and strong regimes (via AdS/CFT).**
- **The computation of anomalous dimensions of local operators in N=4 SYM in planar limit can be reduced to the problem of solving some integrable system.**
- **The regularized amplitudes** $A_n = \langle p_1^{\lambda_1}\ldots p_n^{\lambda_n}|0\rangle$ **(S-matrix) can be also studied in weak/strong coupling regimes. It was realized that likely planar S-matrix in N=4 SYM possesses new type of symmetries - (super)conformal symmetry in momentum space. Algebra of this symmetry together with conventional (super)conformal symmetry can be fused together to infinite dimensional Yangian algebra. It is believed that this symmetries will completely define planar S-matrix of N=4 SYM (*Beisert et al 10*).**

Form factors in N=4 SYM.

- **There is another class of objects in N=4 SYM which is intermediate between completely off-shell correlation functions and completely on-shell amplitudes – form factors (** *van Neerven 85, Selivanov 98* **) :**

$$F_n = \langle p_1^{\lambda_1} \dots p_n^{\lambda_n} | \mathcal{O}(x) | 0 \rangle$$

Some n-particle external state

Local gauge invariant operator

What is the structure of such objects in general ? Can symmetries of N = 4 SYM fix the structure of such objects ? Is there a dual description in terms of Wilson loops ? e c.t.

To answer this questions computations in first several orders of PT are required (*Kazakov D.I. et al 10-11 Brandhuber et al 10-11, Bork 12***).**

Off-shell N=4 supermultiplets.

In the case of the form factors one would like to construct analog of super MHV amplitude at tree level (super form factor), and then use in in unitarity based computations.

Which parameterization (superspace) one should use to describe and what supermultiplet of operators one can chose ?

$$F_n = \langle p_1^{\lambda_1} \dots p_n^{\lambda_n} | \mathcal{O}(x) | 0 \rangle$$

$$\boxed{|\Omega_i\rangle = \left(g_i^+ + \eta^A \Gamma_{i,A} + \frac{1}{2!}\eta^A\eta^B\phi_{i,AB} + \frac{1}{3!}\eta^A\eta^B\eta^C\varepsilon_{ABCD}\bar{\Gamma}_i^D + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D\varepsilon_{ABCD}g_i^- \right)|0\rangle}$$

<u>on-shell momentum superspace</u> (*Nair 88*, *Korchemsky 08*)

SL(2,C) commuting spinors

$$\boxed{\{\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}, \eta^A\}}$$

Grassmann scalars

$$\mathcal{F}_n = \langle \Omega_1 \dots \Omega_n | \ ??? \ | 0 \rangle$$

Usually N=4 supermultiplets in N=4 coordinate superspace are on-shell, and in addition the coordinate N=4 superspace is none-chiral in contrast to on-shell one.

One can sensible set of supermultiplets if one will consider chiral truncation of N=4 supermultiplet itself.

Off-shell N=4 supermultiplets. Harmonic superspace.

N=4 supermultiplet of fields can be described superfield in coordinate superspace

$$W^{AB}(x,\theta^A,\bar{\theta}_A)=\phi^{AB}(x)+\ldots$$

This superfield satisfies the following constraints: $\overline{W}^{AB}(x,\theta,\bar{\theta})=\frac{1}{2}\epsilon^{ABCD}W_{CD}(x,\theta,\bar{\theta})$

and most importantly:

$$D^{\alpha}_{C}W^{AB}(x,\theta,\bar{\theta})=-\frac{2}{3}\delta^{[A}_{C}D^{\alpha}_{L}W^{B]L}(x,\theta,\bar{\theta})$$

$$\bar{D}^{\dot{\alpha}(C}W^{A)B}(x,\theta,\bar{\theta})=0,$$

This constraints can be "solved" in terms of harmonic projected superfield:

$$W^{AB}\rightarrow W^{AB}u^{+a}_{A}u^{+b}_{B}=\epsilon^{ab}W^{++}\quad W^{++}(x,0,0,u)=\phi^{++},\ \phi^{++}=\frac{1}{2}\epsilon_{ab}u^{+a}_{A}u^{+b}_{B}\phi^{AB}$$

Where $(u^{+a}_{A},\ u^{-\dot{a}}_{A})$ are $\frac{SU(4)}{SU(2)\times SU(2)\times U(1)}$ harmonics (*Ogievetsky et al in 80's*).

