An Asymptotically Free SU(5) Model of Grand Unification with Four Generations

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Introduction

The SU(5) model of grand unification in its initial form[1] was rejected by later experiments[2] and required a certain modification.[3] In reference 4, a modified model was suggested that not only granted a vacancy for the τ lepton, but also had all its coupling constants asymptotically vanishing; this provided mathematical consistency in the theory suggested. This model, however, contained, as a matter of fact, two and a half generations.

In view of the recent experimental data,[5] it is of interest to introduce extra generations into the model of Reference 4.

The goal of our work was to construct a realistic asymptotically free SU(5) model of grand unification with no fewer than three generations of particles; we succeeded in doing so with four generations.

The model constructed contains rather many superfluous particles, such as the heavy Dirac ($\underline{24} + \underline{10} + \underline{5^*}$)-plet, but correctly describes the low-energy physics. This heavy multiplet was introduced in order to achieve the following goals: (1) to guarantee asymptotic freedom with respect to every coupling constant (in Reference 4 the heavy 24-plet sufficed, whereas the ($\underline{10} + \underline{5^*}$)-plet proved important for us here) and (2) to split the masses of leptons and quarks.

Usually, to achieve the latter goal within an SU(5) scheme, one includes a Higgs multiplet with 45 or more components.[8] The authors of Reference 4 chose to leave the τ lepton without the accompanying light neutrino. Within the present model, the necessary mass difference between leptons and quarks is provided by introducing a heavy (as heavy as 10^{12} Gev in the case under consideration) generation.

The $SU(3)\times SU(2)\times U(1)$ composition of our model is as follows: apart from four usual and one Dirac heavy generations, it contains a heavy (above a few M_W) leptonic triplet, an octet of neutral quarks, and two weak doublets of triplet quarks with 4/3 and 1/3 charges.

1. LAGRANGIAN

We consider an SU(5) model with the following Lagrangian:

$$L = L_G + L_Y + L_S, \tag{1.1}$$

where L_G, L_Y, and L_S are the kinetic part, Yukawa coupling, and interaction of spinless bosons, respectively.

The kinetic part L_G is completely determined by the multiplet composition of the model: the 24-plet of vector bosons V^a_μ, the 24-plet of fermions B^ρ, two $(\underline{5}+\underline{10^*})$-plets of right fermions ${}^j\theta^R_d$ and ${}^j\psi^{df}_R$ $(j=1,2)$, three $(\underline{5^*}+\underline{10})$-plets of left fermions ${}^\alpha\theta^d_L$ and ${}^\alpha\psi^L_{df}$ $(\alpha=1,2,3)$, and one $(\underline{5^*}+\underline{10})$-plet of right fermions η^d_R and K^R_{df}, which, together with ${}^3\theta^d_L$ and ${}^3\psi^L_{df}$, form a heavy generation of Dirac 4-component fermions (therefore, the model contains four chiral and one Dirac generations). It also contains a $(\underline{24}+\underline{5}+\underline{5})$-plet of spinless bosons ϕ^a, M_d, and N_d. The light generations contain the usual fractionally charged quarks u, d, c, and s of all three colors and the usual leptons (e, μ, ν_e, ν_μ).

The Yukawa coupling L_Y is chosen not to be of the most general form, but the particular form taken survives under radiational corrections:

$$\begin{aligned}
L_Y = & -\bar{B}^\alpha[(K_F\mathbb{P}+K_F^*\mathbb{Q})F^{\alpha\beta\gamma} \\
& +(K_D\mathbb{P}+K_D^*\mathbb{Q})D^{\alpha\beta\gamma}]\phi^\beta B^\gamma \\
& -[\nu_\alpha {}^\alpha\bar{\psi}^{df}_L K^R_{d'f'}(\phi^\rho\Gamma^\rho_{dfd'f'}) \\
& +a_\alpha {}^\alpha\bar{\theta}^L_d \eta^{d'}_R(\phi^\rho\lambda^\rho_{d'd}) \\
& +\delta_2^{j*}\bar{B}^\gamma_2 {}^j\theta^R_{d'}(M^{K*}\lambda^\gamma_{Kd'})
\end{aligned}$$

$$+ i\bar{\eta}^R_d B^\gamma_L (N^{K*} \lambda^\gamma_{Kd})$$

$$+ \delta^\alpha_1 {}^\alpha\bar{\theta}^L_d B^\gamma_R (N^{K*} \lambda^\gamma_{Kd})$$

$$+ f^{j\alpha}_1 {}^j\bar{\psi}^R_{df} {}^\alpha\theta^{d'}_L (M^{K*} \mathscr{P}_{dfKd'})$$

$$+ f^{j\alpha}_2 {}^j\bar{\theta}^d_R {}^\alpha\psi^L_{d'f'} (N^{K*} \mathscr{P}_{Kdd'f'})$$

$$+ K_{\alpha j} {}^\alpha\bar{\psi}^{df}_L {}^j\psi^{d'f'}_R (M^{K*} \varepsilon_{dfKd'f'}) + \text{h.c.}] \tag{1.2}$$

Here $\mathbb{P} = (1+\gamma_5)/2$ and $\mathbb{Q} = (1-\gamma_5)/2$ are projectors onto the left and right parts of spinors, respectively, $F^{\alpha\beta\gamma} = if^{\alpha\beta\gamma}$, and $D^{\alpha\beta\gamma} = 2\,\mathrm{tr}\,(\lambda^\alpha\{\lambda^\beta, \lambda^\gamma\})$, where $f^{\alpha\beta\gamma}$ are the SU(5) structure constants, λ^α are generators of the $\underline{5}$ representation normalized as $\mathrm{tr}\,(\lambda^\alpha\lambda^\beta) = \frac{1}{2}\delta_{\alpha\beta}$, the tensor $\mathscr{P}_{abcd}$ is the projector onto the 10-dimensional space of skew symmetrical rank 2 tensors,

$$\mathscr{P}_{abcd} = \tfrac{1}{2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}),$$

and Γ^α are generators of the $\underline{10}$ -representation

$$\Gamma^\alpha_{abcd} = 2\mathscr{P}_{aba'b'}(\lambda^\alpha_{a'c'}\delta_{b'd'})\mathscr{P}_{c'd'cd}.$$

The most general Yukawa coupling may be written if the terms obtained from (1.2) by the interchange $M \leftrightarrow N$ are added to it.

The scalar interaction has been taken in the form

$$L_S = -\tfrac{1}{8}\lambda^2_{\phi^4}(\phi^2)^2 - \tfrac{1}{8}(\phi^\alpha D^{\alpha\rho\beta}\phi^\beta)^2\delta^2_{\phi^4}$$

$$-[\tfrac{1}{2}\lambda^2_{\phi^2M^2}(M^+M)\phi^2 + \tfrac{1}{2}\delta^2_{\phi^2M^2}(\phi^\alpha D^{\alpha\rho\beta}\phi^\beta)$$

$$\times (M^{K*}\lambda^\rho_{Kl}M_l)$$

$$+\tfrac{1}{2}\lambda^2_{M^4}(M^+M)^2 + (M \to N)]$$

$$-\lambda^2_{M^2N^2}(M^+M)(N^+N) - \delta^2_{M^2N^2}(M^+N)(N^+M). \tag{1.3}$$

This self-coupling is not of the most general form either. It could also include the terms

$$[\lambda(M^+N)\phi^2+\text{h.c.}],$$

$$[\lambda(\phi^\alpha D^{\alpha\rho\beta}\phi^\beta)(M^{K*}\lambda^\rho_{Kl}N_l)+\text{h.c.}],$$

$$[\lambda(M^+M)(M^+N)+\text{h.c.}],\quad [\lambda(\mathrm{N}^+\mathrm{N})(\mathrm{N}^+\mathrm{M})+\text{h.c.}]$$

$$[\lambda(M^+N)(M^+N)+\text{h.c.}]. \tag{1.4}$$

However, these terms, if not included, do not reappear after radiational corrections are taken into account once the Yukawa interaction is chosen the way it is. Besides, the cubic terms in the spinless fields,

$$L_{\phi^3}=a_1M^{K*}(\phi^\alpha\lambda^\alpha_{Kl})M_l+a_2N^{k*}(\phi^\alpha\lambda^\alpha_{Kl})\mathrm{N}_l$$
$$+[a_{12}M^{K*}(\phi^\alpha\lambda^\alpha_{Kl})N_l+\text{h.c.}]+bD^{\alpha\delta\gamma}\phi^\alpha\phi^\beta\phi^\gamma, \tag{1.5}$$

may be added to L_S without affecting the asymptotic freedom.[11]

2. RENORMALIZATION GROUP EQUATIONS

The one-loop approximation of the renormalization group equations[11] for the dimensionless couplings of our model is a system of three sets of equations.

