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# Quantum Equivalence of Dual Field Theories

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Motivated by the study of ultraviolet properties of different versions of supergravities duality transformations at the quantum level are discussed. Using the background field method it is proven on shell quantum equivalence for several pairs of dual field theories known to be classically equivalent. The examples considered include duality in chiral model, duality of scalars and second rank antisymmetric gauge tensors, vector duality and duality of the Einstein theory with cosmological term and the Eddington–Schrödinger theory. © 1985 Academic Press, Inc.

## 1. INTRODUCTION

There are examples of field theories having different lagrangians and variables but possessing equivalent equations of motion. If a transformation between these theories can be carried out through the use of the first order lagrangian it is called a dual transformation. For instance, a free scalar is dual (in $d=4$) to a gauge antisymmetric tensor, an abelian vector is dual to a vector, etc. The use of such duality transformations for construction of versions of supergravities, which are different off shell but have equivalent equations of motion, is well known in $d=4$ [1–3] and in $d>4$ [4]. An important issue is quantum equivalence of these dual versions. It was already noted [5, 6] that dual field representations yield different "topological" anomalies. Leaving aside this "topological" difference (which is absent under certain boundary conditions) one can still raise the following question: Is it possible that one of the dual theories has better ultraviolet properties than the other?

Recently [7] we have carried out the explicit calculation of infinities in a $d=4$ theory, which follows via "trivial" reduction from $N=1$, $d=10$ supergravity [8], and have shown that this theory is one-loop infinite as is its dual, namely, $N=4$ supergravity interacting with six $N=4$ vector multiplets [9]. This suggests a negative answer to the above question. The aim of the present paper is to support the negative answer, proving the on shell quantum equivalence of dual theories. Thus no *principal* advantages can be gained by going from one dual version to another at the quantum level. It should be noted here that, asserting the equivalence, we are interested mainly in infinities of $S$-matrix and thus ignore the



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already-mentioned differences in anomalies [5] and also local anomalous finite terms in some amplitudes induced by point transformations on spinors which may accompany duality transformations in supersymmetrical theories [10].

To construct a dual theory one rewrites the initial lagrangian $\mathscr{L}(\varphi)$ in the first order form (linear in derivatives) using the auxiliary (Lagrange multiplier) field $\phi$: $\mathscr{L}_1(\varphi, \phi)$. Eliminating the original variable $\varphi$ on its equation of motion one finds[1] some constraints on $\phi$. Solving the latter $\phi = \phi(\tilde{\varphi})$ we end up with a dual theory $\tilde{\mathscr{L}}(\tilde{\varphi})$, which is classically equivalent to the original one. However, there exist theories which are equivalent on the classical equations but are not related through duality transformation (we shall call them "pseudodual"). For example, it is sometimes possible to rewrite the equations of motion for $\varphi$ in terms of some new set of variables $\varphi'$ and then to restore the corresponding lagrangian $\mathscr{L}'(\varphi')$. Such pseudodual transformations include, in particular, the reparametrizations $\varphi' = F(\varphi)$. With $F$ being nonlocal the equivalence theorem cannot be applied, and thus pseudodual theories may have different $S$-matrices. The point that distinguishes true dual theories is the possibility to relate their path integrals by the chain of formal functional transformations (insertions of $\delta$-functions and gaussian integrations). It is this relation (absent for pseudodual models) that is the main reason for on shell quantum equivalence of the dual theories.

Let us now explain what is understood by "on shell equivalence." One certainly cannot equate Green's functions for dual variables (though correspondences between averages of some composite operators may exist). It is possible, however, to identify the on shell background functionals[2] [11], expressed in terms of the on shell values of all other fields $\psi$ in the theory (with "dual" fields $\varphi$ and $\tilde{\varphi}$ being expressed in terms of $\psi$ using their equations of motion). More explicitly, if

$$e^{-W[\bar{\varphi},\bar{\psi}]} = \int d\varphi \, d\psi \, \exp\left\{ -S(\bar{\varphi}+\varphi, \bar{\psi}+\psi) + \frac{\delta S}{\delta \bar{\varphi}} \varphi + \left(\frac{\delta S}{\delta \bar{\psi}}\right) \psi \right\} \tag{1.1}$$

and $\tilde{W}$ is the analog of $W$ for $S$ substituted by $\tilde{S}$, then[3]

$$W[\varphi_0(\psi_0), \psi_0] = \tilde{W}[\tilde{\varphi}_0(\psi_0), \psi_0], \tag{1.2}$$

where $\varphi_0$, $\tilde{\varphi}_0$ and $\psi_0$ are solutions of the corresponding classical equations. Equation (1.2) is sufficient for the equivalence of the $\psi$-sector $S$-matrices (and hence, in supersymmetrical case, of the complete $S$-matrices), equality of $\beta$-functions, etc. Note that if we are not assuming $\psi$ lying on its mass shell then $W$ and $\tilde{W}$ are gauge dependent and (1.2) in general breaks down.

[1] Note that dual theory is local only if this elimination can be carried out algebraically.

[2] Or "reducible" effective actions, generating reducible connected graphs. Note that at one loop $W$ coincides with the "irreducible" effective action $\Gamma$, defined in [12].

[3] If $S$ is independent of $\psi$ the analog of (1.2) is $W[\varphi_0] = \tilde{W}[\tilde{\varphi}_0(\varphi_0)]$.

To give a heuristic argument for the validity of (1.2) let us assume that the dual theories have equal vacuum (or temperature) partition functions, i.e.,

$$\int d\varphi\, d\psi \exp(-S(\varphi, \psi)) = \int d\tilde{\varphi}\, d\psi \exp(-\tilde{S}(\tilde{\varphi}, \psi)). \quad (1.3)$$

This assumption can be justified by translating all steps of the dual transformation into the path integral language. Then the same functional transformations which prove (1.3) can be used to establish the relation

$$\int d\varphi\, d\psi \exp(-S(\varphi + \varphi_0, \psi + \psi_0)) = \int d\tilde{\varphi}\, d\psi \exp(-\tilde{S}(\tilde{\varphi} + \tilde{\varphi}_0, \psi + \psi_0)), \quad (1.4)$$

where $\varphi_0$, $\tilde{\varphi}_0$, $\psi_0$ again are on shell values. The validity of (1.4) is due to translation invariance of the functional integrals involved in the quantum dual transformation. Now we observe that (1.4) simply coincides with (1.2).

Below we shall justify Eqs. (1.3) and (1.2) for several examples of dual models. We start (Section 2) with a discussion of the dual transformation in the principal chiral model. Employing the parametrization in terms of currents we find that the theory of [13] is a pseudodual one, construct (for $d = 2$) the true dual model along the lines of [14] and prove the quantum equivalence (1.2) of the latter with the original chiral model. The statement of equivalence provides a general explanation to the observation [14], that both dual theories have equal one-loop $\beta$-functions. We also consider the dual transformation for $d > 2$, which leads to a theory of interacting antisymmetric tensors.

In Section 3 we investigate the duality between a scalar $\varphi$ and a gauge antisymmetric tensor $A_{\mu\nu}$, which is important for supergravity. We first check the equivalence of the two quantum theories in external gravitational, vector, etc., fields[4] and then generalize the proof to the case when all fields are quantized. Again our statement of equivalence (1.2) explains the result of [19], where it was found that the one loop on shell infinities in "gravity plus $\varphi$" and "gravity plus $A_{\mu\nu}$" systems are the same. We also remark that (contrary to some claims [20]) no essential complication in quantization of the dual $A_{\mu\nu}$-theory arises in the case of the scalar–vector coupling through the Chern–Simons current.[5]

Two more examples, namely, duality transformations for (abelian and non-abelian) gauge vectors and for Einstein gravity with $\Lambda$-term are analysed in Section 4. The Einstein theory is dual to the Eddington–Schrödinger theory [23] with dual variable being symmetrical connection. The quantum on shell equivalence of

[4] It should be noted here that the correct covariant quantization of the free antisymmetric tensor gauge theory (as a theory with the reducible closed gauge algebra) on a gravitational background was first given in [15] and later in [16]. The general quantization schemes for theories with reducible open (or closed) gauge algebras were worked out in [17, 18], correspondingly, in the lagrangian and hamiltonian frameworks.

