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Quantized String Models

E. S. FRADKIN AND A. A. TSEYTLIN

Department of Theoretical Physics, P. N. Lebedev Physical Institute, USSR Academy of Sciences, Leninsky Pr. 53, Moscow 117924, USSR

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We discuss and compare the Lorentz covariant path integral quantization of the three bose string models, namely, the Nambu, Eguchi and Brink–Di Vecchia–Howe–Polyakov (BDHP) ones. Along with a critical review of the subject with some uncertainties and ambiguities clearly stated, various new results are presented. We work out the form of the BDHP string ansatz for the Wilson average and prove a formal inequivalence of the exact Nambu and BDHP models for any space-time dimension d. The above three models, known to be equivalent on the classical level, are shown to be equivalent in a semiclassical approximation near a minimal surface and also in the leading $1/d$-approximation for the static $\bar{q}q$-potential. We analyse scattering amplitudes predicted by the BDHP string and find that when exactly calculated for $d < 26$ they are different from the old dual ones, and possess a non-linear spectrum which may be considered as free from tachyons in the ground state.

Contents.

1. INTRODUCTION

Strings were invented in an attempt to provide a dynamical background for various dual models (see, e.g., [1–3]) and were considered first as some phenomenological objects. The advent of QCD brought a qualitative picture of string built of glue (closed or with quarks at the ends). An approximate string description today is believed to be valid in the confinement phase of the theory. There were several proposals concerning particular mechanisms of emergence of one-dimensional string-like objects from QCD. More or less recent attempts can be roughly divided on those which do not appeal to the $1/N$ approximation and are based on conjecture that a free string ansatz is valid for the Wilson loop average [4] (with closed strings being elementary excitations in the confining phase) and those based on the large N limit (see, e.g., [5, 6] and especially [7, 8]). Migdal claimed in [7] that the presence of non-planar graphs in the $N = \infty$ limit [9] (i.e., "string instability"[2]) indicates that the "true" QCD-motivated string model is not a free Nambu one but carries self-interaction (e.g., is populated by fermions). However, it is not yet clear whether the

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Migdal string is a unique and physical solution of the large N QCD dynamics. As was pointed out in [8], the Migdal and the Nambu strings may be considered merely as providing two particular ansatz for the $N=\infty$ masterfield.

With this situation in mind it seems reasonable to study the properties of various possible string models (including, e.g., supersymmetric strings with spin [10–12] and spin and charge [13–15]), trying to establish their common features wich may be shared by the "true" QCD string.

The aim of this paper is to discuss the formulation and compare the properties of the three "phenomenological" bosonic string models: the Nambu [16], the generalized Eguchi [17–19] and the Brink–Di Vecchia–Howe–Polyakov (BDHP) [10, 20] ones. It is important to understand that these models are *different* quantum extensions of the same classical theory possibly lying in one equivalence class with respect to some (e.g., semiclassical) properties. Contrary to the statement of Ref. [20] we see no reasons to consider the quantized BDHP model as providing (a unique) correct quantum analog of the classical Nambu string for space-time dimension $d<26$ because one can formally write down the quantum theory functional integral using directly the Nambu (or Eguchi) action. At the same time, the quantum BDHP model seems to be the most simple and tractable one. For example, the well-known factor $(26-d)$ appears *naturally* in the covariant path integral treatment of the BDHP string [20]. Also, this model, using a metric in its formulation, seems to be well suited for a description of processes with changes of world surface topology of the string.

The remarkable feature of the BDHP model [20] is the appearance (through the Weyl symmetry anomaly) of a new quantum degree of freedom absent at the classical level. This resembles the conjectured occurence of the quantum longitudinal mode for the Nambu string for $d\neq 26$. Keeping in mind that the quantum BDHP model is not equivalent to the quantized Nambu string (as will be shown in the text) let us now recall a number of earlier attempts connected with the Nambu string quantization. In the work of Goddard (GGRT) *et al.* [21] it was proposed to use a canonical approach, choosing the Lorentz-non-covariant gauge (see also [1–3])

$$\bar{h}_{ab}\equiv h_{ab}-\tfrac{1}{2}\,\delta_{ab}\,(h_{cd}\,\delta^{cd})=0, \qquad h_{ab}\equiv\partial_a x^\mu\partial_b x^\mu, \qquad \mu=1,\dots,d;\quad a,b=1,2, \tag{1.1a}$$

$$x_+=\tau, \qquad x_\pm\equiv\frac{1}{\sqrt{2}}\,(x_1\pm x_d) \tag{1.1b}$$

(here $\{z^a\}=\{\tau,\sigma\}$) which *completely* fixes the reparametrization symmetry and reduces the 'square root" classical Nambu–Goto action to that of the free $(d-2)$ transverse degrees of freedom $\{x_i^\perp\}$. Then only these classically independent modes are quantized (by means of the operator approach [21] or with the help of the phase space path integral [22]). However, the resulting theory was found to be Lorentz-covariant (in d demensions) only if

$$d=26, \qquad \alpha(0)=-\alpha'\mu_0^2=1 \tag{1.2}$$

(this condition was also necessary to avoid troubles in the loop diagrams when working only with $x_i^{\perp}$ [22]). It was understood that one *can* in principle construct the ghost-free Lorentz covariant theory for $d < 26$ taking into account (in analogy with a dual model case) a quantum longitudinal mode x'' (or a corresponding set of operators) absent at the classical level [23, 3] (note that for $d > 26$ x'' is ghost-like). However, this possibility was not considered as a natural one ("to construct a quantum theory of a string in fewer than 26 dimensions we therefore require extra, non-classical degrees of freedom. Whether such a system should still be regarded as a string is a semantic point" [1, p. 320]) and it was the requirement that the quantized string should have the same number of degrees of freedom as the classical one that led to restriction (1.2). Let us note for completeness that proposals that longitudinal modes may be relevant in order to overcome (1.2) were discussed, e.g., in [24].

From today's point of view the fact that the classical and quantum numbers of degrees of freedom are unequal speaks about the presence of an *anomaly*, i.e., that some classical symmetry is not respected by quantization (i.e., by regularization or functional measure). Then (1.2) is merely the condition of the absence of the anomaly. Recalling that anomalies in general do not respect unitarity ("positivity of the effective action") we may even understand the origin of the no-ghost restriction $d < 26$ as a condition of the physical sign of the action for the anomalous degree of freedom.

The "commutator" anomalies were already found in earlier approaches [1–3] in the operator algebra framework. However, an understanding of anomaly and the role of a longitudinal mode from the Lorentz covariant path integral quantization was lacking. The difficulty is rooted (i) in the non-polinomiality of the Nambu action and (ii) in the necessity to preserve the d-dimensional $O(d)$ (Lorentz) symmetry while fixing the reparametrization invariance on the world surface. The last condition implies the use of *non-linear* gauges which, however, do not fix the coordinate symmetry completely (the conformal coordinate transformations are left unrestricted). One can observe that it is *impossible* to fix the coordinate group completely, preserving the $O(d)$-symmetry (indeed, we are to use the $O(d)$-invariant combination, e.g., $x_\mu x_\mu$, $x_\mu \partial_a x_\mu$, $\partial_a x_\mu \partial_b x_\mu$, etc., but the gauge $\partial_a x_\mu \partial_b x_\mu = \delta_{ab}$ is overcomplete, while (1.1a) incomplete). The preservation of the $O(d)$ and the conformal symmetry are thus connected.

The first attempt to work out the path integral quantization for the Nambu string was due to Gervais and Sakita [25]. They used the gauge (our notations are euclidean)[1]

$$h_{11} - h_{22} = 2ih_{12}, \qquad x_+ = f(z), \tag{1.3}$$

which linearizes the square root in the Nambu action, fixes the coordinate group completely but breaks the global $O(d)$ symmetry. The main advantage of (1.3) is the

[1] Note that the GGRT gauge (1.1a,b) consisting of *three* instead of *two* conditions formally cannot be used in the functional integral.

possibility to evaluate (after some formal manipulations) the partition function exactly (it turns to be equal to that of the $(d-2)$-free transverse modes). But it is not clear in this approach when (if at all) the theory is Lorentz-covariant. Choozing different (coordinate) gauges which *break* the additional global $(O(d))$ symmetry of the action one may obtain different results *if* there is *an anomaly of this global symmetry* (flat space "conformal" or "Lorentz" anomaly). If the gauge is classically complete, i.e., breaks also the anomalous symmetry, we may get a wrong answer (for example, if one tries to quantize the BDHP string in gauge (1.3) it is impossible to define the covariant path integral and to reveal the origin of the $(26-d)$-factor).

It is important to stress that if in the old-fashioned applications of string models (i.e., in derivation of dual amplitudes) one could in principle use non-Lorentz covariant gauges without contradiction with the Neumann boundary conditions on x_μ, in the gauge theory applications (i.e., dealing with a string ansatz for the Wilson loop average $W[c]$) we *must* preserve the $O(d)$-symmetry in view of the $O(d)$-covariant boundary conditions $x^\mu|_{\partial\mathscr{D}} = c^\mu \in C$. Also the general covariance of the formal path integral is needed in order to provide the contour C reparametrization invariance of $W[c]$. That is why we are obliged to use some incomplete gauge ((1.1a) or its metrical analog for the BDHP case $\bar{g}_{ab}=0$ being the most natural ones).[2]

However, quantizing the Nambu string in gauge (1.1a) it appears to be difficult to establish the $(d-26)$-factor and to prove that only free transverse modes are relevant for $d=26$. At the same time, starting with the BDHP action [10, 20] it is possible to work out the $O(d)$-covariant path integral quantization scheme and to trace the origin of the $d<26$ restriction. It is still remains a possiblity that the covariantly quantized Nambu model may also be cast in some tractable form, though at present it is the BDHP model which seems to be the simplest and most appealing one.

Let us now summarize the results and structure of the paper. In Section 2 we present the classical actions and the Wilson loop ansatz for the Nambu, Eguchi and BDHP models and observe that they are equivalent at the classical level. A natural question is whether this equivalence holds at the quantum level.

In Section 3 we first discuss the quantized BDHP string and show that contrary to the case of the formal partition function considered in [20], in the case of the Wilson loop ansatz one cannot in general obtain an explicit form for the conformal metric effective action and thus reveal a particular role of the $d=26$ dimension. We also point out that power divergences may be consistently neglected in the theory and thus no Liouville-type non-linearities are introduced (cf. [20, 15]). As a consequence, it is possible to establish the exact result, e.g., for the formal partition function, and to trace the analogy with the canonically quantized Nambu string (or oscillator model [1]) in degrees of freedom counting. Then we discuss the Lorentz covariant path integral quantization of the Nambu string and show that the resulting complicated theory is *inequivalent* to the BDHP model. It turns out impossible to reveal to origin of the $d<26$ restriction. We conclude that the Nambu and BDHP quantized strings

[2] Note that the resulting path integral will be of course the $O(d)$-covariant gauge independent.

are *different* extensions of the same classical theory, BDHP model being essentially simpler due to the additional Weyl symmetry.

However, in Section 4 we show that these two models (and also the generalized Eguchi one) *are* equivalent when treated in a semiclassical approximation near a minimal surface. We generalize and simplify the previous analysis of [26] of the $\nu = 1$ Eguchi model and obtain a more explicit form of the semiclassical loop functional, using the results established for the BDHP model. We also discuss the semiclassical $(R + 1/R)$-long-range potential which is thus equivalent for the above three models and show that this equivalence holds also for a static potential calculated in the leading $1/d$-approximation (first discussed in [19]).

In Section 5 we consider scattering amplitudes predicted by the BDHP string model. First we discuss the analog of the old Nambu case definition of the on-shell amplitude in terms of the string path integral [25] and show that it straightforwardly leads to the Veneziano model for $d = 26$. When $d < 26$ we obtain a non-dual amplitude with the linear pole trajectory with a "shifted" intercept. Then we analyse the definition of the off-shell amplitude proposed in [20] (see also [27]). It does not reproduce the spectrum of the Veneziano model for $d = 26$, but yields (for $d < 26$) an interesting off-shell amplitude with a non-linear spectrum (possessing the physical branch without a ground-state tachyon). All integrations over metric are done explicitly under the assumption that power divergences may be consistently neglected.

Our notations for geometrical objects on a two-dimensional manifold with boundary are summarized in Appendix A. In Appendix B we list some useful formulae for the Seeley coefficients of second order differential operators on a manifold with boundary. Appendix C is devoted to a simplified derivation (as compared to that of Ref. [28]) of the boundary terms in the BDHP model ghost operator effective action.