Coordinate and oh-shell momentum harmonic superspaces are:

$$\{x^{\alpha\dot{\alpha}},\ \theta^{+a}_{\alpha},\theta^{-\dot{a}}_{\alpha},\ \bar{\theta}^{+a}_{\dot{\alpha}},\bar{\theta}^{-\dot{a}}_{\dot{\alpha}},u\}\ \{\lambda_{\alpha},\tilde{\lambda}_{\dot{\alpha}},\ \eta^{+a},\eta^{-\dot{a}},\ u\}$$

Off-shell N=4 supermultiplets. Harmonic superspace.

Then one can observe that chiral truncation of harmonic projected superfield is off-shell

$$\mathcal{W}^{++} = W^{++}(x, \theta^+, 0, u)$$

Perfect building block for 1/2-BPS :

$$\mathcal{O}^{(k)} = Tr([\mathcal{W}^{++}]^k), \ \mathcal{O}^{(k)} \sim Tr(\phi^k), \ldots$$

and stress tensor supermutiplet of operators

$$\boxed{\mathcal{O}^{(2)} = \mathcal{T} = Tr(\mathcal{W}^{++}\mathcal{W}^{++}), \ \mathcal{T} \sim Tr(\phi^2), Tr(\psi\psi), Tr(FF) + O(g), \ldots}$$

*
$$[Q_-, \mathcal{T}|_{\theta=0}] = 0$$

$$\mathcal{T} = e^{(Q_+\theta^+)}\mathcal{T}|_{\theta=0}$$

So it is natural as the beginning to consider form factors with operators of such type:

$$\mathcal{F}_n = \langle \Omega_1 \ldots \Omega_n | \mathcal{O}^{(k)}(x, \theta^+, u)|0\rangle$$

Arbitrary combination of N=4 on-shell states (particles)

Operators from1\2 BPS or stress tensor chiral truncated supermultiplets

Form factors of N=4 stress tensor supermultiplet. General structure.

Using relations (*) we can see that Grassmannian structure of form factor has the following form

$$\mathcal{F}_n(\{\lambda,\tilde{\lambda},\eta\},q,\theta^+) \;=\; \delta^4(\sum_{i=1}^{n}\lambda_\alpha^i\tilde{\lambda}_{\dot\alpha}^i - q_{\alpha\dot\alpha})e^{(\theta_\alpha^+ q_+^\alpha)}\delta_{GR}^{+4}\left(q_\alpha^+\right)\mathcal{X}_n\left(\{\lambda,\tilde{\lambda},\eta\},q\right)$$

(*Brandhuber et al 11, Bork 12*)

$$\mathcal{X}_n \;=\; \mathcal{X}_n^{(0)} + \mathcal{X}_n^{(4)} + \ldots + \mathcal{X}_n^{(4n-8)}$$

$\boxed{MHV\text{=n-2}}$ $\boxed{NMHV\text{=n-4}}$ $\boxed{\overline{MHV}\text{ =2-n}}$

Here:

$$q_\alpha^\pm = \sum_{i=1}^{n}\lambda_\alpha^i\eta_i^\pm$$

We can also define $\hat{T}[\ldots] = \int d^{-4}\theta\exp(\theta_\alpha^+\sum_{i=1}^{2}\lambda_i^\alpha\eta_+^i)[\ldots]$ such that (*Bork 11*):

$$Z_n(\{\lambda,\tilde{\lambda},\eta\},q,\{\gamma_\alpha^-\}) = \hat{T}[\mathcal{F}_n] = \delta^4(\sum_{i=1}^{n}\lambda_\alpha^i\tilde{\lambda}_{\dot\alpha}^i - q_{\alpha\dot\alpha})\delta_{GR}^{-4}(q_\alpha^- + \gamma_\alpha^-)\delta_{GR}^{+4}\left(q_\alpha^+\right)\mathcal{X}_n$$