[5] As was noted in [21] such coupling appears in gauged $SU(2) \times SU(2)$ $N = 4$ supergravity of [22]. Analogous couplings are present also in $d > 4$ supergravities, e.g., in $d = 10$ [4].

these theories implies that the latter is non-renormalizable, as is the former [24] (cf. [25]).

The practical conclusion of this paper is that it is not possible to improve non-trivially ultraviolet behaviour of the theory by changing one dual model for another. This does not, of course, exclude the importance of dual transformations in some technical questions, e.g., in study of different coupling regimes (cf. [26–29]). Also, the cancellation of anomalies in certain dual versions [3, 6] may be important in some "off shell" problems [21].

## 2. Duality Transformations for Chiral Model

We shall start with the principal chiral model langrangian

$$\mathscr{L}=\frac{1}{2\lambda^2}\operatorname{tr}(\partial_\mu U\,\partial_\mu U^{-1}), \tag{2.1}$$

where $U=e^{\varphi}\in G$ and $G$ is some compact group with the algebra $g$ ($\varphi\in g$). The corresponding classical equations

$$\partial_\mu(U\,\partial_\mu U^{-1})=0 \tag{2.2}$$

can be rewritten as a set of two equations for the current

$$\partial_\mu J_\mu=0, \tag{2.3}$$

$$F_{\mu\nu}(J)\equiv\partial_\mu J_\nu-\partial_\nu J_\mu+[J_\mu, J_\nu]=0. \tag{2.4}$$

Solving (2.4) we get

$$J_\mu=U\,\partial_\mu U^{-1} \tag{2.5}$$

and thus establish the equivalence of (2.2) and (2.3). If, following [13], we solve first (2.3) (taking $d=2$)

$$J_\mu=\varepsilon_{\mu\nu}\,\partial_\nu\zeta, \qquad \zeta\in g, \tag{2.6}$$

and then substitute the result in (2.4) we find an equation for $\zeta$ which can be obtained from the lagrangian [13]

$$\mathscr{L}'=-\frac{1}{2\lambda'^2}\operatorname{tr}\left(\partial_\mu\zeta\,\partial_\mu\zeta+\frac{1}{3}\varepsilon_{\mu\nu}\zeta\,\partial_\mu\zeta\,\partial_\nu\zeta\right). \tag{2.7}$$

As was shown in [13], the quantum theory, based on (2.7), is inequivalent to the initial one, corresponding to (2.1): one loop $\beta$-functions for $\lambda$ and $\lambda'$ have opposite signs and there is particle creation in the theory (2.7), while it is absent for (2.1). Thus (2.7) is "pseudodual" to (2.1).

The reason for inequivalence lies in the impossibility to carry out the transformation from (2.1) to (2.7) at the quantum level. To see that Eq. (2.3) never arises in the chiral model quantum theory (and thus $\zeta$ cannot be introduced) let us rewrite the formal path integral for (2.1) in terms of the current

$$\int d\varphi \exp\left\{-\frac{1}{2\lambda^2}\int d^dx \operatorname{tr}(\partial_\mu e^\varphi \partial_\mu e^{-\varphi})\right\} \tag{2.8}$$

$$= \int dJ_\mu \, d\varphi \, \delta(J_\mu - e^\varphi \partial_\mu e^{-\varphi}) \exp\left\{\frac{1}{2\lambda^2} \operatorname{tr} J_\mu^2\right\} \tag{2.9}$$

$$= C \int dJ_\mu \, \delta(\varepsilon_{\nu\lambda} F_{\nu\lambda}(J)) \exp\left\{\frac{1}{2\lambda^2} \operatorname{tr} J_\mu^2\right\} \tag{2.10}$$

(throughout this paper we neglect all local measure factors, irrelevant in dimensional regularization; also, we often do not indicate explicitly $\int d^dx$ in the exponents). In order to prove (2.10) one is to use the following general formula:

$$\int d^nx_i \, \delta(f_\alpha(x_i)) \, A(x)$$

$$= \int d^{n-m}y_\lambda [\det K_{\lambda\mu}(y)/\det M_{\alpha\beta}(y)]^{1/2} A(\bar{x}(y)) \tag{2.11}$$

where $i = 1, ..., n$, $\alpha = 1, ..., m$, $\lambda = 1, ..., n-m$, $f_\alpha(x) = 0$ has the solution $x^i = \bar{x}^i(y)$ (if there are several solutions, one is to sum over them in the r.h.s. of Eq. (2.11)) and

$$K_{\lambda\mu} = \frac{\partial \bar{x}^i}{\partial y^\lambda} \frac{\partial \bar{x}^i}{\partial y^\mu}, \qquad M_{\alpha\beta} = \left(\frac{\partial f_\alpha}{\partial x_i} \frac{\partial f_\beta}{\partial x_i}\right)_{x=\bar{x}(y)}. \tag{2.12}$$

In the case of our interest

$$x \to J_\mu, \qquad y \to \varphi, \qquad f_\alpha \to \varepsilon_{\mu\nu} F_{\mu\nu}(J), \qquad \bar{x} \to e^\varphi \partial_\mu e^{-\varphi}. \tag{2.13}$$

Taking into account the "pure gauge" nature of $J_\mu$, one finds ($a, b = 1, ..., \dim g$).

$$\det K_{\lambda\mu} \to \det(-\delta_{ab} \Box), \qquad \det M_{\alpha\beta} \to \det(-\delta_{ab} \Box) \tag{2.14}$$

and thus ($\det K/\det M) = 1$. This proves the equivalence of (2.9) and (2.10).

As was already claimed, Eq. (2.10) suggests no connection with the pseudodual model (2.7). Rather it makes it possible to define the true dual model for (2.1) (this was first noted in [14], where, however, the equivalence of (2.10) and (2.8) was not proved). Introducing an auxiliary field $\phi = \phi^a t^a$ ($t^a$ is a basis in the algebra $g$, $\operatorname{tr}(t_a t_b) = -2\delta_{ab}$, $[t_a, t_b] = f_{abc} t_c$) one can rewrite Eq. (2.10) as

$$Z = \int dJ_\mu^a \, d\phi^a \exp\left\{\frac{1}{\lambda^2} (i\phi^a \varepsilon_{\mu\nu} F_{\mu\nu}^a(J) - (J_\mu^a)^2)\right\}. \tag{2.15}$$

Next, one is to note that the (gaussian) integration over $J_\mu$ is easy to carry out exactly with the result

$$Z = \int d\phi \exp(-\tilde{S}(\phi)), \tag{2.16}$$

$$\tilde{\mathscr{L}} = \frac{1}{\lambda^2} (\varepsilon_{\mu\rho} \partial_\rho \phi^a) \, N^{-1}{}^{ab}_{\mu\nu}(\phi)(\varepsilon_{\nu\delta} \partial_\delta \phi^b),$$

$$N^{ab}_{\mu\nu} = \delta^{ab} \delta_{\mu\nu} - i f^{abc} \phi^c \varepsilon_{\mu\nu}, \qquad N \cdot N^{-1} = \mathbb{1}. \tag{2.17}$$