2. Definitions of String Models

Let us list here a number of string models which may be considered in connection with QCD. We begin with a famous Nambu–Goto action [16]

$$I_N = M^2 \int_{\mathscr{D}} \sqrt{h}\, d^2z, \qquad M^{-2} = 2\pi\alpha', \tag{2.1}$$

$$h = \det h_{ab}, \qquad h_{ab} = \partial_a x^\mu \partial_b x^\mu, \qquad \partial_a = \partial/\partial z^a. \tag{2.2}$$

We use the euclidean formulation throughout this paper and hence $\mathscr{D}$ is a bounded region in $\mathbb{R}^2$ (a strip $[\tau_1, \tau_2] \times [0, \pi]$ in the Minkowski variant), x^μ define the map $\mathscr{D} \to \Sigma \subset \mathbb{R}^d$, $\mu, \nu = 1,..., d$; $a, b = 1, 2$. We shall consider only "tree" approximation when Σ is without holes and handles. The generalized Eguchi models are defined as follows:

$$I_E^{(\nu)} = M^{4\nu} \int_{\mathscr{D}} h^\nu\, d^2z. \tag{2.3}$$

The $\nu = 1$ case was proposed in [17] and quantized in [18, 26]; for $\nu > \frac{1}{2}$ models see [19].

Next comes the action first discussed by Brink, Di Vecchia and Howe [10] (see also [11]) and used for quantization by Polyakov [20] (BDHP model for short)

$$I_G = M^2 \int_{\mathscr{D}} \tfrac{1}{2}\, g^{ab} \sqrt{g}\, \partial_a x^\mu \partial_b x^\mu \, d^2 z. \tag{2.4}$$

Here g_{ab} is a positive definite metric on $\mathscr{D}$ considered as an independent variable.

Though in this paper we shall mainly consider the above three bose string models, let us also for completeness write down the actions for the well-known strings with fermions. The first two are the spinning string model [10, 11] and the model with spin and charge [13, 14], quantized in [12] and [15], respectively. The corresponding lagrangians are the $N = 1$ and $N = 2$ locally supersymmetric extensions of (2.4)

$$I_F = M^2 \int_{\mathscr{D}} d^2 z \sqrt{g} \left\{ \frac{1}{2} g^{ab} \partial_a x \cdot \partial_b x + \frac{i}{2} \bar{\psi} \gamma^a \overleftrightarrow{\mathscr{D}}_a \psi + (\partial_a x + \bar{\psi} \chi_a) \bar{\chi}_b \gamma^a \gamma^b \psi \right\}, \tag{2.5}$$

$$I_{CF} = M^2 \int_{\mathscr{D}} d^2 z \sqrt{g} \left\{ \frac{1}{2} g^{ab} \partial_a \varphi \cdot \partial_b \varphi^* + \frac{i}{2} \bar{\psi} \gamma^a \overleftrightarrow{\mathscr{D}}_a \psi \right.$$
$$\left. + A_a \bar{\psi} \gamma^a \psi + (\partial_a \varphi^* + \bar{\psi} \chi_a) \bar{\chi}_b \gamma^a \gamma^b \psi + \text{h.c.} \right\}. \tag{2.6}$$

In (2.5) the d matter multiplets (x_μ, φ_μ) interact with the $N = 1$ supergravity (χ_a is "gravitino") while in (2.6) all corresponding fields are complex (excluding the vector A_a) with a "doubling" $\varphi_\mu = x_\mu + i\phi_\mu$, $\mu = 1, \ldots, d$. Note that (2.6) posesses the $O(d)$ (not $O(2d)$) global symmetry and thus describes a charged fermi string *in d dimensions*. The last example is the Migdal string [7] probably connected with the $N = \infty$ limit of QCD

$$I_M = \int_{\mathscr{D}} (\bar{\psi}_\lambda \gamma^a \partial_a \psi_\lambda + m \bar{\psi}_\lambda \psi_\lambda \sqrt[4]{h})\, d^2 z, \tag{2.7}$$

where ψ_λ is a bispinor field (with some boundary conditions) and for x_μ the orthogonal gauge (1.1a) is assumed ($h_{ab} = \sqrt{h}\, \delta_{ab}$).

Note that in all above actions we omitted possible counter-terms.

The classical equations of motion following from (2.1), (2.3) and (2.4) are, respectively,

$$\partial_a(\sqrt{h}\, h^{ab} \partial_b x_\mu) = 0, \tag{2.8}$$

$$\partial_a(h^\nu h^{ab} \partial_b x_\mu) = 0, \tag{2.9}$$

$$g_{ab} = F^2 h_{ab}, \qquad \partial_a(\sqrt{g}\, g^{ab} \partial_b x_\mu) = 0. \tag{2.10}$$

Equation (2.8) describes the minimal surface enclosed by a curve $C \subset \mathbb{R}^d$, $x^\mu: \partial\mathscr{D} \to C$. As a consequence of (2.9) for $\nu \neq \frac{1}{2}$ we have

$$h = \text{const}, \tag{2.9b}$$

and thus the stationary points of the symplectic invariant action (2.3) are minimal surfaces in the special parametrization, where (2.9b) is valid. According to (2.10), an arbitrary constant F drops from the x_μ-equation and we are again left with minimal surfaces.

We conclude that all three bose models (2.1), (2.3), (2.4) are *equivalent* at the classical level (we leave aside the question about boundary conditions which may be implied by the action, see, e.g., [29]).

Now let us pass to the functional integral quantization. The connection with gauge theory is commonly established through a string-like ansatz for the Wilson loop expectation value. "String propagators" for the Nambu and Eguchi models are

$$W_N[C] = \int_{x|_{\partial\mathscr{D}}=C} [\mathscr{D}x]\, e^{-I_N}, \tag{2.11}$$

$$W_E^{(\nu)}[C] = \int_0^\infty da\, e^{-a/2} \int_{x|_{\partial\mathscr{D}}=C} [\mathscr{D}x]\, e^{-k(\nu)I_E^{(\nu)}}, \tag{2.12}$$

where $k(\nu) = ((2\nu-1)/\nu)^{2\nu-1} \cdot (1/2\nu)$ is needed for the "area law" in the classical limit and $a = \int_{\mathscr{D}} d^2z$ (see [18, 19]). The functional measures include a fixation of the general coordinate gauge in (2.11) and the symplectic gauge in (2.12). The corresponding gauge freedoms are essential for the contour C reparametrization invariance of $W[C]$. Let

$$\delta z^a = \xi^a(z), \qquad \delta x^\mu = \xi^a \partial_a x^\mu \tag{2.13}$$

be a coordinate transformation in $\mathscr{D}$. The boundary condition in (2.11) and (2.12)

$$x^\mu(\tau) = c^\mu(\tau) \in C \tag{2.14}$$

(τ is some parametrization of $\partial\mathscr{D}$) is preserved by (2.13) if

$$\xi_{\vec{n}}|_{\partial\mathscr{D}} = 0, \qquad \xi_{\vec{t}}|_{\partial\mathscr{D}} = \text{arbitrary function of } \tau, \tag{2.15}$$

where $\vec{n}$ and $\vec{t}$ are the flat metric normal and tangential vectors at $\partial\mathscr{D}$ (cf. (A.4), (A.5), (A.10)). It is the invariance of the integrand in (2.11) under (2.13), (2.15) with an arbitrary $\xi_{\vec{t}}$ that establishes the C-reparametrization invariance of (2.11). In an analogous way for symplectic invariance (2.12) we have (h_{ab} is used as a metric)

$$\xi^a = \mathscr{E}^{ab}\partial_b\lambda, \qquad \lambda|_{\partial\mathscr{D}} = 0, \qquad \partial_n\lambda|_{\partial\mathscr{D}} = \text{arbitrary}. \tag{2.16}$$

The analog of (2.11) for the BDHP model (2.4) reads

$$W_G[C] = \int_{x|_{\partial\mathscr{D}}=C} [\mathscr{D}x\, \mathscr{D}g]\, e^{-I_G}. \tag{2.17}$$

Here again some coordinate $O(d)$-invariant gauge is assumed. The important question is whether some boundary conditions on the metric are to be specified. The answer would of course be trival if g_{ab} was merely a Lagrange multiplier (for example, at the classical level $g_{ab}|_{\partial\mathscr{D}}$ is determined by x^μ and thus there is no above problem in the perturbation theory near the minimal surface (2.10)). However, metric becomes dynamical through the conformal anomaly [20] (see Section 3). That is why in a non-perturbative approach one can in principle impose some boundary conditions on g_{ab} but with the requirement that *they do not spoil the C-reparametrization invariance of* (2.17), i.e., do not fix $\xi_t|_{\partial\mathscr{D}}$ as in (2.15) (though they may restrict $\partial_n \xi_t|_{\partial\mathscr{D}}$), e.g.,

$$\left(g_{ab} \frac{dz^a}{d\tau} \frac{dz^b}{d\tau} \right) \bigg|_{\partial\mathscr{D}} = \frac{dc_\mu}{d\tau} \frac{dc_\mu}{d\tau}. \tag{2.18}$$

Probably this question should be studied separately in each special analysis of (2.17). However, it seems that in general one is to average over all metrics at the boundary in order to establish C-reparametrization invariance,

$$\begin{aligned} W_G[C] &= \int df_{ab}(\tau)\, W_G[C,f], \\ W_G[C,f] &= \int_{g|_{\partial\mathscr{D}}=f,\ x|_{\partial\mathscr{D}}=C} [\mathscr{D}x\, \mathscr{D}g]\, e^{-I_G} \end{aligned} \tag{2.19}$$

(it is possible of course to use $\partial_n g_{ab}|_{\partial\mathscr{D}} = f_{ab}$, etc.). It may turn out, however, that some particular "semiclassical" value of f_{ab} provide a good approximation for (2.19).

In the cases of strings with fermions (2.5)–(2.7) one must functionally integrate over all independent fields in the actions with $x_\mu|_{\partial\mathscr{D}} = c_\mu$ and some appropriate boundary conditions for other variables.

As is obvious from the above discussion, (2.11), (2.12) and (2.17) are the quantum versions of the *same* classical theory (corresponding to the "area law"). We are now going to study the question whether the Nambu, Eguchi and BDHP models share some common features at the quantum level.

3. Quantized BDHP and Nambu Strings

Let us try to give a more precise and explicit form to the BDHP ansatz (2.17) trying to proceed as far as possible without using any approximation. Following the method of Ref. [15] we take the following coordinate gauge (cf. [20] and also [30])

$$\bar{g}_{ab} \equiv g_{ab} - \tfrac{1}{2}\,\delta_{ab}(g_{cd}\,\delta^{cd}) = 0, \tag{3.1}$$

i.e.,

$$g_{ab} = \rho^2\,\delta_{ab}, \qquad \rho = e^{\sigma} = \text{arbitrary}. \tag{3.2}$$

Being trivially $O(d)$ (Lorentz) invariant, this gauge does not fix the coordinate group completely: there is still a freedom of conformal coordinate transformations

$$\begin{aligned} \rho'^2(w)\,\frac{\partial w^c}{\partial z^a}\,\frac{\partial w^c}{\partial z^b} &= \rho^2(z)\,\delta_{ab}, \\ \rho'^2(w,\bar{w}) &= \left|\frac{\partial z}{\partial w}\right|^2 \rho^2(z,\bar{z}), \qquad z = z_1 + iz_2, \end{aligned} \tag{3.3}$$

which should be preserved during quantization. This incompleteness of (3.1) is in accordance with the remarks in Section 1. Action (2.4) and also (3.1) are invariant under the Weyl transformations $g'_{ab} = \lambda^2 g_{ab}$. However, this symmetry is broken due to conformal anomaly (if one uses a regularization, preserving the general covariance and thus also (3.3)) and so the classical gauge $\rho = 1$ is not an adimissible one at the quantum level. As a result, ρ represents a new "quantum" degree of freedom which could be missed if only independent classical variables were quantized.