The first set is the sole equation for the gauge coupling constant

$$\dot{g}=-\tfrac{5}{2}g^3. \tag{2.1}$$

Here the dot stands for differentiation with respect to $t=\log K/16\pi^2$.

This equation is readily solved to give

$$g^2(t)=g_0^2\cdot[1+5g_0^2t]^{-1}; \tag{2.2}$$

all its solutions are asymptotically free, in the sense that they tend to zero as $t\to\infty$.

The second set of equations is formed by those for the Yukawa coupling constants

$$\dot{K}F=\tfrac{42}{5}iK_D\,\mathrm{Im}\,K_F^*\cdot K_D-\tfrac{1}{2}t\delta_1^+a$$

$$+K_F[-30g^2+20|K_F^2|+\tfrac{42}{5}|K_D^2|+a^+a+3v^+v$$

$$+\tfrac{1}{4}(\delta_1^+\delta_1+\delta_2^+\delta_2+|t|^2)],$$

$$\dot{K}_D=10iK_F\,\mathrm{Im}\,K_FK_D^*$$

$$+K_D[-30g^2+10|K_F^2|+\tfrac{76}{5}|K_D^2|+a^+a+3v^+v$$

$$+\tfrac{1}{4}(\delta_1{}^+\delta_1+\delta_2{}^+\delta_2+|t|^2],$$

$$\dot{a}=(-5K_F+\tfrac{21}{5}K_D)t^*\delta_1+a[-\tfrac{72}{5}g^2+10|K_F^2|+\tfrac{42}{5}|K_D^2|$$

$$+\tfrac{16}{5}a^+a+3v^+v+\tfrac{6}{5}|t|^2]+[f_1^+f_1+\tfrac{6}{5}\delta_1\delta_1^+]a,$$

$$\dot{t}=(-5K_F+\tfrac{21}{5}K_D)a^+\delta_1+t[-\tfrac{111}{5}g^2+\tfrac{77}{20}|t|^2+\tfrac{5}{2}|K_F^2|$$

$$+\tfrac{21}{10}|K_D^2|+\tfrac{6}{5}a^+a+\tfrac{12}{5}\delta_1^+\delta_1+\tfrac{1}{4}\delta_2^+\delta_2+2\,\mathrm{tr}\,(f_2^+f_2)],$$

$$\dot{v}=v[-\tfrac{108}{5}g^2+10|K_F^2|+\tfrac{42}{5}|K_D^2|+a^+a+\tfrac{44}{5}v^+v]$$

$$+[\tfrac{1}{2}f_2^+f_2+6KK^+]v,$$

$$\dot{\delta}_1=(-5K_F^*+\tfrac{21}{5}K_D^*)at+\delta_1[-\tfrac{111}{5}g^2+\tfrac{5}{2}|K_F^2|+\tfrac{21}{10}|K_D^2|$$

$$+\tfrac{77}{20}\delta_1^+\delta_1+\tfrac{12}{5}|t|^2+2\,\mathrm{tr}\,(f_2^+f_2)]+[\tfrac{6}{5}aa^++f_1^+f_1]\delta_1,$$

$$\dot{\delta}_2=\delta_2[-\tfrac{111}{5}g^2+\tfrac{5}{2}|K_F^2|+\tfrac{21}{10}|K_D^2|+\tfrac{1}{20}|t|^2+\tfrac{77}{20}\delta_2{}^+\delta_2$$

$$+2\,\mathrm{tr}\,(f_1^+f_1)+24\,\mathrm{tr}\,(K^+K)]+f_2f_2^+\delta_2,$$

$$\dot{f}_1=f_1[-18g^2+\tfrac{12}{5}\delta_2^+\delta_2+2\,\mathrm{tr}\,(f_1^+f_1)+24\,\mathrm{tr}\,(K^+K)]$$

$$+\tfrac{3}{2}f_1f_1^+f_1+f_1[\tfrac{6}{5}aa^++\tfrac{6}{5}\delta_1\delta_1^+]-18K^+Kf_1,$$

$$\dot{f}_2=f_2[-18g^2+\tfrac{12}{5}|t|^2+\tfrac{12}{5}\delta_1^+\delta_1+2\operatorname{tr}(f_2^+f_2)]$$

$$+\tfrac{3}{2}f_2f_2^+f_2+f_2[\tfrac{9}{5}vv^++6KK^+]+\tfrac{6}{5}\delta_2\delta_2^+f_2,$$

$$\dot{K}=K[-\tfrac{108}{5}g^2+\tfrac{12}{5}\delta_2^+\delta_2+2\operatorname{tr}(f_1^+f_1)+24\operatorname{tr}(K^+K)]$$

$$+12KK^+K-\tfrac{3}{2}Kf_1f_1^++[\tfrac{9}{5}vv^++\tfrac{1}{2}f_2^+f_2]K. \tag{2.3}$$

We look for solutions to this set in the form[6] $\omega(t)=\bar{\omega}(t)\cdot g(t)$. The differential equations for $\bar{\omega}(t)$ have nonzero stationary solutions $\bar{\omega}(t)=\text{const}$.

It is an easy matter to check that the neighborhoods of many of these stationary points contain solutions $\bar{\omega}(t)$ that tend to these stationary points. The existence of these solutions allows one to obtain a mass hierarchy of fermions different from that which originates from the hierarchy of the vacuum expectation values of Higgs fields. We discuss this question in more detail below.

The third set of equations serves as the constants of scalar self-coupling

$$\dot{\lambda}^2_{\phi^4}=32\lambda^4_{\phi^4}+\tfrac{84}{5}\lambda^2_{\phi^4}\delta^2_{\phi^4}+\tfrac{168}{25}\delta^4_{\phi^4}+10(\lambda^4_{\phi^2M^2}+\lambda^4_{\phi^2N^2})$$

$$+24g^4-32|K_F^4|-\tfrac{672}{25}|K_D^4|+4z_\phi\lambda^2_{\phi^4}$$

$$-\tfrac{18}{5}(v^+v)^2-\tfrac{4}{5}(a^+a)^2,$$

$$\dot{\delta}^2_{\phi^4}=8\delta^4_{\phi^4}+12\lambda^2_{\phi^4}\delta^2_{\phi^4}+(\delta^4_{\phi^2M^2}+\delta^4_{\phi^2N^2})+15g^2$$

$$-20|K_F^4|-120|K_F^2K_D^2|+\tfrac{28}{5}|K_D^4|$$

$$+4z_\phi\delta^2_{\phi^4}-\tfrac{2}{3}(v^+v)^2-2(a^+a)^2,$$

$$\dot{\lambda}^2_{\phi^2M^2}=4\lambda^4_{\phi^2M^2}+\tfrac{42}{25}\delta^4_{\phi^2M^2}+(26\lambda^2_{\phi^4}+\tfrac{42}{5}\delta^2_{\phi^4}$$