More explicitly, $N^{-1}{}_{\mu\nu} = C\delta_{\mu\nu} + i\varepsilon_{\mu\nu} \mathscr{D}$, $C^{ab} = \delta^{ab} + \cdots$, $\mathscr{D}^{ab} = -f^{abc}\phi^c + \cdots$, we get

$$\tilde{\mathscr{L}} = \frac{1}{\lambda^2} \{ C^{ab}(\phi) \, \partial_\mu \phi^a \, \partial_\mu \phi^b + i\varepsilon_{\mu\nu} \mathscr{D}^{ab}(\phi) \, \partial_\mu \phi^a \, \partial_\nu \phi^b \}. \tag{2.18}$$

The classical equivalence of (2.1) and (2.18) follows from the observation that the action in (2.15) is the first order action for (2.18) and at the same time, when calculated on shell ($J_\mu = U\partial_\mu U^{-1}$), coincides with the chiral model action ($\hat{S}(\phi, J)_{J(\varphi)} = S(\varphi)$, $\hat{S}(\phi, J)_{J(\phi)} = \tilde{S}(\phi)$). Varying the former with respect to $\phi$ and $J$ one finds

$$\varepsilon_{\mu\nu} F_{\mu\nu}(J) = 0, \qquad J_\mu = e^\varphi \, \partial_\mu \, e^{-\varphi}, \tag{2.19}$$

$$J^a_\mu = i\varepsilon_{\mu\nu} \mathscr{D}^{ab}_\nu(J) \phi^b \qquad \text{or} \qquad J^a_\mu = iN^{-1}{}^{ab}_{\mu\nu}(\phi) \, \varepsilon_{\nu\rho} \, \partial_\rho \, \phi^b, \tag{2.20}$$

where $\mathscr{D}^{ab}_\nu = \delta^{ab} \partial_\nu + f^{acb} J^c_\nu$. Equation (2.20) defines the classical correspondence between the dual variables $\varphi^a$ and $\phi^a$. Being nonlocal (like the one given by (2.6)), this transformation (in contrast to $\varphi \to \zeta$) can be carried out in the path integral.

Now let us prove the quantum equivalence of (2.1) and (2.18). As was aleady stressed in the Introduction, it is inadequate to try to compare the standard Green's functions for $\varphi$ and $\phi$. In fact, the dual transformation is impossible if one adds a source term $ij\varphi$ in the exponent in (2.8). If the source is introduced for the current, $ij_\mu(e^\varphi \, \partial_\mu \, e^{-\varphi})$, then the same chain of transformations which led from (2.8) to (2.16) gives

$$Z[j_\mu] = \int d\phi \, e^{-\tilde{S}(\phi, j)}, \qquad \tilde{S}(\phi, j) = \tilde{S}(\phi)|_{\partial_\mu\phi \to \partial_\mu\phi + (1/2)\lambda^2 \varepsilon_{\mu\nu} j_\nu}. \tag{2.21}$$

Thus the correlation functions for the currents in the $\varphi$-theory can be expressed in terms of expectation values of some non-linear composite operators in the $\phi$-theory. This off shell correspondence is obviously inconvenient. A more direct one, advocated in this paper, is possible if one goes on shell and compares the background functionals (1.1) in both theories. It is easy to prove the analog of (1.2), namely

$$W[\varphi_0] = \tilde{W}[\phi_0(\varphi_0)], \tag{2.22}$$

where $\varphi_0$ is a solution of (2.2) and $\phi_0$ is the corresponding solution of (2.19), (2.20). We have already proved this equality in the vacuum sector ($\varphi_0=0$). To switch on non-trivial backgrounds ($\varphi_0$ and $\phi_0$) one is to use the translational invariance ($\varphi \to \varphi+\varphi_0$, $\phi \to \phi+\phi_0$) of the integrals involved. Then all functional transformations go just in the same way as in the vacuum sector (one introduces the background values for the auxiliary fields $J_\mu$ and $\phi$, which for consistency must be taken to be their classicl values, etc). Equation (2.22) implies the identity of $\beta(\lambda)$-functions and correspondence of $S$-matrices in both dual theories. Thus any explicit checks of equivalence (cf. [14]) are now unnecessary.

We shall end this section with several remarks about generalizations of the duality transformation described above. Let us first discuss the generalization to higher dimensional ($d>2$) case. Solving (2.3),

$$J^a_\mu = \partial_\rho B^a_{\rho\mu}, \qquad B_{\mu\nu} = -B_{\nu\mu}, \tag{2.23}$$

and substituting this in (2.4)

$$\partial_\mu \partial_\rho B^a_{\rho\nu} - \partial_\nu \partial_\rho B^a_{\rho\mu} + f^{abc} \partial_\rho B^b_{\rho\mu} \partial_\sigma B^c_{\sigma\nu} = 0, \tag{2.24}$$

we get a pseudodual theory in terms of a set of antisymmetric tensors $B^a_{\mu\nu}$

$$\mathscr{L}' = \partial_\mu B^a_{\mu\nu} \partial_\rho B^a_{\rho\nu} + f^{abc}(B^a_{\mu\nu} \partial_\rho B^b_{\rho\mu} \partial_\sigma B^c_{\sigma\nu} + \text{terms of analogous structure}). \tag{2.25}$$

Note that (2.24) is invariant under

$$\delta B^a_{\mu\nu} = \partial_\rho \Lambda^a_{\rho\mu\nu}, \qquad \Lambda_{\rho\mu\nu} = \Lambda_{[\rho\mu\nu]}, \tag{2.26}$$

while (2.25) is not. This is an interesting example of a theory with gauge-invariant equations of motion but non-invariant lagrangian. Note that it is the interaction term in (2.25) that is not invariant under (2.26).[6] This may imply inconsistency of the theory at the quantum level (even if we neglect its power counting non-renormalizability).

To find the dual theory we start again with (2.8) and repeat the argument for equivalence of (2.9) and (2.10) (now with $\delta(\varepsilon_{\mu\nu}F_{\mu\nu}) \to \prod_{\mu<\nu} \delta(F_{\mu\nu})$). The simplest reasoning for the triviality of the jacobian factor $(\det K/\det M)^{1/2}$ (see (2.11)–(2.14)) is based on the invariance of the measure $dJ_\mu \prod_{\mu<\nu} \delta(F_{\mu\nu}(J))$ under the gauge transformations of $J_\mu$, implying that no non-trivial ($\partial_\mu \varphi$-dependent) terms can appear in (2.9) if one starts with (2.10). It should be noted that the whole argument is at least formal since the number of $\delta$-functions in (2.10) is greater than the number of $J_\mu$-integrals for $d>3$. Now instead of (2.15) we get

$$Z = \int dJ^a_\mu \, dA^a_{\mu\nu} \exp\left\{\frac{1}{\lambda^2}\left[iA^a_{\mu\nu}F^a_{\mu\nu}(J) - (J^a_\mu)^2\right]\right\}. \tag{2.27}$$

[6] This is easy to understand from the absence of the corresponding Noether identity, which would follow from (2.24) if (2.25) were invariant.

Integrating out $J^a_\mu$ we are left with the second order dual lagrangian (cf. (2.16)) with the dual variables being antisymmetric tensors $A^a_{\mu\nu}$

$$\tilde{\mathscr{L}} = \frac{1}{\lambda^2} \{\partial_\rho A^a_{\rho\mu} N^{-1\,ab}_{\mu\nu}(A)\, \partial_\sigma A^b_{\sigma\nu}\},$$
$$N^{ab}_{\mu\nu} = \delta^{ab}\, \delta_{\mu\nu} - if^{abc} A^c_{\mu\nu}. \tag{2.28}$$

The classical equivalence of this model and the principal chiral model (2.1) was first demonstrated in [30]. Here we assert that the equivalence (cf. (2.22)) holds also at the quantum level. It is interesting to note that because of $\mathscr{D}_{[\mu} F_{\rho\delta]} = 0$ Eq. (2.27) possesses a new gauge invariance absent in $d=2$ case (cf. (2.26))

$$\delta A^a_{\mu\nu} = \mathscr{D}^{ab}_\rho(J)\, \Lambda^b_{\rho\mu\nu}. \tag{2.29}$$

It leads to a non-linear gauge invariance of (2.28), namely, (2.29) with $J^a_\mu \to iN^{-1\,ab}_{\mu\nu}\, \partial_\rho A^b_{\nu\rho}$. However, the corresponding algebra closes only on the mass shell [30], producing complications in straightforward quantization of the theory (cf. [30, 31]). These complications are probably irrelevant if we are interested not in the Green's functions for $A_{\mu\nu}$ but only in on shell background functional (which is gauge independent).