A naive application of the Faddeev–Popov procedure to (3.1) gives the following ghost operator:

$$\begin{aligned} \begin{pmatrix} \eta_1 & -\eta_2 \\ \eta_2 & \eta_1 \end{pmatrix} &= (G^{ab}{}_c\,\xi^c), \\ G^{ab}{}_c &= \rho^2(\delta_{ab}\partial_c - \delta_{ac}\partial_b - \delta_{bc}\partial_a), \end{aligned} \tag{3.4}$$

$$\begin{aligned} (\det G)^{-1} &\sim \int d\xi_1\,d\xi_2\,\delta[\rho^2(\partial_2\xi_2 - \partial_1\xi_1)]\,\delta[\rho^2(\partial_2\xi_1 + \partial_1\xi_2)] \\ &= \int d\xi\,d\bar{\xi}\,\delta[\rho^2\partial\bar{\xi}]\,\delta[\rho^2\bar{\partial}\xi], \end{aligned} \tag{3.5}$$

$$\xi = \xi_1 + i\xi_2, \qquad \partial = \partial_1 - i\partial_2, \qquad \bar{\partial} = \partial_1 + i\partial_2. \tag{3.6}$$

However, in order to preserve the general covariance one must first "forget" about condition (3.2) and use the following trick,

$$\delta\bar{g}_{ab} = \nabla_a\xi_b + \nabla_b\xi_a - g_{ab}\nabla_c\xi_c \equiv G_{abc}\,\xi^c, \tag{3.7}$$

$$\det G = \sqrt{\det \Delta_{1ab}}, \qquad \Delta_{1ab} = G^{cd}{}_a\,G^{cd}{}_b, \tag{3.8}$$

$$\xi \Delta_1 \xi = \int d^2z \sqrt{g}\, \xi^a \Delta_{1ab} \xi^b,$$
$$\Delta_{1ab}\xi^b = -\nabla^b G_{abc} \xi^c = -(\nabla_c \nabla^c + R/2)\, \xi_a, \tag{3.9}$$

where we used (A.1). The resulting functional measure in (2.17) is

$$[\mathscr{D}x\, \mathscr{D}g] = \{\mathscr{D}g\, \delta[\bar{g}_{ab}] \sqrt{\det \Delta_1}\}\{\mathscr{D}x\, \mu[g]\}. \tag{3.10}$$

Here $\det \Delta_1$ is defined in covariant way and under the δ-function condition may be written as [20] (cf. (3.5))

$$\det{}' L_1, \qquad L_1 = \rho^{-4} \partial \rho^2 \bar{\partial}. \tag{3.11}$$

Note that L_1 possesses conformal zero modes ($\bar{\partial}\xi_0 = 0$), coming from the residual symmetry (3.3). They should be omitted in $\det' L_1$.

The local factor $\mu[g]$ in (3.10) is necessary for correspondence with the canonical path integral quantization [31]. If one considers (2.4) as a lagrangian for d scalar fields in an external metric, the canonical momentum integration results in $\mu[g] = \prod_z [\sqrt{g}\, g^{00}]^d$. As was pointed out in [31], the analogous factors provide the cancellation of the leading (momentum space volume) divergences in the g_{ab}-effective action (which of course may be considered as a sort of a normal ordering prescription). In (3.10) is assumed that integration goes over non-negative metrics, i.e., $g_+ \geqslant 0$, $g_\pm = g_{11} \pm g_{22}$,

$$\int \prod_{a<b} dg_{ab}\, \delta(\bar{g}_{11})\, \delta(\bar{g}_{12}) \cdots = \int_0^\infty dg_+ \int_{-\infty}^\infty dg_- \int_{-\infty}^\infty dg_{12}\, \delta(g_-)\, \delta(g_{12}) \cdots.$$

As a result, the only integration left is over $\rho^2 = \frac{1}{2} g_+$ or $\sigma = \frac{1}{2} \log \rho^2$, $-\infty < \sigma < \infty$ (the factor $\prod_z e^{2\sigma}$ is absorbed in a definition of $\mathscr{D}g_{ab}$).

Integrating over x_μ in (2.17) we get

$$W_G[C] = \int d\sigma \exp(-I_{\text{eff}}[\sigma, C]), \tag{3.12}$$

$$I_{\text{eff}} = d \cdot \Gamma(\Delta_{0c}) - \Gamma(\Delta_1), \tag{3.13}$$

$$\Gamma(\Delta) \equiv \tfrac{1}{2} \log \det \Delta,$$
$$\Delta_0 = -\frac{1}{\sqrt{g}} \partial_a (g^{ab} \sqrt{g}\, \partial_b) = -e^{-2\sigma} \Box, \tag{3.14}$$

where the subscript c indicates that $\det \Delta_0$ is calculated with the boundary condition (2.14) (it is through this place that C-dependence of (3.12) comes from).

Let us first consider $\Gamma(\Delta_1)$, which can be calculated in an explicit form. The necessary boundary conditions on ξ_a (2.15) are, however, insufficient for a well-posed boundary problem for Δ_1. The most natural sufficient ones are the mixed boundary conditions (cf. [28])

$$\xi_n|_{\partial\mathscr{D}} = 0, \qquad (\partial_n \xi_t)|_{\partial\mathscr{D}} = 0, \tag{3.15}$$

preserving the conformal invariance (3.3) of the problem (cf. (3.5)). Then (see Appendixes B, C)

$$\Gamma(\Delta_1) = \Gamma_\infty^{(1)} + \Gamma_\sigma^{(1)} + \Gamma_0^{(1)}, \tag{3.16}$$

$$\Gamma_\infty^{(1)} = 2\left(-\frac{L^2}{8\pi}\int_{\mathscr{D}} e^{2\sigma}\, d^2z - \frac{1}{12}\chi \log\frac{L^2}{\mu^2}\right), \tag{3.17}$$

$$\Gamma_\sigma^{(1)} = 26\left(-\frac{1}{12\pi}\int_{\mathscr{D}} \frac{1}{2}(\partial\sigma)^2\, d^2z - \frac{1}{12\pi}\int_{\partial\mathscr{D}} K_T\cdot\sigma\cdot dz\right), \tag{3.18}$$

$$\Gamma_0^{(1)} = \tfrac{1}{2}\log\det(-\Box)_D + \tfrac{1}{2}\log\det(-\Box)_N. \tag{3.19}$$

Here $L\to\infty$, $dz = \sqrt{dz^a\, dz^a}$, K_T is defined in (A.9), (A.10), the Euler number χ (A.12) is 1 if $\mathscr{D}$ is simply connected and subscripts D and N indicate the Dirichlet and Neumann boundary conditions. The "volume" part of (3.18) was first given in [20] while the boundary term was established in [28] (see also Appendix C). Note that (3.16) is invariant under (3.3).

Now let us discuss $\Gamma(\Delta_{0_c})$. It is important to stress that its dependence on σ can be explicitly calculated ("through the anomaly") only *under some limited class of boundary conditions.* Really, if $\Delta_0\varphi = \lambda\varphi$, $\int_{\mathscr{D}}\varphi^2\sqrt{g}\, d^2z = 1$, under a variation of σ we have

$$\delta\Delta_0 = -2\,\delta\sigma\,\Delta_0, \qquad \delta\lambda + \lambda\int_{\mathscr{D}} d^2z\,\sqrt{g}\,2\cdot\delta\sigma\cdot\varphi^2 = \Sigma,$$
$$\Sigma = -2\int_{\partial\mathscr{D}} dz\,(\varphi\partial_{\tilde n}\,\delta\varphi - \delta\varphi\partial_{\tilde n}\varphi). \tag{3.20}$$

Only if $\Sigma = 0$ do we get the desired simple result (see Appendixes A, B for notations)

$$\delta(\operatorname{tr} e^{-t\Delta_0}) = -2t\frac{\partial}{\partial t}\int_{\mathscr{D}} d^2z\,\sqrt{g}\,\delta\sigma(z)\langle z|e^{-t\Delta}|z\rangle$$

and so

$$\delta\Gamma_\sigma^{(0)} = -\mathscr{A}_2[\delta\sigma], \tag{3.21}$$

where we put

$$\Gamma(\Delta_{0_c}) = \Gamma_\infty^{(0)} + \Gamma_\sigma^{(0)} + \Gamma_0^{(0)}, \tag{3.22}$$

$$\Gamma_0^{(0)} = \tfrac{1}{2}\log\det(-\Box)_c. \tag{3.23}$$

Σ is, of course, zero for the Dirichlet and Neumann conditions (if, e.g., $\varphi|_{\partial\mathscr{D}}=0$, then $\delta\varphi|_{\partial\mathscr{D}}=0$) and also for the σ-independent Robin conditions $(\partial_{\tilde{n}}+\tilde{\psi})\,\varphi|_{\partial\mathscr{D}}=0$, $\delta\tilde{\psi}/\delta\sigma=0$, cf. (B.9). However, in the general case of (2.14), Σ depends on $(c^\mu\partial_{\tilde{n}}\,\delta x^\mu)|_{\partial\mathscr{D}}$ and thus (3.21) is not valid. As a result, one *cannot separate the anomalous dependence on σ* and on the contour C as it was naively assumed in (3.22), (3.23) and so $\Gamma(\Delta_{0_c})$ *cannot be in general calculated "through the anomaly."* This conclusion is rather evident in view of the requirement of the C-reparametrization invariance of (3.12). If σ and C dependences were separated as in (3.22) one may simply integrate over σ in (3.12) with the non-covariant result

$$W_G[C]\sim\exp\left(-\frac{d}{2}\log\det(-\Box)_c\right),\tag{3.24}$$

which is obviously dependent on parametrization of $\partial\mathscr{D}$ and C. In the correct expression the reparametrization invariance is restored by the (possibly non-local) *boundary terms*, involving σ as well as c^μ. Note that if we assume some restrictive boundary conditions on σ (e.g., $\sigma|_{\partial\mathscr{D}}=0$, $\partial_n\sigma|_{\partial\mathscr{D}}=0$) these terms may vanish, again leading to a non-invariant result (3.24) (cf. the discussion after (2.17)).

Observing that the local "volume" part of $\Gamma(\Delta_{0_c})$ is of course independent of c_μ and any boundary conditions, it is useful to write down the expression for $\Gamma(\Delta_0)$ assuming for concreteness that the Dirichlet or the Neumann conditions are imposed on x_μ (see (B.3), (B.13), (B.14) and also [26, 15])

$$\Gamma^{(0)}_\infty=-\frac{1}{8\pi}L^2\int_{\mathscr{D}}e^{2\sigma}\,d^2z\pm\frac{1}{8\sqrt{\pi}}L\int_{\partial\mathscr{D}}e^\sigma\,dz-\frac{1}{12}\chi\log\frac{L^2}{\mu^2},\tag{3.25}$$

$$\Gamma^{(0)}_\sigma=-\frac{1}{12\pi}\int_{\mathscr{D}}\frac{1}{2}(\partial_a\sigma)^2\,d^2z-\frac{1}{12\pi}\int_{\partial\mathscr{D}}K_T\cdot\sigma\cdot dz\mp\frac{1}{8\pi}\int_{\partial\mathscr{D}}\partial_{\tilde{n}}\sigma\,dz+\mathscr{O},\tag{3.26}$$

where according to (A.10), (A.11) $\tilde{n}^a=\varepsilon^{ab}z'_b$, $z'^a=dz^a/dz$, $K_T=\varepsilon_{ab}z'^a z''^b$ and $\mathscr{O}$ is the Neumann case zero mode contribution

$$\mathscr{O}=\begin{cases}D: 0\\ N: \log A,\end{cases}\qquad A=\int_{\mathscr{D}}\sqrt{g}\,d^2z.\tag{3.27}$$

There is a close similarity between results (3.17), (3.18) and (3.25), (3.26): sign–changing terms cancel in the Δ_1–case due to the mixed boundary conditions (3.15).

As was already pointed out, an appropriate choice of the local measure in (3.10) subtracts the L^2-divergences in (3.17), (3.15) (note that L^2 and L dependent terms are automatically absent in the dimensional or ζ-function regularization). Saying in other words, we may always put the renormalized values of μ and λ in the corresponding counter-terms (cf. [20, 27])

$$\mu^2 \int_{\mathscr{D}} e^{2\sigma} \, d^2 z + \lambda \int_{\partial \mathscr{D}} e^{\sigma} \, dz \tag{3.28}$$

equal to zero in a consistent way, because the above infinite terms exhaust all divergences of this type in the theory (one need not worry about "next order" corrections, etc.). The conditions $\mu = \lambda = 0$ simplify the discussion of string scattering amplitudes (Section 5) and are assumed in the following.

At the same time, it seems interesting to note the necessity of the $\theta \cdot \chi$ ($\theta = \text{const}$) bare term in the string action in order to renormalize the logarithmic divergences. This term leads to the anomalous dimension ($W_G \sim \mu^{-2\theta \cdot \chi}$) in the Wilson factor and may be important in next to the leading order approximations in $1/N$, where one must average over surfaces M, $\partial M = C$, with holes and handles and thus is led to the sum over topologies of the type $W[C] = \sum_\chi e^{-\theta\chi} W^{(\chi)}[C]$.