Using BCFW we can see that $Z_n^{tree,MHV} = \delta^4(\sum_{i=1}^{n}\lambda_i^\alpha\tilde{\lambda}_i^{\dot\alpha} + q^{\alpha\dot\alpha})\dfrac{\delta_{GR}^{-4}\left(q_\alpha^- + \gamma_\alpha^-\right)\delta_{GR}^{+4}\left(q_\alpha^+\right)}{\langle 12\rangle\ldots\langle n1\rangle}$

Form factors of N=4 stress tensor supermultiplet at 1 loop. Generalized unitarity.

On general grounds at one loop form factors in N=4 SYM can be decomposed in terms of scalar Integrals as:

$$
\begin{aligned}
Z_n^{(1)} &= \sum_i C_i^{4m} B_i^{4m} + C_i^{3m} B_i^{3m} + C_i^{2mh} B_i^{2mh} + C_i^{2me} B_i^{2me} + C_i^{1m} B_i^{1m} \\
&+ \sum_j C_j^{3m} T_j^{3m} + C_j^{2m} T_j^{2m} + C_j^{1m} T_j^{1m} + perm.
\end{aligned}
$$

Here B – is box type scalar integral, T – is triangle type scalar integral. All of these integrals are known. The Grassmannian structure of the coefficients is:

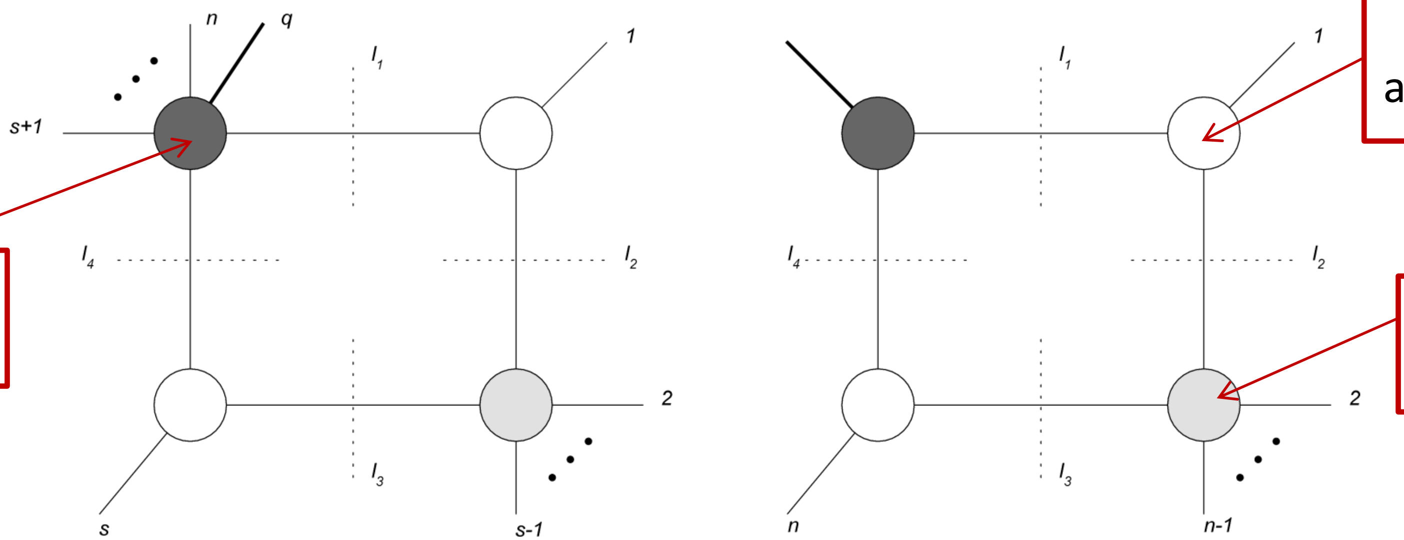
$$
C_k \sim \delta_{GR}^{-4}\left(q_\alpha^- + \gamma_\alpha^-\right) \delta_{GR}^{+4}\left(q_\alpha^+\right) \left(\mathcal{C}_k^{(0)} + \mathcal{C}_k^{(4)} + \ldots + \mathcal{C}_k^{(4n-8)}\right)
$$