Our next remark is about generalization on the $G/H$-chiral model case. Let $h$ be an invariant subalgebra of $g$ and $H_\mu \in h$. Then instead of (2.27) we have

$$Z = \int dJ^a_\mu\, dH^i_\mu \exp\left\{\frac{1}{\lambda^2} [iA^a_{\mu\nu} F^a_{\mu\nu}(J) - (J^i_\mu + H^i_\mu)^2 - (J^\alpha_\mu)^2]\right\}, \tag{2.30}$$

where $i = 1,\ldots, \dim h$, $\alpha = 1,\ldots, \dim g - \dim h$ (we have split the basis $t_a$ on $t^i$ and $t^\alpha$). Integrations over $H^i_\mu$ and $J^a_\mu$ are easily carried out. The resulting dual action depends on $A^a_{\mu\nu}$ and is analogous to (2.28). It generates the quantum theory, which is on shell equivalent to the $G/H$-chiral model theory.

## 3. Duality Transformations for Antisymmetric Gauge Tensors

Here for simplicity we shall consider only the example of scalar–second rank antisymmetric tensor duality in $d=4$ (generalizations to other cases are straightforward).

The construction of classically equivalent dual model is based on the first order formulation. Suppose that the lagrangian for the scalar field $\varphi$ depends only on its derivatives, $\mathscr{L}(\partial_\mu \varphi, \psi)$ ($\psi$ stands for all other fields). Then introducing two auxiliary fields we have (the imaginary unit is inserted in order to keep the analogy with quantum path integral case)

$$\mathscr{L}_1 = \mathscr{L}(f_\mu, \psi) + i\lambda_\mu(f_\mu - \partial_\mu \varphi). \tag{3.1}$$

Eliminating $\varphi$ on its classical equations

$$\partial_\mu \lambda_\mu = 0, \qquad \lambda_\mu = \varepsilon_{\mu\alpha\beta\gamma}\, \partial_\alpha A_{\beta\gamma}, \tag{3.2}$$

we get

$$\tilde{\mathscr{L}}_1 = \mathscr{L}(f_\mu, \psi) + i\varepsilon_{\mu\alpha\beta\gamma}\, \partial_\alpha A_{\beta\gamma} f_\mu. \tag{3.3}$$

"Integrating" over $f_\mu$ (substituting its classical value) we finish with the dual lagrangian

$$\tilde{\mathscr{L}} = \tilde{\mathscr{L}}(F_{\mu\nu\rho}, \psi), \qquad F_{\mu\nu\rho} = \partial_{[\mu} A_{\nu\rho]}. \tag{3.4}$$

Note that in order for $\tilde{\mathscr{L}}$ to be local, $\mathscr{L}(f_\mu, \psi)$ must be an algebraic function of $f_\mu$, i.e., $\mathscr{L}$ must depend only on first derivatives of $\varphi$.

Let us now specialize the form of $\mathscr{L}$ to

$$\mathscr{L} = [\tfrac{1}{2}(\partial_\mu \varphi)^2 + \partial_\mu \varphi J_\mu(\psi)]\, \sqrt{g}. \tag{3.5}$$

Such a (pseudo) scalar lagrangian is found, e.g., in $SU(4)$ (or gauged $SU(2) \times SU(2)$) supergravity [1, 22] (any additional factors in the kinetic term can be absorbed in a redefinition of the metric). Here $J_\mu$ depends on all other fields in the theory, e.g., vectors, spinors, etc. Following steps (3.1)–(3.3) we get the dual lagrangian [3]

$$\mathscr{L}_1 = [\tfrac{1}{2} f_\mu^2 + f_\mu J_\mu + i\lambda_\mu(f_\mu - \partial_\mu \varphi)]\, \sqrt{g}, \tag{3.6}$$

$$\tilde{\mathscr{L}} = \tfrac{1}{2}(F_\mu^* - iJ_\mu)^2 \sqrt{g}, \qquad F_\mu^* \equiv \varepsilon_{\mu\alpha\beta\gamma}\, \partial_\alpha A_{\beta\gamma}. \tag{3.7}$$

An apparent paradox with (3.7) is that it depends on the full "current" $J_\mu$, while (3.5) depends only on its longitudinal part

$$J_\mu^{\|} = \nabla_\mu \nabla^{-2} \nabla_\nu J_\nu \equiv J_\mu - J_\mu^{\perp}, \qquad \nabla_\mu = \partial_\mu + \left\{ {\lambda \atop \rho\mu} \right\}.$$

The resolution of this "paradox" is provided by the observation that in view of the trivial invariance of (3.6) under $\delta J_\mu = a_\mu^{\perp}$, $\delta\lambda_\mu = ia_\mu^{\perp}$, we can always rewrite (3.7) as

$$\tilde{\mathscr{L}} = \tfrac{1}{2}(\bar{F}_\mu^* - iJ_\mu^{\|})^2 \sqrt{g}, \tag{3.8}$$

$$\bar{A}_{\mu\nu} = A_{\mu\nu} + a_{\mu\nu}(J), \qquad J_\mu^{\perp} \equiv \varepsilon_{\mu\alpha\beta\gamma} \nabla_\alpha a_{\beta\gamma}.$$

Thus any of the equivalent models (3.7) and (3.8) can be called dual to (3.5). The only advantage of (3.7) is its explicit locality. At the same time, Eq. (3.8) can be simplified to

$$\tilde{\mathscr{L}} = [\tfrac{1}{2}(\bar{F}^*_\mu)^2 - \tfrac{1}{2}(J^{||}_\mu)^2]\sqrt{g}. \tag{3.9}$$

The analogous factorized form of (3.5) is

$$\begin{aligned} \mathscr{L} &= [\tfrac{1}{2}(\partial_\mu \bar{\varphi})^2 - \tfrac{1}{2}(J^{||}_\mu)^2]\sqrt{g}, \\ \bar{\varphi} &= \varphi + \nabla^{-2}\nabla_\nu J_\nu. \end{aligned} \tag{3.10}$$

The above observation also helps to clarify the following point. If $\varphi$ interacts with a gauge vector through the $\varphi F_{\mu\nu} F^*_{\mu\nu}$-term, then $J_\mu$ in (3.5) contains the Chern–Simons current

$$J_\mu = i\varepsilon_{\mu\alpha\beta\gamma}(A^a_\alpha \partial_\beta A^a_\gamma + \tfrac{1}{3} f^{abc} A^a_\alpha A^b_\beta A^c_\gamma), \tag{3.11}$$

$$\nabla_\mu J_\mu = \frac{i}{2} F^a_{\mu\nu} F^{*a}_{\mu\nu}, \qquad J^{||}_\mu = \frac{i}{2} \nabla_\mu \nabla^{-2}(F^a_{\mu\nu} F^{*a}_{\mu\nu}). \tag{3.12}$$

It is $J^{||}_\mu$ that is invariant under vector gauge transformations. Hence the lagrangian (3.8) or (3.9) is explicitly gauge invariant under the independent transformations of $A^a_\mu$ and $\bar{A}_{\mu\nu}$

$$\delta A^a_\mu = \mathscr{D}^{ab}_\mu(A_\lambda)\,\varepsilon^b, \qquad \delta \bar{A}_{\mu\nu} = 0, \tag{3.13}$$

$$\delta \bar{A}_{\mu\nu} = \nabla_{[\mu} \Lambda_{\nu]}, \qquad \delta A^a_\mu = 0. \tag{3.14}$$