Let us finally give the total expression for the σ-effective action (3.13) *as if* the Dirichlet or the Neumann boundary conditions were imposed on x^μ. The first case corresponds to the functional integral (2.17) treated as a formal partition function while the second is useful for the discussion of the strings Green's functions. Summing (3.18) and (3.25) we get

$$\begin{aligned} I_{\text{eff}}[\sigma] = \frac{26-d}{12\pi} \left[\int_{\mathscr{D}} \frac{1}{2} (\partial\sigma)^2 \, d^2 z + \int_{\partial\mathscr{D}} K_T \cdot \sigma \cdot dz \right] \\ \pm \frac{d}{8\pi} \int_{\partial\mathscr{D}} \partial_{\vec{n}} \sigma \, dz + d \cdot \mathscr{O} \end{aligned} \tag{3.29}$$

(for the "volume" term see [20], while for the boundary terms see Ref. [28], where, however, the Neumann case zero mode contribution (3.27) was omitted). We conclude that if $26 - d > 0$, σ is a physical degree of freedom (this is the analog of the no-ghost theorem for the Nambu string [3]). If $d = 26$ *and* the boundary condition

$$\partial_{\vec{n}} \sigma |_{\partial\mathscr{D}} = 0 \tag{3.30}$$

is imposed on σ, we conclude that there is no Weyl symmetry anomaly in the formal function integral for the BDHP model. This, however, *does not imply* that (2.17) with the *correct* boundary conditions (2.14) is simplified for $d = 26$ because Eq. (3.29) cannot be used in this case.

Let us remark that using (3.29) is possible[3] to obtain the exact result for the

[3] In contrast with the approach of Ref. [20], where μ in (3.28) was assumed to be non-zero.

formal partition function, neglecting the boundary terms and differences connected with boundary conditions (recall (3.19), (3.23))

$$Z = \int d\sigma \, e^{-I_{\text{eff}}} = [\det(-\Box)]^{-(1/2)N}, \tag{3.31}$$

$$N = d-1, \qquad d < 26,$$
$$= d-2, \qquad d = 26.$$

Here N is the number of "quantum" degrees of freedom. We again observe the similarity with the Nambu string which is known [1–3] to have equal numbers of classical (transverse) and quantum degrees of freedom only at the critical dimension, while for $d < 26$ one must also take into account the quantum longitudinal mode, here effectively represented by σ.

It is probably worth noting that the expressions, analogous to (3.29) and (3.31), can be obtained also for the supersymmetric strings (2.5) and (2.6). Let us present the corresponding result, e.g., for the charged spinning string (2.6), obtained in our work [15],

$$I_{\text{eff}} = \frac{2-d}{4\pi} \int d^2 z \left\{ \frac{1}{2} (\partial_a \sigma)^2 - \frac{1}{2} (\partial_a \mathscr{H})^2 + \frac{i}{2} \bar{\lambda} \gamma^a \vec{\partial}_a \lambda \right\} \tag{3.32}$$

in the "incomplete" quantum gauge, where

$$\chi_a = \tfrac{1}{2} \gamma_a \lambda, \qquad A_a = \tfrac{1}{2} \varepsilon_{ab} \partial^b \mathscr{H}. \tag{3.33}$$

Here σ, $\mathscr{H}$ and λ terms correspond to the Weyl, chiral and superconformal anomalies. Note that different signs of the first two terms in (3.32) are connected with the fact that anomalies do not in general respect unitarity. Thus, the theory is ghost and anomaly free only for $d = 2$ in agreement with the result of [13]. For the neutral spinning string (2.5) one must put $\mathscr{H} = 0$ and $(2-d) \to (10-d)/2$ (see [12]). When $d < 10$, σ and λ are analogous to longitudinal degrees of freedom of the Neveu–Schwarz model [1–3].

Let us now develop *the Lorentz covariant* path integral quantization for the Nambu string trying to follow the same way as in the case of the BDHP model. First of all, we want to show that the Wilson loop ansatz in the case of the Nambu (2.11) and BDHP (2.17) strings *are formally inequivalent*. Substituting the identity

$$1 = \int \mathscr{D} g_{ab} \, \delta(g_{ab} - F^2 h_{ab}(x)) \tag{3.34}$$

in (2.11) and noting that according to (2.1) and (2.4), $I_G[g_{ab} = F^2 h_{ab}] = I_N[h]$, we get (2.11) in the form

$$W_N[C] = \int_{x|_{\partial \mathscr{D}} = C} [\mathscr{D} g \, \mathscr{D} x] \, \delta(g_{ab} - F^2 \partial_a x \cdot \partial_b x) \, e^{-I_G[g,x]}. \tag{3.35}$$

Comparing (2.17) and (3.35) we conclude that the functional integration in the Nambu case goes over the surface $g_{ab}=F^2h_{ab}$ in the (g,x)-space. However, this condition holds for the BDHP string only in the classical (and also semiclassical; see Section 4) limit (2.10). The fixation of the coordinate gauge of course does not eliminate this discrepance, which is connected with a peculiar role of g_{ab} in the quantized BDHP model (2.17). Indeed, the dependence on the metric is a non-linear one and g_{ab} cannot be treated as a simple Lagrange multiplier (which can be, e.g., eliminated before the x_μ-integration).

To illustrate this point let us rewrite (3.35) introducing the tensor density auxiliary field α^{ab} (with integration limits $\pm i\infty$) and assuming $F=1$

$$W_N=\int [\mathscr{D}g\,\mathscr{D}x\,\mathscr{D}\alpha]\exp\Big\{-M^2\int d^2z \times\left(\tfrac{1}{2}\alpha^{ab}\partial_a x_\mu\partial_b x_\mu-\tfrac{1}{2}\alpha^{ab}g_{ab}+\sqrt{g}\right)\Big\}. \tag{3.36}$$

Supposing that the result of the g_{ab} integration (being a local factor) can be absorbed in the measure $\mathscr{D}\alpha$, we are left with the expression, analogous to (2.17). However, we cannot put $\alpha^{ab}=\tilde{g}^{ab}\sqrt{\tilde{g}}$ with $\tilde{g}_{ab}$ playing the role of the metric in (2.17), because in general $\det\alpha\neq 1$. If we now choose the gauge (cf. (3.1))

$$\alpha^{ab}=\alpha\delta^{ab},\qquad \bar{\alpha}^{ab}=0 \tag{3.37}$$

and formally give up the condition of general covariance, we obtain for the product of the ghost and x_μ-determinants (cf. (3.5), (3.11))

$$[\det(\alpha\partial\alpha\bar{\partial})]^{1/2}\cdot[\det(\partial_a\alpha\partial_a)]^{-(1/2)d}. \tag{3.38}$$

Though the α-dependence of the first factor can be explicitly evaluated (the corresponding operator has a product structure), this is not valid for the $\partial_a\alpha\partial_a$ operator, leading to a non-local dependence on α. Thus possible conjecture that α plays the role of ρ^2 in (3.2) in the BDHP case and that the result, analogous to (3.29) can therefore be obtained for the Nambu string, is incorrect. This is probably due to the fact that the BDHP action (2.4) possesses the additional Weyl symmetry which is absent in the auxiliary action in (3.36).

We obtain a useful explicit form of the Nambu model path integral if we choose the orthogonal gauge $\bar{h}_{ab}=0$ (1.1a), and introduce a collective variable $h^2=\partial_a x_\mu\partial_a x_\mu$,

$$W_N=\int [\mathscr{D}x_\mu\,\mathscr{D}h^2\,\mathscr{D}\alpha\,\mathscr{D}\bar{\alpha}^{ab}]\exp(-\tilde{I}_N), \tag{3.38}$$

$$\tilde{I}_N=M^2\int_{\mathscr{D}} d^2z\,\Big\{\frac{1}{2}(\alpha\delta^{ab}+\bar{\alpha}^{ab})\,\partial_a x_\mu\partial_b x_\mu + h^2\left(1-\frac{\alpha}{2}\right)+\frac{1}{M^2}26\cdot\frac{1}{24\pi}(\partial_a\log h)^2\Big\}, \tag{3.39}$$

where used the explicit form of the ghost determinant contribution available from (3.18) under substitution $e^{\sigma} \to h$ and omitted boundary terms.

We conclude that quantizing the Nambu string in the covariant ("incomplete") gauge we do not obtain the $(d-26)$-factor (i.e., we cannot say that for $d=26$ only free transverse modes contribute). The resulting theory is *inequivalent* and *essentially more complicated* than that of the BDHP model. However, it is possible to prove that both theories *are* equivalent in the semiclassical approximation.

4. Semiclassical Approximation

Let us begin with a semiclassical approximation for te Nambu (2.11) and Eguchi (2.12) models. Expanding near a minimal surface we have

$$x^\mu = \varphi^\mu + \varepsilon\phi^\mu, \qquad \varphi^\mu|_{\partial\mathscr{D}} = c^\mu, \qquad \phi^\mu|_{\partial\mathscr{D}} \sim \frac{dc^\mu}{d\tau}, \tag{4.1}$$

$$\delta h_{ab} = 2\varepsilon\partial_{(a}\varphi^\mu\partial_{b)}\phi^\mu + \varepsilon^2\partial_a\phi_\mu\partial_b\phi_\mu, \tag{4.2}$$

$$\delta(\det h)^\nu = \nu\,\delta h^a_a - \frac{\nu}{2}\,\delta h_{ab}\,\delta h^{ab} + \frac{\nu^2}{2}\,(\delta h^a_a)^2. \tag{4.3}$$

We consider fluctuations ϕ^μ which leave the contour C fixed up to a reparametrization, i.e., $(\varphi^\mu + \varepsilon\phi^\mu)_{\partial\mathscr{D}} = c^\mu(\tau + \mathscr{E})$. All indices are contracted with a background minimal surface metric

$$h_{ab} = \partial_a\varphi_\mu\partial_b\varphi_\mu, \qquad \partial_a(\sqrt{h}\,h^{ab}\partial_b\varphi_\mu) = 0. \tag{4.4}$$

The $O(\varepsilon^2)$ term in the Eguchi action (2.3) can be written in the form

$$I_E^{(\nu)}(\varepsilon^2) = \varepsilon^2 M^{4\nu}\int d^2z\, h^\nu\partial_a\phi^\mu M^{ab}_{\mu\nu}\,\partial_b\phi^\nu, \tag{4.5}$$

$$M^{ab}_{\mu\nu} = M^{\perp ab}_{\mu\nu} + 2(\nu - \tfrac{1}{2})\,M^{\parallel ab}_{\mu\nu}, \qquad M^{\parallel ab}_{\mu\nu} = \partial^a\varphi_\mu\partial^b\varphi_\nu, \tag{4.6}$$

$$M^{\perp ab}_{\mu\nu} = h^{ab}\delta_{\mu\nu} + \partial^a\varphi_\mu\partial^b\varphi_\nu - \partial^a\varphi_\nu\partial^b\varphi_\mu - h^{ab}\partial_c\varphi_\mu\partial^c\varphi_\nu. \tag{4.7}$$

For $\nu = \frac{1}{2}$, (4.5) and (4.7) give the result for the Nambu case (2.1). Introducing

$$\phi_\mu = \phi^\perp_\mu + \phi^\parallel_\mu, \qquad \phi^\perp\cdot\phi^\parallel = 0, \qquad \phi^\parallel_\mu = \xi^a\partial_a\varphi_\mu, \tag{4.8}$$

we have (with the help of the classical equations (4.4))

$$\partial\phi\,M\,\partial\phi = \partial\phi^\perp\,M^\perp\,\partial\phi^\perp + 2(\nu - \tfrac{1}{2})\,\partial\phi^\parallel M^\parallel\,\partial\phi^\parallel, \tag{4.9}$$

$$\partial\phi\,M^\perp\,\partial\phi = \partial\phi^\perp\,M^\perp\,\partial\phi^\perp \tag{4.10}$$

(note that (4.10) is a consequence of the general covariance of the Nambu action and the fact that $\phi^{\parallel}$ describes the infinitesimal coordinate transformation; cf. (2.13)). In view of (4.8), $\mathscr{D}\phi = \mathscr{D}\phi^{\parallel}\,\mathscr{D}\phi^{\perp}$ and we formally get the following semiclassical results for (2.11) and (2.12):

$$W_N[C] \sim e^{-M^2 A(C)} Z^{\perp}, \qquad A(C) = \int_{\mathscr{D}} \sqrt{h}\, d^2 z, \tag{4.11}$$

$$W_E^{(\nu)}[C] \sim e^{-M^2 A(C)} Z^{\perp} \cdot Z^{\parallel}, \tag{4.12}$$

$$Z^{\parallel} = [\det_{\parallel} \Delta_{\parallel}]^{-1/2}, \qquad Z^{\perp} = [\det_{\perp} \Delta_{\perp}]^{-1/2}, \tag{4.13}$$

$$\Delta_{\parallel(\perp)\mu\nu} = -\frac{1}{\sqrt{h}} \partial_a (\sqrt{h}\, M^{\parallel(\perp)ab}_{\mu\nu} \partial_b) \tag{4.14}$$

(we used (2.9b) for $\nu \neq \frac{1}{2}$ in order to "covariantize" all determinants and thus trivially integrate over the area a in (2.12)).

