New glance at the special role of (3+1)-dimensional De Sitter space-time

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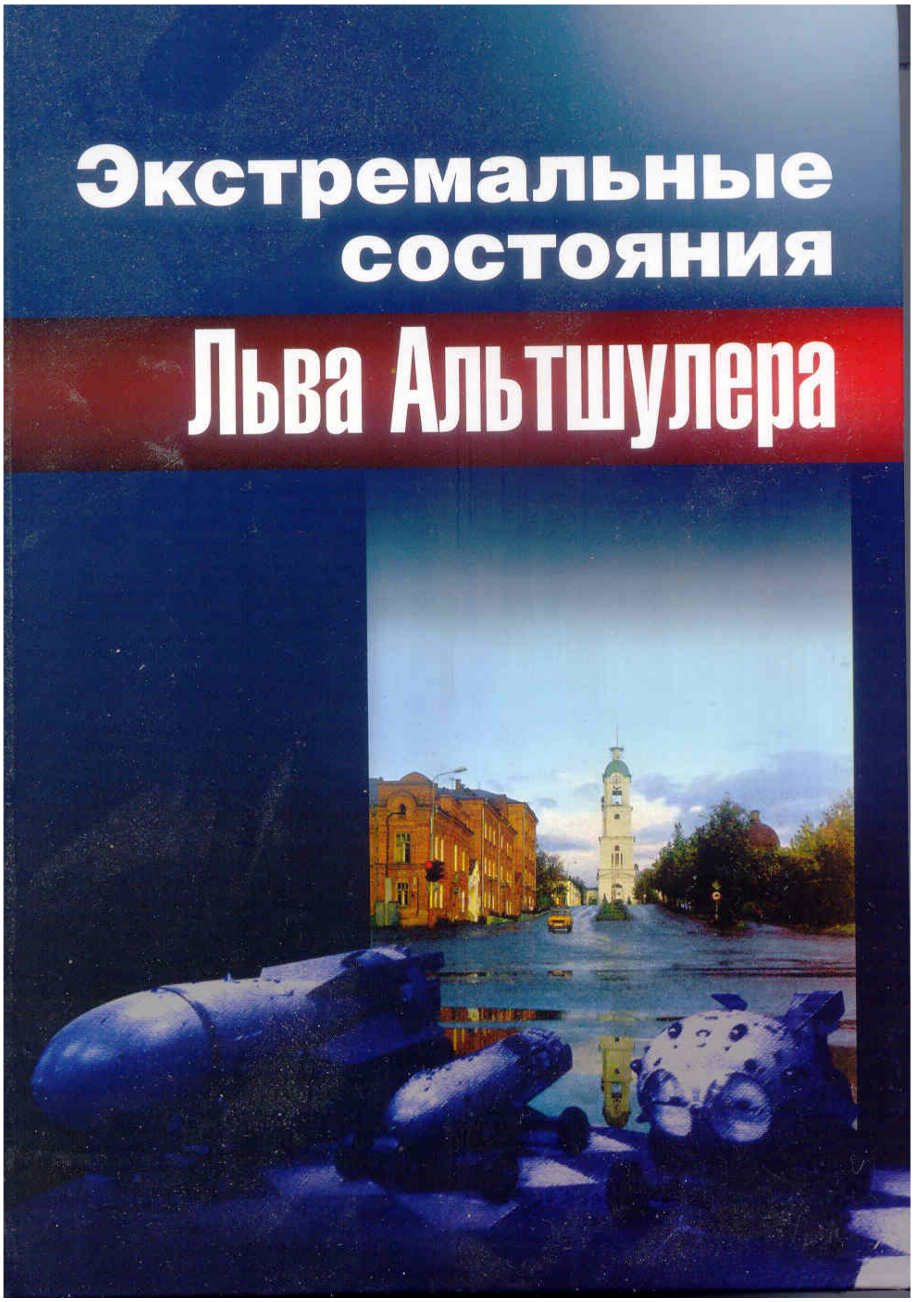
1 First history note

