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CONFORMALLY INVARIANT SOLUTION OF QUANTUM FIELD THEORY EQUATIONS (1)

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The exact conformally-invariant solution of renormalized quantum field theory equations is considered. Dynamical equations form an infinite system of equations for vertices. This system does not determine the vertices uniquely; it must be supplemented by axiomatic requirements (locality, crossing symmetry etc.). The latter were not taken into consideration in the course of the solution of equations, therefore the solution obtained contains a number of arbitrary functions. The role of the mentioned additional requirements for the construction of the unique solution is discussed. The vertices are considered, containing conserved external lines (the current and the energy-momentum tensor). It is shown that the generalized Ward identity makes it possible to obtain these vertices explicitly up to an arbitrary transverse part.

1. Introduction

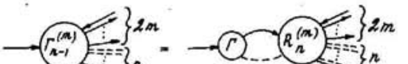
The problem of internal consistency of local quantum field theory required the development of a calculational technique considerably proceding beyond the scope of the usual perturbation theory. By the present paper we begin the systematic account of non-peturbational solutions of the exact quantum field theory equations. The solution of the problem of internal consistency supposes two successive stages: firstly the asymptotic behaviour of the Green function at small distances (near the cone) must be obtained and secondly this solution at small distances must be sewed together with the solution at large distances. This second task is necessary for the transition to the mass shell.

In the present paper we make an attempt of a partial solution of the problem of the Green function asymptotic behaviour near the cone: we assume that at small distances the conformal symmetry is dynamically relaized.

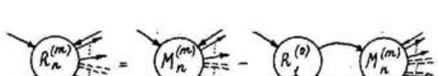
It is important that the basic system of unrenormalized Schwinger-Dyson equations for the Green functions is not a suitable tool for dealing with our problem, because the appearance of the dynamical symmetry of the asymptotic solution is closely connected with the infiniteness of the renormalization. Therefore it is necessary that all the renormalization and the functional resolution of the uncertainty of the type $0 \times \infty$ caused by the so-called "overlapping" divergences, should be carried out directly within the basic system of equations for the Green functions. This programme was realized for the Schwinger-Dyson system of equations by one of the present authors in refs [1,2] (see also ref. [3]) where the completely renormalized system of equations for the Green functions was obtained. This system of renormalized equations will serve as the starting point of the further investigation. Since the mass corrections are neglegible when one considers the asymptotic behaviour near the cone, we may confine ourselves to the study of equations in the limit of zero experimental mass. The system of renormalized equations in this case is known as a system of bootstrap equations recently studied by several authors in refs. [4–8]. It was shown that these equations possess a conformally invariant solution.

Such a solution takes place under some particular choice of dimensionless coupling constants; it wad realized in refs. [4–9] in a form of a skeleton expansion for the Green functions. The higher-order n-point Green functions ($n \geqslant 4$) were represented by expansions in skeleton diagrams constructed from conformally invariant vertices and propagators. Equations for the one-particle and vertex Green functions (in a three-vertex approximation) play the role of self-consistency conditions and are used to determine the coupling constants and scale dimensions of fundamental fields. In the present work we make an attempt to solve the system of renormalized equations with zero experimental mass (bootstrap equations) exactly, without the use of skeleton expansions for the higher-order Green functions. Since the exact system of renormalized equations contains an infinite number of relations connecting vertices with different number of particles, we have to deal with an infinite system of integro-differential equations. However, a remarkable property of these renormalized equations is that the assumption about the conformal invariance of the solution makes it possible to diagonalize these equations [10–13], and as we shall show, to obtain the exact solution.

We shall consider for definiteness the system of renormalized equations of the theory with the pseudo-scalar interaction $L_{\text{int}} = \lambda \bar{\psi} \gamma_5 \psi$, which was also considered in ref. [2]. By putting $Z_1 = 0$ we obtain the system of bootstrap equations:



(1.1)



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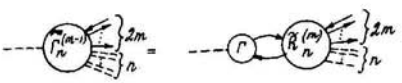
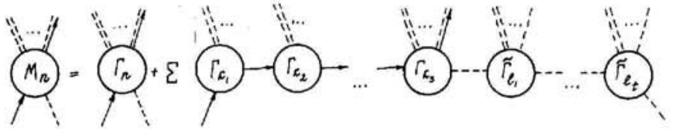


(1.3)

 (1.4)

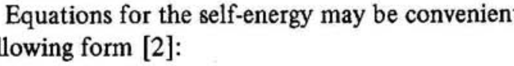
which form the basis of our approach. The vertex $\Gamma_n^{(m)}$ here is the one-particle irreducible vertex. It may be obtained from the Green function by subtraction of all diagrams which may be divided by cutting a line in any direction, see for example (3.4). The vertices $M_n^{(m)}$ and $\widetilde{M}_n^{(m)}$ do not contain diagrams which may be divided by cutting a line in the transverse direction. In particular at $m = 0$ one has for the vertex $M_n^{(0)}$

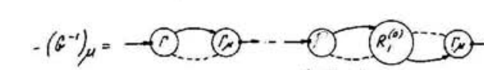
 (1.5)

where $M_n \equiv M_n^{(0)}$, $\Gamma_n \equiv \Gamma_n^{(0)}$. The sign Σ denotes the summation over all possible decompositions of external lines into groups $K_1 \dots K_s$ and $l_1 \dots l_t$ and the symmetrization over these indices. Since the field $\varphi(x)$ is psuedoscalar the vertices Γ_{l_i} may have only an even number of external lines.

In the skeleton theory the vertices entering eqs. (1.1)–(1.4) possess the following interpretation: $\Gamma_n^{(m)}$ is the sum of diagrams which cannot be divided by cutting a line in any direction; $R_n^{(m)}$ (or $\widetilde{R}_n^{(m)}$) is the sum of diagrams which cannot be divided by cutting a fermion and a boson (or two fermion) lines in the transverse direction; the vertices $M_n^{(m)}$ and $\widetilde{M}_n^{(m)}$ are sums of diagrams which cannot be divided by cutting a line in the transverse direction. The simplest equations of the system ($m = 0, n = 1$) are given in sect. 3.

Equations for the self-energy may be conveniently (see eq. (3.16)) written in the following form [2]:

 (1.6)

 (1.7)

where G and D are Green functions (2.1), the dots denote the amputation at the corresponding external lines, see (2.11), (2.12), the index μ means the derivative with respect to the external momentum.

Eqs. (1.1), (1.3), (1.6), (1.7) are the dynamical equations, while eqs. (1.2), (1.4) serve as the definition of vertices $R_n^{(m)}$ and $\widetilde{R}_n^{(m)}$. Eqs. (1.1), (1.2), (1.5), (1.6) determine the fermion field. They conserve the number of fermion legs and fall into independent groups, characterized by a definite value of m. Eqs. (1.3), (1.4) determine the boson field and connect with each other the groups of equations with different m.

The system of equations (1.1)–(1.7) must be supplemented by axiomatic requirements (spectrality, positivity, locality, etc.), also by the boundary conditions on the mass shell. The latter are not needed within the framework of the conformally invariant theory where the two- and three-point Green functions are known explicitely. Eqs. (1.1)–(1.7) together with these additional requirements must determine not only the Green functions but also the coupling constant (the normalization of the three-point vertex) and the dimensions of the fields ψ and φ. The present work solves only a part of the problem – the general conformally invariant solution of the dynamical equations is obtained (sects. 3–5). Eqs. (1.1)–(1.7) as they are do not determine the Green functions uniquely, the solution obtained contains a considerable arbitrariness (sect. 5). In particular it admists any kind of symmetry compatible with the conformal one. This arbitrariness may be partially excluded with the aid of general principles of the quantum field theory discussed in sects. 5–7. The generalized Ward identities for the current and the energy-momentum tensor, which reflect the conservation laws and commutation rules, are of special importance for the exclusion of the mentioned arbitrariness (sect. 4). The Euclidean formulation of the quantum field theory suggested in ref. [14] will be used throughout the present paper.

The paper is organized as follows. It is shown in sect. 3 that the assumption about the conformal invariance of the theory makes it possible to eliminate the auxiliary vertices $M_n^{(m)}$ and $R_n^{(m)}$ from the system of equations.

This is possible due to the fact that the conformal group determines the three-point vertices up to a constant factor. The system (1.1)–(1.7) in its final version (eqs. (3.15), (3.18)–(3.20)) consists of equations connecting only the vertices $\Gamma_n^{(m)}$ with different number of external lines. It is important for the derivation of these equations that the vertices $\Gamma_n^{(m)}$ and not the Green functions, enter eqs. (1.1)–(1.7). If one has used the system of equations for the Green functions instead of eqs. (1.1)–(1.7) the mentioned programme would not have been realizable because the Green functions, contrary to the $\Gamma_n^{(m)}$, contain the diagrams which may be divided by cutting a line.

In sect. 4 the conformal expansion of the vertices $\Gamma_n^{(m)}$ (analogous to Fourier-transformation) is considered, diagonalizing the equations obtained in sect. 3. Similar expansions for the conformally invariant functions were suggested in refs. [10, 15,16] and [11,17], where they were well-founded. Contrary to ref. [11] we shall use the expansion in all the arguments (see eq. (4.11)). It may be well-founded only for the vertices $\Gamma_n^{(m)}$ (because of absence of diagrams which may be divided by cutting a line). Each vertex is associated with its image $\rho_n^{(m)}(\sigma_1 \dots \sigma_{n+2m})$ which makes it possible to rewrite the equations for $\Gamma_n^{(m)}$ in a form of algebraic equations for $\rho_n^{(m)}$ and to solve them (sect. 5). Sect. 6 deals with the restrictions following from the crossing-symmetry requirement. In the general case this requirement leads to homogeneous linear integral equations for the functions $\rho_n^{(m)}$ (see also refs. [15,11]). It is shown that these equations are compatible with the general expression for $\rho_n^{(m)}$ obtained in sect. 5. The vertex with two spinor and two meson lines is considered. The crossing-symmetry does not lead to any considerable limitations in this case. The general solution of integral equations in the case of the vertex with four meson lines is discussed. A simplest example of a vertex is considered which may be determined by the crossing-symmetry requirements.

Sec. 7 is devoted to the analysis of generalized Ward identities. It is shown that in the framework of the conformally-invariant theory these identities permit us to express the spectral density of the longitudinal part of a vertex with an arbitrary number of external conserved lines (the current or energy-momentum tensor lines) in terms of the corresponding lower-order Green function without the conserved external lines.

As an example the four-point vertex functions are considered, including the current or the energy-momentum tensor. The higher-order vertices were considered in our paper [13].