Changing the variables $\phi^{\parallel} \to \xi$ according to (4.8) we have

$$Z^{\parallel} = \int d\xi_a \exp \left\{ - \int_{\mathscr{D}} (\nabla_a \xi^a)^2 \sqrt{h}\, d^2 z \right\}. \tag{4.15}$$

In view of (4.1), $d\varphi^{\mu}|_{\partial\mathscr{D}} = dc^{\mu}$ or $\partial_t \varphi^{\mu}|_{\partial\mathscr{D}} \sim \phi^{\mu}|_{\partial\mathscr{D}}$, implying the following boundary conditions (cf. (2.15), (3.15)):

$$\xi_n|_{\partial\mathscr{D}} = 0, \qquad \phi^{\perp}_{\mu}|_{\partial\mathscr{D}} = 0. \tag{4.16}$$

Splitting ξ_a,

$$\xi_a = \partial_a \zeta + \mathscr{E}_{ab} \partial^b \lambda, \tag{4.17}$$

we see that (4.16) is satisfied if (cf. (A.4), (A.5))

$$\partial_n \zeta|_{\partial\mathscr{D}} = 0, \qquad \lambda|_{\partial\mathscr{D}} = 0, \tag{4.18}$$

and so

$$Z^{\parallel} = [\det \Delta_{0_D}]^{1/2} \cdot [\det \Delta_{0_N}]^{-1/2}. \tag{4.19}$$

Here (and in (4.17)) we omitted the harmonic zero made and also the infinity due to the symplectic zero mode (represented by λ; cf. (2.16)). Subscripts D and N in (4.19) indicate the type of the boundary condition for $\Delta_0 = -(1/\sqrt{h})\partial_a(\sqrt{h}\, h^{ab}\partial_b)$. For the derivation it is sufficient to note that $(\nabla_a \xi^a)^2 = (\Delta_0 \zeta)^2$ and that the Jacobian of transformation (4.17) is $[\int d\zeta\, d\lambda \exp(-\int d^2 z\, \xi_a^2 \sqrt{h})]^{-1}$.

Result (4.12), (4.19) for $\nu = 1$ Eguchi model was first obtained in [26], but our derivation seems more general and straightforward. However, the $Z^{\perp}$ factor (4.13) was written in [26] in a not very explicit form. Let us now work out a simple representation of $Z^{\perp}$ which is useful in order to make contact with the BDHP model. The main observation is that according to (4.7)

$$\partial\phi\, M^{\perp}\,\partial\phi = h^{ab}\partial_a\phi\cdot\partial_b\phi - \tfrac{1}{2}(\delta_1\bar{h}_{ab})^2, \tag{4.20}$$

$$\delta_1\bar{h}_{ab} \equiv \partial_a\varphi\cdot\partial_b\phi + \partial_a\phi\partial_b\varphi - h_{ab}\partial_c\varphi\cdot\partial_c\phi, \tag{4.21}$$

and thus $\delta_1\bar{h}_{ab}=0$ may be taken as *a coordinate gauge*. Averaging in a standard manner over a class of analogous gauges one can cancel the last term in (4.20) and then integrate over ϕ_μ (we use integral representation of $Z^{\perp}$ and (4.10))

$$Z^{\perp} = [\det(\Delta_0)_{\phi_\mu}]^{-1/2}\cdot[\det\Delta_{1ab}]^{1/2}. \tag{4.22}$$

Here the Δ_1-factor stands for the Faddeev–Popov determinant, which is the same as in the BDHP case (cf. (3.1), (3.7)–(3.9) with $g_{ab}\to h_{ab}$) because under the coordinate transformation

$$\delta\phi_\mu = \eta^a\partial_a\varphi_\mu, \qquad \delta(\delta_1\bar{h}_{ab}) = \nabla_a\eta_b + \nabla_b\eta_a - h_{ab}\nabla_c\eta_c. \tag{4.23}$$

Thus Δ_1 is defined on η_a with boundary conditions being (cf. (3.15))

$$\eta_n|_{\partial\mathscr{D}} = 0, \qquad \partial_n\eta_t|_{\partial\mathscr{D}} = 0. \tag{4.24}$$

Now we note that $\det\Delta_1$ *is explicitly calculable* according to (3.16)–(3.19) which are valid in coordinate system, where $h_{ab}=e^{2\sigma}\delta_{ab}$ (in arbitrary coordinates we need appropriate Green's function insertions, e.g., $\sigma\Box\sigma\to R\Box^{-1}R$, etc.). If we formally neglect a possible "twist" of fluctuation on the boundary and put $\phi^\mu|_{\partial\mathscr{D}}=0$, the first factor in (4.22) is also calculable and we get

$$Z^{\perp} \sim \exp\left\{-\frac{26-d}{12\pi}\int_{\mathscr{D}} d^2z\,\frac{1}{2}(\partial_a\sigma)^2 - \frac{2-d}{12\pi}\int_{\partial\mathscr{D}} K_T\cdot\sigma\cdot dz - \frac{d}{8\pi}\int_{\partial\mathscr{D}}\partial_{\tilde{n}}\sigma\, dz\right\}\times Z^{\perp}(0), \tag{4.25}$$

$$Z^{\perp}(0) = [\det(-\Box)_D]^{-(1/2)(d-2)} \tag{4.26}$$

Note also that according to (4.19) (see also [26])

$$Z^{\parallel} \sim \exp\left\{\frac{1}{2}\log A(C) - \frac{1}{4\pi}\int_{\partial\mathscr{D}}\partial_{\tilde{n}}\sigma\cdot dz\right\}. \tag{4.27}$$

Let us now consider the semiclassical approximation for the BDHP string (2.17). Expanding near the classical configuration (h_{ab},φ_μ) (2.10), we get

$$g_{ab} = h_{ab} + \varepsilon\gamma_{ab}, \qquad x_\mu = \varphi_\mu + \varepsilon\phi_\mu, \tag{4.28}$$

$$I_G(\varepsilon^2) = \varepsilon^2 M^2 \int_{\mathscr{D}} d^2 z \sqrt{h}\, \{h^{ab}\partial_a \phi_\mu \partial_b \phi_\mu$$

$$- 2\bar{\gamma}^{ab}\partial_a \varphi_\mu \partial_b \phi_\mu + \tfrac{1}{2}\bar{\gamma}^{ab}\bar{\gamma}_{ab}\}, \tag{4.30}$$

$$\bar{\gamma}_{ab} \equiv \gamma_{ab} - \tfrac{1}{2} h_{ab}\gamma, \qquad \gamma = h^{ab}\gamma_{ab}. \tag{4.30}$$

We see that γ drops from (4.29) due to the "quantum" Weyl invariance which is preserved (in contrast to the background one) in the semiclassical approximation. As a consequence, one can choose $\gamma = 0$ as a quantum Weyl gauge. Integrating over $\bar{\gamma}_{ab}$ we immediately obtain coincidence with the ε^2-part of the Nambu action (4.20).

Therefore after fixing the coordinate gauge ($\delta_1 \bar{h}_{ab} = 0$) and integrating over ϕ_μ we are led to the *same* result (4.11), (4.22) (or (4.25)) as in the Nambu string case. The final expression can also be obtained from the exact result (3.12), (3.13) if we put there $g_{ab} = h_{ab}(\varphi)$. This simply corresponds to fixing the gauge $\bar{\gamma}_{ab} = 0$ instead of $\delta_1 \bar{h}_{ab} = 0$.

It is worth pointing out that this semiclassical equivalence is rather obvious from the comparison of the exact expressions (2.17) and (2.35): one is simply to note that the δ-function in (3.35) is trivial on the classical equations. At the same time it contributes non-trivially in the next ("two loop") order, leading to *different* results for the Nambu and BDHP models.

As a next remark let us mention the emergence of the universal $(d-26)$-factor in a semiclassical approximation for bose string models (cf. (4.25)). As a result, for $d = 26$ "one-loop" corrections are rather trivial. They are also "trivial" if the contour C is a planar one (the minimal surface is flat and $\sigma = 0$ in (4.25)). Then the result is simply proportional to (4.26). A simple exercise then is to derive from (4.11), (4.12), (4.26) a universal static long-range $(R + R^{-1})$-potential [26, 32]

$$V_{\text{semiclass.}} = M^2 R - \frac{\gamma}{R} + \text{const}, \qquad \gamma = \frac{\pi}{12}\cdot\frac{d-2}{2} \tag{4.31}$$

by considering the standard $R \times T$ Wilson loop (and taking $\mathscr{D}$ to be a rectangular region) and noting that the Dirichlet boundary conditions $\phi_\mu(0) = \phi_\mu(R) = 0$ imply that we are dealing with a free Casimir type problem (or the partition function for a one-dimensional gas), i.e.,

$$[\operatorname{tr}\log(-\Box)_D]_{\text{finite}} = \frac{\pi T}{R}\zeta(-1), \qquad \zeta(-1) = -\frac{1}{12} \tag{4.32}$$

(we note in passing that the formal expression (4.31) first appeared in [33]). As a result, the BDHP string leads to the *same* semiclassical long-range potential (4.31) as other reasonable string models (including the Migdal one [34]).

Let us now prove that the equivalence of the Nambu, Eguchi and BDHP models holds also in the leading $1/d$ approximation for the static potential. For the first two models this approximation was discussed in [19] with the result

$$V_{d\to\infty} = M^2 R(1 - R_c^2/R^2)^{1/2},$$
$$R_c^2 = \frac{\pi d}{12 M^2}. \tag{4.33}$$

If we take C to be the $R \times T$ Wilson loop and $\mathscr{D}$ the rectangular $R \times T$ region, choose (instead of (3.1)) the following coordinate gauge for the BDHP model,

$$x_1 = z_1, \qquad x_2 = z_2 \tag{4.34}$$

and then integrate over x_i $(i = 3,..., d; x_i|_{\partial\mathscr{D}} = 0)$ in (2.17), we get the following effective large d action

$$\tilde{I}_G = M^2 \int_{-T/2}^{T/2} dt \int_0^R dr \left\{ \frac{1}{2} g^{ab} \sqrt{g}(\delta_{ab} + \sigma_{ab}) - \frac{1}{2} \alpha^{ab}\sigma_{ab} + \frac{d}{2} \operatorname{tr} \log(-\partial_a \alpha^{ab} \partial_b) \right\},$$
$$W_G \sim \int [\mathscr{D}g\, \mathscr{D}\sigma\, \mathscr{D}\alpha] \exp((-\tilde{I}_G). \tag{4.35}$$

Here we introduced the collective variable $\sigma_{ab} = \partial_a x_i \partial_b x_i$ with the help of the Lagrange multiplier α^{ab} (cf. (3.36)). The leading $1/d$-contribution is obtained by minimizing (4.35) with respect to g_{ab}, σ_{ab} and α^{ab}. Varying g_{ab} we get $g_{ab} \sim (\delta_{ab} + \sigma_{ab})$ (cf. (2.10)). Substituting this result in (4.35) we recover exactly the corresponding large d action for the Nambu model (Eq. (2.25) of Ref. [19] or (4.35) with the first term replaced by $[\det(\delta_{ab} + \sigma_{ab})]^{1/2}$) and *thus get the same result for the potential.*

It is probably rather obvious that the equivalence, observed in the semiclassical approximation, holds also in the leading term of the $1/d$ expansion, because the functional integral over metrics in (2.17) in this case is simply evaluated in a saddle point. It is conceivable that the difference between the Nambu and BDHP models will show up after an actual integration over metric in (3.12). If we formally integrate over σ in (3.12), using (3.29), neglecting boundary terms and assuming the Dirichlet boundary condition for σ, we get one more copy of $\log\det(-\Box)_D$ in (4.26) (cf. (3.31)) and thus are to replace (for $d < 26$) the $(d-2)$-coefficient in (4.31) by the $(d-1)$ one. This result is in analogy with the degrees of freedom counting (3.31) and may be sensible if the "longitudinal mode" σ, appearing through the anomaly, contributes in the long-range potential.

5. Scattering Amplitudes

In this section we are going to consider scattering amplitudes, predicted by string models, using a covariant path integral approach discussed (in the case of the Wilson factor ansatz) in the previous sections. It was observed in the early days of string models that some appropriate quantities, constructed with the help of a formal functional integral for the Nambu string, reproduce the Veneziano or Shapiro–Virasoro dual amplitudes (see, e.g., [1, 2]). The more or less consistent approaches of Refs. [25] and [22] (based on the non-covariant gauges (1.3) and (1.1a, b), respectively),[4] however, worked only with the transverse modes (though the $d = 26$ restriction was not evident in [25]). Treated in this way the Nambu model was equivalent to the corresponding dual models and thus possessed the same shortcomings, e.g. the tachyonic ground state (cf. (1.2)).