$$+12\lambda^2_{M^4})\lambda^2_{\phi^2M^2}+(10\lambda^2_{N^2M^2}+2\delta^2_{N^2M^2})\lambda^2_{\phi^2N^2}$$

$$+2(z_\phi+z_M)\lambda^2_{\phi^2M^2}+6g^4-8(\delta_2^+\delta_2)[\tfrac{1}{2}|K_F^2|$$

$$+\tfrac{21}{50}|K_D^2|-\tfrac{72}{5}v^+KK^+v-\tfrac{8}{5}a^+f_1^+f_1a,$$

$$\dot{\lambda}^2_{\phi^2N^2} = 4\lambda^4_{\phi^2N^2} + \tfrac{42}{25}\delta^4_{\phi^2N^2} + (26\lambda^2_{\phi^4} + \tfrac{42}{5}\delta^2_{\phi^4}$$

$$+ 12\lambda^2_{N^4})\lambda^2_{\phi^2N^2} + (10\lambda^2_{N^2M^2} + 2\delta^2_{N^2M^2})\lambda^2_{\phi^2M^2}$$

$$+ 2(z_\phi + z_N)\lambda^2_{\phi^2N^2} + 6g^4 - \tfrac{48}{25}|t|^2 a^+ a$$

$$- \tfrac{48}{25}(a^+\delta_1\delta_1^+ a) - 8(|t|^2 + \delta_1^+\delta_1)(\tfrac{1}{2}|K_F^2| + \tfrac{21}{50}|K_D^2|)$$

$$- \tfrac{12}{5} v^+ f_2^+ f_2 v - 16\,\mathrm{Re}\,[t^*(a^+\delta_1)(-\tfrac{1}{4}K_F + \tfrac{21}{100}K_D)],$$

$$\dot{\delta}^2_{\phi^2M^2} = (8\lambda^2_{\phi^2M^2} + \tfrac{13}{5}\delta^2_{\phi^2M^2} + 2\lambda^2_{\phi^4} + \tfrac{34}{5}\delta^2_{\phi^4}$$

$$+ 2\lambda^2_{M^4})\delta^2_{\phi^2M^2} + 2\delta^2_{M^2N^2}\delta^2_{N^2\phi^2} + 2(z_\phi + z_M)\delta^2_{\phi^2M^2}$$

$$+ 15g^4 - 8\delta_2{}^+\delta_2[\tfrac{5}{4}|K_F^2| + \tfrac{13}{20}|K_D^2| - \tfrac{5}{2}\,\mathrm{Re}\,K_F K_D^*]$$

$$+ \tfrac{32}{3} v^+ K K^+ v + 2a^+ f_1^+ f_1 a,$$

$$\dot{\delta}^2_{\phi^2N^2} = (8\lambda^2_{\phi^2N^2} + \tfrac{13}{5}\delta^2_{\phi^2N^2} + 2\lambda^2_{\phi^4} + \tfrac{34}{5}\delta^2_{\phi^4}$$

$$+ 2\lambda^2_{N^4})\delta^2_{\phi^2N^2} + 2\delta^2_{M^2N^2}\delta^2_{M^2\phi^2} + 2(z_\phi + z_N)\delta^2_{\phi^2N^2}$$

$$+ 15g^4 - 8(|t|^2 + \delta_1^+\delta_1)(\tfrac{5}{4}|K_F^2| + \tfrac{13}{20}|K_D^2| + \tfrac{5}{2}\,\mathrm{Re}\,K_F K_D^*)$$

$$+ \tfrac{2}{5}(|t|^2 a^+ a + a^+\delta_1\delta_1{}^+ a) - \tfrac{8}{3} v^+ f_2^+ f_2 v$$

$$+ \tfrac{16}{5}\,\mathrm{Re}\,[K_D t^*(a^+\delta_1)],$$

$$\dot{\lambda}^2_{M^4} = 18\lambda^4_{M^4} + 10\lambda^4_{M^2N^2} + 2\delta^4_{M^2N^2} + 4\lambda^2_{M^2N^2}\delta^2_{M^2N^2}$$

$$+ 24\lambda^4_{\phi^2M^2} + \tfrac{42}{25}\delta^4_{\phi^2M^2} + 4z_M\lambda^2_{M^4} + \tfrac{198}{25}g^4$$

$$- 4\,\mathrm{tr}\,(f_1 f_1^+ f_1 f_1^+) - 192\,\mathrm{tr}\,(KK^+KK^+) - \tfrac{116}{25}(\delta_2^+\delta_2)^2,$$

$$\dot{\lambda}^2_{N^4} = 18\lambda^4_{N^4} + 10\lambda^4_{M^2N^2} + 2\delta^4_{M^2N^2} + 4\lambda^2_{M^2N^2}\delta^2_{M^2N^2}$$

$$+ 24\lambda^4_{\phi^2N^2} + \tfrac{42}{25}\delta^4_{\phi^2N^2} + 4z_N\lambda^2_{N^4} + \tfrac{198}{25}g^4$$

$$- 4\,\mathrm{tr}\,(f_2 f_2^+ f_2 f_2^+) - \tfrac{116}{25}(|t|^4 + (\delta_1^+\delta_1)^2),$$

$$\dot{\lambda}^2_{M^2N^2}=(\lambda^2_{M^4}+\lambda^2_{N^4})(12\lambda^2_{M^2N^2}+2\delta^2_{M^2N^2})+4\lambda^4_{M^2N^2}$$

$$+2\delta^4_{M^2N^2}+24\lambda^2_{\phi^2M^2}\lambda^2_{\phi^2N^2}-\tfrac{21}{50}\delta^2_{\phi^2M^2}\delta^2_{\phi^2N^2}$$

$$+2(z_2+z_N)\lambda^2_{M^2N^2}+\tfrac{81}{50}g^4-\tfrac{26}{5}|t|^2\delta_2^+\delta_2$$

$$-24\,\mathrm{tr}\,(K^+Kf_2f_2^+)-\tfrac{19}{5}(\delta_1^+f_1^+f_1\delta_1+\delta_2^+f_2^+f_2\delta_2),$$

$$\dot{\delta}^2_{M^2N^2}=2\delta^2_{M^2N^2}(\lambda^2_{M^4}+\lambda^2_{N^4})+8\lambda^2_{M^2N^2}\delta^2_{M^2N^2}$$

$$+10\delta^4_{M^2N^2}+\tfrac{21}{10}\delta^2_{\phi^2M^2}\delta^2_{\phi^2N^2}+2(z_M+z_N)\delta^2_{M^2N^2}$$

$$+\tfrac{63}{10}g^4+\tfrac{2}{5}|t|^2\delta_2^+\delta_2+24\,\mathrm{tr}\,(f_2f_2^+KK^+)$$

$$-\tfrac{1}{5}(\delta_1^+f_1^+f_1\delta_1+\delta_2^+f_2f_2^+\delta_2),$$

where

$$z_\phi=10|K_F^2|+\tfrac{42}{5}|K_D^2|+3v^+v+a^+a-15g^2,$$

$$z_M=\tfrac{12}{5}\delta_2^+\delta_2+2\,\mathrm{tr}\,(f_1^+f_1)+24\,\mathrm{tr}\,(K^+K)-\tfrac{36}{5}g^2,$$

$$z_N=\tfrac{12}{5}(|t|^2+\delta_1^+\delta_1)+2\,\mathrm{tr}\,(f_2^+f_2)-\tfrac{36}{5}g^2.$$

For some solutions of the equations for the Yukawa coupling constants, (2.4) have solutions of the form

$$\omega^2(t)=\bar{\omega}^2\cdot g^2(t). \tag{2.5}$$

The existence of such solutions of the set (2.4) imposes restrictions on the choice of solutions of the set (2.3).