2. Notations and normalization conditions

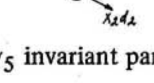
Let $D_\delta(x)$ and $G_d(x)$ be the Euclidian Green functions of the pseudoscalar and spinor fields (δ and d are dimensions). The renormalization invariance makes it possible to normalize them independently. Let us put

$$D_\delta(x) = \frac{4^\delta}{(4\pi)^2} \frac{\Gamma(\delta)}{\Gamma(2-\delta)} (x^2)^{-\delta} , \qquad G_d(x) = -i \frac{4^d}{(4\pi)^2} \frac{\Gamma(d+\frac{1}{2})}{\Gamma(\frac{5}{2}-d)} \frac{\hat{x}}{(x^2)^{d+\frac{1}{2}}} . \tag{2.1}$$

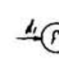
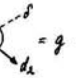
With this normalization one has:

$$D_\delta^{-1}(x) = D_{4-\delta}(x) , \qquad G_d^{-1}(x) = G_{4-d}(x) . \tag{2.2}$$

The Euclidian Green function of three fields

$$\Gamma^{d_1 d_2 \delta}_{(x_1 x_2 x_3)} = \langle 0 | T \psi_{d_1}(x_1) \bar{\psi}_{d_2}(x_2) \varphi_\delta(x_3) | 0 \rangle =$$  (2.3)

may be expressed through the γ_5 invariant part [8,9] of the three-point function

 $= g$  $\equiv g\, C_+{}^{d_1 d_2 \delta}_{(x_1 x_2 x_3)}$ (2.4)

Here g is the coupling constant and

$$C_+^{d_1 d_2 \delta}(x_1 x_2 x_3)$$

$$= N \Gamma(\tfrac{1}{2}(d_1 + d_2 + \delta) - 2) S_{\frac{1}{2}(d_1 - d_2 + \delta)}(x_{13}) \gamma_5 S_{\frac{1}{2}(d_2 - d_1 + \delta)}(x_{32}) \widetilde{\Delta}_{\frac{1}{2}(d_1 + d_2 - \delta)}(x_{12}) , \tag{2.5}$$

where $x_{ik} = x_i - x_k$,

$$S_d(x) = -\frac{4^d}{(4\pi)^2}\,\Gamma(d+\tfrac{1}{2})\,\frac{\hat{x}}{(x^2)^{d+1/2}}\,, \qquad \widetilde{\Delta}_\delta(x) = \frac{4^d}{(4\pi)^2}\,\Gamma(\delta)(x^2)^{-\delta}\,, \quad (2.6)$$

$$N = \{\Gamma(\tfrac{1}{2}(d_1+d_2+\delta)-2)\Gamma(4-\tfrac{1}{2}(d_1+d_2+\delta))\Gamma(\tfrac{1}{2}(d_2-d_1+\delta)+\tfrac{1}{2})\Gamma(\tfrac{5}{2}-\tfrac{1}{2}(d_2-d_1+\delta))$$
$$\times\,\Gamma(\tfrac{1}{2}(d_1+d_2-\delta))\Gamma(2-\tfrac{1}{2}(d_1+d_2-\delta))\Gamma(\tfrac{1}{2}(d_1-d_2+\delta)+\tfrac{1}{2})\Gamma(\tfrac{5}{2}-\tfrac{1}{2}(d_1-d_2+\delta))\}^{-1/2}\,. \quad (2.7)$$

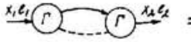
The normalized γ_5 non-invariant function $C_-^{d_1 d_2 \delta}(x_1 x_2 x_3)$ is given in sect. 4. The functions (2.6) satisfy the relations [18,9]:

$$\int \mathrm{d}x_4\, S_{d_1}(x_{14}) S_{d_2}(x_{42}) \widetilde{\Delta}_\delta(x_{34}) = (4\pi)^2 S_{2-d_2}(x_{13}) S_{2-d_1}(x_{32}) \widetilde{\Delta}_{2-\delta}(x_{12})\,, \quad (2.6a)$$

$$\int \mathrm{d}x_4\, \widetilde{\Delta}_{\delta_1}(x_{14}) \widetilde{\Delta}_{\delta_2}(x_{24}) \widetilde{\Delta}_{\delta_3}(x_{34}) = (4\pi)^2 \widetilde{\Delta}_{2-\delta_1}(x_{23}) \widetilde{\Delta}_{2-\delta_2}(x_{13}) \widetilde{\Delta}_{2-\delta_3}(x_{12})\,, \quad (2.6b)$$

$$\widetilde{\Delta}_{4-\delta}(x) = \Gamma(2-\delta)\Gamma(\delta-2)\widetilde{\Delta}_\delta^{-1}(x)\,, \qquad S_{4-d}(x) = -\Gamma(\tfrac{5}{2}-d)\Gamma(d-\tfrac{3}{2}) S_d^{-1}(x)\,. \quad (2.6c)$$

The normalization factor (2.7) is fixed by the orthogonality conditions (see sect. 4 for the details):



$$= i\pi g^2 [\mu_{1/2}(l_1)]^{-1}\, \delta_{l_1 l_2} G_{l_1}(x_{12})\,, \quad (2.8)$$

$$= i\pi g^2 [\mu_0(l_1)]^{-1} \{\delta_{l_1 l_2} D_{l_1}(x_{12}) + \delta_{l_1, 4-l_2}\, \delta(x_1 - x_2)\}\,, \quad (2.9)$$

where $\mu_{1/2}$ and μ_0 are weight functions entering eq. (4.4)–(4.6):

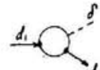
$$\mu_{1/2}(l) = \tfrac{1}{4}(4\pi)^6\, \frac{\Gamma(l+\frac{1}{2})\Gamma(\frac{9}{2}-l)}{\Gamma(\frac{5}{2}-l)\Gamma(l-\frac{3}{2})}\,, \quad (2.10)$$

$$\mu_0(l) = \tfrac{1}{16}(4\pi)^6\, \frac{\Gamma(l)\Gamma(4-l)}{\Gamma(l-2)\Gamma(2-l)}\,. \quad (2.10a)$$

They may be calculated with the aid of eqs. (2.6a–c).

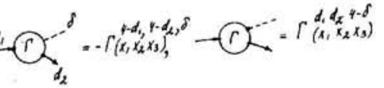
Let us introduce the notation for amputated vertices

$= g$



$$= \int \mathrm{d}x_1' \mathrm{d}x_2' G_{d_1}^{-1}(x_1 - x_1') \Gamma^{d_1 d_2 \delta}(x_1' x_2' x_3) G_{d_2}^{-1}(x_2' - x_2)\,, \quad (2.11)$$

and similarly for the pseudoscalar leg. Using eq. (2.6a,b) one finds



(2.12)

The vertices Γ with an amputated spinor leg may be expressed in terms of the γ_5 non-invariant function C_- (see sect. 4).

With the normalization (2.2), (2.12), all the equations are manifestly invariant under the substitution $d \to 4-d, \delta \to 4-\delta$, which is the consequence of the similar invariance of the Casimir operators of the conformal group. According to eq. (2.2), (2.12) this substitution corresponds to two different ways of formulation of the theory: in terms of complete Green functions and inverse propagators for internal lines, or with the aid of amputated Green functions. The first way is used everywhere except for sect. 8.

In sect. 3 the functions (2.5) with dimensions $d_2 = d, d_1 = l$ will be used, where d is the dimension of the spinor field entering eq. (1.1)–(1.5) and l is arbitrary. As to the dimensions of fields it is supposed that they are limited within the range

$$\tfrac{3}{2} < d < \tfrac{5}{2}\,, \qquad 1 < \delta < 3\,. \tag{2.13}$$

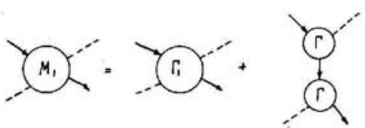
like in the skeleton [8,9].

3. The system of renormalized equations in conformally invariant field theory

Let us show that the assumption about the conformal invariance makes it possible to eliminate the additional vertices $R_n^{(m)}, \widetilde{R}_n^{(m)}, M_n^{(m)}, \widetilde{M}_n^{(m)}$ from the system (1.1)–(1.7). We shall illustrate this statement first in an example of the simplest equations of the system. One has:

(3.1)

(3.2)



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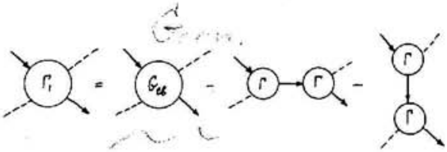
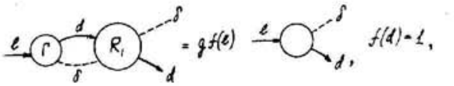


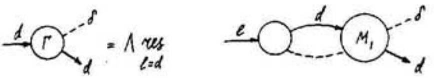
(3.4)

where $R_1 \equiv R_1^{(0)}$. Consider the quantity $\Gamma^{ld\delta} D_\delta^{-1} G_d^{-1} R_1$, where l is an arbitrary dimension. In virtue of the conformal invariance this quantity is proportional to $C_+^{ld\delta}$ defined in sect. 2:



(3.5)

where $f(l)$ is a new function. It follows from (3.1) that $f(d) = 1$. Using eqs. (3.5) and (3.2) we obtain the known result [10] that the quantity $\Gamma^{ld\delta} D_\delta^{-1} G_d^{-1} M_1$ as the function of the dimension l possesses the pole at $l = d$. Therefore we may rewrite eq. (3.1) as:

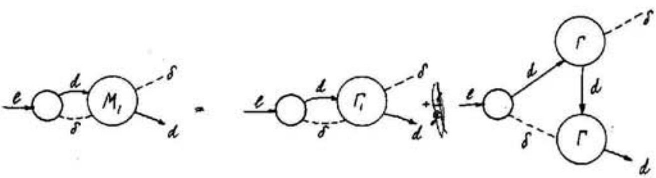


(3.6)

where

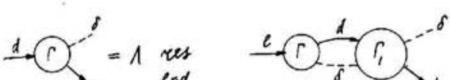
$$\Lambda = -g \frac{\mathrm{d}f(l)}{\mathrm{d}l} \bigg|_{l=d} . \tag{3.7}$$

Under substitution of (3.3) into (3.6) one has



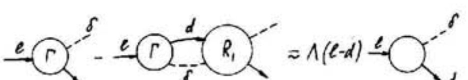
(3.8)

l being arbitrary. The second term is finite in the limit $l \to d$ because of (2.13), so it does not contribute to eq. (3.6). It follows that the first term has the pole at $l = d$ with the same residue as the left-hand side has. Finally the system (3.1)–(3.3) takes the form of one equation connecting directly the vertices Γ and Γ_1:



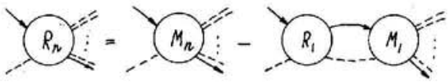
(3.9)

Let us represent the other equations of the system in an analogous form. Without any loss of generality we shall confine ourselves to the consideration of eq. (1.12) at $m = 0$. The final results will be given for the complete system (3.1–4). From (3.5), (3.7) one has at $l \to d$