I suppose I met Vitaly Ginzburg earlier than any other participant of this Conference since this happened in 1939 - few months after my birth. I confess I do not remember sufficiently well this history event but Ginzburg said me much later that he really observed me in my cradle. And who knows - perhaps this turned me to theoretical physics after all :)

Vitaly Ginzburg and my father Lev Altshuler were good friends beginning from 1932 and all the life through. The third friend was Veniamin Tsukerman, who was the Head of the X-Ray Lab where 16 years old Vitaly started working in 1932. In 2006 as a gift to 90 years of Vitaly Ginzburg I wrote an Article "Three friends: Altshuler, Ginzburg, Tsukerman", its English-language version is placed at the UFN/tribuna web site.

Vitaly Ginsburg, Lev Altshuler, Veniamin Tsukerman in front of Tsukerman's cottadge in Nuclear Center Arzamas-16 (Town Sarov), 1955

Also the book "Extremal States of Lev Altshuler" was published in Fizmatlit Publ in 2011 where there is also the contribution by Vitaly Ginzburg and plenty of documents and witnesses.



2 Second History Note. Problem of Dimensionality. Integral form of Einstein equations

There are no dynamical answer yet to the questions: why observed Universe is 3+1 dimensional? Why, if we suppose higher dimensions of equal rights at the Big Bang, only 3 space dimensions expand to large volume? Beginning from the Ehrenfest pioneer work (1920) a number of important observations about the privileged character of 3+1 space-time were made. In frames of string theory and branes' dynamics interesting attempts to explain the expansion of 3 dimensions based upon the observations that $2+2<5$ and $4+4<10$ were made by

- Brandenberger R.N. and Vafa C., , Nucl. Phys., vol. B316, p.391 (1989).

and

- Karch A. and Randall L., *Relaxing to Three Dimensions*, Phys. Rev. Lett., 95:161601 (2005), *Preprint* hep-th/0506053

correspondingly, etc., etc. Here the special role of (3+1) is revealed in totally different context.

Altshuler (1966, 1985) and independently Lynden-Bell (1967) proposed the integral representation of Einstein equations. This "integral" approach was also developed by Sciama, Weylen, Gilman (1969, 1970) and Raine (1981).

- Altshuler B.L., *"Integral form of Einstein equations and covariant formulation of Mach's principle"*, ZhETP, vol. 51, p. 1143 (1966). [Sov. Phys. JETP, vol. 24, p. 766 (1967)].

- Lynden-Bell D., *"On the Origins of Space-Time and Inertia"*, Monthly Notices Roy. Astron. Soc., vol. 135, 4, p. 413-428 (1967).

- Sciama D.W., Waylen P.C., Gilman R.G., *"Generally Covariant Integral Formulation of Einstein's Field Equations"*, The Physical Review, vol. 187, p. 1762-1766 (1969).

- Gilman R.G., *"Machian theory of inertia and gravitation"*, The Physical Review, vol. D2, 1400-1410 (1970).

- Raine D.J., *"Mach's principle and space-time structure"*, Rep. Prog. Phys., vol. 44, p. 1151 (1981).

- Altshuler, Boris L., *"Mach's Principle. Part I. Initial state of the Universe"*, International Journal of Theoretical Physics, vol. 24, 1, p. 99-107 (1985).

$$R_{AB} - \frac{1}{2} g_{AB} R = \kappa T_{AB} \tag{1}$$

$$g_{AB}(x) = \kappa N \int G^{ret\ PQ}_{AB}(x, y|g) T_{PQ}(y) \sqrt{-g}\, d^N y. \tag{2}$$

which we write down here for space-time of arbitrary dimensionality N.

This integral form is the vivid formulation of Mach's principle as Einstein put it: "Space-time must be totally created by matter".