Coefficients before scalar integrals can be computed in the unitarity based methods by comparing the analytical properties of both sides of the relation viewed as the functions of Mandelstam kinematical invariants of momenta of external particles

Form factors of N=4 stress tensor supermultiplet at 1 loop. MHV sector.

$$\boxed{Z_n^{(1),MHV} = \sum_i C_i^{2me} B_i^{2me} + \sum_j C_j^{2m}\ T_j^{2m} + perm.}$$

MHV form factor



$\overline{MHV}$ amplitude

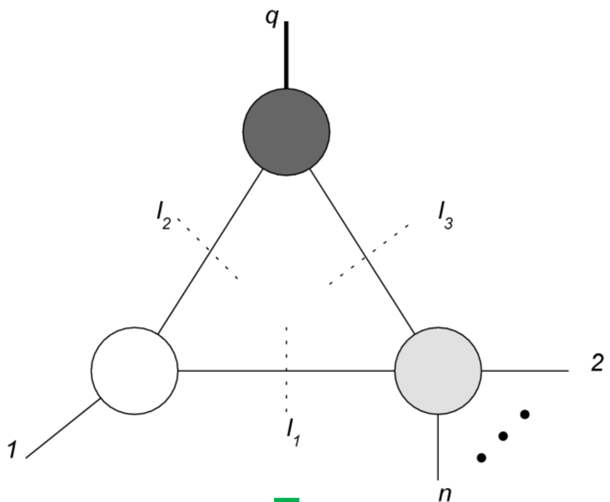
MHV amplitude

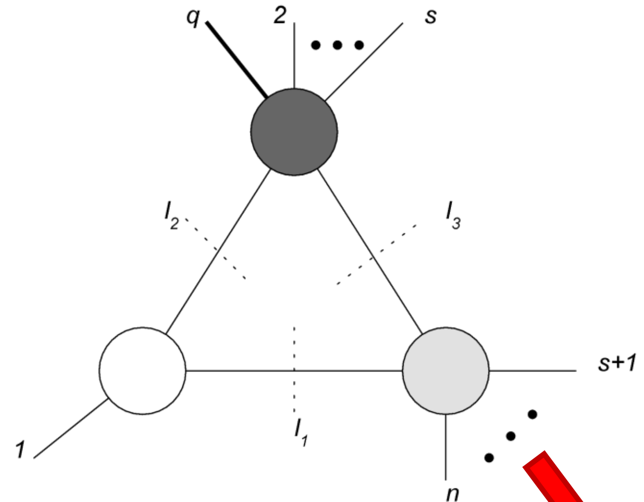
$$C^{2me}_{s_{2\ldots s-1},s_{1\ldots s}} = Z_n^{tree,MHV}\frac{1}{2}\Delta^{2me}_{s_{2\ldots s-1},s_{1\ldots s}}$$

$$\Delta^{2me}_{s_{2\ldots s-1},s_{1\ldots s}} = x^2_{2s}x^2_{1s+1} - x^2_{1s}x^2_{2s+1}$$

$$\boxed{x_{rs}^{\alpha\dot\alpha} = \sum_{i=r}^{s-1} p_i^{\alpha\dot\alpha}}$$

Form factors of N=4 stress tensor supermultiplet at 1 loop. MHV sector.





No triangles of such type. Valid for NMHV sector as well

$$
\begin{aligned}
C^{2m}_{s_{2\ldots n},q^2} &= Z_n^{tree,MHV}\frac{1}{2}\Delta^{2m}_{s_{2\ldots n},q^2} \\
\Delta^{2m}_{s_{2\ldots n},q^2} &= x^2_{2n+1} - x^2_{1n+1}
\end{aligned}
$$

$$\boxed{x^{\alpha\dot{\alpha}}_{rs} = \sum_{i=r}^{s-1} p_i^{\alpha\dot{\alpha}}}$$

Form factors of N=4 stress tensor supermultiplet at 1 loop. MHV sector. Example.