To provide the invariance of (3.7) one is to note that under (3.13) $\delta J_\mu = a^\perp_\mu$, $a^\perp_\mu = \varepsilon_{\mu\alpha\beta\gamma}\, \partial_\alpha \Omega_{\beta\gamma}$ and thus non-invariance of $J_\mu$ can be compensated by the additional transformation of $A_{\mu\nu}$

$$\delta A_{\mu\nu} = i\Omega_{\mu\nu}, \qquad \Omega_{\mu\nu} = 2iF^a_{\mu\nu}\varepsilon^a. \tag{3.15}$$

Note that $\bar{A}_{\mu\nu}$ is invariant under (3.15) and thus is a proper gauge invariant variable. Using (3.8) or (3.9) we never confront the problem of the modification (3.15) of the gauge algebra and thus anticipate no specific problems in quantization of the theory. It should be stressed that the possibility of introducing the gauge invariant variable here is due to a trivial (bilinear) dependence of the action on $A_{\mu\nu}$. Thus the situation is similar to that in the theory of quantum abelian vector in external gravitational field where one can easily split the variables on the "longitudinal" (which drops from the action) and gauge invariant "transverse" ones. Just the same separation of variables applies to (3.7).[7] Having said all this, in what

[7] That is why the complicated quantization scheme, developed for (3.7) in Ref. [20], seems to be unnecessary and artificial.

follows we shall not specify the particular structure of $J_\mu$ in (3.5) since it is irrelevant for the proof of quantum equivalence of dual models.

Before turning to the quantum theory we want to add a remark on the classical equivalence of (3.5) and (3.7). Writing the field equations for (3.6)

$$f_\mu + J_\mu + i\lambda_\mu = 0, \qquad f_\mu - \partial_\mu \varphi = 0, \qquad \nabla_\mu \lambda_\mu = 0, \tag{3.16}$$

we find the on shell "interpolation relation" between the dual variables $\varphi$ and $A_{\mu\nu}$

$$\partial_\mu \varphi + J_\mu + i\varepsilon_{\mu\alpha\beta\gamma}\,\partial_\alpha A_{\beta\gamma} = 0. \tag{3.17}$$

The two consequences of (3.17)

$$\nabla^2\varphi + \nabla_\mu J_\mu = 0, \qquad \varepsilon_{\mu\nu\rho\sigma}\nabla_\rho(F^*_\sigma - iJ_\sigma) = 0, \tag{3.18}$$

are recognized as the classical equations for (3.5) and (3.7). It should be understood that (3.17) is senseless off the mass shell defined by (3.18) (in fact, it states that an exact one-form is equal to a closed one, but such an equality is non-trivial only for harmonic forms).

Let us now discuss the quantum equivalence of the theories (3.5) and (3.7), starting with the simplest case when gravity and fields $\psi$ in $J_\mu$ are external. Thus we are to compare the "partition functions"

$$Z[g, J] = \int d\varphi\, \exp\{-[\tfrac{1}{2}(\partial_\mu\varphi)^2 + \partial_\mu\varphi J_\mu]\sqrt{g}\}, \tag{3.19}$$

$$\tilde{Z}[g, J] = \int dA_{\mu\nu}\, \exp\{-\tfrac{1}{2}(\varepsilon_{\mu\alpha\beta\gamma}\,\partial_\alpha A_{\beta\gamma} - iJ_\mu)^2\sqrt{g}\}. \tag{3.20}$$

Integration over $\varphi$ in (3.19) gives

$$Z = \exp[\tfrac{1}{2}(J^{\parallel}_\mu)^2\sqrt{g}](\det \Delta_0)^{-1/2}, \qquad \Delta_0 = -\nabla^2. \tag{3.21}$$

To integrate over $A_{\mu\nu}$ in (3.20) it is not necessary to use the general procedures for quantization of gauge antisymmetric tensors (cf. [15–18]). It is sufficient to split $A_{\mu\nu}$ on gauge invariant and pure gauge parts with respect to the background metric

$$A_{\mu\nu} = A^{(H)}_{\mu\nu} + A^{\perp}_{\mu\nu} + \nabla_\mu\xi^{\perp}_\nu - \nabla_\nu\xi^{\perp}_\mu, \qquad \nabla_\mu A^{\perp}_{\mu\nu} = 0 \tag{3.22}$$

($A^{(H)}$ is a harmonic two-form; we assume the space-time to be compact), to integrate over $A^{\perp}$ and to *define* the integral over pure gauge variables. The corresponding splitting of the measure is

$$dA = dA^{(H)}\, dA^{\perp}\, d\xi^{\perp}(\det{}' \Delta_{1\perp})^{1/2}, \qquad \Delta_{1\mu\nu} = -\nabla^2_{\mu\nu} + R_{\mu\nu} \tag{3.23}$$

(the prime means that zero modes are omitted). Then (cf. (3.8), (3.9))

$$\tilde{Z} = (\det' \Delta_{1\perp})^{1/2} \int dA^{(H)}\, d\xi^{\perp}\, dA^{\perp} \times \exp\{\tfrac{1}{2}(J^{\|}_{\mu})^2 - \tfrac{1}{2}(\varepsilon_{\mu\alpha\beta\gamma}\partial_{\alpha}\bar{A}^{\perp}_{\beta\gamma})^2\}\sqrt{g}. \tag{3.24}$$

Shifting the variable $A^{\perp} \to \bar{A}^{\perp}$ (observing that $a^{\|}_{\mu\nu}$ drops from (3.8)) and integrating over $\bar{A}^{\perp}$ we get the additional factor

$$(\det' \Delta_{2\perp})^{-1/2}, \qquad \Delta_{2\,\alpha\beta}^{\,\mu\nu} = -\nabla^{2\,\mu\nu}_{\ \alpha\beta} + 2R^{\mu}_{[\alpha}\,\delta^{\nu}_{\beta]} - 2R^{\mu\nu}_{\alpha\beta}. \tag{3.25}$$

The integral $\int d\xi^{\perp}$ is defined using the relation

$$d\xi_{\mu} = d\xi^{(H)}\, d\xi^{\perp}\, d\phi (\det' \Delta_0)^{1/2}, \qquad \xi_{\mu} = \xi^{(H)}_{\mu} + \xi^{\perp}_{\mu} + \partial_{\mu}\phi,$$
$$\int d\xi^{\perp} \sim \left\{\int d\xi^{(H)} (\det' \Delta_0)^{1/2}\right\}^{-1} \times \text{"group volume."} \tag{3.26}$$

Then

$$\tilde{Z} = \{\exp[\tfrac{1}{2}(J^{\|}_{\mu})^2\sqrt{g}](\det \Delta_0)^{-1/2}\} \times \{[\det' \Delta_{1\perp}/\det' \Delta_{2\perp}]^{1/2}\} \times \left\{\int dA^{(H)}_{\mu\nu} \int d\phi^{(H)} \Big/ \int d\xi^{(H)}_{\mu}\right\}. \tag{3.27}$$

The first bracket here is identical to $Z$ in (3.21). It is easy to prove that the second bracket is equal to one (we substitute $A^{\perp}_{\mu\nu}$ by $\varepsilon_{\mu\nu\alpha\beta}\,\partial_{\alpha} B^{\perp}_{\beta}$ and check that $dA^{\perp}_{\mu\nu} = dB^{\perp}_{\mu}(\det \Delta_{1\perp})^{1/2}$, $A^{\perp}\Delta_2 A^{\perp} = B^{\perp}(\Delta_1)^2 B^{\perp}$. The third bracket is the contribution of harmonic zero modes which produce additional anomalous scale dependence of $\tilde{Z}$ (cf. [5])

$$\tilde{Z} \sim \mu^{\chi/2} Z, \qquad \chi = 2(b_2 - b_1 + b_0) \tag{3.28}$$

($\chi$ is the Euler number and $b_n$ are the Betti numbers). The conclusion is that (3.19) is equivalent to (3.20) up to topological anomaly (3.28) (which is absent for asymptotically flat boundary conditions).