Here we list some solutions of the sets (2.3, 2.4):

$\vert\bar{K}_F^2\vert=$	1.28	1.22	1.26	1.27	1.34
$\vert\bar{K}_D^2\vert=$	1.63	3.13	2.35	1.37	2.41
$\vert\bar{f}_1^{11}\vert^2=$	0	2.27	0	2.22	1.77

$|\bar{f}_1^{22}|^2 = 3.31 \quad 0 \quad 2.82 \quad 2.22 \quad 1.77$

$|\bar{f}_2^{11}|^2 = 3.60 \quad 0 \quad 1.87 \quad 2.37 \quad 0$

$|\bar{f}_2^{22}|^2 = 0 \quad 4.43 \quad 3.07 \quad 2.78 \quad 4.43$

$|\bar{v}_1|^2 = 0.514 \quad 0.787 \quad 0 \quad 0 \quad 0$

$|\bar{v}_2|^2 = 0 \quad 0 \quad 0.563 \quad 0.566 \quad 0$

$\bar{\lambda}^2_{\phi^4} = 0.596 \quad 0.594 \quad 0.582 \quad 0.592 \quad 0.594$

$\bar{\delta}^2_{\phi^4} = 0.985 \quad 0.785 \quad 0.967 \quad 0.979 \quad 1.25$

$\bar{\lambda}^2_{\phi^2 M^2} = 0.0545 \quad 0.153 \quad 0.112 \quad 0.0220 \quad 0.125$

$\bar{\lambda}^2_{\phi^2 N^2} = -0.0709 \quad -0.165 \quad -0.0602 \quad -0.0606 \quad -0.153$

$\bar{\delta}^2_{\phi^2 M^2} = 0.115 \quad 0.878 \quad 0.539 \quad -0.0564 \quad 0.507$

$\bar{\delta}^2_{\phi^2 N^2} = -0.876 \quad -1.13 \quad -0.600 \quad -0.596 \quad -1.03$

$\bar{\lambda}^2_{M^4} = 1.16 \quad 1.21 \quad 1.14 \quad 0.940 \quad 0.951$

$\bar{\lambda}^2_{N^4} = 1.36 \quad 1.64 \quad 1.15 \quad 1.15 \quad 1.64$

$\bar{\lambda}^2_{M^2 N^2} = 0.493 \quad -0.0796 \quad 0.311 \quad 0.248 \quad -0.0447$

$\bar{\delta}^2_{M^2 N^2} = -0.274 \quad -0.199 \quad -0.201 \quad -0.229 \quad -0.235$

The rest of the charges are equal to zero. Note that, since the equations for the Yukawa couplings are invariant under the transformations

$$\begin{aligned} &f_1 \to u_1 f_1 u_2^+, \quad f_2 \to u_3 f_2 u_4^+, \quad K \to e^{i\phi_K} u_4 K u_1^+, \\ &a \to u_2 a, \quad v \to e^{i\varphi_v} u_4 v, \quad \delta_1 \to e^{i\varphi_1} u_2 \delta_1, \\ &\delta_2 \to e^{i\varphi_2} u_3 \delta_2, \\ &K_F \to e^{i\varphi_F} K_F, \quad K_D \to e^{i\varphi_F} K_D, \quad t \to e^{i(\varphi_F + \varphi_1)} t \end{aligned} \tag{2.6}$$

with u_1 and u_3 unitary 2×2 matrices and u_2 and u_4 unitary 3×3 matrices, we are able to create a whole family of solutions by applying (2.6) to any solution of the set (2.3) found.

3. SOLUTIONS FOR YUKAWA CONSTANTS THAT HAVE THE ASYMPTOTIC FORM $\omega(t)=\bar{\omega}_0 \cdot g(t)$

The set (2.3) of equations for the Yukawa coupling has the form

$$\omega_i = f_i(\omega). \tag{3.1}$$

By introducing the new functions

$$\bar{\omega}_i(t) = \frac{\omega_i(t)}{g(t)} \tag{3.2}$$

and the new variable x in place of t

$$x = -\tfrac{2}{5}\log g, \tag{3.3}$$

it is reduced to the form

$$\frac{\mathrm{d}\,\bar{\omega}_i}{\mathrm{d}\,x} = f_i(\bar{\omega}) + \tfrac{5}{2}\bar{\omega}_i \tag{3.4}$$

(certainly, $\bar{g}(t) \equiv 1$).

Some stationary points of this set were written in the preceding section. Here we are going to study their stability.

The behavior of solutions of (3.4) near the point $\bar{\omega}(t) \equiv \bar{\omega}_0$ is subject to the equation

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\tilde{\omega} = A\tilde{\omega} + \tfrac{5}{2}\tilde{\omega}, \tag{3.5}$$

where $\tilde{\omega} = \bar{\omega} - \bar{\omega}_0$ and

$$A_{ij} = \left.\frac{\partial f_i}{\partial \omega_j}\right|_{\bar{\omega}_0} \tag{3.6}$$

It may be easily verified that the matrix A that governs this behavior has the property $A_{ij} = A_{ji} = 0$ if $\bar{\omega}_{i0} = 0$ while $\bar{\omega}_{j0} \neq 0$.

We are interested in the behavior of the deflections $\tilde{\omega}_i$ for all i such that

$\bar{\omega}_{i0}=0$. Hence we consider only the corresponding matrix block of A. By applying transformations (2.6), we can come to the situation when the matrices $\bar{f}_{10}$ and $\bar{f}_{20}$ are diagonal and the value of $\bar{K}_{F0}$ is real.

For

$$\bar{K}_0^{\alpha j}=0, \quad \bar{t}_0=0, \quad \bar{\delta}_{10}=0, \quad \bar{a}_0=0, \bar{K}_{D0}=0 \tag{3.7}$$

(it is only for these values that we succeeded in finding asymptotically free solutions of (2.4) for the scalar self-couplings), the part of the matrix A that corresponds to vanishing $\bar{\omega}_{i0}$ is diagonal with some diagonal elements λ_i.

From (3.7), it follows directly that the solutions $\tilde{\omega}_i(t)$ that correspond to vanishing $\bar{\omega}_{i0}$ behave at large t as

$$\tilde{\omega}_i \sim \exp\left[(\lambda_i+\tfrac{5}{2})\log g^{-2/5}\right]=g^{\alpha i}, \tag{3.8}$$

where

$$\alpha_i=-\tfrac{2}{5}\lambda_i-1. \tag{3.9}$$

The values of the powers α_i for the solutions listed in the preceding section are the following:

i	α_i	α_i	α_i	α_i	α_i
$\mathrm{Im}\, K_F$	4.59	4.37	4.55	4.60	4.86
$\mathrm{Re}\, K_D$	9.71	9.25	9.59	9.68	10.22
a_1	−0.98	−1.97	−0.96	−1.89	−1.31
a_2	−2.30	−1.06	−2.08	−1.89	−1.31
a_3	−0.98	−1.06	−0.96	−1.00	−0.60
v_1	—	—	0.24	0.09	2.28
v_2	0.71	−0.90	—	—	1.39
v_3	0.71	−0.01	0.62	0.57	2.28
t	3.56	2.80	2.43	2.35	2.76
δ_1^1	3.72	2.21	2.67	1.60	2.29
δ_1^2	2.40	3.12	1.54	1.60	2.29
δ_1^3	3.72	3.12	2.67	2.49	3.00
δ_2^1	—	—	—	—	—

i	α_i	α_i	α_i	α_i	α_i
δ_2^2	1.44	−1.75	−0.48	−0.16	−1.78
f_1^{ij}	1.99	1.38	1.69	1.33	1.05
f_2^{11}	—	0.59	—	—	1.50
f_2^{21}	2.95	2.09	2.25	2.08	2.66
f_2^{12}	2.54	1.15	0.71	1.01	1.50
f_2^{22}	3.32	—	—	—	—
f_2^{13}	2.54	1.15	1.12	1.42	1.50
f_2^{23}	3.32	2.66	2.25	2.08	2.66
K^{11}	2.34	3.61	2.75	3.63	3.56
K^{21}	3.43	3.30	2.11	3.14	2.67
K^{31}	3.43	4.18	3.13	4.10	3.56
K^{12}	4.32	2.25	4.45	3.63	3.56
K^{22}	5.41	1.93	3.80	3.14	2.67
K^{32}	5.41	2.82	4.82	4.10	3.56

Initial values for those ω_i whose α_i is positive may be taken from a small neighborhood of the curve $\bar{\omega}_{i0}g(t)$, whereas those for ω_i with negative α_i should lie on the special curve $\bar{\omega}_{i0}g(t)$. We exploit this possibility below.