Combining coefficients obtained for MHV form factor with the scalar integrals for n=3 we can obtain:

$$Z_3^{(1),MHV}/Z_3^{tree,MHV}\Big|_{fin} = -(1+\mathbb{P}+\mathbb{P}^2)\left(2\mathrm{Li}_2\left(1-\frac{x_{14}^2}{x_{13}^2}\right)+\mathrm{Log}^2\left(\frac{x_{13}^2}{x_{24}^2}\right)+\frac{\pi}{3}\right)$$

Permutation operator

(*Brandhuber et al 11, Bork 12*)

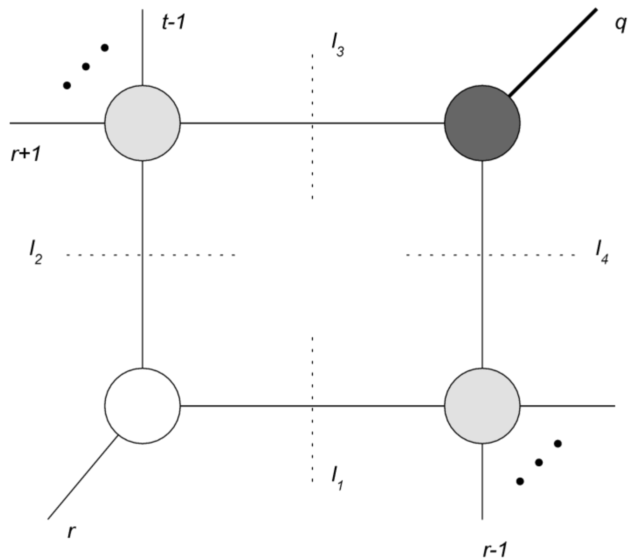
In general we expect that the following structure for the IR part of form factor holds

$$Z_n^{(1)}\Big|_{IR} = Z_n^{tree}\frac{1}{\epsilon^2}\sum_{i=1}^{n}\left(\frac{s_{ii+1}}{\mu^2}\right)^{\epsilon}$$

We will use it in the NMHV sector to fix some triangle coefficients without their explicit computations.

Form factors of N=4 stress tensor supermultiplet at 1 loop. NMHV sector. General Structure.

$$Z_n^{(1),NMHV} = \sum_i C_i^{2me} B_i^{2me} + C_i^{2mh} B_i^{2mh} + C_i^{3m} B_i^{3m} + \sum_j C_j^{2m} T_j^{2m} + C_j^{3m} T_j^{3m} + perm.$$



Dual coordinates

$$\langle\theta_{rt}| = \sum_{i=r}^{t-1} \eta_i \langle i| = \sum_{i=r}^{t-1} \eta_i \lambda_i$$

$$C^{3m}_{s_{r+1\ldots t-1}, s_{t\ldots r-1}, q^2} = Z_n^{tree,MHV} \tilde{R}^{(1)}_{rtt} \frac{1}{2} \Delta_{s_{r+1\ldots t-1}, s_{t\ldots r-1}, q^2}$$

$$\tilde{R}^{(1)}_{rtt} = \frac{\langle t t-1\rangle \hat{\delta}^4 \left(\langle\theta_{tr}|x_{1n+1}x_{rt}|r\rangle + \langle\theta_{rs}|x_{1n+1}x_{tr}|r\rangle\right)}{x^4_{1n+1}\langle r|x_{rt}x_{1n+1}|t\rangle\langle r|x_{tr}x_{1n+1}|t-1\rangle\langle r|x_{tr}x_{1n+1}|r\rangle}$$

$$\Delta_{s_{r+1\ldots t-1}, s_{t\ldots r-1}, q^2} = x^2_{rt}x^2_{rt+1} - x^2_{tr}x^2_{r+1t}$$

Form factors of N=4 stress tensor supermultiplet at 1 loop. NMHV sector. General Structure.