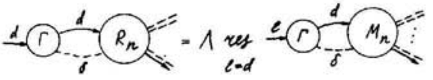
(3.10)

Keeping this in mind one uses the equations:



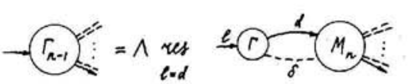
(3.10a)

to find the following relation



(3.11)

The existence of a pole at $l = d$ of the quantity on the right-hand side follows from eqs. (1.1), which may be rewritten similarly to eq. (3.6) as

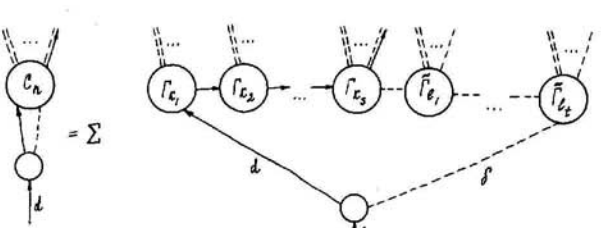


(3.12)

The next step consists in the elimination of the vertices M_n, determined by eqs. (1.5). Eqs. (1.5) yield:

(3.13)

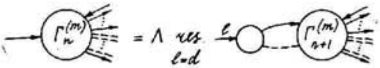
where C_n denotes the set of diagrams summed up by means of the symbol Σ in eqs. (1.5). According to (3.12) the left-hand side of eq. (3.13) possesses the pole at $l = d$. The right-hand side consists of two terms, the first one possessing the same pole. The second term is regular like the corresponding term of (3.8), because the integrals



(3.14)

converge. The regularity of this term may be easily proved in the skeleton theory, where the integrals (3.14) converge in the range of dimensions (2.13). The convergence of these integrals in our approach is secured also by the asumption about the convergence of conformal expansions for vertices Γ_n and M_n, see (4.8), (4.11). It is important that any diagram entering (3.14) may be divided by cutting two lines in such a way that both resulting subdiagrams are of the same structure as the diagrams which are summed in eq. (1.5). Therefore the integrals (3.14) converge provided that (4.9) is satisfied.

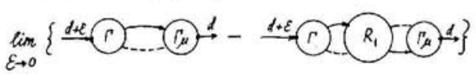
The regularity of the second term in (3.13) gives us the possibility of rewriting eq. (3.12) in a form similar to that of eq. (3.9). One has at arbitrary m:



(3.15)

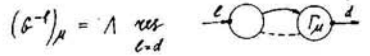
This system constitutes the final result: it does not contain the additional vertices $R_n^{(m)}$ and $M_n^{(m)}$, Its exact solution will be obtained in sect. 5.

The constant Λ remains to be found. It may be calculated from the equation for the self-energy. This equation in a form (1.6) contains the uncertainty $0 \times \infty$. In order to resolve this uncertainty we shall use the trick suggested in ref. [19]: the right-hand side of eq. (1.6) will be considered as the limit of the expression



(3.16)

Using (3.10) one finds



(3.17)

The integral on the right-hand side may be calculated with the aid of eqs. (2.6a–b) and the following relation:

$$\int \mathrm{d}y\,(x_1 - y)_\mu \widetilde{\Delta}_{\delta_1}(x_1 - y)\widetilde{\Delta}_{\delta_2}(y - x_2)$$
$$= \frac{\Gamma(3-\delta_1)\Gamma(2-\delta_2)}{\Gamma(5-\delta_1-\delta_2)}\,(x_1 - x_2)_\mu \widetilde{\Delta}_{\delta_1+\delta_2-2}(x_1 - x_2)\,.$$

The calculation gives ($\epsilon = l - d$)

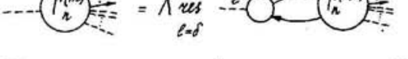
$$\tfrac{1}{4}\Gamma(\tfrac{1}{2}\epsilon)\{\mu_{1/2}(d)\}^{-1}\,(x_1 - x_2)_\mu\, G^{-1}(x_1 - x_2)\,.$$

Passing to the limit in eq. (3.17) we obtain finally

$$g\Lambda = -2\mu_{1/2}(d)\,, \tag{3.18}$$

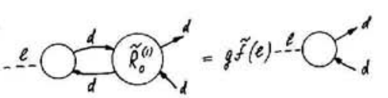
where $\mu_{1/2}(d)$ is the weight function (2.10).

The system (1.3), (1.4), (1.7) may be considered in an analogous manner. Eliminating the additional vertices $\widetilde{R}_n^{(m)}$ and $\widetilde{M}_n^{(m)}$ in the same way as in the course of derivation of eq. (3.15) we obtain:



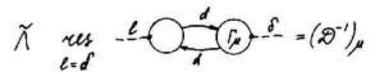
(3.19)

where $\widetilde{\Lambda} = -g\,\mathrm{d}f/\mathrm{d}l|_{l=\delta}$, and the function $\widetilde{f}(l)$ is defined similarly to the function $f(l)$:



(3.19a)

Resolution of the uncertainty in eq. (1.7), similarly to eq. (3.16), gives



(3.19b)

whence it follows that

$$g\widetilde{\Lambda} = 2\mu_0(\delta)\,. \tag{3.20}$$

where $\mu_0(\delta)$ is the weight function (2.10a).

4. Conformal expansion

The rigorous foundation of the conformal expansion in Euclidean coordinates, the conformal group being SO(5.1), is given in refs. [11,17]. It is important that the conformal group structure in the Euclidean space is relatively simple. The similar analysis in the pseudoeuclidean space is hampered by the complicated geometry of the SO(4,2) group, more precisely of its universal covering group. For example a large part of the non-degenerate representations of the latter group has infinite component with respect to spin. However the usage of the spectrality and positivity axioms makes it possible to obtain the conformal expansion in this case too [16]. This question will be considered in more detail in another work. The irreducible representations of the SO(5,1) group are labeled by three numbers

$$\sigma = (l, j_1, j_2) \oplus (l, j_2, j_1)\,, \tag{4.1}$$

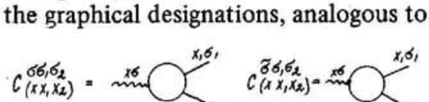
l being the dimension and j_1, j_2 the quantum numbers of the SO(4) group (the Euclidean Lorentz Group). Let us introduce the invariant two- and three-point functions $\Delta_\sigma(x)$ and $C^{\sigma_1\sigma_2\sigma_3}(x_1x_2x_3)$. Their normalization will be fixed by conditions analogous to (2.2), (2.8), (2.12)

$$\Delta_\sigma^{-1}(x) = \Delta_{\widetilde{\sigma}}(x)\,, \qquad \int dx_1'\,\Delta_{\widetilde{\sigma}_1}(x_1 - x_1')\,C^{\sigma_1\sigma_2\sigma_3}(x_1'x_2x_3) = C^{\widetilde{\sigma}_1\sigma_2\sigma_3}(x_1x_2x_3)\,, \tag{4.2}$$

where

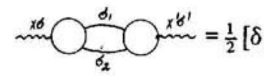
$$\widetilde{\sigma} = (4 - l, j_2, j_1) \oplus (4 - l, j_1, j_2). \tag{4.1a}$$

Representations (4.1) and (4.1a) are equivalent. The amputated functions $C^{\sigma_1\sigma_2\sigma_3}(x_1x_2x_3)$ correspond to representations (4.1a), see also sect. 2. Let us introduce the graphical designations, analogous to (2.5), (2.11)



(4.2a)

The wavy external line will correspond to the quantum numbers of representations contributing to the expansion (4.6); the solid external lines will correspond to the quantum numbers of fields. The orthogonality relation for the functions (4.2a) may be put into the form:



$$= \tfrac{1}{2}\left[\delta_{\sigma\sigma'}\delta(x-x') + \delta_{\sigma\tilde{\sigma}'}\Delta_{\sigma}(x-x')\right]. \qquad (4.3)$$

Here the symbol $\delta_{\sigma\sigma'}$ is defined on the integration contour $l = 2 + i\nu$, $\infty < \nu < \infty$, by the following relation:

$$\sum_{\sigma} f(\sigma')\delta_{\sigma\sigma'} = f(\sigma), \qquad \sum_{\sigma} = \frac{1}{2\pi i}\sum_{j_1 j_2}\int_{2-i\infty}^{2+i\infty} \mu_j(l)\,\mathrm{d}l\,; \qquad (4.4)$$

$\mu_j(l)$ is the weight function with $j = (j_1, j_2)$. Its explicit form for several important cases is found in sect. 2, see eqs. (2.10), (2.10a). The internal lines in eq. (4.3) correspond to functions $\Delta_{\tilde{\sigma}_1}$ and $\Delta_{\tilde{\sigma}_2}$.

As an example we shall give the expression for the normalized three-point function with spinor legs $\sigma_{1,2} = (l_{1,2}, \frac{1}{2}, 0) \oplus (l_{1,2}, 0, \frac{1}{2})$ and a scalar $\sigma_3 = (l_3, 0, 0)$. In this case the functions $\Delta_{\sigma}(x)$, which satisfy eq. (4.2) are equal to (2.1) and for $C^{\sigma_1\sigma_2\sigma_3}(x_1x_2x_3)$ one has

$$C^{\sigma_1\sigma_2\sigma_3}(x_1x_2x_3) = C_+^{l_1l_2l_3}(x_1x_2x_3) \dotplus C_-^{l_1l_2l_3}(x_1x_2x_3)\,. \qquad (4.5)$$

Here $C_+^{l_1l_2l_3}$ is given by eqs. (2.5)–(2.7) and $C_-^{l_1l_2l_3}$, is of the form:

$$C_-^{l_1l_2l_3}(x_1x_2x_3) =$$

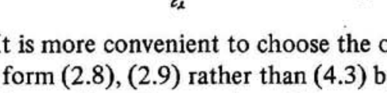
$$= iN\Gamma(\tfrac{1}{2}(l_2+l_2+l_3)-\tfrac{3}{2})\,S_{\frac{1}{2}(l_1+l_2-l_3)}(x_{12})\gamma_5\widetilde{\Delta}_{\frac{1}{2}(l_1-l_2+l_3)}(x_{13})\widetilde{\Delta}_{\frac{1}{2}(l_2-l_1+l_3)}(x_{23})\,,$$

where N denotes the normalization factor (2.7). Using (2.6a,b) one may easily verify that

$$\int \mathrm{d}x_1'\, G_{4-l_1}(x_1-x_1')\, C_\pm^{l_1l_2l_3}(x_1'x_2x_3) = C_\mp^{4-l_1,l_2,l_3}(x_1x_2x_3)\,,$$

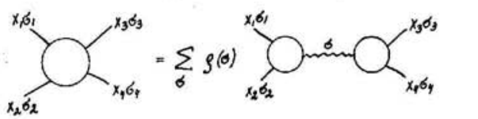
$$\int \mathrm{d}x_3'\, D_{4-l_3}(x_3-x_3')\, C_\pm^{l_1l_2l_3}(x_1x_2x_3') = C_\pm^{l_1,l_2,4-l_3}(x_1x_2x_3)\,,$$

so that the function (4.5) satisfies eq. (4.2). Graphically:

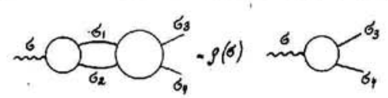
 (4.2b)

It is more convenient to choose the orthogonality condition for this function in the form (2.8), (2.9) rather than (4.3) because of the γ_5 invariance of the theory.