Solutions of this kind were used earlier to build an asymptotically supersymmetrical E_8 GUM.[7]

4. VACUUM EXPECTATION VALUES OF SPINLESS FIELDS

Let us look for the vacuum expectation values (VEVs) of spinless fields within the tree approximation. The potential, as written in terms of the matrix

$$w=\langle\phi^\alpha\rangle\lambda^\alpha \tag{4.1}$$

and the vectors v_1 and v_2,

$$v_{1a}=\langle M_a\rangle, \quad v_{2a}=\langle N_a\rangle, \tag{4.2}$$

is, in the three approximation, equal to

$$V(w, v_1, v_2)=-\frac{m_\phi^2}{2}\operatorname{tr} w^2-m_M^2 v_1^+ v_1-m_N^2 v_2^+ v_2$$

$$
\begin{aligned}
&-(m_{12}v_1^+v_2+\text{h.c.})+\tfrac{1}{4}(\tfrac{1}{2}\lambda^2_{\phi^4}-\tfrac{1}{5}\delta^2_{\phi^4})(\operatorname{tr}w^2)^2\\
&+\tfrac{1}{4}\delta^2_{\phi^4}\operatorname{tr}w^4+\big[\tfrac{1}{2}(\lambda^2_{\phi^2M^2}-\tfrac{1}{5}\delta^2_{\phi^2M^2})\operatorname{tr}w^2\cdot v_1^+v_1\\
&+\tfrac{1}{2}\delta^2_{\phi^2M^2}v_1^+w^2v_1+\tfrac{1}{2}\lambda^2_{M^4}(v_1^+v_1)^2\\
&+(M\to N, v_1\to v_2)\big]\\
&+\lambda^2_{M^2N^2}(v_1^+v_1)(v_2^+v_2)+\delta^2_{M^2N^2}(v_1^+v_2)(v_2^+v_1)\\
&+a_1v_1^+wv_1+a_2v_2^+wv_2+(a_{12}v_1^+wv_2+\text{h.c.})\\
&+\tfrac{1}{3}b\operatorname{tr}w^3. \qquad (4.3)
\end{aligned}
$$

We are looking for the VEVs in the form

$$
w=\begin{bmatrix} \frac{s}{3} & & & & \\ & \frac{s}{3} & & & \\ & & \frac{s}{3} & & \\ & & & -s+\frac{r}{2} & \\ & & & & -\frac{r}{2} \end{bmatrix},\quad v_1=\begin{bmatrix}0\\0\\0\\0\\\rho\end{bmatrix},\quad v_2=\begin{bmatrix}0\\0\\0\\0\\\sigma\end{bmatrix}. \qquad (4)
$$

Parameters ρ, σ, and s may be used as independent ones instead of the masses m_ϕ^2, m_M^2, and m_N^2. For $t=r/s$, it is easy to obtain an equation that has the form

$$
(t-1)(t+\tfrac{2}{3})\left[\delta^2_{\phi^4}\frac{s^2}{3}-\frac{b}{2}\right]+\frac{t}{2}(\delta^2_{\phi^2M^2}|R|^2+\delta^2_{\phi^2N^2}|S|^2)
$$
$$
=a_1|R|^2+a_2|S|^2+2\operatorname{Re}(a_{12}R^*S) \qquad (4.5)
$$

in terms of the dimensionless variables $R=\rho/s$ and $S=\sigma/s$. This quadratic in the t equation gives a solution for VEVs.

5. MASSES OF VECTOR BOSONS

Since the symmetry group and the Higgs sector of our model are the same as in Reference 4, the vector boson masses acquired due to spontaneous symmetry breaking must also be the same. First of all, one readily separates massless vector fields: the octet of gluons, i.e., the fields of the form

$$G^a_\mu = \left[\begin{array}{ccc|cc} & * & & 0 & 0 \\ & & & 0 & 0 \\ \hline & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad (a=1,2,\ldots,8)$$

and the electromagnetic field (photon) of the form

$$A_\mu = \begin{bmatrix} -\frac{1}{3} & & & & \\ & -\frac{1}{3} & & & \\ & & -\frac{1}{3} & & \\ & & & 1 & \\ & & & & 0 \end{bmatrix}.$$

The mass matrix of the other vector fields splits into blocks that correspond to the eigensubspaces of the operators $\hat{A}_\mu$, $\hat{G}^3_\mu$, and $\hat{G}^8_\mu$, where $G^3_\mu = \operatorname{diag}(1, -1, 0, 0, 0)$, $G^8_\mu = \operatorname{diag}(1, 1, -2, 0, 0)$, and $\hat{X}(Y) = [X, Y]$.

Since all charged particles are contained in one-dimensional eigensubspaces of $\hat{A}_\mu$, $\hat{G}^3_\mu$, and $\hat{G}^8_\mu$, one immediately finds their masses to be

$$m_v^2 = g^2 \Big[[\operatorname{tr} V^2]^{-1} \cdot [\operatorname{tr}([V, w] \cdot [V, w]^+) \Big] + \Big[\tfrac{1}{2} |(V + V^+) v_1|^2 + \tfrac{1}{2} |(V + V^+) v_2|^2] \cdot [\operatorname{tr}(V + V^+)^2]^{-1} \Big]. \quad (5.1)$$

Hence it follows that the leptoquark masses are

$$m^2_{(\pm 4/3, \mp 1, \mp 1)} = m^2_{(\pm 4/3, \pm 1, \mp 1)} = m^2_{(\pm 4/3, 0, \mp 2)} = g^2 (\tfrac{r}{2} - \tfrac{4}{3} s)^2, \quad (5.2)$$

$$m^2_{(\pm 1/3, \mp 1, \mp 1)} = m^2_{(\pm 1/3, \pm 1, \mp 1)} = m^2_{(\pm 1/3, 0, \mp 2)} = g^2 [(\tfrac{r}{2} + \tfrac{s}{3})^2 + \tfrac{1}{2}(|\rho|^2 + |\sigma|^2)], \quad (5.2)$$

and the mass of the charged $W_\pm$ boson is equal to

$$m_W^2 = m_{(\pm 1,0,0)}^2 = g^2[(r-s)^2 + \tfrac{1}{2}(|\rho|^2 + |\sigma|^2)]. \tag{5.3}$$

In order to find the mass of the neutral Z boson it suffices to note that we know the masses of the three out of the four neutral bosons:

$$A_\mu, G_\mu^3, \text{and } G_\mu^8.$$

The Z boson must be orthogonal to them,

$$Z = \alpha \cdot \mathrm{diag}(1, 1, 1, 1, -4),$$

and, according to (5.1), have the mass

$$m_Z^2 = \tfrac{2}{5} g^2 (|\rho|^2 + |\sigma|^2). \tag{5.4}$$

It is seen from the expressions for the vector boson masses (5.2–5.4) that, so long as we want to obtain the necessary mass hierarchy, the following estimates are to be held:

$$m_W = g[(r-s)^2 + \tfrac{1}{2}(|\rho|^2 + |\sigma|^2)]^{1/2} \sim 100\,\mathrm{GeV},$$

$$m_{\mathrm{L.\text{-}Q.}} \sim 10^{14}\text{–}10^{15}\,\mathrm{GeV},$$

(the subscript L.-Q. indicates leptoquarks), so that the inequalities

$$|g(r-s)| \lesssim 100\,\mathrm{GeV}$$

and

$$|g\rho| \lesssim 100\,\mathrm{GeV}$$

$$|g\sigma| \lesssim 100\,\mathrm{GeV}$$

are valid. These inequalities imply the following estimates:

$$\left|\frac{r}{s} - 1\right| \lesssim 10^{-12},$$

$$\left|\frac{\rho^2}{s^2}\right| \lesssim 10^{-24}, \quad \left|\frac{\sigma^2}{s^2}\right| \lesssim 10^{-24}. \tag{5.5}$$