Kernel $G^{ret\ PQ}_{AB}(x, y|g)$ in integral formulation (indices A,B refer to the point x; P, Q - to y) is a bi-tensor Green function on the background space-time described by the same metric and is the retarded solution of the equation

$$E^{CD}_{AB} G_{CD}{}^{PQ}(x, y) = \delta^P_A \delta^Q_B \frac{\delta^N(x - y)}{\sqrt{-g}}, \tag{3}$$

where covariant differential operator E^{CD}_{AB} is defined on the same background g_{AB}.

This operator must satisfy simple condition:

$$E^{CD}_{AB} g_{CD} = N(R_{AB} - \frac{1}{2} g_{AB} R), \tag{4}$$

then its action upon integral form immediately gives Einstein equations.

There are many doubts and questions as regards to integral formulation of Einstein equations. One of questions was put to the author by John Archibald Wheeler (in 1968 at the Second International Gravitational Conference in Tbilisi) who said: "Why don't you include energy-momentum of gravitational waves in the source in the RHS of integral form?". I came back now to this old stuff not because I found out an answer, but because for the naturally defined (see below) differential operator E^{CD}_{AB} the demand of validity of integral representation of solutions of Einstein equations unexpectedly proves to be a selection rule for the dimensionality of space-time.

3 Mach operator

We define the differential operator E_{AB}^{CD} taking the second variation of the Einstein Action $\int R\sqrt{-g}$:

$$
\begin{aligned}
&E_{AB,CD}h^{CD} = \frac{2}{\sqrt{-g}}\frac{\delta^2(R\sqrt{-g})}{\delta g^{AB}\delta g^{CD}}h^{CD} = \\
&\frac{2}{\sqrt{-g}}\frac{\delta[\sqrt{-g}(R_{AB}-\frac{1}{2}\,g_{AB}R)]}{\delta g^{CD}}\,h^{CD} = \\
&\Big[-g_{AC}g_{BD}\nabla^2 - 2R_{AC,BD} + R_{AC}g_{BD} + R_{BC}g_{AD} + \frac{1}{2}g_{AB}g_{CD}\nabla^2 + \\
&R_{AB}g_{CD} + g_{AB}R_{CD} - g_{AC}g_{BD}R - \frac{1}{2}g_{AB}g_{CD}R\Big]\,h^{CD},
\end{aligned}
\tag{5}
$$

where $\nabla^2 = g^{PQ}\nabla_P\nabla_Q$ is D'Alambertian, and h_{CD} are small variations of metric

$$g_{CD} \to g_{CD} + h_{CD} \tag{6}$$

subject to the transverse gauge condition

$$\nabla_B(h_A^B - \frac{1}{2}\delta_A^B h_C^C) = 0. \tag{7}$$

First four terms of operator $E_{AB,CD}$ are the standard Lichnerowicz operator $\triangle_L$ which gives the variation of Ricci tensor in the transverse gauge: $2\delta R_{AB} = (\triangle_L h)_{AB} = -\nabla^2 h_{AB} - 2R_A{}^C{}_B{}^D h_{CD} + R_A^C h_{BC} + R_B^C h_{AC}$.

Action of E_{AB}^{CD} upon g_{CD} (instead of h_{CD}) immediately gives LHS of Einstein Equations, multiplied by N. Thus Green function defined by $EG = \delta$ may be used in presenting Einstein equations in integral form. Let us for short call the differentaial operator defined above *Mach operator*.

In what follows the solutions of the homogeneous equation

$$E_{AB}^{CD}u_{CD} = 0, \tag{8}$$

are studied on some elementary backgrounds.

It is worthwhile to note immediately that in this paper, following

- Gibbons G. and Hartnoll S.A., *"Gravitational instability in higher dimensions"*, Phys. Rev., vol. D66, (2002) 064024, *Preprint* hep-th/0206202.