The same conclusion can be reached by direct functional transformation of (3.19). Rewriting the action in the first order form and integrating over $\varphi$ one finds

$$Z = \int d\lambda_{\mu}\, \delta(\nabla_{\mu}\lambda_{\mu}) \exp\{-\tfrac{1}{2}(\lambda_{\mu} - iJ_{\mu})^2\sqrt{g}\}. \tag{3.29}$$

Splitting then $\lambda_{\mu}$ on harmonic, exact and coexact parts

$$\lambda_{\mu} = \lambda^{(H)}_{\mu} + \partial_{\mu}\phi + \varepsilon_{\mu\alpha\beta\gamma}\,\partial_{\alpha} A^{\perp}_{\beta\gamma} \tag{3.30}$$

we obtain

$$Z=\int d\lambda^{(H)}\exp\{-\tfrac{1}{2}(\lambda_\mu^{(H)}-iJ_\mu^{(H)})^2\sqrt{g}\}\int d\phi(\det{}'\Delta_0)^{1/2}\,\delta(\Delta_0\phi)$$

$$\times\int dA^\perp_{\mu\nu}(\det{}'\Delta_{2\perp})^{1/2}\exp\{-\tfrac{1}{2}(\varepsilon_{\mu\alpha\beta\gamma}\,\partial_\alpha A^\perp_{\beta\gamma}-iJ_\mu)^2\sqrt{g}\}. \qquad (3.31)$$

Integration over $\lambda^{(H)}$ and $\phi$ brings us back to (3.20), (3.23)[8] (with $d\xi^\perp\sim(\det{}'\Delta_0)^{-1/2}$) but now with the zero modes' contribution subtraced.

Now let us extend the proof of equivalence to the case when gravity is quantized (but $J_\mu=0$). According to (1.2) we are to prove that

$$W[g_0,\varphi_0(g_0)]=\tilde{W}[g_0,A_0(g_0)], \qquad (3.32)$$

where $g_{0\mu\nu}$, $\varphi_0$ and $A_{0\mu\nu}$ are solutions of the classical equations (here we neglect harmonic zero mode contributions assuming space-time to be topologically trivial). We have already discussed the case of $\varphi_0=0$ and $A_{0\mu\nu}=0$. To switch on the backgrounds one makes the shift $\varphi\to\varphi+\varphi_0$, $g\to g_0+h$, $A\to A+A_0$ and proceeds in the same manner as in Sections 1 and 2. Namely, under the on shell condition all terms linear in quantum fields cancel and thus simple transformations of path integrals go just as in the vacuum case (cf. (3.6), (3.29)). We have symbolically

$$e^{-W[g_0,\varphi_0]}=\int[dh]\,d\varphi\,\exp\{-S_G(g_0+h)-\tfrac{1}{2}(\partial_\mu\varphi+\partial_\mu\varphi_0)^2G(g_0+h)\}, \qquad (3.33)$$

where $S_G$ is an action for gravity and $G$ stands for $g^{\mu\nu}\sqrt{g}$. Inserting $1=\int df_\mu\,d\lambda_\mu\exp(i\lambda_\mu(f_\mu-\partial_\mu\varphi-\partial_\mu\varphi_0))$ and integrating over $\varphi$ one at the same time is to assume a non-trivial background for $\lambda_\mu$ (satisfying (3.17) in order not to spoil the consistency, i.e., the cancellation of all terms linear in quantum fields). Treating $g_0+h$ as an effective background metric, finally we are left with

$$e^{-W[g_0,\varphi_0]}=\int[dh]\,e^{-S_G(g_0+h)}\int[dA_{\mu\nu}]$$

$$\times\exp\{-\tfrac{1}{2}[\varepsilon_{\mu\alpha\beta\gamma}\,\partial_\alpha(A_{\beta\gamma}+A_{0\beta\gamma})]^2\,\tilde{G}(g_0+h)\}=e^{-\tilde{W}[g_0,A_0]}, \qquad (3.34)$$

where $\tilde{G}$ stands for the metric factor and $[dA]$ is a proper gauge fixing measure, depending on $g_0$ and $g_0+h$ (we assume that the gravitational gauge is independent of $A_{\mu\nu}$).[9] The correctness of all formal manupulations with the path integral can be directly checked in the one-loop approximation. Thus our result (3.32) provides the general explanation for the equivalence of on shell infinities for gravity plus scalar and gravity plus antisymmetric tensor theories, first found in explicit one-loop calculation in [19].

[8] We again make use of the relation $\det{}'\Delta_{1\perp}=\det{}'\Delta_{2\perp}$.

[9] This assumption is of purely technical nature since $W$ is gauge independent on shell.

If $J_\mu \neq 0$ all reasoning goes analogously (for simplicity hare we ignore gravity)

$$e^{-W[\varphi_0(\psi_0),\psi_0]} = \int d\psi\, d\varphi \exp\{-S_\psi(\psi+\psi_0) - \tfrac{1}{2}[(\partial_\mu \varphi + \partial_\mu \varphi_0)^2 + (\partial_\mu \varphi + \partial_\mu \varphi_0) J_\mu(\psi+\psi_0)]\}$$
$$= \int d\psi [dA_{\mu\nu}] \exp\{-S_\psi(\psi+\psi_0) - \tfrac{1}{2}[F^*_\mu(A+A_0) - iJ_\mu(\psi+\psi_0)]^2\}$$
$$= \exp\{-\tilde{W}[A_0(\psi_0), \psi_0]\}, \tag{3.35}$$

where $\varphi_0$ and $A_0$ satisfy (3.17), (3.18), while $\psi_0$ is a solution of the classical equation $\delta S_\psi/\delta\psi + \frac{1}{2}\partial_\mu \varphi_0\, \delta J_\mu/\delta\psi = 0$ ($S_\psi$ is an action for $\psi$).

Finally, let us comment on an alternative attempt to relate the generating functionals for Green's functions in both theories. Starting, for example, with

$$Z[j_\mu] = \int d\varphi\, d\psi \exp\{-S_\psi - [\tfrac{1}{2}(\partial_\mu \varphi)^2 + (J_\mu(\psi) + j_\mu)\, \partial_\mu \varphi]\} \tag{3.36}$$

and repeating the chain of functional transformations, we end with

$$Z[j_\mu] = \int d\psi \exp\{-S_\psi - \tfrac{1}{2}\partial_\mu(J_\mu(\psi) + j_\mu)\, \Box^{-1}\, \partial_\rho(J_\rho(\psi) + j_\rho)\}$$
$$= \int d\psi [dA_{\mu\nu}] \exp\{-S_\psi - \tfrac{1}{2}[\varepsilon_{\mu\alpha\beta\gamma}(\partial_\alpha A_{\beta\gamma}) - iJ_\mu - ij_\mu]^2\}. \tag{3.37}$$

Changing the variable $A \to \bar{A}$ as in (3.8), (3.9) we see that $j_\mu$ decouples from $\bar{A}_{\mu\nu}$, i.e., that $Z[j_\mu]$ does not generate Green's functions for $\bar{A}_\mu$ (cf. [32]). We conclude once more that the comparison of on shell background functionals seems the most natural way of demonstrating the equivalence of dual theories.