If now we take (4.5) into account, from which it follows that

$$\frac{r}{s}\sim 1-\frac{9}{10}\frac{|\rho^2|\delta^2_{\phi^2 M^2}+|\sigma|^2\delta^2_{\phi^2 N^2}}{\delta^2_{\phi^4}s^2},$$

we can make the preceding estimate more accurate:

$$\left|\frac{r}{s}-1\right|\gtrsim 10^{-24},$$

$$g\cdot|r-s|\gtrsim 10^{-9}\,\mathrm{GeV}, \tag{5.6}$$

and come to the following values for the weak vector boson masses:

$$m_W\approx\frac{g}{\sqrt{2}}(|\rho|^2+|\sigma|^2)^{1/2},$$

$$m_Z\approx g\sqrt{\frac{2}{5}}(|\rho|^2+|\sigma|^2)^{1/2}, \tag{5.7}$$

with their ratio

$$\frac{m_W}{m_Z}=\frac{\sqrt{5}}{2}=\sqrt{2}\cos\theta_W$$

being determined by the following value of the Weinberg angle θ_W, i.e., the angle between A and the hyperplane orthogonal to $(W_\pm, [W_+, W_-])$:

$$\sin^2\theta_W=\frac{[\mathrm{tr}\,(AZ_0)]^2}{\mathrm{tr}\,A^2\cdot\mathrm{tr}\,Z_0^2}=\tfrac{3}{8},$$

where $Z_0=[W_+, W_-]$.

In concluding this section, the following remark is in order. The estimates (5.6) were obtained under the assumption that the coefficients by the cubic terms in the scalar potentials are small. If, on the contrary, they are large, only the weaker estimates (5.5) are valid. In the latter case, the scalar 24-plet may break SU(5) down not to SU(3) × SU(2) × U(1), but to SU(3) × U(1), which is then broken down to SU(3) × U(1) by the scalar 5-plets. An investigation of this sequence of breakdowns of SU(5) will be performed elsewhere.

6. MASSES OF FERMIONS

The mass matrix of fermions splits into blocks that correspond to invariant subspaces of the operators $\hat{A}$, $\hat{G}^3$, and $\hat{G}^8$.

They total nine:

1, 2. Charge $\pm 4/3$:

$$M_{\pm 4/3} = m_B \pm K_F\left(\frac{r}{2} + \frac{4}{3}s\right) + K_D\left(-\frac{r}{2} - \frac{2}{3}s\right).$$

3. Quarks with a $-2/3$ charge:

$$M_{(-2/3,0,+2)} = \begin{array}{cc} {}^3\psi^L_{12} & {}^i\psi^L_{12} \end{array}$$
$$\begin{bmatrix} m_K + \frac{8}{3}v_3^* s & \frac{8}{3}v_i^* s \\ 4K^{3j*}\rho & 4K^{ij*}\rho \end{bmatrix} \begin{array}{l} \bar{K}^{12}_R \\ {}^j\bar{\psi}^R_{34}. \end{array}$$

4. *Quarks with a* $+2/3$ charge:

$$M_{(+2/3,0,-2)} = \begin{array}{cc} {}^3\psi^L_{34} & {}^i\psi^L_{34} \end{array}$$
$$\begin{bmatrix} m_K + (2r - \frac{8}{3}s)v_3^* & (2r - \frac{8}{3}s)v_i^* \\ 4K^{3j*}\rho & 4K^{ij*}\rho \end{bmatrix} \begin{array}{l} K^{34}_R \\ {}^j\bar{\psi}^R_{12} \end{array}$$

5. Quarks with a $-1/3$ charge:

$$M_{(-1/3,1,1)}=\begin{array}{c}\begin{array}{ccc} B_L & {}^3\psi^L_{15} & {}^i\psi^L_{15}\end{array}\\ \left[\begin{array}{ccc} m_B-K_F(\frac{r}{2}+\frac{s}{3})+K_D(\frac{s}{3}-\frac{r}{2}) & 0 & 0\\ 0 & m_K+4v_3^*(\frac{8}{3}-\frac{r}{2}) & 4v_i(\frac{s}{3}-\frac{r}{2})\\ \frac{1}{\sqrt{2}}\delta^j_2\rho & -f^{j3}_2\sigma & -f^{ji}_2\sigma \end{array}\right]\end{array}\begin{array}{c} \\ \bar{B}_R\\ \bar{K}^{15}_R\\ {}^j\bar{\theta}^1_R\end{array}$$

6. Quarks with a $+1/3$ *charge*:

$$M_{(1/3,-1,-1)}=\begin{array}{c}\begin{array}{ccc} B_L & {}^3\theta^1_L & {}^i\theta^1_L\end{array}\\ \left[\begin{array}{ccc} m_B+K_F(\frac{s}{3}+\frac{r}{2})+K_D(\frac{s}{3}-\frac{r}{2}) & \frac{1}{\sqrt{2}}\delta^{3*}_1\sigma & \frac{1}{\sqrt{2}}\delta^{i*}_1\sigma\\ \frac{1}{\sqrt{2}}t\sigma^* & m_\eta+a_3^*\frac{s}{3} & a_i^*\frac{s}{3}\\ 0 & -f^{j3}_1\rho & -f^{ji}_1\rho \end{array}\right]\end{array}\begin{array}{c} \\ \bar{B}_R\\ \bar{\eta}^R_{1\cdot}\\ {}^j\bar{\psi}^R_{15}\end{array}$$

7. Quarks with a -1 *charge*:

$$M_{(-1,0,0)}=\begin{array}{c}\begin{array}{ccc} B_L & {}^3\theta^4_L & {}^i\theta^4_L\end{array}\\ \left[\begin{array}{ccc} m_B-K_F(r-s)-K_Ds & \frac{1}{\sqrt{2}}\delta^{3*}_1\sigma & \frac{1}{\sqrt{2}}\delta^{i*}_1\sigma\\ \frac{1}{\sqrt{2}}t\sigma^* & m_\eta+a_3^*(\frac{r}{2}-s) & a_i^*(\frac{r}{2}-s)\\ 0 & -f^{j3}_1\rho & -f^{ji}_1\rho \end{array}\right]\end{array}\begin{array}{c} \\ \bar{B}_R\\ \bar{\eta}^R_{4\cdot}\\ {}^j\bar{\psi}^R_{45}\end{array}$$

8. Leptons with a $+1$ *charge*:

$$
M_{(+1,0,0)} = \begin{array}{c} \\ \\ \\ \end{array}
\begin{array}{ccc}
B_L & {}^3\psi^L_{45} & {}^i\psi^L_{45}
\end{array}
$$

$$
M_{(+1,0,0)}=\begin{bmatrix} m_B+K_F(r-s)-K_D s & 0 & 0 \\ 0 & m_K-4v_3^* s & -4v_i^* s \\ \frac{1}{\sqrt{2}}\delta^j_2\rho & -f_2^{j3}\sigma & -f_2^{ji}\sigma \end{bmatrix}\begin{array}{c} \bar{B}_R \\ \bar{K}_R^{45}. \\ {}^j\bar{\theta}_R^4 \end{array}
$$