The conformal expansion of four-point functions is of the form:

 (4.6)

Here the integration over coordinates and summation over spinor indices of the wavy line corresponding to Δ_σ^{-1} is assumed. Σ_σ is defined by eq. (4.4) and $\rho(\sigma)$ is the spectral function. With the regard to (4.3) the definition of this function reads



(4.7)

The properties of the spectral function depend on the normalization of $C^{\sigma_1\sigma_2\sigma_3}(x_1x_2x_3)$. In the normalization (4.2), one has:

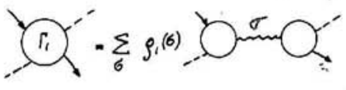
$$\rho(\sigma) = \rho(\widetilde{\sigma}) . \tag{4.7a}$$

The spectrum of the numbers $j_{1,2}$ contributing to (4.6) is determined by the quantum numbers of the fields. In the case of two scalar fields ($\sigma_{1,2} = (\delta_{1,2}, 0, 0)$) only the tensor representations $j_1 = j_2 = j$ contribute. If one of the fields is scalar and the other one is the spinor field, the spin-tensor representations contribute to (4.6). Finally, if the tensor fields correspond to the external lines (for example it may be the current or the energy-momentum tensor) then the expansion (4.6) must be modified. Several independent sets of functions $C^{\sigma_1\sigma_2\sigma_3}$ may arise in this case, corresponding to the transverse and longitudinal parts of tensor functions. The details may be found in sect. 8 and in ref. [13].

Relations (4.3), (4.7) are rigorously founded only for the value of σ belonging to the integration contour in (4.4). The functions $C^{\sigma_1\sigma_2\sigma_3}$ entering eq. (4.6) must be interpreted as the Glebsch-Gordan kernels. They may be defined for arbotrary complex l by analytic continuation from this contour provided that the analytic properties of $\rho(\sigma)$ permit such a continuation.

It will be supposed that all the spectral functions possess the needed analytic properties.

The theory $\lambda\overline{\psi}\gamma_5\psi\varphi$ must be invariant under the γ_5 transformations: $\psi \to \gamma_5\psi$, $\overline{\psi} \to \overline{\psi}\gamma_5$, $\varphi \to -\varphi$. The functions $C^{\sigma_1\sigma_2\sigma_3}$ and consequently the expansion (4.6) do not generally possess this property. The function (4.5) serves as an example. Therefore it is necessary to single out the γ_5 invariant part of (4.6) (for more details see ref. [13]). It is important that the relation (4.7a) will be still well-founded, notwithstanding that the γ_5 invariant part of $C^{\sigma_1\sigma_2\sigma_3}$ may not satisfy eq. (4.2). Let us illustrate this statement in an example of the expansion of the vertex Γ_1



(4.6a)

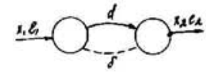
Consider the spinor contribution $\sigma = (l, \frac{1}{2}, 0) \oplus (l, 0, \frac{1}{2})$. The condition (4.7a) will be invalidated, the γ_5 invariant functions $C_+^{dl\delta}$ being used in the expansion from the very beginning. Therefore we shall start from the expansion including the full functions (4.5). The relation (4.7a) is then satisfied, but the γ_5 invariance is broken and must be restored with the aid of the displacement:

$$\Gamma_1 \to \tfrac{1}{2}(\Gamma_1 + \gamma_5\Gamma_1\gamma_5) .$$

Since

$$\gamma_5 C_\pm^{ld\delta}\gamma_5 = \pm C_\pm^{ld\delta} ,$$

only the terms $C_+^\dagger G^{-1}C_+$ and $C_-^\dagger G^{-1}C_-$ survive in the expansion (4.6a). The terms $C_+^\dagger G^{-1}C_-$ and $C_-^\dagger G^{-1}C_+$ are excluded. Using eq. (4.2b) and performing the displacement $\sigma \to \widetilde{\sigma}$ in the second term we obtain with the regard for (4.7a) the term $2C_+^\dagger G^{-1}C_+$ instead of the corresponding two terms. The extra factor 2 may be compensated by means of a proper choice of the orthogonality relation for the functions $C_+^{ld\delta}$ (the factor $\frac{1}{2}$ on the right-hand side of (2.8,9) is additional in comparison to eq. (4.3)), so that the spectral function $\rho_1(\sigma)$ is determined by eq. (4.7) as before. It must be noticed that the second term is absent from the orthogonality relation (2.8) for the functions $C_+^{ld\delta}$, because this term is connected with the γ_5 non-invariant part of $C^{ld\delta}$. This term will arise if one writes the orthogonality relation as

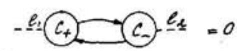
 $= i\pi[\mu_{1/2}(l_1)]^{-1}\delta_{l_1 l_2}\delta^{(4)}_{(x_1-x_2)}$. (2.8a)

We used eq. (4.2b) when passing from (2.8) to (2.8a). The orthogonality relation for the full function (4.5) includes both mentioned terms.

In the expansion of the four-fermion vertex $\Gamma_0^{(1)}$ two terms may be present $C_+^\dagger D_l^{-1}C_+$ and $C_-^\dagger D_l^{-1}C_-$, which may not be converted into each other. The second term does not contribute to the equation

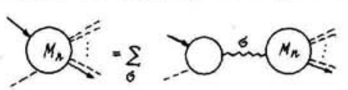
 (4.7b)

since

 (4.7c)

after the trace in the spinor line is calculated.

Consider the expansion of higher-order vertices. One has for the vertex M_n:

 (4.8)

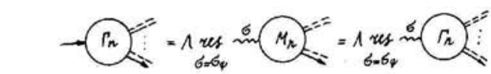
where

 (4.8a)

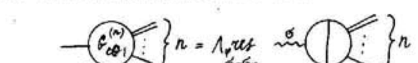
Similar relations hold for the vertices Γ_n and R_n. The expansion (4.8) is justified if the integrals

 (4.9)

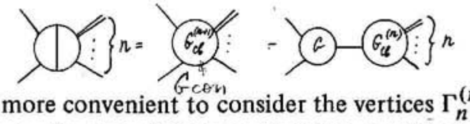
converge [11]. They do converge because the vertices M_n (also Γ_n and R_n) do not contain diagrams which may be divided by cutting a line in the transverse direction. Using eq. (4.8a) one may rewrite eqs. (3.12), (3.15) at $m = 0$ as


(4.9a)

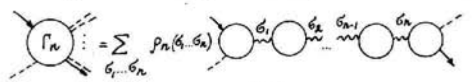
where σ_ψ are the spinor field quantum numbers (see sect. 5). The similar equations for Green functions were obtained in ref. [11] for the $\lambda\varphi^3$ theory:


(4.10)

where the conformal harmonic stands on the right-hand side. This harmonic is defined in parallel to (4.8a) with the aid of the $(n + 2)$-point vertex


(4.10a)

It is more convenient to consider the vertices $\Gamma_n^{(m)}$ rather than Green functions, because in the case of these vertices it is possible to eliminate completely the additional vertices. Besides, the vertices $\Gamma_n^{(m)}$ (contrary to the functions entering (4.10, 10a) and the functions $\widetilde{R}_n^{(m)}, R_n^{(m)}$ and $\widetilde{M}_n^{(m)}, M_n^{(m)}$ do not contain diagrams which may be divided by cutting a line. They may be represented as conformal expansions in all the arguments. In particular, at $m = 0$ one has


(4.11)

The expansion of vertices $\Gamma_n^{(m)}$, m being arbitrary, is considered in refs. [12,13]. The spectral function $\rho_n(\sigma_1 \dots \sigma_n)$ must satisfy the homogeneous integral equations securing the symmetry of the vertices Γ_n in the arguments of the meson legs (sect. 6). In order that these expansions were justified it is necessary that the integrals of the type (4.9) for the vertices Γ_n with an arbitrary number of internal lines should converge (there must be not less than two external meson legs). These integrals do converge in the skeleton theory. It is noteworthy that the similar integrals for Green functions are infinite because of diagrams which may be divided by cutting a line.

It is supposed in parallel to (4.6) that the spectral function $\rho_1(\sigma_1 \dots \sigma_n)$ may be analytically continued to the whole complex plane in every argument. Therefore the "Hermiticity" relation.

$$\rho_n^*(\sigma_1 \dots \sigma_n) = \rho_n(\sigma_n^* \dots \sigma_1^*) , \tag{4.12}$$

which is easily proved for the points of the integration contour, takes place for arbitrary complex dimensions $l_1 \dots l_n$.

5. Solution of the system

Using an expansion of the type (4.11) for vertices $\Gamma_n^{(m)}$, one may diagonalize the system of eqs. (3.15), (3.20) and solve it.