## 4. More Examples of Duality Transformations

We start with duality transformations for abelian vector theories. Let us consider a prototypical example of vector lagrangian which appears in extended supergravities

$$\mathscr{L} = \tfrac{1}{2}[f_1(\varphi)\, F_{\mu\nu} F_{\mu\nu} + if_2(\varphi)\, F_{\mu\nu} F^*_{\mu\nu}]. \tag{4.1}$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $F^*_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}$, and $f_1$ and $f_2$ are functions of the scalar field $\varphi$ (in a realistic context there are several vector and scalar fields, $f_i$ are matrices and there are also terms linear in $F_{\mu\nu}$ and bilinear in fermions). The first order lagrangian for (4.1) is

$$\mathscr{L}_1 = \tfrac{1}{2}[f_1 F^2_{\mu\nu} + if_2 F_{\mu\nu} F^*_{\mu\nu}] + iG^{\mu\nu}(F_{\mu\nu} - 2\partial_\mu A_\nu). \tag{4.2}$$

Integrating out $A_\mu$ we get a constraint

$$\partial_\mu G^{\mu\nu} = 0 \tag{4.3}$$

with a solution

$$G_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta} B_{\alpha\beta}, \qquad B_{\alpha\beta} = \partial_\alpha B_\beta - \partial_\beta B_\alpha. \tag{4.4}$$

Elimination of $F_{\mu\nu}$ then gives the lagrangian for the dual variable, i.e., abelian gauge vector $B_\mu$

$$\begin{gathered}\tilde{\mathscr{L}} = \tfrac{1}{2}[\tilde{f}_1 B_{\mu\nu}^2 + i\tilde{f}_2 B_{\mu\nu} B^*_{\mu\nu}],\\ \tilde{f}_1 = -\frac{f_1}{f_1^2 + f_2^2}, \qquad \tilde{f}_2 = \frac{f_2}{f_1^2 + f_2^2}.\end{gathered} \tag{4.5}$$

The quantum equivalence of (4.1) and (4.5) (understood again as in (1.2), i.e., $W[\varphi_0, A_\mu(\varphi_0)] = \tilde{W}[\varphi_0, B_\mu(\varphi_0)]$) can be demonstrated in complete analogy with the examples discussed above in Sections 3 and 4. As a by product, we conclude that inequivalence of the finite parts of $S$-matrices in $SO(4)$ and $SU(4)$ supergravities,[10] observed in [10], is completely due to anomalous redefinitions of spinors and not to the vector duality transformation. It is worth noting also that the abelian vector duality transformations were discussed at the classical level in Ref. [33], where a remark was made that these transformations should leave invariant the $S$-matrix because they leave invariant the equations of motion and the energy–momentum tensor (and hence the hamiltonian).

Next we consider the case of non-abelian vector gauge field. The analog of (4.2) now looks like

$$\begin{aligned}\mathscr{L}_1 = {}& \tfrac{1}{2}[f_1 F^a_{\mu\nu} F^a_{\mu\nu} + if_2 F^a_{\mu\nu} F^{*a}_{\mu\nu}]\\ &+ iA^a_{\mu\nu}(F^a_{\mu\nu} - 2\partial_\mu A^a_\nu - gf^{abc} A^b_\mu A^c_\nu).\end{aligned} \tag{4.6}$$

Integrating over $A^a_\mu$ (formally defining the corresponding gaussian path integral by an analytic continuation) we obtain the dual lagrangian in terms of the dual variable—now the antisymmetric tensor $A^a_{\mu\nu}$

$$\begin{aligned}\tilde{\mathscr{L}} = {}& \tfrac{1}{2}[\tilde{f}_1 A^a_{\mu\nu} A^a_{\mu\nu} + i\tilde{f}_2 A^a_{\mu\nu} A^{*a}_{\mu\nu}]\\ &+ \frac{i}{g}(\partial_\mu A^a_{\mu\rho})\, M^{-1\,ab}_{\rho\sigma}(\partial_\nu A^b_{\nu\sigma}),\\ M^{ab}_{\mu\nu} = {}& f^{abc} A^c_{\mu\nu}, \qquad M \cdot M^{-1} = 1,\end{aligned} \tag{4.7}$$

where $\tilde{f}_1$ and $\tilde{f}_2$ are the same as in (4.5). The limit $g \to 0$ brings us back to the abelian case with $A^a_{\mu\nu}$ having a potential. Analogous duality transformation for a

[10] Which differ on the vector duality transformation plus point transformation of spinors and scalars [1].

non-abelian gauge theory was considered several times in the literature [27–29]. Here we want to stress the on shell quantum equivalence of the dual theory (4.7) and the original Yang–Mills one (for example, both theories are to have the same $\beta$-functions for the gauge coupling $g$). It is also interesting to compare (4.7) with the lagrangian (2.28) of the theory, dual to the chiral model (2.1). The invariance of (4.7) which is the direct analog of (2.29) is generated by

$$\delta A^a_{\mu\nu} = (\delta^{ab}\,\partial_\rho - M^{-1}{}^{dc}_{\rho\sigma}(A)\, f^{adb}\,\partial_\nu A^c_{\nu\sigma})\, \Lambda^b_{\rho\mu\nu}. \tag{4.8}$$

This gauge algebra should be taken into account in an attempt to calculate Green's functions, corresponding to (4.7).

Our last example is provided by the Einstein gravity with a cosmological constant

$$\mathscr{L} = -\frac{1}{k^2}(R - 2\Lambda)\sqrt{g}, \tag{4.9}$$

which is classically equivalent to the Eddington–Schrödinger theory [23]

$$\begin{gathered} \tilde{\mathscr{L}} = 1/\lambda\,\sqrt{\det R_{\mu\nu}(\Gamma)}, \qquad \Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}, \\ R_{\mu\nu}(\Gamma) = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\rho\lambda}\Gamma^\rho_{\mu\nu} - \Gamma^\lambda_{\rho\nu}\Gamma^\rho_{\lambda\mu}. \end{gathered} \tag{4.10}$$

To demonstrate the classical equivalence of (4.9) and (4.10) we rewrite (4.9) in the first order form, introducing $R_{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$ as independent fields

$$\begin{aligned} \mathscr{L}_1 = &-\frac{1}{k^2}(g^{\mu\nu}R_{\mu\nu} - 2\Lambda)\sqrt{g} + iN^{\mu\nu}_\lambda\left(\Gamma^\lambda_{\mu\nu} - \begin{Bmatrix}\lambda \\ \mu\nu\end{Bmatrix}\right) \\ &+ iG^{\mu\nu}(R_{\mu\nu} - R_{\mu\nu}(\Gamma)) \end{aligned} \tag{4.11}$$

($N$ and $G$ are Lagrange multipliers). The corresponding equations of motion can be rewritten as

$$G^{\mu\nu} = -\frac{i}{k^2}\sqrt{g}\, g^{\mu\nu}, \qquad N^{\mu\nu}_\lambda = 0, \qquad \Gamma^\lambda_{\mu\nu} = \begin{Bmatrix}\lambda \\ \mu\nu\end{Bmatrix}, R_{\mu\nu} = R_{\mu\nu}(\Gamma), \tag{4.12}$$

$$R_{\mu\nu}(\Gamma) = \Lambda g_{\mu\nu}, \qquad \nabla_\lambda(\Gamma)\, g^{\mu\nu} = 0 \tag{4.13}$$

(note that $N = 0$ is the origin of the Palatini principle). Here we recognize (4.13) as the equations of motion for (4.10) written in the first order form (with $g_{\mu\nu}$ being auxiliary field). The corresponding first order lagrangian is a natural starting point in the proof of quantum equivalence

$$Z = \int dR\, dG\, d\Gamma \exp(-\tilde{S}_1), \tag{4.14}$$

$$\tilde{\mathscr{L}}_1 = \frac{1}{\lambda}\sqrt{\det R_{\mu\nu}} + iG^{\mu\nu}(R_{\mu\nu} - R_{\mu\nu}(\Gamma)), \tag{4.15}$$