At the same time one may hope that if the longitudinal mode is properly taken into account, some more realistic amplitudes may emerge for $d = 4$. This implies the use of the Lorentz-covariant incomplete gauges with the correct account of the anomaly. However, at present it seems difficult to realize this program working directly with the Nambu action. As it is evident from Sections 3 and 4, the BDHP action (2.4) provides a more simple quantum extension of the same classical theory. That is why, it is the BDHP model we shall consider mainly in this section.

The basic point is how to *define* a scattering amplitude given a covariant string path integral. One may try to follow the old strings interaction formalism [25, 22], in particular the definition given in Ref. [25]. However, it implicitly assumes that the spectrum of the model is already known (e.g., from the operator formalism) and thus gives only the on-shell amplitude. On the other hand, one may propose some *heuristic* definition of the off-shell amplitude (appealing to the obvious differences in formulations of the Nambu and BDHP models) with a belief that for $d < 26$ it may cure the troubles of old dual models. Such a definition was proposed by Polyakov [20] and non-trivially interpreted and extended in Ref. [27], where it was shown that using a saddle point approximation (with $d \to -\infty$) for the BDHP model it is possible to recover not only the amplitude but also the spectrum of the standard Veneziano model. There was expressed a hope that next $1/d$ corrections will give more realistic dual-like amplitudes. The main obstacle which precluded verification of the above conjecture was the inclusion in Refs. [20, 27] of the non-linear "area" and "length" counter-terms (3.28). As was pointed out in Section 3, it seems consistent to put (in this or another way) renormalized values of μ and λ to zero. Then all functional integrations are carried out explicitly and we obtain the exact expression for the amplitude (see below).

Let us start with the analog of the "old" definition of Ref. [25]. This is justified by some uncertainties present in the Polyakov's heuristic definition to be discussed later. Basing mainly on a "correspondence principle" with the $d = 26$ Nambu string case,

[4] Note that the phase space path integral of Ref. [22] was not of course a true configuration space integral with the GGRT gauge but rather a direct analog of the operator formalism.

let us define the N reggeons (open strings) scattering amplitude (we consider throughout only tree amplitudes) as (cf. [25])

$$V(p_1,\dots,p_N)=\int_{-\infty}^{\infty} d\mu_{KN}\prod_{i=1}^{N}|z_{i+1}-z_i|^{-\alpha'\mu^2}\, Z[J_0(z)], \tag{5.1}$$

$$d\mu_{KN}=\prod_{j\neq a,b,c}^{N} dz_j\,|z_a-z_b|\,|z_b-z_c|\,|z_c-z_a|\prod_{i=1}^{N}|z_{i+1}-z_i|^{-1}, \tag{5.2}$$

$$J_0^{\mu}(z)=\sum_{i=1}^{N} p_i^{\mu}\,\delta^{(2)}(z-z_i), \qquad z_i\in\partial\mathscr{D},\quad \operatorname{Im} z_i=0, \tag{5.3}$$

$$Z[J]=\langle e^{iJ\cdot x}\rangle_{\partial_n x|_{\partial\mathscr{D}}=0}, \qquad J\cdot x=\int_{\mathscr{D}} d^2z\, J_\mu x_\mu, \tag{5.4}$$

where all particles are supposed to be in one state $p_j^2=-\mu^2$ (p_j are the euclidean momenta), $d\mu_{KN}$ is the projective invariant Koba-Nielsen measure (see, e.g., [1]), the domain $\mathscr{D}$ is taken to be the upper half-plane with variables z_j lying on the real axis $\partial\mathscr{D}$. Averaging in (5.4) is assumed to be done with the help of the string functional integral with the *Neumann* boundary conditions $\partial_n x_\mu|_{\partial\mathscr{D}}=0$ imposed. We omitted complications connected with a limiting procedure in the current (5.3) and self-energy factors (see [25, 35]). Let us also remind that (5.1) may be obtained by a conformal transformation from the qualitatively more obvious integral over "interaction times" τ_i (with $\mathscr{D}$ being a (σ,τ) strip with cuts; see, e.g., [1, 2]).

In the closed string case (pomeron or glueball scattering; see, e.g., [36]) interaction is possible along the whole length of the string (or the whole cylindric world surface) and so integration in (5.1) is to be carried out over all complex plane (i.e., $z_j\in\mathbb{C}$ in the current (5.3) and in (5.4))

$$\tilde{V}(p_1,\dots,p_N)=\int d\tilde{\mu}_{KN}\,\tilde{Z}[J_0(z)]\prod_{i=1}^{N}|z_{i+1}-z_i|^{-\alpha'\mu^2/2}, \tag{5.5}$$

$$d\tilde{\mu}_{KN}=\prod_{j\neq a,b,c} d^2z_j\,|z_a-z_b|^2\,|z_b-z_c|^2\,|z_c-z_a|^2\prod_{i=1}^{N}|z_{i+1}-z_i|^{-2}. \tag{5.6}$$

In the case of the Nambu model with $d=26$ definitions (5.1) and (5.5) were shown [25, 37, 36] to reproduce correspondingly the Veneziano and the Virasoro–Shapiro amplitudes with ground states being $\alpha'\mu_0^2=-1$ and $\alpha'\mu_0^2=-4$ and with, e.g.,

$$Z_{N,d=26}[J_0]=\prod_{i<j}|z_i-z_j|^{2\alpha' p_i p_j}, \qquad z_i\in\mathbb{R}. \tag{5.7}$$

Note that it is the momentum dependent integrand (5.7) of the amplitude that arises from the string path integral while the procedure of establishing measure (5.2) or (5.6) is not very compelling and straightforward even in Mandelstam's approach [22, 36]. As was already mentioned above, the derivation of (5.7) in [25] is not completely satisfactory one (for example, the non-covariant gauge (1.3) used depends on the source and thus the result was not manifestly factorizable). Let us now show that in the case of the covariantly quantized BDHP model expression (5.7) is the immediate consequence of taking $d=26$, while for $d<26$ a natural generalization emerges.

Using (2.4), (2.17) (now with the boundary condition $\partial_n x_\mu|_{\partial\mathscr{D}}=0$) and thus (3.29) (for the Neumann case) we get

$$Z_G[J]=\int d\sigma \exp\left\{-I_{\text{eff}}[\sigma]+\frac{1}{2M^2}\int d^2z\, d^2z' \times J(z)\, G(z,z'\mid\sigma)\, J(z')\right\}, \qquad M^{-2}=2\pi\alpha', \tag{5.8}$$

where G is the Neumann problem Green's function of the covariant Laplacian $\Delta_0=-\nabla_a\nabla^a$ (3.14)

$$\begin{aligned} &\Delta_{0_z} G(z,z')=-\delta(z,z')\equiv-\frac{1}{\sqrt{g(z)}}\,\delta^{(2)}(z-z'),\\ &\partial_{n_z} G(z,z')=0, \end{aligned} \tag{5.9}$$

and J is a tensor density like (5.3). For $\mathscr{D}$ being a halfplane and $g_{ab}=\delta_{ab}$

$$\begin{aligned} G(\sigma=0)=G_0&=\frac{1}{2\pi}\ln(|z'-z|\cdot|z'-\bar z|), && z\neq z'\\ &=\frac{1}{2\pi}\ln\varepsilon+\frac{1}{2\pi}\ln|z-\bar z|, && z'=z+\varepsilon,\quad \varepsilon\to 0. \end{aligned} \tag{5.10}$$

When $g_{ab}=e^{2\sigma}\delta_{ab}$ we are to use a covariant regularization:

$$(z'^a-z^a)(z'^b-z^b)\,g_{ab}=\varepsilon^2 \qquad \text{or} \qquad |z'-z|=e^{-\sigma}\varepsilon.$$

As a consequence of preserving the covariance we obtain the result of Refs. [20, 27]

$$\begin{aligned} G(z,z'\mid\sigma)&=G_0, && z\neq z',\\ &=G_0-\frac{1}{2\pi}\sigma(z), && z\to z'. \end{aligned} \tag{5.11}$$

Substituting (5.3) and (5.11) in (5.8) we get

$$Z_G[J_0]=Z'\cdot Z_{0_N}, \qquad Z_{0_N}\sim\exp\left(-\frac{1}{2M^2}J_0G_0J_0\right), \tag{5.12}$$

$$Z'=\int d\sigma\cdot e^{-I_{\text{eff}}+\mathscr{T}\cdot\sigma}, \qquad \mathscr{T}=-\frac{\alpha'}{2}\sum_j p_j^2\,\delta^{(2)}(z-z_j), \tag{5.13}$$

where Z_{0_N} is of course equivalent to (5.7), i.e., to the result of the analogue model [37] or that of the naive transverse mode treatment of the Nambu model. If $d = 26$ and the Neumann boundary condition (3.30) is imposed on σ, I_{eff} is trivial ($K_T = 0$ for a flat $\partial\mathscr{D}$, cf. (A.11), and we omit the zero mode contribution in (3.29)); i.e., there is no conformal anomaly in the theory and that is why the Weyl gauge $\sigma = 0$ must be imposed. As a result, we are straightforwardly led to (5.7) as the exact prediction of the BDHP model for $d = 26$. This is indeed a trivial consequence of the fact that for $d = 26$ the gauge $g_{ab} = \delta_{ab}$ is an admissible one and thus (2.4) is just the action of the analogue model. Comparing the above derivation of (5.7) with that of Ref. [25] for the Nambu model we once more are convinced that the BDHP model is a simpler one.

In order to evaluate (5.13) for $d < 26$ we need some boundary condition on σ (cf. the discussion in Section 2). A natural choice is again the Neumann one (3.30) and we get

$$Z' = C' e^{-(1/2\nu)\mathscr{J} G_0 \mathscr{J}}, \qquad \nu = \frac{26 - d}{12\pi} > 0. \tag{5.14}$$

The resulting amplitude is (5.1) with

$$Z_G \sim \prod_{i<j} |z_i - z_j|^{2\gamma_{ij}},$$

$$\gamma_{ij} = \alpha' p_i p_j - \frac{24}{26 - d} \left(\frac{\alpha'}{4}\right)^2 p_i^2 p_j^2 = \alpha' p_i p_j - \mathscr{H}, \tag{5.15}$$

$$\mathscr{H} = \frac{3}{2} \cdot \frac{\alpha'^2}{26 - d} \cdot \mu^4, \qquad p_i^2 = -\mu^2. \tag{5.16}$$

The whole effect of the integration over σ is thus a finite shift $\mathscr{H}$ in (5.16). It is now a standart exercise (see, e.g., [1]) to work out the poles of the corresponding amplitude

$$\alpha(s) \equiv \alpha(0) - \alpha' s = n, \qquad n = 0, 1, \ldots, \qquad s = \Big(\sum_{i<\tau} p_i\Big)^2, \tag{5.17}$$

$$\alpha(0) = -\alpha' \mu^2 + \mathscr{H}, \tag{5.18}$$

(recall that our momenta are euclidean). Thus we have a linear Regge trajectory and the Veneziano condition $\alpha(0) = 1$ implies that the ground state mass is $\alpha'\mu^2 \simeq 15.6$ or $\alpha'\mu^2 \simeq -0.94$. As a result, the ground state is not necessarily tachyonic (note that the tachyonic value is near to that of the Veneziano model). The amplitude is dual (i.e., the integrand of (5.1) is projectively invariant) only when $\mathscr{H} = 0$, i.e., only for $d = 26$, and therefore it is not clear whether we are to impose the condition $\alpha(0) = 1$ or rather to assume $\alpha'\mu^2 = -1$ in accordance with $d = 26$ case. In any case it should be noted that deviations from the Veneziano model are rather small for $d=4$ and probably do not contradict the common wisdom.