- Kodama H. and Ishibashi A., *"A master equation for gravitational perturbations of maximally symmetric black holes in higher dimensions"*, Prog. Theor. Phys., vol. 110, p. 701-722 (2003), *Preprint* hep-th/0305147.

-Ishibashi A. and Kodama H., *"Stability of Higher-Dimensional Schwarzschild Black Holes"*, Prog. Theor. Phys., vol. 110, p. 901-919, *Preprint* hep-th/0305185.

the gauge-invariant tensor variations of the stationary background metrics of class

$$ds^2 = g_{ab}dx^a dx^b + r^2(x)d\sigma_n^2; \tag{9}$$

are considered. Here x^a are coordinates of the m-dimensional space-time, $a = 0, 1 \ldots (m-1)$; $d\sigma_n^2 = \gamma_{ij}dx^i dx^j$ is the metric of the n-dimensional G_n-invariant space with normalized constant sectional curvature $K = 0, \pm 1$. So the dimension of the whole space-time is $N = m + n$.

And since tensor modes are transverse and traceless by their definition the conclusions of this paper do not depend on the choice of transverse gauge condition.

4 Ghosts of Mach operator select 4 dimensions. Einstein universe background

$$ds^2 = -dt^2 + r_0^2 d\Omega_n^2. \tag{10}$$

And let us consider this metric as a background in Eq. $E_{AB}^{CD} u_{CD} = 0$ written for tensor modes of u_{AB} "living" on sphere S^n:

$$\tilde{u}_i^j = f(t) v_i^j (x^k), \tag{11}$$

$v_i^j(x^k)$ are tensor eigenmodes of the Laplace-Beltrami operator on n-sphere of unit radius: $\triangle_{S^n} v_i^j = l(l+n-1) - 2$; $l = 1, 2 \ldots$; $n = 3, 4 \ldots$ (tensor modes do not exist on 2-sphere).

Then Eq. $Eu = 0$ comes to:

$$\left[\frac{d^2}{dt^2} + \mu^2 \right] f(t) = 0, \;\; \mu^2 r_0^2 = l(l+n-1) - 2 - n(n-3), \tag{12}$$

here $l \geq 1$ and $n \geq 3$.

For the most "ghosts-threatening" lowest value of momentum number ($l = 1$) we have:

$$\mu^2 r_0^2 = 4n - 2 - n^2 \tag{13}$$

which is positive for $n = 3$ and negative for $n \geq 4$. Thus Mach operator 'selects' (1+3)-dimensional Einstein Universe.

5 Ghosts of Mach operator select 4 dimensions. De Sitter background

For this highly symmetric background ($R_{AB} = c(N-1)g_{AB}$) Eq-s $E^{CD}_{AB}\tilde{u}_{CD} = 0$ written for traceless modes $\tilde{u}^C_C = 0$ looks as:

$$(\triangle_L \tilde{u})^B_A - cN(N-1)\tilde{u}^B_A = (-\nabla^2 - cN(N-3))\tilde{u}^B_A = 0, \tag{14}$$

It is seen that on the De Sitter background ($c > 0$) mass squared of traceless modes of Mach operator becomes negative for space-time dimension $N \geq 4$. Actually, as we'll show now, the ghost-problems of Mach operator at this background begin for $N \geq 5$.

The only difference here from the analogous equation for tensor variations of Einstein equations is in the term $cN(N-1)$ in the LHS. For the "mass shell" variations of De Sitter space-time this term must be changed by $2c(N-1)$ (cf. Eq. (24) in Gibbons and Hartnoll (2002)); hence there mass squared of traceless modes is non-negative and de Sitter space is stable as expected.

Let us now look at the ghosts of Mach operator written on the De Sitter background which metric we - again in parallel with analyses of Gibbons and Hartnoll or Ishibashi and Kodama - take here in a form:

$$ds^2 = -(1 - c\,r^2)dt^2 + \frac{dr^2}{1 - c\,r^2} + r^2 d\Omega^2_n, \tag{15}$$

($d\Omega^2_n$ is metric of round sphere S^n).