Let us consider the example of eq. (3.15) at $m = 0$. Under substitution of (4.11) into (3.15) we obtain for $n \geqslant 1$ using the analyticity of ρ_n and the orthogonality relation (4.3):

$$\rho_n(\sigma_1 \dots \sigma_n) = \Lambda \operatorname*{res}_{\sigma=\sigma_\psi} \rho_{n+1}(\sigma\sigma_1 \dots \sigma_n) , \tag{5.1a}$$

where $\sigma_\psi = (d, \frac{1}{2}, 0) \oplus (d, 0, \frac{1}{2})$ are quantum numbers of the field $\psi(x)$. Using (4.12) eq. (5.1a) may be rewritten in another form

$$\rho_n(\sigma_1 \dots \sigma_n) = \Lambda \operatorname*{res}_{\sigma=\sigma_\psi} \rho_{n+1}(\sigma_1 \dots \sigma_n\sigma) . \tag{5.1b}$$

Here the residue is taken with respect to the last argument. Besides, one has from (3.9):

$$\operatorname*{res}_{\sigma=\sigma_\psi} \rho_1(\sigma) = g\Lambda^{-1} . \tag{5.2}$$

Eqs. (5.1a, 1b) and (5.2) form an infinite system of algebraic equations which may be easily solved. The solution contains a considerable arbitrariness (which is discussed below), because not only the functions ρ_n but also their residies enter these equations. Let us introduce the normalized function:

$$\overline{\rho}_1(\sigma) = g^{-1}\Lambda\rho_1(\sigma) ,$$

and represent $\rho_2(\sigma_1\sigma_2)$ as:

$$\rho_2(\sigma_1\sigma_2) = \overline{\rho}_1(\sigma_1) f_2(\sigma_1\sigma_2) \overline{\rho}_1(\sigma_2) ,$$

where $f_2(\sigma_1\sigma_2)$ is arbitrary. The substitution of this into eq. (5.1) at $n = 1$ yields with the regard for (5.2):

$$f_2(\sigma_1\sigma_\psi) = f_2(\sigma_\psi\sigma_1) = g\Lambda^{-2} .$$

The latter is the only confinement for the function $f_2(\sigma_1\sigma_2)$ imposed by eqs. (5.1). With the introduction of a normalized function

$$\overline{\rho}_2(\sigma_1\sigma_2) = g^{-1}\Lambda^2 f_2(\sigma_1\sigma_2) ,$$

one has finally:

$$\rho_2(\sigma_1\sigma_2) = g\Lambda^{-2}\overline{\rho}_1(\sigma_1)\overline{\rho}_2(\sigma_1\sigma_2)\overline{\rho}_1(\sigma_2) .$$

By means of the representation of all the $\rho_n(\sigma_1 \dots \sigma_n)$ in an analogous form and introduction of new arbitrary functions $\overline{\rho}_n(\sigma_1 \dots \sigma_n)$, one may obtain the general solution of the system. It reads:

$$\rho_n(\sigma_1 \dots \sigma_n) = g\Lambda^{-n} \left\{ \prod_{i=1}^{n} \overline{\rho}_1(\sigma_i) \right\} \times$$

$$\times \left\{ \prod_{i=1}^{n-1} \overline{\rho}_2(\sigma_i\sigma_{i+1}) \right\} \left\{ \prod_{i=1}^{n-2} \overline{\rho}_3(\sigma_i\sigma_{i+1}\sigma_{i+2}) \right\} \dots \overline{\rho}_n(\sigma_1 \dots \sigma_n) , \tag{5.3}$$

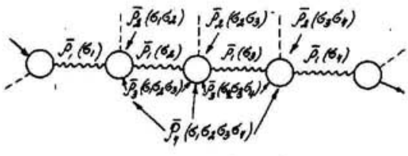


Fig. 1.

where $\overline{\rho}_k(\sigma_1 \dots \sigma_k)$ are arbitrary functions satisfying the following conditions:

$$\operatorname*{res}_{\sigma=\sigma_\psi} \overline{\rho}_1(\sigma) = 1 , \qquad \overline{\rho}_k(\sigma_1 \dots \sigma_k)|_{\sigma_1=\sigma_\psi} = \overline{\rho}_k(\sigma_1 \dots \sigma_k)|_{\sigma_k=\sigma_\psi} = 1 , \qquad k \geqslant 2 . \tag{5.4}$$

Besides, $\overline{\rho}_1(\sigma)$ must decrease rapidly enough when $l \to \infty$ and $j_{1,2} \to \infty$ in order that (4.11) converge, and each of the functions $\overline{\rho}_k(\sigma_1 \dots \sigma_k)$ must satisfy eq. (4.12).

Let us consider the structure of the solution in more detail. It follows from (5.3) that the first n fucntions from the infinite set

$$\overline{\rho}_1(\sigma_1), \overline{\rho}_2(\sigma_1\sigma_2), \dots , \overline{\rho}_n(\sigma_1 \dots \sigma_n) , \dots \tag{5.5}$$

correspond to the vertex Γ_n according to the following rule: each internal line in the expansion (4.11) corresponds to the function $\overline{\rho}_1(\sigma)$, each internal vertex corresponds to $\overline{\rho}_2(\sigma_1\sigma_2)$, each pair of vertices to $\overline{\rho}_3(\sigma_1\sigma_2\sigma_3)$, etc. For example, fig. 1 above. Such a structure of functions ρ_n is closely connected with the crossing-symmetry requirement. The crossing-symmetry would be broken, if any one of the functions $\overline{\rho}_k(\sigma_1 \dots \sigma_k)$ was put equal to unity (sect. 6). Suppose that we had written down the expansion (4.11) in the form (5.3) from the very beginning before the substitution into the equations, and that each vertex Γ_n had been put into correspondence with a set of functions: $\overline{\rho}_1^{(n)}(\sigma_1), \overline{\rho}_2^{(n)}(\sigma_1\sigma_2), \dots , \overline{\rho}_n^{(n)}(\sigma_1 \dots \sigma_n)$. Then as a result of solving the equations one would find that the functions $\overline{\rho}_k^{(n)}$ corresponding to different vertices Γ_n, coincide:

$$\begin{array}{lllll}
\Gamma_1 \qquad \Gamma_2 & \Gamma_3 & \Gamma_4 & & \dots \\
\overline{\rho}_1^{(1)}(\sigma) = \overline{\rho}_1^{(2)}(\sigma) & = \overline{\rho}_1^{(3)}(\sigma) & = \overline{\rho}_1^{(4)}(\sigma) & & = \dots \\
 & \overline{\rho}_2^{(2)}(\sigma_1\sigma_2) = \overline{\rho}_2^{(3)}(\sigma_1\sigma_2) & = \overline{\rho}_2^{(4)}(\sigma_1\sigma_2) & & = \dots \\
 & & \overline{\rho}_3^{(3)}(\sigma_1\sigma_2\sigma_3) = \overline{\rho}_3^{(4)}(\sigma_1\sigma_2\sigma_3) & & = \dots \\
 & & & \dots\dots\dots\dots\dots , &
\end{array}$$

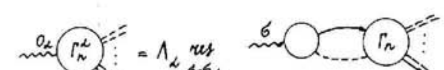
i.e. all the information contained in the dynamical equations is exhausted by the equality of the functions, placed on one and the same line.

The normalization factor of the vertex Γ_n may be related to the residue of the function $\rho_n(\sigma_1 \dots \sigma_n)$ with respect to all the arguments at the poles $\sigma_i = \sigma_\psi$ corresponding to the fundamental spinor field. One has from (5.3), (5.4):

$$g_n \equiv \underset{\sigma_1=\sigma_\psi}{\text{res}} \ \underset{\sigma_2=\sigma_\psi}{\text{res}} \dots \underset{\sigma_n=\sigma_\psi}{\text{res}} \ \rho_n(\sigma_1 \dots \sigma_n) = g\Lambda^{-n} .$$

The coefficients g_n serve as the analogy of the mass-shell values of the vertices Γ_n in the conventional relativistic theory: the transition to the spectral function $\rho_n(\sigma_1 \dots \sigma_n)$ is analogous to the Fourier transformation, while the value $\sigma = \sigma_\psi$ is the analogy of the mass-shell condition: $p^2 = m^2$.

Relations (4.11) and (4.3,4) determine the general conformaly-invariant solution of the renormalized Schwinger-Dyson equations. The greatest arbotrariness admitted by these equations consists in the possibility of defining arbitrarily an infinite set of functions (5.5) limited by conditions (5.4). This arbitrariness originates from the fact that we solved the system of connected equations, which is arranged in such a way that each vertex contains the full information about the lower vertices. In order to obtain a unique solution it is necessary to indicate a way to close the system up. It is natural to think that the closing takes place when the initial term is properly taken into consideration. This is clear in an example of the self-energy equations (in the course of their derivation the equation $z_1\gamma = \Gamma - \Gamma G^{-1} D^{-1} R$ was used, and the uncertainty $0 \times \infty$ arose at $z_1 \to 0$) and the Ward identities, see sect. 7. In both cases we obtain the non-trivial limitations for the spectral function. Although the current and the energy-momentum tensor play a special role in the theory, because they are the concerved quantities, one may expect that consideration of other tensor fields (non-conserved) will bring an additional information about the initial term. Such tensor or spin-tensor fields, for example



(5.6)

where σ_α are the quantum numbers of the field O_α, are connected with the poles [10–13,15] of the functions ρ_n in the variables $l_1, \dots, l_n$. It is necessary therefore to investigate their connection with the fundamental fields, which is a difficult task and will be considered elsewhere.

Some confinements for the dimensions l_α of the fields O_α follow from axiomatic requirements. The axioms of spectrality and positivity impose the following limitations upon l_α:

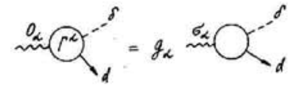
$$l_\alpha \geqslant 2 + j_1^{(\alpha)} + j_2^{(\alpha)} , \qquad \text{or} \qquad l_\alpha > 1 + j_1^{(\alpha)} , \qquad \text{if} \qquad j_2 = O ,$$

as shown in refs. [16,21]. These limitations arise in the course of analysis of expansions like (4.6), (4.6a) in Minkowski space [16].

Note that the constants Λ_α may be obtained from the equations for the self-energy similarly to (3.18), (3.20). One makes sure (see also [11]) that

$$g_\alpha \Lambda_\alpha = -2\mu(\sigma_\alpha)\,, \tag{5.7}$$

where $\mu(\sigma)$ is the weight function entering into eq. (4.6), (4.6a) and g_α is the coupling constant of the field O_α and the fundamental fields. This constant is determined in parallel to (2.4) by the relation:



(5.8)

At $n = 1$ one has from (5.6):

$$\operatorname*{res}_{\sigma=\sigma_\alpha} \bar{\rho}_1(\sigma) = \Lambda g_\alpha / \Lambda_\alpha g$$

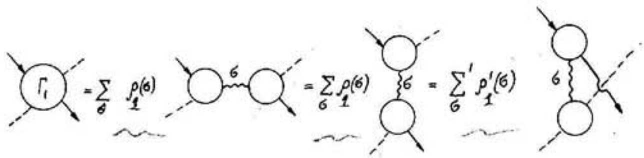
or with the regard for (3.18) and (5.7):

$$\operatorname*{res}_{\sigma=\sigma_\alpha} \bar{\rho}_1(\sigma) = \frac{\mu(\sigma_\psi)}{\mu(\sigma_\alpha)} \frac{g_\alpha^2}{g^2}\,.$$

Therefore the problem of obtaining the function $\bar{\rho}_1(\sigma)$, i.e. its poles and the corresponding residies, may be reduced to that of obtaining the dimensions and coupling constants of the fields O_α. Crossing symmetry is the strongest axiomatic requirement. In some case it gives the possibility to obtain the constants g_α, if the dimensions of O_α are known, and *vice versa*.