Similar answers can be obtained for different configurations of external legs for coefficients before box scalar integrals:

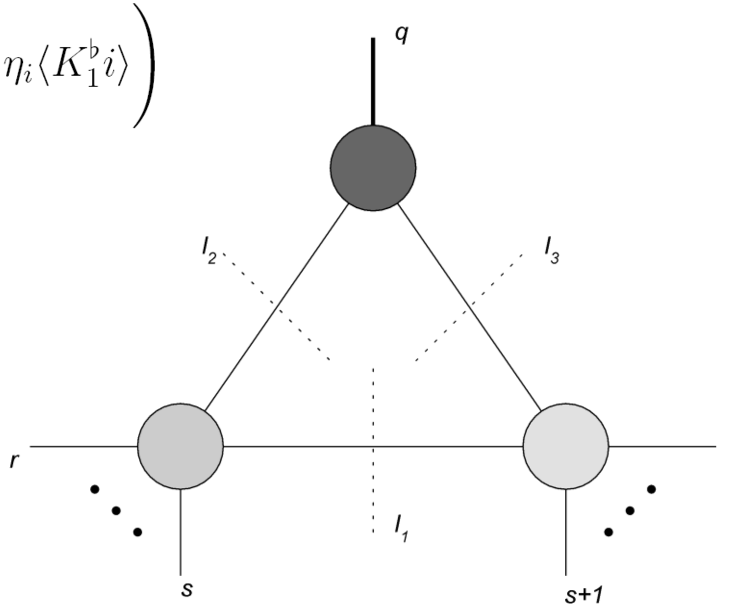
$$
\begin{aligned}
C^{3m}_{s_{s-1\ldots r+1}s_{t\ldots r-1}s_{t\ldots s-1}} &= Z_n^{tree,MHV} R^{(1)}_{rst}\frac{1}{2}\Delta_{s_{s-1\ldots r+1}s_{t\ldots r-1}s_{t\ldots s-1}},\\
R^{(1)}_{rst} &= \frac{\langle s-1 s\rangle\langle t-1 t\rangle\hat{\delta}^4\left(\langle\theta_{tr}|x_{ts}x_{rs}|r\rangle+\langle\theta_{rs}|x_{ts}x_{tr}|r\rangle\right)}{x_{ts}^2\langle r|x_{rs}x_{ts}|t-1\rangle\langle r|x_{rs}x_{ts}|t\rangle\langle r|x_{tr}x_{ts}|s-1\rangle\langle r|x_{tr}x_{ts}|s\rangle}\\
\Delta_{s_{s-1\ldots r+1}s_{t\ldots r-1}s_{t\ldots s-1}} &= s_{r\ldots s-1}s_{r\ldots t}-s_{t\ldots r-1}s_{r+1\ldots s-1}=x_{rs}^2x_{rt+1}^2-x_{tr}^2x_{r+1s}^2
\end{aligned}
$$

$$
\begin{aligned}
C^{3m}_{s_{r\ldots s}s_{t..r-1}s_{t\ldots r-1}} &= Z_n^{tree,MHV} R^{(2)}_{rst}\frac{1}{2}\Delta_{s_{r\ldots s}s_{t..r-1}s_{t\ldots r-1}},\\
R^{(2)}_{rst} &= \frac{\langle s-1 s\rangle\langle t-1 t\rangle\hat{\delta}^4\left(\langle\theta_{tr}|x_{st}x_{sr}|r\rangle+\langle\theta_{rs}|x_{st}x_{tr}|r\rangle\right)}{x_{st}^2\langle r|x_{sr}x_{st}|t-1\rangle\langle r|x_{sr}x_{st}|t\rangle\langle r|x_{tr}x_{st}|s-1\rangle\langle r|x_{tr}x_{st}|s\rangle}\\
\Delta_{s_{r\ldots s}s_{t..r-1}s_{t\ldots r-1}} &= s_{s\ldots r-1}s_{t\ldots r}-s_{r\ldots t-1}s_{s\ldots r}=x_{sr}^2x_{tr+1}^2-x_{rt}^2x_{sr+1}^2
\end{aligned}
$$

Form factors of N=4 stress tensor supermultiplet at 1 loop. NMHV sector. General Structure.