We consider tensor modes on a sphere:

By the standard changing variable r to Regge-Wheeler type dimensionless coordinate y and rescaling field φ:

$$dy = \frac{\sqrt{c}dr}{1 - cr^2}, \quad r = \frac{1}{\sqrt{c}}\tanh y, \quad \varphi = r^{-n/2}\Phi \tag{16}$$

($0 < y < \infty$) finally we come to a Schroedinger-type equation

$$-\frac{d^2\Phi}{dy^2}+\left[\frac{4l(l+n-1)+n^2-2n}{4\sinh^2 y}-\frac{\beta(n)}{4\cosh^2 y}\right]\Phi=\frac{1}{c}E^2\Phi. \tag{17}$$

Coefficient $\beta(n)$ in potential in square brackets is crucial. It is function of dimensionality n. And this "Schredinger Equation" actually embraces two different cases of interest with different dependences $\beta(n)$:

$\beta_T = 5n^2 + 6n$;
$\beta_T^{(e)} = n^2 + 2n$.

Here $\beta_T(n)$ is received from Mach operator and $\beta_T^{(e)}(n)$ - from variation of Einstein equations.

Potential $V(y) = [\cdots]$ in square brackets is, as expected, non-negative for $\beta = \beta_T^{(e)}$, i.e. for tensor variations of de-Sitter background. But it is not the case for corresponding eigenmodes of Mach operator.

Undesirable ghost exists if for $E^2 < 0$ in "Schredinger equation" the normalization condition is valid:

$$\int \varphi^2 \frac{r^n}{1-cr^2}\, dr = \int_0^\infty \Phi^2 dy < \infty. \tag{18}$$

Ghost appears if negative potential well of $V(y) = [\cdot]$ is sufficiently deep. Fortunately there is exact solution of which clarifies the word "sufficiently":

$$\Phi = (\tanh y)^{l+n/2}(\cosh y)^{-\gamma},\ \ \gamma = \frac{1}{2}\sqrt{1+\beta} - \left(l + \frac{n}{2} + \frac{1}{2}\right), \tag{19}$$

with the ghost-like negative energy squared $E^2 = -c\gamma^2 < 0$.

This solution meets normalization condition if here $\gamma > 0$. This is evidently not the case for "Einstein" values of $\beta = \beta_T^{(e)}$, hence there are no normalized ghost modes among Einstein variations of de Sitter metric. For tensor modes of Mach operator $\beta = \beta_T$ normalization condition $\gamma > 0$ looks as:

$$\sqrt{5n^2 + 6n + 1} > 2l + n + 1 \ (l \geq 1, \ n \geq 3). \qquad (20)$$

In particular for $n = 3$ (i.e. for the 5 dimensional space-time this condition comes to $2 - l > 0$, hence normalization condition is fulfilled for the ghost-like tensor mode with $l = 1$.

Thus there are ghosts among tensor modes of Mach operator written on the 5-dimensional (and higher than 5 dimensions) De Sitter backgrounds. In particular this means that Integral form written for tensor modes on De-Sitter background of five and more dimensions is plagued by the ghosts of the retarded Green function G.

6 $AdS_m \times S^n$ background

Let us mention first the pure AdS. In this case there are no ghost problems of traceless modes.

It is possible to show also that in RS model the presence of the Z_2-symmetric co-dimension one brane results in the ghost bound state of tensor mode of Mach operator in the δ-function negative well potential of the brane.

More interesting is to consider the properties of Mach operator when the background is the Freund-Rubin $AdS_m \times S^n$ space-time:

$$ds^2 = dz^2 + e^{-2Hz}\eta_{\mu\nu}dx^\mu dx^\nu + r_0^2 d\Omega_n^2, \tag{21}$$

which stability was investigated in DeWolfe O., Freedman D.Z., Gubser S.S., Horowitz G.T., and Mitra I., *"Stability of $AdS_p \times M_q$ Compactifications Without Supersymmetry"*, Physical Review D, 65 (6). Art. No. 064033 (2002), *Preprint* hep-th/0105047.