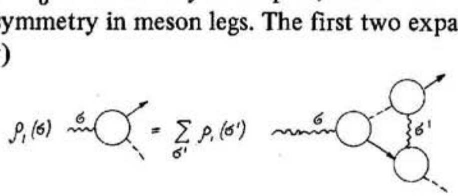
6. Crossing symmetry

The vertices Γ_n are symmetrical in the coordinates of meson legs and antisymmetrical in the coordinates of nucleon legs. This leads to the homogeneous integral equations [15,11] for the spectral functions. Consider the vertex Γ_1. Its expansion may be written down in the three different ways



(6.1)

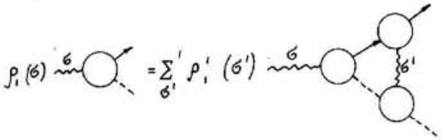
where Σ'_σ includes only even spins, because odd spins in the last term would lead to antisymmetry in meson legs. The first two expansions yield the integral equation for $\rho_1(\sigma)$



(6.2)

while the function $\rho'_1(\sigma)$ remains arbitrary. Therefore the general solution of eq.

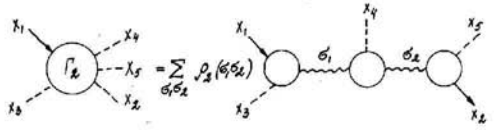
(6.2) may be written down as



(6.2a)

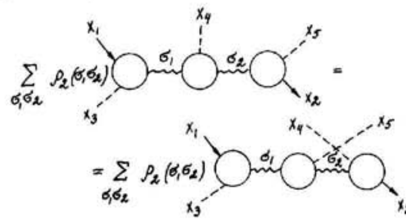
where $\rho_1'(\sigma')$ is an arbitrary function.

Let us show that the crossing-symmetry does not contradict the general solution (5.34). For example let us consider the vertex Γ_2:



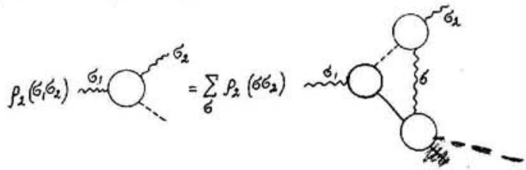
(6.2b)

The symmetry with respect to the replacement $x_4 \rightleftarrows x_5$ gives



(6.2c)

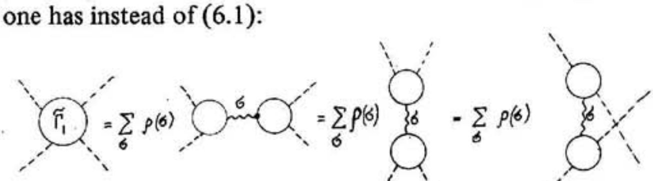
Using the arthogonality relation we obtain



(6.3)

Substitution of the expression (5.3) for $\rho_2(\sigma_1\sigma_2)$ and cancellation of the factor $\overline{\rho}_1(\sigma_2)$ yield the equation for the product $\overline{\rho}_1(\sigma_1)\overline{\rho}_2(\sigma_1\sigma_2)$ where the function $\overline{\rho}_1(\sigma_1)$ satisfies eq. (6.2) expressing the crossing symmetry of Γ_1. If we put $\sigma_2 = \sigma_\psi$ in eq. (6.3) after the cancellation of $\overline{\rho}_1(\sigma_2)$, then eq. (6.2) will be reestablished, provided that eq. (5.4): $\overline{\rho}_2(\sigma_1\sigma_\psi) = 1$ is taken into consideration. Thus the crossing symmetry agrees with the general expression (5.3), (5.4).

Stronger limitations arise for the vertices containing only neutral fields. In this case one has instead of (6.1):



(6.4)

The solution of the integral equations for $\rho(\sigma)$ which arises here, may be found using the known representation of the vertices in the form of the Mellin integrals with the symmetric kernel $f(\alpha\beta\gamma)$, see ref. [9].

$$\widetilde{\Gamma}_1(x_1x_2x_3x_4) = (x_{12}^2x_{34}^2)^{-\delta} \int \mathrm{d}\alpha \mathrm{d}\beta\, f(\alpha, \beta, -\alpha-\beta-\delta)$$
$$\times (x_{12}^2x_{34}^2)^{-\alpha-\beta}(x_{14}^2x_{23}^2)^{\alpha}(x_{13}^2x_{24}^2)^{\beta}\,, \tag{6.5}$$

where the integration contour in α and β passes along the imaginary axis. Crossing symmetry gives:

$$f(\alpha\beta\gamma) = f(\alpha\gamma\beta) = f(\beta\alpha\gamma)\,. \tag{6.6}$$

A function $\rho(\sigma)$ satisfying eq. (6.4) may be represented as an integral of an arbitrary function (6.6). Let us consider the simplest example when the usage of this representation makes it possible to find the vertex exactly. Let us require that only the scalar representations should enter the expansion (6.4): $\widetilde{\rho}(l, s) = 0$, when $s \neq 0$, $\widetilde{\rho}(l, 0)$ is arbitrary. The expansion (6.4) in this case takes the form:

$$\widetilde{\Gamma}_{s=0}(x_1x_2x_3x_4) = \int \mathrm{d}l\, \widetilde{\rho}'(l)\, (x_{12}^2)^{\frac{1}{2}l-\delta}(x_{34}^2)^{2-\frac{1}{2}l-\delta} \tag{6.7}$$
$$\int \mathrm{d}z\, \{(x_1-z)^2(x_2-z)^2\}^{-\frac{1}{2}l}\{(x_3-z)^2(x_4-z)\}^{-\frac{1}{2}l-2}\,.$$

All the normalization factors depending on l are included into $\widetilde{\rho}'(l)$. The calculation of the integral over z with the aid of the standard technique [18] gives:

$$(6.7) \sim (x_{12}^2x_{34}^2)^{-\delta}$$
$$\times \int \mathrm{d}\alpha \mathrm{d}\beta \{\Gamma^2(-\alpha)\Gamma^2(-\beta) \int \mathrm{d}l\, \widetilde{\rho}'(l) \frac{\Gamma(\frac{1}{2}l+\alpha+\beta)\Gamma(2-\frac{1}{2}l+\alpha+\beta)}{\Gamma^2(\frac{1}{2}l)\Gamma^2(2-\frac{1}{2}l)}$$
$$\times (x_{12}^2x_{34}^2)^{-\alpha-\beta}(x_{23}^2x_{14}^2)^{\alpha}(x_{13}^2x_{24}^2)^{\beta}\,.$$

This expression is of the form (6.5), where $f(\alpha, \beta, -\alpha-\beta-\delta)$ consists of the terms in the curly brackets. If one requires eq. (6.6) now, this will fix $\widetilde{\rho}'(l)$ so that:

$$\int \mathrm{d}l\, \widetilde{\rho}'(l) \frac{\Gamma(\frac{1}{2}l+\gamma-\delta)\Gamma(2-\frac{1}{2}l+\gamma-\delta)}{\Gamma^2(\frac{1}{2}l)\Gamma^2(2-\frac{1}{2}l)} = A\Gamma^2(-\gamma)\,,$$

where A is an arbitrary constant. As to the function $f(\alpha\beta\gamma)$ we obtain:

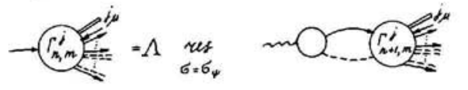
$$f(\alpha\beta\gamma) = A\Gamma^2(-\alpha)\Gamma^2(-\beta)\Gamma^2(-\gamma)\,.$$

7. Ward identity

In the framework of the unrenormalized theory the Ward identities are the consequences of dynamical equations provided that the fields satisfy the canonical commutation relations. On the other hand the canonical commutation relations may be considered as consistency conditions [22] provided the validity of Lagrange and Hamilton equations is required independently. In this case the commutation relations as they are do not contain any additional limitations for the Green functions. Both approaches are equivalent before the renormalization.

The situation is quite different in the conformally invariant theory. The conformal symmetry originates as a result of infinite renormalization, completely destroying the bare term. As a consequence of the renormalizations some information contained in canonical commutation rules is lost and consequently the solution of the postulated egeneralized Ward identity [23] gives additional information.

The vertices with the current and the energy-momentum tensor are determined by equations of the type (5.6), if j_μ or $T_{\mu\nu}$ are taken as the field O_α. In the case of equations for vertices with $T_{\mu\nu}$ it is necessary to take into account the contributions of nucleon and boson fields. The corresponding spectral functions possess the poles with the quantum numbers of the current or the energy-momentum tensor. No other limitations for spectral functions arise. Using the equations defining the vertices with j_μ and $T_{\mu\nu}$ and eqs. (3.15) we find:



(7.1)

and similarly for the energy-momentum tensor. Another system of equations of the same type is to be obtained from eq. (3.19).

We shall confine ourselves to the consideration of two vertices with the current: Γ^j and Γ^j_1. The higher-order vertices with the current may be considered in an analogous way, see ref. [13]. One has [7,9]:

$$\Gamma^{dd}_{j_\mu}(x_1 x_2 1 x) = A\left\{\tfrac{1}{2}(B+1)S_{3/2}(x_1-x)\gamma_\mu S_{3/2}(x-x_2)\widetilde{\Delta}_{d-3/2}(x_1-x_2)\right.$$
$$\left. + \frac{1}{d-\frac{3}{2}}(B-1)S_{d-1}(x_1-x_2)\widetilde{\Delta}_1(x_1-x)\widetilde{\Delta}_1(x_2-x)\lambda^x_\mu(x_1x_2)\right\}, \qquad (7.2)$$

where S_d and $\widetilde{\Delta}_\delta$ are defined in (2.6), and B is an arbitrary constant

$$\lambda^x_\mu(x_1x_2) = \frac{(x_1-x)_\mu}{(x_1-x)^2} - \frac{(x_2-x)_\mu}{(x_2-x)^2}. \qquad (7.2a)$$

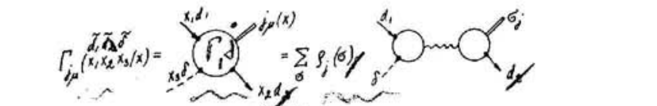
The constant A is determined by the Ward identity

$$\partial^x_\mu \Gamma^{dd}_{j_\mu}(x_1x_2|x) = -ie[\delta(x_1-x)-\delta(x_2-x)]\,G_d(x_1-x_2)$$

and is equal to:

$$A = (4\pi)^2 e\,\frac{\Gamma(d+\frac{1}{2})}{\Gamma(d-\frac{3}{2})\Gamma(\frac{5}{2}-d)}. \qquad (7.3)$$

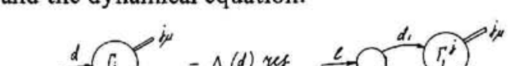
The vertex Γ^j_1 must be written in the form:


(7.4a)

where σ_j are the current quantum numbers, d_1 and d are the dimensions of the spinor fields ψ_{d_1} and ψ_d. ($\widetilde{d}, \widetilde{\delta}$ denotes the amputation in the corresponding argument, see (2.11), (2.12) and (4.2)). The vertex Γ_1^j satisfies the generalized Ward identity [23]

$$\partial_\mu^x \Gamma_1^{j_\mu, \widetilde{d}_1 \widetilde{d} \widetilde{\delta}}(x_1 x_2 x_3 | x) = -ie\,[\delta(x_1 - x) - \delta(x_2 - x)]\,\Gamma^{\widetilde{d}_1 \widetilde{d} \widetilde{\delta}}(x_1 x_2 x_3) \qquad (7.5)$$

and the dynamical equation:


(7.1a)

which follows from (7.1) at $m = 1, n = 0$. Let us substitute (7.4) into (7.5). Using the orthogonality relation (4.3) and taking (2.4) and (2.12) into consideration we find:

$$\rho_j(\sigma)\,\partial_\mu^x C_\mu^{\sigma d \sigma_j}(x_1 x_2 x) = -ieg \int \mathrm{d}y\; C^{\sigma d_1 \delta}(x_1 x y)\, C_+^{\widetilde{d}_1 d \widetilde{\delta}}(x x_2 y)\,. \qquad (7.6)$$

The second term on the right-hand side of (7.5) does not contribute to (7.6) because the function $\rho_j(\sigma)$ is defined for real l by the analytic continuation from the integration contour in (4.4), where the mentioned term is equal to zero by virtue of the orthogonality relation. Eq. (7.6) completely determines the spectral function $\rho_j(\sigma)$ provided that the derivative on the left-hand side is not zero. The latter is not always so. Let us consider for example the spinor contribution to (7.4)

$$\sigma = (l, \tfrac{1}{2}, 0) \oplus (l, 0, \tfrac{1}{2})\,. \qquad (7.7a)$$

The unnormalized Clebsch-Gordon kernels in this case may be represented as

$$C_\mu^{l d \sigma_j}(x_1 x_2 x) = N_j(l,d)\,\{\tfrac{1}{2}[F(l,d) + 1]\,S_{3/2 + \frac{1}{2}(l-d)}(x_1 - x)\,\gamma_\mu\, S_{3/2 - \frac{1}{2}(l-d)}(x - x_2)$$
$$\widetilde{\Delta}_{\frac{1}{2}(l+d-3)}(x_1 - x_2) + 2\,[1 - (\tfrac{1}{2}(l-d)^2]\,(l + d - 3)^{-1}\,[F(l,d) - 1]$$
$$S_{\frac{1}{2}(l+d)-1}(x_1 - x_2)\,\widetilde{\Delta}_{1 + \frac{1}{2}(l-d)}(x_1 - x)\,\widetilde{\Delta}_{1 - \frac{1}{2}(l-d)}(x_2 - x)\,\lambda_\mu^x(x_1 x_2)\}\,, \qquad (7.7)$$

where N_j is the normalization coefficient and $F(l,d)$ is an arbitrary function. One has from (7.7)

$$\partial_\mu^x C_\mu^{l d \sigma_j}(x_1 x_2 x)$$
$$= -\tfrac{1}{2}(l-d)\,N_j(ld)\,S_{\frac{1}{2}(l+d)-2}(x_{12})\,\widetilde{\Delta}_{2 + \frac{1}{2}(l-d)}(x_1 - x)\,\widetilde{\Delta}_{2 - \frac{1}{2}(l-d)}(x_2 - x)\,. \qquad (7.8)$$

The function $F(l, d)$ does not contribute to (7.8) and consequently to the Ward identity (7.5) and (7.6). Thus the Ward identity determines the vertex Γ_1^j up to the arbitrary function $F(l, d)$ in (7.7). This arbitrariness may be represented in another form: one may put $F(l, d) = 0$ in (7.7) and simultaneously add the term

(7.9)

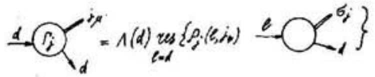
and $\rho_j'(l, j_\psi) \sim F(l, d)$, to the right-hand side of (7.4). In this case two independent sets of Clebsch-Gordan kernels and consequently two spectral functions $\rho_j(\sigma)$ and $\rho_j'(\sigma)$ arise in the expansion. The Ward identity determined only the function $\rho_j(\sigma)$.

Let us find this function for the spinor contribution (7.7a). Substituting (7.8) into (7.6) and evaluating the integral on the right-hand side with the aid of eqs. (2.6a, b) we find

$$\begin{aligned} &N_j(l, d)\, \rho_j(l, j_\psi) \\ &= -\frac{2}{(4\pi)^4}\, egN(d_1 d\delta)\Gamma(4 - \tfrac{1}{2}(d_1 + d + \delta))\Gamma(2 - \tfrac{1}{2}(d + d_1 - \delta)) \\ &\times \Gamma(\tfrac{5}{2} - \tfrac{1}{2}(d_1 - d + \delta))\Gamma(\tfrac{1}{2}(d - d_1 + \delta) + \tfrac{1}{2})\, \frac{1}{l - d} N(ld\delta) \\ &\times \frac{\Gamma(\frac{1}{2}(l + d_1 + \delta) - 2)\Gamma(\frac{1}{2}(d_1 - l + \delta) + \frac{1}{2})\Gamma(\frac{1}{2}(d_1 + l - \delta))\Gamma(\frac{5}{2} - \frac{1}{2}(l - d_1 + \delta))}{\Gamma(2 - \frac{1}{2}(l - d))\Gamma(2 + \frac{1}{2}(l - d))\Gamma(\frac{9}{2} - \frac{1}{2}(l + d))}, \end{aligned} \tag{7.10}$$

where $N(ld\delta)$ is given by eq. (2.7) and $j_\psi = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$.

Let us consider the dynamical equation (7.1a). Substitution of (7.4) into (7.1a) gives:



(7.11)

Using (7.2) and (7.7) we obtain:

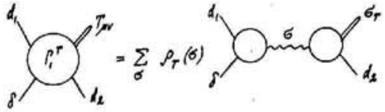
$$\Lambda(d) \operatorname*{res}_{l=d} \{N_j(l, d)\, \rho_j(l, j_\psi)\} = A(d)\,, \tag{7.12a}$$

$$F(d, d) = B\,. \tag{7.12b}$$

Relation (7.12a) was obtained from (7.11) by taking a derivative ∂_μ^x (as a result the terms with $F(l, d)$ and B disappear). Then eq. (7.1a) becomes equivalent to (7.12a). Substituting eqs. (3.18), (7.10) and (7.8) into eq. (7.12a) we find that the latter is satisfied identically at any value of the fundamental field dimensions.

Eqs. (7.1) for higher-order vertices may be considered in an analogous manner. The Ward identities give the possibility to express the longitudinal part of the vertex with the current in terms of the corresponding vertex without the current as it was in the case $n = 1$, while the dynamical equations (7.1) are satisfied at any values of the fields dimensions.

Let us consider the vertices including the energy-momentum tensor. We shall confine ourselves to the case of scalar fields. Let $\varphi_{d_1}(x_1), \varphi_{d_2}(x_2), \varphi_\delta(x_3)$ be scalar fields in the D-dimensional Euclidian space. One has



(7.13)

Here Γ_1^T is the one-particle irreducible vertex; $\sigma = (l, s)$,,where s denotes the spin; $\sigma_T = (D, 2)$ are the quantum numbers of the energy-momentum tensor. The Ward identity for the vertex Γ_1^T amputated in arguments x_1, x_2 and x_3 reads [7]:

$$-\partial_\mu^x \Gamma_1^{T_{\mu\nu},\tilde{d}_1\tilde{d}_2\tilde{\delta}}(x_1x_2x_3|x)$$
$$= [\delta(x-x_1)\partial_\nu^{x_1} + \delta(x-x_2)\partial_\nu^{x_2} + \delta(x-x_3)\partial_\nu^{x_3}]\Gamma^{\tilde{d}_1\tilde{d}_2\tilde{\delta}}(x_1x_2x_3)$$
$$+ \frac{d_1-D}{D}\partial_\nu^x\delta(x-x_1) + \frac{d_2-D}{D}\partial_\nu^x\delta(x-x_2)$$
$$+\frac{\delta-D}{D}\partial_\nu^x\delta(x-x_3)]\Gamma^{\tilde{d}_1\tilde{d}_2\tilde{\delta}}(x_1x_2x_3)\,. \tag{7.14}$$

Let us substitute (7.13) into (7.14). We obtain:

$$-\frac{1}{g}\rho_T(\sigma)\partial_\mu^x C_{\mu\nu}^{\sigma d_2\sigma_T}(x_1x_2x) = \int \mathrm{d}y\, C^{\sigma d_1\delta}(x_1xy)\partial_\nu^x C^{\tilde{d}_1\tilde{\delta}d_2}(xyx_2)$$
$$+ \int \mathrm{d}y\, C^{\sigma d_1\delta}(x_1yx)\partial_\nu^x C^{\tilde{d}_1\tilde{\delta}d_2}(yxx_2)$$
$$+ \frac{d_1-D}{D}\partial_\nu^x \int \mathrm{d}y\, C^{\sigma d_1\delta}(x_1xy)C^{\tilde{d}_1\tilde{\delta}d_2}(xyx_2)$$
$$+ \frac{\delta-D}{D}\partial_\nu^x \int \mathrm{d}y\, C^{\sigma d_1\delta}(x_1yx)C^{\tilde{d}_1\tilde{\delta}d_2}(yxx_2)\,, \tag{7.15}$$

where $C^{\sigma d_1\delta}$ and $C^{d_1\delta d_2}$ are Clebsch-Gordan kernels normalized by conditions (4.2), (4.3):

$$C^{\sigma d_1d_2}(xx_1x_2) = \Gamma(\tfrac{1}{2}(d_1+d_2+l+s)-h)\,\frac{\Gamma(\frac{1}{2}(d_1-d_2+l+s))\Gamma(\frac{1}{2}(d_2-d_1+l+s))}{\Gamma(\frac{1}{2}(d_1-d_2+l-s))\Gamma(\frac{1}{2}(d_2-d_1+l-s))}$$
$$N_s\widetilde{\Delta}_{\frac{1}{2}(d_1+d_2-l+s}(x_1-x_2)\widetilde{\Delta}_{\frac{1}{2}(d_1-d_2+l-s)}(x_1-x)\,\widetilde{\Delta}_{\frac{1}{2}(d_2-d_1+l-s)}(x_2-x)$$
$$\{\lambda_{\mu_1}^x(x_1x_2)\ldots\lambda_{\mu_s}^x(x_1x_2) - \text{traces}\}\,, \tag{7.16}$$

where $h = \frac{1}{2}D$,

$$\widetilde{\Delta}_\delta(x) = \frac{4^\delta}{(4\pi)^h}\,\Gamma(\delta)\,\frac{1}{(x^2)^\delta}\,,$$