Trading $G^{\mu\nu}$ for $g_{\mu\nu}$ by the point transformation

$$G^{\mu\nu} = -\frac{i}{k^2}\sqrt{g}\, g^{\mu\nu}, \qquad \lambda = -\Lambda k^2, \tag{4.16}$$

we identify the $G^{\mu\nu}R_{\mu\nu}(\Gamma)$ term in (4.15) as being the Einstein lagrangian in the first order (Palatini) form. Therefore, the result of the gaussian integration over $\Gamma^{\lambda}_{\mu\nu}$ is well known (see, e.g., [34]): it reproduces the Einstein action in the metric representation, $-(1/k^2)\int R_{\mu\nu}(\{\ \})\, g^{\mu\nu}\sqrt{g}\, d^4x$. To carry out the integration over $R_{\mu\nu}$ we expand it near the classical extremum

$$R_{\mu\nu} = R^{(0)}_{\mu\nu} + b_{\mu\nu}, \qquad R^{(0)}_{\mu\nu} = \Lambda g_{\mu\nu},$$

$$Z = \int dg_{\mu\nu} \exp\left\{-\frac{1}{k^2}\int (R(g) - 2\Lambda)\sqrt{g}\, d^4x\right\}$$
$$\times \int db_{\alpha\beta} \exp\left\{-\frac{1}{8\lambda}\int \sqrt{g}\, d^4x\, [(b^{\mu}_{\mu} b^{\rho}_{\rho} - 2b_{\mu\nu}b^{\mu\nu}) + O(b^3)]\right\}, \tag{4.17}$$

where the integral over $b_{\mu\nu}$ contributes only in a local measure. The proof of equivalence of the on shell background functionals

$$W[g_0] = \tilde{W}\left[\Gamma^{\lambda}_{\mu\nu} = \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}_{g_0}\right] \tag{4.18}$$

goes analogously. A natural consequence of on shell equivalence of (4.9) and (4.10) is non-renormalizability of the latter theory [24]. This conclusion contradicts the claim made in [25].

Finally, we note that analogous quantum equivalence considerations can be applied to various other kinds of dual theories (for example, to those recently discussed in [35]).

*Note added in proof.* Historically, the first time duality-chirality relations were noted in supergravity was in [36]. A lagrangian similar to (2.28) first appeared in [37]. Duality between scalar and antisymmetric tensor was first observed in [38]. Some interesting aspects of duality transformations were recently discussed in [39, 40].

## References


1. E. Cremmer, J. Scherk, and S. Ferrara, *Phys. Lett. B* **74** (1978), 61.
2. E. Cremmer and B. Julia, *Nucl. Phys. B* **159** (1979), 141.
3. H. Nicolai and P. K. Townsend, *Phys. Lett. B* **98** (1981), 257.
4. A. H. Chamseddine, *Phys. Rev. D* **24** (1981), 3065; E. Bergshoeff, M. de Roo, B. de Wit, and P. van Nieuwenhuizen, *Nucl. Phys. B* **195** (1982), 97; G. Chapline and N. Manton, *Phys. Lett. B* **120** (1982), 105.
5. M. J. Duff and P. van Nieuwenhuizen, *Phys. Lett. B* **94** (1980), 197.

6. M. J. DUFF, *in* "Superspace and Supergravity" (S. W. Hawking and M. Roček Eds.) Cambridge Univ. Press, New York/London, 1982.
7. E. S. FRADKIN AND A. A. TSEYTLIN, *Phys. Lett. B* **137** (1984), 357.
8. A. H. CHAMSEDDINE, *Nucl. Phys. B* **185** (1981), 403.
9. M. FISCHLER, *Phys. Rev. D* **20** (1979), 396.
10. M. T. GRISARU, *Phys. Lett. B* **79** (1978), 225.
11. B. S. DE WITT, "Dynamical Theory of Groups and Fields," Gordon & Breach, New York, 1965.
12. G. 'T HOOFT, *in* "Functional and Probabilistic Methods in Quantum Field Theory," Acta Univ. Wratisl. No. 368, 1976; B. S. DE WITT, *in* "*Quantum Gravity* II" (C. Isham, Ed.), Oxford Univ. Press, London/New York, 1982.
13. C. NAPPI, *Phys. Rev.* D **21** (1980), 418.
14. B. E. FRIDLING AND A. JEVICKI, *Phys. B* **134** (1984), 70.
15. A. S. SCHWARZ, *Lett. Math. Phys.* **2** (1978), 247; *Commun. Math. Phys.* **67** (1979), 1.
16. J. THIERRY–MIEG, Harvard preprint HUTMP-B86, 1979; W. SIEGEL, *Phys. Lett. B* **93** (1980), 170; T. KIMURA, *Prog. Theor. Phys.* **65** (1981), 338; U. N. OBUKHOV, *Phys. Lett. B* **109** (1982), 195.
17. I. A. BATALIN AND G. A. VILKOVISKY, *Phys. Lett.* B **120** (1983), 166.
18. I. A. BATALIN AND E. S. FRADKIN, *Phys. Lett. B* **122** (1983), 157.
19. E. SEZGIN AND P. VAN NIEUWENHUIZEN, *Phys. Rev. D* **22** (1980), 301.
20. L. BAULIEU, *Nucl. Phys. B* **227** (1983), 157.
21. E. S. FRADKIN AND A. A. TSEYTLIN, *Nucl. Phys. B* **234** (1984), 472.
22. D. Z. FREEDMAN AND J. H. SCHWARZ, *Nucl. Phys. B* **137** (1978), 333.
23. A. S. EDDINGTON, "The Mathematical Theory of Relativity," Cambridge Univ. Press, New York/London, 1924; E. SCHRODINGER, *Proc. Royal Irish Academy LI, Sect. A* (1947), 163.
24. A. A. TSEYTLIN, *Yad. Fiz.* **40** (1984), 1363.
25. M. MARTELLINI, *Phys. Rev. Lett.* **51** (1983), 152.
26. R. SAVIT, *Rev. Mod. Phys.* **52** (1980), 452.
27. I. A. BATALIN AND E. S. FRADKIN, *Ann. Phys.* (N.Y.) **83** (1974), 367.
28. M. B. HALPERN, *Phys. Rev. D* **19** (1979), 512.
29. K. SEO AND M. OKAWA, *Phys. Rev. D* **21** (1980), 1614; Y. KAZAMA AND R. SAVIT, *Phys. Rev. D* **21** (1980), 2916; L. DURAND AND M. MENDEL, *Phys. Rev. D* **26** (1982), 1368; W. MECKLENBURG AND L. MIZRACHI, *Phys. Rev. D* **27** (1983), 1922.
30. D. Z. FREEDMAN AND P. K. TOWNSEND, *Nucl. Phys. B* **177** (1981), 282.
31. L. BAULIEU AND J. THIERRY–MIEG, *Nucl. Phys. B* **228** (1983), 259.
32. W. MECKLENBURG AND L. MIZRACHI, *Phys. Rev. D* **29** (1984), 1709.
33. M. K. GAILLARD AND B. ZUMINO, *Nucl. Phys. B* **193** (1981), 221.
34. A. A. TSEYTLIN, *J. Phys. A* **15** (1982), L105.
35. W. SIEGEL, *Phys. Lett. B* **134** (1984), 318.
36. P. VAN NIEUWENHUIZEN AND J. A. VERMASEREN, *Phys. Lett. B* **65** (1976), 263.
37. S. DESER, *Phys. Rev.* **187** (1969), 1931.
38. V. I. OGIEVETSKY AND I. V. POLUBARINOV, *Yad. Fiz.* **4** (1968), 210.
39. P. K. TOWNSEND AND P. VAN NIEUWENHUIZEN, *Phys. Lett. B* **109** (1982), 179.
40. P. K. TOWNSEND, K. PILCH AND P. VAN NIEUWENHUIZEN, *Phys. Lett. B* **136** (1984), 38.