For this metric we have:

$$R_{ab} = -(m-1)H^2 g_{ab}, \;\; R_{ij} = \frac{n-1}{r_0^2}g_{ij} = \frac{(m-1)^2}{n-1}H^2 g_{ij}, \tag{22}$$

and total scalar curvature is given by:

$$R = \frac{m-1}{n-1}(m-n)H^2. \tag{23}$$

Homogeneous Eqs. $E_{AB}^{CD}u_{CD} = 0$ on the background $AdS_m X S^n$ for two types of tensor modes:

- spherical tensor modes $u_i^j = \varphi_{(n)}(z, x^\mu)v_i^j(x^k)$ on S^n and - gravitational waves on AdS_m which are spherical scalar modes $u_\mu^\nu = \varphi_{(m)}(z, x^\mu)v_\mu^\nu(x^k)$ (we again omit everywhere the spherical momentum index l) come to two equations for scalars $\varphi_{(n)}$ and $\varphi_{(m)}$ correspondingly:

$$(\triangle(m) - M^2_{(n),(m)})\varphi_{(n),(m)} = 0, \tag{24}$$

Where $\triangle(m) = g^{ab}\nabla_a\nabla_b$ is D'Alambertian on AdS_m and effective masses $M_{(n)}$ and $M_{(m)}$ are:

$$M_{(n)}^2 = \frac{l(l+n-1) - 2 + 2n}{r_0^2} - R =$$
$$\left[\frac{(m-1)^2}{(n-1)^2}(l(l+n-1) - 2 + 2n) - \frac{m-1}{n-1}(m-n)\right] H^2, \qquad (25)$$

here $l = 1, 2\ldots;\quad n \geq 3$;

$$M_{(m)}^2 = \frac{l(l+n-1)}{r_0^2} - 2(m-1)H^2 - R =$$
$$\left[\frac{(m-1)^2}{(n-1)^2}\, l(l+n-1) - 2(m-1) - \frac{m-1}{n-1}(m-n)\right] H^2, \qquad (26)$$

here $l = 0, 1\ldots;\quad n \geq 2$.

It is easily seen that Breitenlohner-Freedman condition for AdS_m $M^2 \geq [-(m-1)^2/4]H^2$ which guarantees absence of the ghosts-solutions is always fulfilled for $M_{(n)}^2$, that is for tensor modes of S^n.

Whereas for for gravity waves on AdS_m, i.e. for $M_{(m)}^2$ it gives for lower spherical mode $l = 0$:

$$mn + 9 \geq 5(m+n). \qquad (27)$$

Minimal dimension of space-time AdS_m permitted by by this expression is $m = 6$, in this case we have $n \geq 21$, i.e. for total dimension $N = m + n \geq 27$.

For $m = n$, i.e. for the $AdS_n \times S^n$ background, this condition gives $n \geq 9$. In this case $R = 0$ and for tensor modes under consideration Mach operator comes to the Lichnerowicz operator $\triangle_L$. Thus $n \geq 9$ (i.e. $N = 2n \geq 18$) is a condition of absence of ghosts of the AdS_m tensor modes of Lichnerowicz operator written on the $AdS_n \times S^n$ background.

7 Discussion

Main results of this paper are given by Formulae which show that (3+1)-dimensional "Einstein universe" and De Sitter space-time are singled out by the demand of absence of ghosts of the naturally defined gravity differential operator. However this is just a mathematical observation. To connect this result for De Sitter universe with the possible dynamical answer to the nagging question "Why only 3 space dimensions expand during inflation?" is an open problem.