$\lambda^x_\mu(x_1x_2)$ is given by the formula (7.2a). The normalization of functions (7.16) has been evaluated in ref. [17], see also [11,13]:

$$N_s(d_1d_2l) = \{\Gamma(\tfrac{1}{2}(d_1+d_2+l+s)-h)\Gamma(\tfrac{1}{2}(d_1+d_2-l+s))\Gamma(\tfrac{1}{2}(d_1-d_2+l+s))$$
$$\times\,\Gamma(\tfrac{1}{2}(d_2-d_1+l+s))\Gamma(D-\tfrac{1}{2}(d_1+d_2+l-s))\Gamma(h-\tfrac{1}{2}(d_1+d_2-l-s))$$
$$\times\,\Gamma(h-\tfrac{1}{2}(d_1-d_2+l-s))\Gamma(h-\tfrac{1}{2}(d_2-d_1+l-s))\}^{-1/2}\,. \tag{7.17}$$

Let us consider the scalar contribution to (7.13). In this case $\sigma = (l, 0)$ and the quantity $C^{\sigma d\sigma_T}(xx_1x_2)$ is determined by eq. (7.16) with $l = D$, $s = 2$. Contrary to (7.7) this quantity is determined uniquely and, consequently, the scalar contribution to (6.13) does not contain the terms of the type (7.9). The same also refers to the vector contribution $\sigma = (l, 1)$, see the considerations below. Evaluation of integrals in (7.15) gives:

$$\rho_T(l, s=0) = \frac{8D}{D-1}\,\frac{g}{(4\pi)^{2h}}\,\frac{N_0(l,d_1\delta)N_0(d_1d_2\delta)}{N_2(ld_2D)}$$
$$\times\,\frac{1}{(l-d_2)(D-l-d_2)}\,\{\Gamma(\tfrac{1}{2}(l+d_2)+1)\Gamma(D-\tfrac{1}{2}(l+d_2)+1)$$
$$\times\,\Gamma(h+\tfrac{1}{2}(l-d_2)+1)\Gamma(h-\tfrac{1}{2}(l-d_2)+1)\}^{-1}\,\{\Gamma(\tfrac{1}{2}(l+d_1+\delta)-h)$$
$$\times\,\Gamma(\tfrac{1}{2}(d_1+\delta-l))\Gamma(h-\tfrac{1}{2}(d_1-d_2+\delta))\Gamma(D-\tfrac{1}{2}(d_1+d_2+\delta))\}$$
$$\times\,\{\tfrac{1}{2}(\delta-d_1)(f_1-f_2)(D-d_1-\delta)-\tfrac{1}{2}(f_1+f_2)l(D-l)$$
$$+\,(1/2D)(f_1d_1+f_2\delta)(l-d_2)(D-l-d_2)\}\,, \tag{7.18}$$

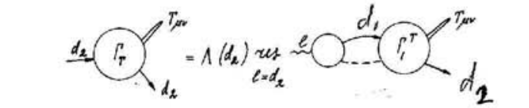
where

$$f_1 = \Gamma(\tfrac{1}{2}(l+d_1-\delta))\Gamma(h-\tfrac{1}{2}(l-d_1+\delta))\Gamma(\tfrac{1}{2}(d_2-d_1+\delta))\Gamma(h-\tfrac{1}{2}(d_1-\delta+d_2))\,,$$

and f_2 may be obtained from f_1 by the replacement $d_1 \rightleftarrows \delta$. In accrodance with (4.7a) one has:

$$\rho_T(l, 0) = \rho_T(D-l, 0)\,.$$

It is not difficult to verify that the spectral function (7.18) satisfied the dynamical equation. In the $\lambda\varphi_{d_1}\varphi_{d_2}\varphi_\delta$ theory in the D dimensional space we have:

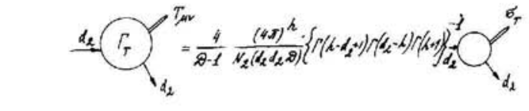


(7.19)

where

$$\Lambda(d) = -2g^{-1}\mu_0^D(d)\,, \qquad \mu_0^D(d) = \tfrac{1}{4}(4\pi)^{3h}\,\Gamma(h)\,\frac{\Gamma(l)\Gamma(D-l)}{\Gamma(h-l)\Gamma(l-h)}\,,$$

(the additional factor $\frac{1}{4}$ in (3.10a) appears because of the trace over the spinor indices). The Green function on the left-hand side equals:



(7.19a)

Evaluation of $\mathrm{res}_{l=d_2}\,\rho_T(l,0)$ from (7.18) shows that eq. (7.19) is satisfied identically for any value of dimensions d_1, d_2, δ.

Consider eq. (7.15), s being arbitrary. The three-point function, including the traceless symmetrical tensor $\sigma_1 = (d_1 s)$ of rank s, the scalar $\sigma_2 = (d_2, 0)$ and the traceless symmetrical tensor of the second rank $\sigma_3 = (d_3, 0)$ is equal to

$$\begin{aligned}
&C^{\sigma_1\sigma_2\sigma_3}_{M_1\ldots M_s;\nu_1\nu_2}(x_1x_2x_3)\\
&= \widetilde{\Delta}_{\frac{1}{2}(d_1+d_3-d_2-s-2)}(x_{13})\widetilde{\Delta}_{\frac{1}{2}(d_2+d_3-d_1+s-2)}(x_{23})\widetilde{\Delta}_{\frac{1}{2}(d_1+d_2-d_3+2-s)}(x_{12})\\
&\times\Bigg\{A\,[\lambda^{x_1}_{\mu_1}(x_2x_3)\ldots\lambda^{x_1}_{\mu_s}(x_2x_3) - \text{traces}]\\
&\times\left[\lambda^{x_3}_{\nu_1}(x_1x_2)\lambda^{x_3}_{\nu_2}(x_1x_2) - \frac{1}{D}g_{\nu_1\nu_2}\frac{x_{12}^2}{x_{13}^2x_{23}^2}\right]\\
&+\frac{1}{x_{13}^2}B\Bigg[\sum_k \lambda^{x_1}_{\mu_1}(x_2x_3)\ldots\lambda_{\hat{\mu}_k}\ldots\lambda^{x_1}_{\mu_s}(x_2x_3)\Big(g_{\mu_k\nu_1}(x_{13})\,\lambda^{x_3}_{\nu_2}(x_1x_2)\\
&+g_{\mu_k\nu_2}(x_{13})\,\lambda^{x_3}_{\nu_1}(x_1x_2) - \frac{2}{D}g_{\nu_1\nu_2}g_{\mu_k\nu}(x_{13})\,\lambda^{x_3}_{\nu}(x_1x_2)\Big) - \text{traces in }\mu\Bigg]\\
&+\frac{1}{(x_{13}^2)^2}C\Bigg[\sum_{k\neq l}\lambda^{x_1}_{\mu_1}(x_2x_3)\ldots\lambda_{\hat{\mu}_k}\ldots\lambda_{\hat{\mu}_l}\ldots\lambda^{x_1}_{\mu_s}(x_2x_3)\\
&\times\left(g_{\mu_k\nu_1}(x_{13})g_{\mu_l\nu_2}(x_{13}) - \frac{1}{D}g_{\nu_1\nu_2}g_{\mu_k\mu_l}\right) - \text{traces}\Bigg]\Bigg\}\,, \qquad (7.20)
\end{aligned}$$

where A, B and C are arbitrary coefficients, $x_{ik} = x_i - x_k$, $g_{\mu\nu}(x) = g_{\mu\nu} - 2x_\mu x_\nu/x^2$, the symbol $\hat{\mu}_k$ denotes the missing corresponding index. Putting $\sigma_3 = \sigma_T$, i.e. $d_3 = D$, in (7.20) and taking the derivative $\partial^{x_3}_{\nu_1}$ we find that the left-hand side of (7.15) equals:

$$\partial^{x_3}_{\kappa_1} C^{\sigma_1\sigma_2\sigma_T}_{\mu_1\ldots\mu_s;\nu_1\nu_2}(x_1x_2x_3) = \frac{x_{12}^2}{x_{13}^2 x_{23}^2}\,\widetilde{\Delta}_{h-\frac{1}{2}(d_2-d_1+s+2)}(x_{13})$$
$$\widetilde{\times}\,\widetilde{\Delta}_{h-\frac{1}{2}(d_1-d_2-s+2)}(x_{23})\widetilde{\Delta}_{\frac{1}{2}(d_1+d_2-s+2)-h}(x_{12})$$
$$\times\Bigg\{A_1\lambda^{x_3}_{\nu_2}(x_1x_2)\,[\lambda^{x_1}_{\mu_1}(x_2x_3)\ldots\lambda^{x_1}_{\mu_s}(x_2x_3) - \text{traces}]$$
$$+\frac{1}{x_{13}^2}B_1\Big[\sum_k g_{\mu_k\nu_2}(x_{13})\lambda^{x_1}_{\mu_1}(x_2x_3)\ldots\lambda_{\hat{\mu}_k}\ldots\lambda^{x_1}_{\mu_s}(x_2x_3) - \text{traces in }\mu\Big]\Bigg\}\,, \quad (7.21)$$

where

$$A_1 = \left[\frac{D-1}{D}(d_2-d_1) - \frac{s}{D}\right]A + 2s\frac{D-2}{D}(h - \tfrac{1}{2}(d_1-d_2-s))B\,,$$

$$B_1 = \frac{1}{D}A + \left[d_2 - d_1 + \frac{2}{D}s\right]B + 2(s-1)(h - \tfrac{1}{2}(d_1-d_2-s+2))C\,.$$

At $s \geqslant 2$ the quantity (7.21) may be turned into zero by appropriate choice of the coefficients A, B and C. This means that the terms of the type (7.9) enter eq. (7.13) at $s \geqslant 2$.

In order to obtain $\rho_T(l, s)$ at $s \geqslant 1$ it is necessary to calculate integrals on the right-hand side of (6.12). This may be conveniently done with the aid of the relation [13]:

$$\int \mathrm{d}x\,\widetilde{\Delta}_{\delta_1}(x_1-x)\widetilde{\Delta}_{\delta_2}(x_2-x)\widetilde{\Delta}_{\delta_3}(x_3-x)\,\{\lambda^{x_1}_{\mu_1}(xx_3)\ldots\lambda^{x_1}_{\mu_s}(xx_3) - \text{traces}\}$$
$$= (4\pi)^h\,\frac{\Gamma(h-\delta_3+s)\Gamma(\delta_1)}{\Gamma(h-\delta_3)\Gamma(\delta_1+s)}\,\widetilde{\Delta}_{h-\delta_1}(x_{23})\widetilde{\Delta}_{h-\delta_2}(x_{13})\widetilde{\Delta}_{h-\delta_3}(x_{12})$$
$$\times\,\{\lambda^{x_1}_{\mu_1}(x_2x_3)\ldots\lambda^{x_1}_{\mu_s}(x_2x_3) - \text{traces}\}\,,$$

where $\delta_1 + \delta_2 + \delta_3 = D$. We shall not write down the final calculations because they lead to rather cumbersome expressions.

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