For three mass triangle coefficients we have:

$$
\begin{aligned}
C^{3m}_{s_{r\ldots s},s_{s+1\ldots t},q^2} &= Z_n^{MHV,tree}\frac{1}{2}\mathcal{R}_{rst}\Delta_{s_{r\ldots s},s_{s+1\ldots t},q^2},\\
\mathcal{R}_{rst} &= \sum_{\pm S}\frac{\gamma}{K_1^2K_2^2}\left(\frac{K_1^2}{\gamma}-1\right)^{-3}\frac{\langle rt\rangle\langle ss+1\rangle}{\prod_{i=1}^n\langle iK_1^\flat\rangle}\\
&\times\ \hat{\delta}^4\left(K_1^2/\gamma\sum_{i=r}^{s}\eta_i\langle K_1^\flat i\rangle+(K_1^2/\gamma-1)\sum_{i=s+1}^{t}\eta_i\langle K_1^\flat i\rangle\right)\\
\Delta_{s_{r\ldots s},s_{s+1\ldots t},q^2} &= q^2=x^2_{1n+1}
\end{aligned}
$$



Where $K_1=p_{s+1\ldots t}=x_{s+1\ldots t-1},\ K_2=q=x_{1n+1}$

and $K_i^\flat$ is a massless projection

Form factors of N=4 stress tensor supermultiplet at 1 loop. NMHV sector. Examples.

Combining together obtained information about the coefficients we can get for n=3

$$Z_3^{tree,NMHV} = Z_3^{MHV,tree}\frac{1}{3}(1+\mathbb{P}+\mathbb{P}^2)\tilde{R}_{211}^{(1)} \qquad Z_3^{(1),NMHV}/Z_3^{tree,NMHV} = Z_3^{(1),MHV}/Z_3^{tree,MHV}$$

While for n=4 we have: $Z_4^{NMHV,tree} = Z_4^{MHV,tree}\frac{1}{2}(1+\mathbb{P}+\mathbb{P}^2+\mathbb{P}^3)(\tilde{R}_{311}^{(1)}+R_{241}^{(1)})$ and

$$\begin{aligned} Z_4^{NMHV,(1)}/Z_4^{tree,MHV}\Big|_{fin} &= (1+\mathbb{P}+\mathbb{P}^2+\mathbb{P}^3)\frac{1}{2}(\tilde{R}_{311}^{(1)}+R_{241}^{(1)})V_4 \\ &+ (1+\mathbb{P}+\mathbb{P}^2+\mathbb{P}^3)\mathcal{R}_{124}W_4 \end{aligned}$$

Where:

$$\begin{aligned} V_4 &= -2\sum_{i=1}^{2}\left(\mathrm{Li}_2\left(1-\frac{s_{ii+1}}{s_{123}}\right)+\mathrm{Li}_2\left(1-\frac{s_{123}}{s_{ii+1}}\right)\right)-4\mathrm{Li}_2\left(1-\frac{q^2}{s_{123}}\right) \\ &+ \sum_{i=1}^{2}\left(-L^2\left(\frac{s_{ii+1}}{s_{123}}\right)+L\left(\frac{s_{ii+1}}{s_{i+2i+3}}\right)L\left(\frac{q^2}{s_{i+2i+3}}\right)\right)-L^2\left(\frac{s_{12}}{s_{23}}\right)-\frac{\pi^2}{3} \end{aligned}$$

$$\begin{aligned} W_4 &= \frac{1}{Q}\left(2\mathrm{Li}_2(-xR)+2\mathrm{Li}_2(-yR)+L(xR)L(yR)+L\left(\frac{y}{x}\right)L\left(\frac{1+yR}{1+xR}\right)+\frac{\pi^2}{3}\right) \\ Q &= \left((1-x-y)^2-4xy\right)^{1/2},\ R=2(1-x-y+R)^{-1},\ x=\frac{s_{12}}{q^2},\ y=\frac{s_{34}}{q^2} \end{aligned}$$

Relations between form factors and amplitudes.