9. Neutral leptons:

$$
\begin{array}{cccc}
B_L^{15} & B_L^{24} & {}^3\theta_L^5 & {}^i\theta_L^5
\end{array}
$$

$$
M_{(0,0,0)}=\begin{bmatrix} m_B+K_D(\frac{3}{4}r-\frac{4}{3}s) & \frac{K_D}{\sqrt{15}}(2s-\frac{3}{4}r) & 0 & 0 \\ \frac{K_D}{\sqrt{15}}(2s-\frac{3}{4}r) & m_B-\frac{3}{4}K_D r & -\frac{2}{\sqrt{10}}\delta_1^{3*}\sigma & -\frac{2}{\sqrt{10}}\delta_1^{i*}\sigma \\ 0 & -\frac{2}{\sqrt{10}}t\sigma & m_\eta-a_3^*\frac{r}{2} & -a_i^*\frac{r}{2} \\ 0 & -\frac{2}{\sqrt{10}}\delta^j_2\rho & 0 & 0 \end{bmatrix}\begin{array}{c} \bar{B}_R^{15} \\ \bar{B}_R^{24} \\ \bar{\eta}_5^R \\ {}^j\bar{\theta}_R^5 \end{array}
$$

10. The octet of neutral quarks contained in the multiplet B,

$$m_8 = m_B + \tfrac{2}{3}K_D s.$$

Here m_B is the mass of the 24-plet and m_η and m_K are the bare masses of the 5- and 10-plet, respectively. It is rather obvious that there are massless neutrinos in the model within the tree approximation and that, by choosing the parameters m_B, m_η, and m_K appropriately, we can obtain the required mass difference between the leptons with the ± 1 charge and quarks with the $\pm 1/3$ charge.

7. MASSES OF FERMIONS FOR SPECIFIC VALUES OF YUKAWA COUPLINGS

To be specific, consider the third of the above-listed solutions of (2.3) for the Yukawa couplings: $\bar{K}_F = 1.12$, $\bar{\delta}_2^1 = 1.53$, $\bar{f}_1^{22} = 1.70$, $\bar{f}_2^{11} = 1.37$, $\bar{f}_2^{22} = 1.75$, $\bar{v}_2 = 0.75$, and $a_i \equiv 0$, with all the other quantities ω/g being small, and set $r = s$ in every mass matrix, this being possible because of the estimates (5.6).

To separate the light fermion blocks, multiply the matrices $M_{-2/3}$, $M_{+2/3}$, $M_{-1/3}$, M_{+1} to the right by the unitary matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & n_3/\sqrt{1+n_3^2} & 0 & -1/\sqrt{1+n_3^2} \\ 0 & n_1/\sqrt{1+n_3^2} & -n_2 & n_1 n_3/\sqrt{1+n_3^2} \\ 0 & n_2/\sqrt{1+n_3^2} & n_1 & n_2 n_3/\sqrt{1+n_3^2} \end{bmatrix},$$

where

$$n_1 = -v_1/\sqrt{(v_1)^2 + (v_2)^2},$$

$$n_2 = -v_2/\sqrt{(v_1)^2 + (v_2)^2},$$

$$n_3 = \left(\frac{m_K}{cs} - v_3\right) \Big/ \sqrt{(v_1)^2 + (v_2)^2},$$

where the number c, on which n_3 depends, is equal to $-8/3$ for the matrix $M_{-2/3}$, to $2/3$ for $M_{+2/3}$ and $M_{-1/3}$, and to 4 for M_{+1}. In addition, multiply the matrix M_{+1} to the left by the unitary matrix

$$\begin{bmatrix} m_3/\sqrt{1+m_3^2} & 0 & m_1/\sqrt{1+m_3^2} & m_2/\sqrt{1+m_3^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -m_2 & m_1 \\ -1/\sqrt{1+m_3^2} & 0 & m_1 m_3/\sqrt{1+m_3^2} & m_2 m_3/\sqrt{1+m_3^2} \end{bmatrix},$$

where

$$m_1 = \delta_2^1/\sqrt{(\delta_2^1)^2+(\delta_2^2)^2},$$

$$m_2 = \delta_2^2/\sqrt{(\delta_2^1)^2+(\delta_2^2)^2},$$

$$m_3 = \sqrt{2}\,(m_B - K_D s)/(\rho\sqrt{(\delta_2^1)^2+(\delta_2^2)^2}.$$

Then the blocks, lighter than $g\rho \sim 100$ GeV (and hence lighter than $gs \sim 10^{14}$ GeV) are separated:

$$M^l_{\pm 2/3} = 4\rho \begin{bmatrix} -K^{11}n_2 + K^{21}n_1 & \dfrac{(-K^{31}+n_1 n_3 K^{11} + n_2 n_3 K^{21})}{\sqrt{1+n_3^2}} \\ \Big\downarrow & \\ -K^{12}n_2 + K^{22}n_1 & \dfrac{(-K^{32}+n_1 n_3 K^{12} + n_2 n_3 K^{22})}{\sqrt{1+n_3^2}} \end{bmatrix},$$

where $n_3 = n_3(-8/3)$ for $M^l_{-2/3}$ and $n_3 = n_3(2/3)$ for $M^l_{+2/3}$;

$$M^l_{-1/3} = \sigma \begin{bmatrix} -f_2^{11}n_2 + f_2^{12}n_1 & \dfrac{(-f_2^{13}+n_1 n_3 f_2^{11} + n_2 n_3 f_2^{12})}{\sqrt{1+n_3^2}} \\ & \\ -f_2^{21}n_2 + f_2^{22}n_1 & \dfrac{(-f_2^{23}+n_1 n_3 f_2^{21} + n_2 n_3 f_2^{22})}{\sqrt{1+n_3^2}} \end{bmatrix},$$

where $n_3 = n_3(2/3)$; and

$M^l_{+1} =$

$$\sigma \begin{bmatrix} -m_2 & m_1 \\ \dfrac{m_1 m_3}{\sqrt{1+m_3^2}} & \dfrac{m_2 m_3}{\sqrt{1+m_3^2}} \end{bmatrix} \cdot \begin{bmatrix} -f_2^{11} n_2 + f_2^{12} n_1 & \dfrac{-f_2^{13} + n_1 n_3 f_2^{11} + n_2 n_3 f_2^{12}}{\sqrt{1+n_3^2}} \\ -f_2^{21} n_2 + f_2^{22} n_1 & \dfrac{-f_2^{23} + n_1 n_3 f_2^{21} + n_2 n_3 f_2^{22}}{\sqrt{1+n_3^2}} \end{bmatrix}.$$

We have v_1, $v_3 \ll v_2$ and $\delta_2^2 \ll \delta_2^1$; hence,

$$n_1 \approx -v_1/v_2, \qquad m_1 \approx 1$$

$$n_2 \approx -1, \qquad m_2 \approx \frac{\delta_2^2}{\delta_2^1}$$

$$n_3 \approx \frac{m_K}{csv_2} - \frac{v_3}{v_2} \ll 1, \qquad m_3 \approx \frac{\sqrt{2}}{\rho \delta_2^1}(m_B - K_D s).$$

Consider first an approximation for $M^l_{-1/3}$ where only maximal terms are kept in each element:

$$M^l_{-1/3} \approx g\sigma \begin{bmatrix} 1.37 & -\tilde{f}_2^{13} \\ \tilde{f}_2^{21} - 1.75\dfrac{v_1}{v_2} & -\tilde{f}_2^{23} - 1.75 n_3(\tfrac{2}{3}) \end{bmatrix}$$