The above approach has a drawback of not revealing the spectrum of states (e.g., we are to *assume* $\alpha(0)=1$ as in the Veneziano model). Let us therefore consider the following definition of the off-shell amplitude suggested in [20] (cf. (5.1)–(5.6)):

$$\Gamma(p_1,..., p_N)=\left\langle \prod_{j=1}^{N} \int_{\mathscr{D}} d^2 z_j \sqrt{g(z_j)}\, e^{ip_j \cdot x(z_j)} \right\rangle_{\partial_n x|_{\partial\mathscr{D}}=0}. \tag{5.19}$$

Poles of (5.19) are to define the mass spectrum, while the residues in these poles are the scattering amplitudes. It was claimed in [20] that for $d=26$ one obtains from (5.19) the standard dual model in the Koba–Nielsen form. However, this proposal is somewhat ambiguous and unclear. Really, only *closed* surfaces were considered in [20] and so $\{z_j\}$ in (5.19) are supposed to lie on a closed one. Then we recognize that the author of Ref. [20] was probably speaking about the scattering of short *closed* strings or pomerons (imagine the closed membrane with tubes coming out of it). The fact that *surface* (and not boundary) z_j-integrations appear in (5.19) is now justified by the analogy with the Virasoro–Shapiro amplitude (cf. (5.5), (5.6)). However, it seems impossible to obtain (for $d=26$) the whole spectrum and the Koba–Nielsen measure for the Virasoro–Shapiro model starting only with (5.19).

This heuristic definition (5.19) was given rather different (and more precise) interpretation in Ref. [27], where (5.19) was identified with the off-shell *reggeon* (meson) scattering amplitude (with $\mathscr{D}$ being a half-plane) with the idea to reproduce the on-shell amplitude (where $z_j \in \partial\mathscr{D}$) taking the residues in the poles (i.e., for $z_j \to \bar{z}_j$). This idea *does not work* for the closed string case, where integration goes over the complex plane already in the on-shell amplitude (cf. (5.5), (5.6)). Restricting to the open strings, let us study this proposal, assuming first that two counter-terms (3.28) (with $\mu, \lambda \neq 0$) are added in (3.29). Observing that (5.19) is closely related to (5.1), (5.12), (5.13) (now with $z_j \in \mathscr{D}$) we get (cf. [20, 27, 28])

$$\Gamma(p_1,..., p_N)=C\int \prod_{j=1}^{N} d^2 z_j \int d\sigma \exp(-\tilde{I}_{\rm eff}[\sigma]+\tilde{\mathscr{F}}\sigma)$$

$$\times \exp\left\{\pi\alpha' \sum_{i,j=1}^{N} p_i p_j \cdot G_0(z_i, z_j)\right\}, \tag{5.20}$$

$$\tilde{I}_{\rm eff}[\sigma]=\frac{26-d}{12\pi}\left[\int_{\mathscr{D}} \left(\frac{1}{2}(\partial_a \sigma)^2+\tilde{\mu}^2 e^{2\sigma}\right) d^2 z\right.$$

$$\left.+\int_{\partial\mathscr{D}} (K_T \sigma+\tilde{\lambda} e^{\sigma})\, dz\right]+\frac{d}{8\pi}\int_{\partial\mathscr{D}} \partial_{\tilde{n}} \sigma\, dz,$$

$$(\mu \sim \tilde{\mu}, \lambda \sim \tilde{\lambda}), \tag{5.21}$$

$$\tilde{\mathscr{F}}=2\sum_{j=1}^{N} \delta^{(2)}(z-z_j)\left(1-\frac{\alpha'}{4} p_j^2\right), \tag{5.22}$$

where we separated all σ-dependent terms, using (5.11). If we put at this point $d=26$, taking the Weyl gauge $\sigma=0$, we get the amplitude of the general structure

$$\Gamma(p_1,..., p_N) = \prod_{j=1}^{N} \int_{\mathscr{D}} d^2 z_j \prod_{i<j}^{N} (|z_i - z_j|\,|z_i - \bar{z}_j|)^{\gamma_{ij}} \prod_{k=1}^{N} |z_k - \bar{z}_k|^{\delta_k}, \quad (5.23)$$

where for $d=26$

$$\gamma_{ij}^{(0)} = \alpha' p_i p_j, \qquad \delta_k^{(0)} = \frac{1}{2} \gamma_{kk}^{(0)} = \frac{\alpha'}{2} p_k^2. \quad (5.24)$$

According to Ref. [27] the poles of the amplitude like (5.23) occur when $z_j \to \bar{z}_j$ (integrations in the residues go over the boundary $\operatorname{Im} z = 0$), i.e., the mass spectrum generating condition is

$$\delta_k = -n, \qquad n = 1, 2,... . \quad (5.15)$$

Using (5.24), (5.25) we get the following spectrum for $d=26$,

$$\alpha' p_k^2 = -2n, \quad (5.26)$$

which *does not coincide* with that of the Veneziano model ($\alpha' p_k^2 = -n+2$), e.g., there is no ground-state tachyon. This result implies that the above definition (5.19) *does not satisfy the correspondence principle* with the basic fact (obvious when starting with (5.1)) that for $d=26$ the amplitude is the Veneziano one.

However, in [27] the Veneziano model was claimed to follow from (5.20), (5.21) in rather unphysical $d \to -\infty$ saddle point approximation. Being unable to calculate the integral over σ in (5.20) (for $\tilde{\mu}, \tilde{\lambda} \neq 0$ in (5.21)) these authors evaluated it at the saddle point

$$\sigma(z) = \frac{1}{2} \log \frac{2}{\tilde{\mu}^2 |z-\bar{z}|^2}, \qquad \tilde{\lambda} = \sqrt{2}\,\tilde{\mu}, \quad (5.27)$$

i.e., a solution of the classical equations corresponding to (5.21), which is singular on the boundary ($\sigma|_{\partial\mathscr{D}} \to \infty$, $\partial_n \sigma|_{\partial\mathscr{D}} \to \infty$). In this case one again is led to (5.23) but now with

$$\tilde{\delta}_k^{(0)} = -2 + \alpha' p_k^2 \quad (5.28)$$

yielding (through (5.25)) the proper Veneziano spectrum ($\alpha' p_k^2 = 1, 0, -1,...$). At the same time it is not quite clear if the proper Koba–Nielsen measure (5.2) can be naturally obtained after taking $z_j \to \bar{z}_j$ in (5.23).

The above result is not a very appealing one: we got the same Veneziano model (with all its drawbacks) as an outcome of rather non-trivial approximation essentially based on $\tilde{\mu}, \tilde{\lambda} \neq 0$. The only merit is a possibility to obtain the off-shell amplitude. Let us now see if the situation can be improved by putting the renormalized values of $\tilde{\mu}$

and $\tilde{\lambda}$ in (5.21) equal to zero and integrating over σ in (5.20) exactly. Again (cf. (5.13)) we confront the question about a boundary condition on σ (note that this question would not arise if one develops a perturbation theory near (5.27), which automatically dictates singular boundary behaviour). When $\tilde{\mu}=\tilde{\lambda}=0$ the most obvious choice is the Neumann one. This condition is even necessary in order to integrate $\exp\{-\int[(\partial\sigma)^2+\mathscr{F}\sigma]\,d^2z\}$ over σ by the shift $\sigma\to\sigma+\sigma_0$, $\sigma_0\sim\square^{-1}\mathscr{F}$ (one must equate $\int_{\partial\mathscr{D}}dz(\sigma\partial_{\tilde{n}}\sigma_0)$ to zero in order to cancel the linear in σ terms). In complete analogy with a previous treatment (5.13)–(5.16) we get (5.23) with

$$\gamma_{ij}=\alpha' p_i p_j-\zeta_{ij},\qquad \delta_k=\frac{1}{2}\gamma_{kk}=\frac{\alpha' p_k^2}{2}-\frac{1}{2}\zeta, \tag{5.29}$$

$$\zeta_{ij}=\frac{24}{26-d}\left(1-\frac{\alpha'}{4}p_i^2\right)\left(1-\frac{\alpha'}{4}p_j^2\right),\qquad \zeta=\zeta_{ii}. \tag{5.30}$$

When all particles are in the same state with $p_j^2=-\mu^2$

$$\zeta=\zeta_{ij}=\frac{24}{26-d}\left(1+\frac{\alpha'}{4}\mu^2\right)^2. \tag{5.31}$$

According to (5.25) for the mass spectrum we get

$$-\frac{\alpha'\mu^2}{2}-\frac{12}{26-d}\left(1+\frac{\alpha'}{4}\mu^2\right)^2=-n, \tag{5.32}$$

$$\alpha'\mu^2=-\frac{34}{3}\pm\sqrt{\left(\frac{34}{3}\right)^2+16\left(\frac{11}{6}n-1\right)},\qquad d=4. \tag{5.33}$$

As a result, for each $n=1,2,\ldots$ we have a physical as well as a tachyonic state. If we omit the tachyonic trajectory (as "unphysical" one), we approximately have for small n $\alpha'\mu^2\simeq 1.3n-0.7$, while in general the physical trajectory is a non-linear one.

Summarizing the discussion of this section we conclude that in the framework of the first definition of the scattering amplitude (5.1)–(5.4) the BDHP model naturally gives the old Veneziano (or Shapiro–Virasoro) result for $d=26$, while for $d<26$ (e.g., $d=4$) it predicts a slightly modified non-dual amplitude (5.1), (5.15), (5.16) with the linear Regge trajectory with a "shifted" intercept (5.18) (providing a possibility of having a non-tachyonic ground state). Starting with the off-shell definition (5.19) we *do not obtain the spectrum of the conventional dual model* for $d=26$ (cf. (5.26)). This probably may be considered as a drawback of this heuristic definition. When $d<26$ we are led to some off-shell amplitude with a non-linear spectrum (5.33) possessing a physical branch without tachyonic ground state.

Though in the absence of some reasonable QCD-motivated definition of strings scattering amplitudes the discussed results are certainly incomplete and preliminary, it is the possibility itself that the covariantly quantized BDHP string leads (for $d=4$) to a non-linear mass spectrum and non-standard amplitudes that seems very interesting and may show the way from the old "ideal" dual models to more realistic ones expected to follow from QCD.

APPENDIX A: NOTATIONS FOR GEOMETRICAL OBJECTS ON TWO-DIMENSIONAL MANIFOLD WITH BOUNDARY

Let M^2 be a two-dimensional manifold with boundary ∂M^2. We shall mainly consider the simplest case when M^2 can be covered by one coordinate system, i.e., is diffeomorphic to a domain $\mathscr{D} \subset \mathbb{R}^2$ with boundary $\partial\mathscr{D}$. The corresponding coordinates are $\{z^a\}$, $a = 1, 2$. If g_{ab} is the (euclidean signatured) metric on M^2, the following relations are valid for the curvature:

$$R^a_{\ bcd} = \frac{R}{2}(\delta^a_c g_{bd} - \delta^a_d g_{bc}), \qquad R_{ab} = \frac{R}{2} g_{ab},$$
$$R^a_{\ bcd} = \partial_c \Gamma^a_{bd} - \cdots, \qquad R_{ab} = R^c_{\ acb}, \qquad R = R_{ab} g^{ab}. \tag{A.1}$$

In the conformal coordinates we have

$$g_{ab} = \rho^2 \delta_{ab}, \qquad \rho = e^\sigma, \tag{A.2}$$

$$R = -2e^{-2\sigma}\Box\sigma, \qquad \Box = \partial_a \partial_a. \tag{A.3}$$

Let t^a and n^a be some vector fields in a neighbourhood of the boundary, coinciding on ∂M with the unite tangent vector and outward normal to the boundary

$$t^a = \frac{dz^a}{ds}, \qquad ds^2 = g_{ab}\, dz^a\, dz^b, \tag{A.4}$$

$$n_a = \mathscr{E}_{ab} t^b, \qquad \mathscr{E}^{ab} = \frac{1}{\sqrt{g}} \varepsilon^{ab}, \qquad \varepsilon^{12} = +1, \tag{A.5}$$

$$t^a t_a = 1, \qquad n^a n_a = 1, \qquad t^a n_a = 0,$$
$$g_{ab} = t_a t_b + n_a n_b \tag{A.6}$$

(all indices are raised and lowered with g_{ab}).