One can note that:

(*Zhiboedov 11, Brandhuber et al 11, Bork 11*)

$$Z_n^{tree,MHV}(\{\lambda,\tilde{\lambda},\eta\},0,\{0\}) = \mathcal{A}_n^{tree,MHV}(\lambda,\tilde{\lambda},\eta)$$

Accident ? – No. Lagrangian of N=4 SYM in chiral form belongs to chiral truncation of N=4 stress tensor supermultiplet:

$$g\frac{\partial A_n}{\partial g} = \langle p_1^{\lambda_1}\dots p_n^{\lambda_n}|\mathcal{L}(q=0)|0\rangle$$

$$\mathcal{L}_{chiral}^{\mathcal{N}=4} = \int d^{-4}\theta_\alpha^c \mathcal{T}(q,\theta^+)$$

$$\mathcal{L}_{chiral}^{\mathcal{N}=4}(q) = \hat{T}[\mathcal{T}](q,0)$$

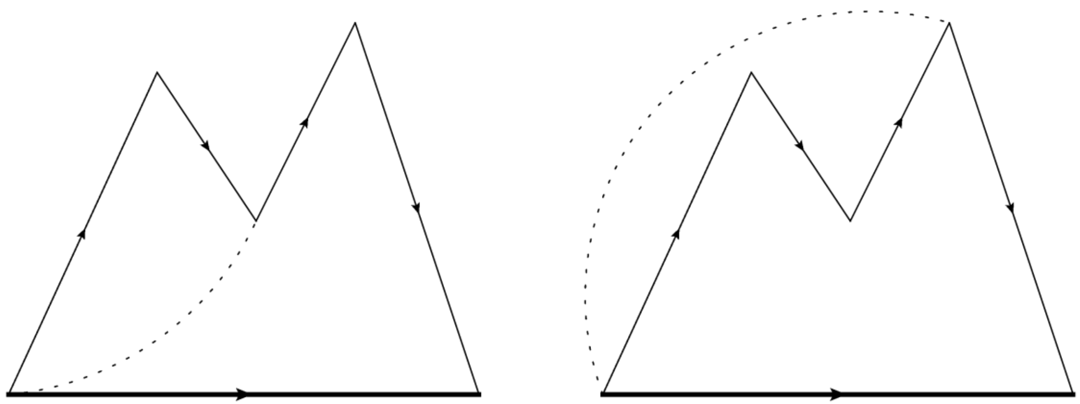
$$\boxed{Z_n(\{\lambda,\tilde{\lambda},\eta\},0,\{0\}) = \hat{T}[\mathcal{F}_n](\{\lambda,\tilde{\lambda},\eta\},0,\{0\}) = g\frac{\partial \mathcal{A}_n(\lambda,\tilde{\lambda},\eta)}{\partial g}}$$

Such relation is expected to hold at all loops except some degenerate configurations of external momenta.

Relations between form factors and Wilson loops.

It was pointed out that MHV form factors is likely dual to periodic Wilson loops (*Brandhuber et al 11*). What about other than MHV sectors ?

At first glance this might work. Lets consider for example NMHV n=4 case at tree level:



+perm.

Number of contributing diagrams exactly matches the number of R coefficients:

$$Z_4^{NMHV,tree} = Z_4^{MHV,tree}\frac{1}{2}(1+\mathbb{P}+\mathbb{P}^2+\mathbb{P}^3)(\tilde{R}^{(1)}_{311}+R^{(1)}_{241})$$

Conclusions.

- **The powerful on-shell methods in N=4 SYM can be applied to partially off-shell objects as well.**
- **The form factors in N=4 SYM share lots of similarities with amplitudes – factorization of IR divergences, similar super space formulations, dual description in terms of Wilson line-like objects, maximal trancendentality, ec.t.**
- **There are some hints that rich symmetries of N=4 SYM amplitudes (S-matrix) may (partially) survive for form factors as well.**
- **There is hope that N=4 SYM is integral theory and its S-matrix can be completely fixed by the symmetry arguments. If this is the case, then one may expect even more rich structure for form factors (like in 2D sin-gordon models).**