It is obvious that, up to the higher-order terms, one has

$$\mathrm{tr}(M^l_{-1/3} M^{l+}_{-1/3}) = (1.37\, g\sigma)^2,$$

$$\det M^l_{-1/3} = (g\sigma)^2 \cdot \left[\tilde{f}_2^{21} \tilde{f}_2^{13} - \frac{1.75}{0.75} \tilde{v}_1 \tilde{f}_2^{13} - 1.37 \tilde{f}_2^{23} - 1.75 \cdot 1.37 \cdot n_3(\tfrac{2}{3}) \right].$$

Thus, we have obtained the masses of two quarks with $-1/3$ charges:

$$m^{(1)}_{-1/3} \approx g\sigma[-\tilde{f}^{23}_{2} - 1.7\,\tilde{v}_1 \tilde{f}^{13}_{2} + 0.73\,\tilde{f}^{21}_{2} \tilde{f}^{13}_{2} - 1.75\,n_3(\tfrac{2}{3})],$$

$$m^{(2)}_{-1/3} \approx 1.37\, g\sigma. \tag{7.1}$$

Consider now M^l_{+1}. It is clear that

$$\det M^l_{+1} = \frac{m_3}{\sqrt{1+m_3^2}} \det M^l_{-1/3}\bigg|_{n_3=n_3(4)}.$$

But

$$\det M^l_{-1/3}\bigg|_{n_3=n_3(c)} + \frac{1.75}{0.75}(g\sigma)^2 \frac{m_K}{gs}\left(\frac{3}{2} - \frac{1}{c}\right);$$

hence,

$$\det M^l_{+1} = \frac{m_3}{\sqrt{1+m_3^2}}\left[\det M^l_{-1/3} + 2.92\,(g\sigma)^2 \frac{m_K}{gs}\right].$$

When m_3 is not small ($m_3 \geqslant 1$),

$$\operatorname{tr}(M^l_{+1} M^{l+}_{+1}) \approx \left(1.37\, g\sigma \frac{m_3}{\sqrt{1+m_3^2}}\right)^2$$

and we obtain the masses of the two leptons with $+1$ charges:

$$m^{(1)}_{+1} \approx \left| m^{(1)}_{-1/3} + 2.13\, g\sigma \frac{m_K}{gs} \right|,$$

$$m^{(2)}_{+1} \approx m^{(s)}_{-1/3} \cdot \frac{m_3}{\sqrt{1+m_3^2}}. \tag{7.2}$$

For large m_3, the light sectors of the matrices $M_{+1/3}$ and M_{-1} look simple. Taking, for simplicity, the f_1^{ij} diagonal and large m_η, we obtain

$$m^{(1)}_{+1/3} = m^{(1)}_{-1} = f_1^{11}\rho,$$

$$m^{(2)}_{+1/3} = m^{(2)}_{-1} = f_1^{22}\rho. \tag{7.3}$$

We now identify the particles whose bare masses are $m^{(1)}_{-1}$, $m^{(1)}_{+1/3}$, $m^{(1)}_{+1}$, $m^{(1)}_{-1/3}$, $m^{(2)}_{+1}$, and $m^{(2)}_{-1/3}$ with the electron, d quark, μ meson, s quark, τ lepton, and b quark, respectively. The radiational corrections must convert (7.3) into the inequalities

$$m^{(1)}_{1/3} > m^{(1)}_{-1},$$

$$m^{(2)}_{1/3} > m^{(2)}_{-1}.$$

The above identification leads to the estimates

$$\begin{aligned} m_e &= f_1^{11} g\rho \sim 0.5\,\mathrm{MeV}, \\ m_\tau &= 1.37\, g\sigma \sim 1.8\,\mathrm{GeV}, \end{aligned} \tag{7.4}$$

i.e.

$$g\sigma \sim 1\,\mathrm{GeV} \tag{7.5}$$

and then

$$g\rho \approx \sqrt{2}\, m_W, \tag{7.6}$$

and the case now under consideration leads to a prediction of a lepton with the mass $m^{(2)}_{-1} = 2.4\, m_W$. (7.6)

Now, the use of (7.1–2) for the bare masses of the μ meson and s quark results in the conclusion that

$$\begin{aligned} m_\mu &= m_s \pm 2.13\, g\sigma \frac{m_K}{gs} \sim 100\,\mathrm{MeV}, \\ m_s &\sim 12\,\mathrm{MeV}. \end{aligned} \tag{7.7}$$

Therefore,

$$\frac{m_K}{gs} \sim 4.10^{-2},$$

whence

$$m_K \sim 4.10^{12}\,\text{GeV}. \tag{7.8}$$

Let us inspect now the quarks with $\pm 2/3$ charges. By having identified the $\pm 1/3$ charge quarks with known particles, we agreed that the lightest of the two quarks whose masses are determined by $M^l_{-2/3}$ is the u quark, while the other one is the top quark from the fourth generation (the h quark). The lightest in $M^l_{+2/3}$ is the c quark and the heaviest is the t quark.

Taking the bare masses of these quarks as†

$$m_u \sim 0.3\,\text{MeV},$$

$$m_h \sim 10\text{–}100\,\text{GeV},$$

$$m_c \sim 0.5\,\text{GeV},$$

$$m_t \sim 10\text{–}30\,\text{GeV},$$

we come to the conditions on the Yukawa coupling constants $\tilde{K}^{ij}$:

$$\det M^l_{-2/3} \sim (0.003\text{–}0.03)(\text{GeV})^2,$$

$$\operatorname{tr}(M^l_{-2/3} M^{l+}_{-2/3}) \sim (10^2\text{–}10^4)(\text{GeV})^2,$$

$$\det M^l_{+2/3} \sim (5\text{–}15)(\text{GeV})^2,$$

$$\operatorname{tr}(M^l_{+2/3} M^{l+}_{+2/3}) \sim (100\text{–}1000)(\text{GeV})^2.$$

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†The radiational corrections are expected to make the quark masses several times larger than their bare values: approximately thrice for masses of about a few GeVs and ten times for those of about a few MeVs.

With the estimate of the quantity m_K/gs obtained above, one may show that these conditions can be satisfied with a $\tilde{K}^{ij}$ of about 0.1.

In conclusion, we present the values of the neutrino masses. Since $\det M_{(0,0,0)}=0$, the presence of the four-component massless neutrino is necessary.

Besides, there are two heavy neutrinos with masses of about m_B, one heavy neutrino with a mass of the order of m_η and one light neutrino with a mass much less than $g\sigma \sim 1$ GeV: $0 \leqslant m_{\nu_\tau} \ll g\sigma$.

CONCLUSION

Here we list briefly the main results obtained.

We have constructed an SU(5) model of the strong, weak, and electromagnetic interaction with the following properties:

(1) it is asymptotically free in every coupling constant,

(2) the set of Higgs multiplets provides the breaking of the SU(5) symmetry down to $SU(3)\times U(1)$; the leptoquarks are given masses above 10^{14} GeV and the W boson one of about 100 GeV,

(3) there are four light (up to 100 GeV) chiral generations, one heavy Dirac generation, and one heavy 24-plet of spinors in the model,

(4) the mass difference $(m_\mu/m_e)/(m_s/m_d)\approx 8.5$ between leptons and quarks is provide without introducing an additional Higgs 45-plet,

(5) within the tree approximation, electronic and muonic neutrinos form a single massless four-component particle via the Mahmound–Konopinski scheme,[10]

(6) the τ lepton has a light or massless neutrino,

(7) apart from the known particles the model predicts, in one of its versions, a lepton with a mass of $2.4\,m_W$.

Investigations of the physical consequences of the model and its modifications are of interest. E.g., the preliminary estimates[12] indicate that the renormalized Weinberg angle in the model is approximately equal to $\sin^2\theta_W \approx 0.22$ and that the proton lifetime exceeds 10^{31} years. If the terms (1.4) are added to the potential of the scalar self-coupling (1.3), a heavy axion will appear in our model; this will provide a good framework for searching for a solution to the strong CP violation[9] problem.

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