There is direct connection between absence of ghosts of this operator and validity of integral form of Einstein Equations if the retarded Green function is used there. "t what is the physics behand the integral form - the physics which will justify its use as a selection rule (for dimensionalities in particular)?

The ideas of "gravity without gravity" (rephrasing Weeler's favourite saying) or of "space-time totally created by matter" (which comes up to Mach's idea of relativity of accelerated movements) look quite dynamical. And hystorically Mach's ideas inspired Einstein for creation of General relativity, which however did not exclude empty ("non-machian") solutions of Einstein equations.

In string theory graviton is a dynamical excitation of more fundamental object and background space-time is Bose condensate of these excitations, and Einstein's gravity Action comes up as an effective one. However string theory suffers from plethora of admissible backgrounds which deprives it of physical predictability. String theory evidently needs additional selection rules. Can integral form of Einstein Equations be among such rules?

so nice condition for dimensionalities m, n of $AdS_m \times S^n$ received above is so far just a numerology of dimensionalities which may become science in case dynamical grounds for integral formulation of Einstein equations will be found out.

Boundary conditions imposed by integral form (??) upon solutions of Einstein equations (??) are easily received if we express energy-momentum tensor in the RHS of (??) from (??), (??): $N\kappa T_{PQ} = E^{CD}_{PQ} g_{CD}$, and then integrate (??) by parts. This (with account $\nabla_M g_{CD} \equiv 0$) gives

$$\oint \left[\sqrt{-g(y)} \nabla_{N_y} G^{ret\,Q}_{AB}{}_Q(x,y)\right] dS^{N_y} = 0 \tag{28}$$

which is the integral over boundary of space-time. Fulfillment of (??) guarantees the validity of (??).

Now we come to the formulae which will be used in the bulk of this paper.

Essentially more strong conditions than are imposed upon metric satisfying integral representation (??) if we demand that "neighboring" solutions of Einstein equations also are purely inhomogeneous. Thus we present integral form for the small variations of metric $\delta g_{AB} = h^{(e)}_{AB}$ (symbol (e) means that this is a solution of linear variation of Einstein equations (??) on the background metric g_{AB} satisfying (??)). Variation of (??) gives (symbolically) $\delta g = \delta G \cdot T + G \cdot \delta T$. The first term is calculated from variation of (??) preserving the retarded nature of the Green function: $\delta G = -G \cdot \delta E \cdot G$, thus with account of (??): $\delta G \cdot T = -G \cdot \delta E \cdot g$. Second term is received from variation of (??), (??): $G \cdot \delta T = G \cdot \delta E \cdot g + G \cdot E \cdot \delta g$. This chain of variations gives finally:

$$h^{(e)}_{AB}(x) = \int G^{ret\,CD}_{AB}(x,y|g) E^{PQ}_{CD} h^{(e)}_{PQ}(y) \sqrt{-g}\, d^N y. \tag{29}$$

Here, according to the definition of E^{CD}_{AB} in (??):

$$E^{CD}_{AB} h^{(e)}_{CD} = \frac{2}{\sqrt{-g}} \delta(\sqrt{-g}\kappa T_{AB}), \tag{30}$$

which is just the variation of Einstein equations (??).

This Eq. comes to identity if we act upon it with differential operator E^{CD}_{AB}. "Machian" absence of the "free term" in the RHS of (??) means the fulfillment, in analogy with (??), of the following boundary condition:

$$\oint \{ \sqrt{-g(y)} [(\nabla_{N_y} G^{ret\,PQ}_{AB}(x,y)) \, (h^{(e)}_{PQ} - \frac{1}{2} g_{PQ} h^{(e)K}_{K}) -$$

$$G_{AB}^{ret\ PQ}(x,y)\nabla_{N_y}(h_{PQ}^{(e)} - \frac{1}{2}g_{PQ}h_K^{(e)K})]\}\, dS^{N_y} = 0. \tag{31}$$