The second fundamental form of the boundary and its trace are defined as follows

$$K_{ab} = \nabla_c n_d \gamma^c_{(a} \gamma^d_{b)}, \qquad K = K_{ab} g^{ab}, \tag{A.7}$$

where $\gamma_{ab} = g_{ab} - n_a n_b$ is the induced metric on ∂M and ∇_c is the g_{ab}-covariant derivative. Using (A.6) we get

$$K = \nabla_a n^a \qquad \text{or} \qquad K = -n_a \nabla_t t^a,$$
$$\nabla_t = t^a \nabla_a. \tag{A.8}$$

One can always split $K\,ds$ on metric-dependent and metric-independent ("topological") parts

$$\sqrt{\gamma}\,K = \sqrt{\gamma}\,K_R + K_T. \tag{A.9}$$

(If τ is an arbitrary parametrization of ∂M, $ds = \sqrt{\gamma}\,d\tau$). In conformal coordinates (A.2) we have

$$\begin{gathered} ds = \rho\,dz, \qquad dz = \sqrt{dz_a\,dz_a}, \\ K\,ds = dz\,\partial_{\tilde{n}}\sigma + K_T\,dz, \qquad \partial_{\tilde{n}} = \tilde{n}_a \partial_a, \\ K_T = -\tilde{n}_a \partial_{\tilde{t}} \tilde{t}^a = \partial_a \tilde{n}_a, \qquad \tilde{n}_a \tilde{n}_a = 1, \\ \tilde{t}_a \tilde{t}_a = 1, \qquad n^a = \tilde{n}^a \cdot \rho^{-1}, \end{gathered} \tag{A.10}$$

where dz, $\tilde{n}^a$ and $\tilde{t}^a$ are the flat metric counterparts of ds, n^a and t^a. Note also that in an arbitrary parametrization of the boundary

$$\rho K = \varepsilon_{ab} z'^b \partial_a \sigma + \frac{\varepsilon_{ab} z'^a z''^b}{z'^2}, \tag{A.11}$$

where

$$z'^a = \frac{dz^a}{d\tau}, \qquad \tilde{t}^a = \frac{z'^a}{|z'|}, \qquad K_T = \varepsilon_{ab} \tilde{t}^a \frac{d}{d\tau} \tilde{t}^b.$$

The Euler number of M^2 is

$$\chi = \chi_V + \chi_S = \frac{1}{4\pi}\left(\int_M R\,\sqrt{g}\,d^2z + \int_{\partial M} 2K\,\sqrt{\gamma}\,d\tau\right). \tag{A.12}$$

As a topological invariant it must be independent of g_{ab} in the case of M^2 being homeomorphic to $\mathscr{D}$. This is really true as one can see from (A.3), (A.9), (A.10): $\int_M R\,\sqrt{g}\,d^2z$ and $\int_{\partial M} K_R\,\sqrt{\gamma}\,d\tau$ simply cancel ($R\,\sqrt{g} = -2\Box\sigma$) and as a result

$$\chi = \frac{1}{4\pi}\int_{\partial\mathscr{D}} 2K_T\,d\tau = \frac{1}{2\pi}\int_{\partial\mathscr{D}} \partial_a \tilde{n}_a\,d\tau, \tag{A.13}$$

implying that $\chi(M) = 1 - N_H$, N_H = number of holes (one is to use that for a unit circle $\tilde{n}^a = (\cos\theta, \sin\theta)$, $\partial_a \tilde{n}_a = 1$ and $\chi = 1/2\pi \int_0^{2\pi} d\theta = 1$). The general case of a "non-planar" M^2 is treated by cutting it on a number of "planar" parts and using (A.12) (and the Gauss theorem) for each part. As a result, $\chi(\partial M \neq \phi) = 1 - N_H - 2N_h$, $\chi(\partial M = \phi) = 2 - N_H - 2N_h$, where N_h is the number of handles.

APPENDIX B: DIVERGENCES OF log det Δ AND SEELEY COEFFICIENTS

Let Δ be the following elliptic differential operator,

$$\Delta = -\frac{1}{\sqrt{g}} \mathscr{D}_a g^{ab} \sqrt{g} \mathscr{D}_b + X, \tag{B.1}$$

where $\mathscr{D}_a = \partial_a + B_a$ (B_a and x are internal space matrices). By definition

$$\Gamma = \frac{1}{2} \log \det \Delta = -\frac{1}{2} \int_\varepsilon^\infty \frac{dt}{t} \operatorname{tr} e^{-t\Delta}, \qquad \varepsilon \to +0. \tag{B.2}$$

The infinite part of Γ is given by ($L \to \infty$)

$$\Gamma_\infty = -\frac{1}{2} \left(A_0 L^2 + 2A_1 L + A_2 \log \frac{L^2}{\mu^2} \right), \tag{B.3}$$

where

$$(\operatorname{tr} e^{-t\Delta})_{t \to 0} \simeq \sum_{k=0}^{2} A_k t^{(k-2)/2} + O(\sqrt{t}), \tag{B.4}$$

$$A_k = \int_M b_k \sqrt{g}\, d^2 z + \int_{\partial M} c_k \sqrt{\gamma}\, d\tau, \qquad b_{2p+1} = 0. \tag{B.5}$$

Equation (B.4) is a consequence of the general expansion

$$\int_M \langle z | e^{-t\Delta} | z \rangle f(z) \sqrt{g}\, d^2 z \,\bigg|_{t \to 0} = \sum_{k=0}^{2} \mathscr{A}_k[f]\, t^{(k-2)/2} + O(\sqrt{t}), \tag{B.6}$$

where f is a smooth function on the closure of $\mathscr{D}$. As a result,

$$A_k = \operatorname{tr} \mathscr{A}_k[f = 1]. \tag{B.7}$$

The values of $\mathscr{A}_k$ depend on the boundary conditions assumed in the definition of $\det \Delta$. Let us consider the Dirichlet and the generalized Neumann (or Robin) boundary problems

$$\Delta_D \phi = \lambda \phi, \qquad \phi|_{\partial\mathscr{D}} = 0, \tag{B.8}$$

$$\Delta_R \phi = \lambda \phi, \qquad (\partial_n + \psi)\phi|_{\partial\mathscr{D}} = 0 \tag{B.9}$$

(ψ is a given matrix function). For the Dirichlet case one has (see, e.g., [26])

$$4\pi \mathscr{A}_0 = I \int_{\mathscr{D}} f \sqrt{g}\, d^2 z, \qquad 4\pi \mathscr{A}_1 = -\frac{\sqrt{\pi}}{2} \cdot I \cdot \int_{\partial\mathscr{D}} f\, ds, \tag{B.10}$$

$$4\pi\mathscr{A}_2 = \int_{\mathscr{D}} f\sqrt{g}\left(\frac{R}{6}I - X\right) d^2z + \frac{1}{6} I \int_{\partial\mathscr{D}} 2Kf\,ds + \frac{1}{2} I \int_{\partial\mathscr{D}} \partial_n f\,ds, \tag{B.11}$$

where $ds = \sqrt{\gamma}\,d\tau$ and I is a unite internal space matrix. The analogous result for the Robin case is

$$4\pi\mathscr{A}_0 = I\int_{\mathscr{D}} f\sqrt{g}\,d^2z, \qquad 4\pi\mathscr{A}_1 = +\frac{\sqrt{\pi}}{2} I \int_{\partial\mathscr{D}} f\,ds,$$
$$4\pi\mathscr{A}_2 = \int_{\mathscr{D}} f\sqrt{g}\left(\frac{R}{6}I - X\right) d^2z + \frac{1}{6} I \int_{\partial\mathscr{D}} 2Kf\,ds \tag{B.12}$$
$$+ 2\int_{\partial\mathscr{D}} (B_a n^a - \psi) f\,ds - \frac{1}{2} I \int_{\partial\mathscr{D}} \partial_n f\,ds.$$

Thus the difference between (B.11) and (B.12) is in the change of two signs and in the new $(B_n - \psi)$-term. The corresponding Seeley coefficients are (B.7), (B.5)

$$\bar{b}_p \equiv 4\pi b_p, \qquad \bar{c}_p \equiv 4\pi c_p,$$
$$\Delta_D, \Delta_R\colon \bar{b}_0 = \mathrm{tr}\, I, \qquad \bar{b}_2 = \mathrm{tr}\left(I\cdot\frac{R}{6} - X\right), \tag{B.13}$$

$$\Delta_D\colon \bar{c}_0 = 0, \qquad \bar{c}_1 = -\frac{\sqrt{\pi}}{2}\,\mathrm{tr}\, I,$$
$$\bar{c}_2 = \frac{1}{6}\cdot 2K\cdot \mathrm{tr}\, I, \tag{B.14}$$

$$\Delta_R\colon \bar{c}_0 = 0, \qquad \bar{c}_1 = +\frac{\sqrt{\pi}}{2}\,\mathrm{tr}\, I,$$
$$\bar{c}_2 = \frac{1}{6} 2K\,\mathrm{tr}\, I + 2\,\mathrm{tr}(B_a n^a - \psi). \tag{B.15}$$

Results (B.13), (B.14) were obtained by McKean and Singer [38] (see also [39, 26]). The Robin case (for $B_a = 0$) was discussed in [40]. Finally, the Neumann case ($\psi = 0$) results of Ref. [38] were corrected by the B_n-term in [28]. It should, however, be pointed out that problem (B.9) is a "well-posed" one only if $\psi = B_n$ or $n^a\mathscr{D}_a\phi|_{\partial\mathscr{D}} = 0$ (and so in this "well-posed" case the results of [39] are correct). Really, only under this condition the operator Δ is a symmetrical one: $(\phi_1, \Delta\phi_2) = (\phi_2, \Delta\phi_1)$ and the boundary condition is covariant under the internal gauge transformations. At the same time, the expression for c_2 with $(B_n - \psi)$-term is sometimes useful in the formal discussion of non-self-adjoint operators.

It is easy to check the consistency of the $(B_n - \psi)$-combination. For example, consider the case of Δ being the flat Laplacian defined on scalars and $\psi = \partial_n \log \rho$. Then the change $\phi \to \rho\phi$ results in the Neumann problem for $\Delta = -\mathscr{D}_a \mathscr{D}_a$, $\mathscr{D}_a = \partial_a - \partial_a \log \rho$. Thus the results for c_2 in both cases are the same.

APPENDIX C: Effective Action for the Ghost Determinant of the BDHP Model

Establishing expressions (3.17)–(3.18) for $\Gamma(\Delta_1)$ in (3.13) is a non-trivial problem, solved for the "volume" part in [20] (see also [15]) and for the boundary one in [28]. However, the discussion of Ref. [28] seems to be too complicated and slightly incomplete. Boundary terms actually were found there for a half-plane case, where $K_T = 0$, while the K_T-dependent terms were then reconstructed demanding the conformal invariance of the Δ_1-effective action. The condition of conformal invariance (under (3.3)) of the result must be the consequence not only of invariance the ("incompleteness") of gauge (3.1) *but* also the conformal invariance of *the boundary conditions* on ξ_a (3.15). We want to show here how to obtain the correct result working only with covariant operators (B.1) on a *general domain* $\mathscr{D}$, defined on scalars with the appropriate Dirichlet and Robin boundary conditions, automatically establishing the conformal invariance of the result and thus using only results (B.10)–(B.12) (with $B_a = 0$) which were known before the analysis of the B_a-contribution in (B.12) made in [28].

We begin by noting that [15] (see Appendix B for notations, cf. (3.21))

$$\delta\Gamma(\Delta_1) = -2\mathscr{A}_2^{(\Delta_1)}[\delta\sigma] + \mathscr{A}_2^{(\hat{\Delta}_1)}[\delta\sigma], \tag{C.1}$$

where

$$\Delta_{1ab} = (-\nabla_c^2 - R/2)\, g_{ab}, \qquad \hat{\Delta}_{1ab} = (-\nabla_c^2 + R)\, g_{ab}, \tag{C.2}$$

and the boundary conditions for Δ_1 and $\hat{\Delta}_1$ are assumed to be of the mixed type, respecting the coordinate invariance and thus the residual conformal one (cf. (3.3) and discussion in [28]). By the appropriate rotation of ξ_a we get

$$\det \Delta_{1ab} = \det \Delta_{1D} \det \Delta_{1R}, \qquad \det \hat{\Delta}_{1ab} = \det \hat{\Delta}_{1D} \det \hat{\Delta}_{1R}, \tag{C.3}$$

where

$$\Delta_1 = -\nabla_c^2 - R/2, \qquad \hat{\Delta}_1 = -\nabla_c^2 + R \tag{C.4}$$

are defined on scalars with the Dirichlet or Robin boundary conditions

$$\Delta_{1D}\colon \phi|_{\partial\mathscr{D}} = 0, \qquad \Delta_{1R}\colon (\partial_n + K)\phi|_{\partial\mathscr{D}} = 0, \tag{C.5}$$

$$\hat{\Delta}_{1D}\colon \hat{\phi}|_{\partial\mathscr{D}} = 0, \qquad \hat{\Delta}_{1R}\colon (\partial_n - 2K)\hat{\phi}|_{\partial\mathscr{D}} = 0, \tag{C.6}$$

where K is defined in (A.8) (and so, e.g., $\Delta_{1R}\colon \nabla_a(n^a\phi)|_{\partial\mathscr{D}}=0$). If $g_{ab}=e^{2\sigma}\delta_{ab}$ and $K_T=0$, $K=\partial_n\sigma$ and $\Delta_{1R}\colon \partial_n(e^{\sigma}\phi)|_{\partial\mathscr{D}}=0$ which should be compared with $\partial_n\xi_t=\partial_n(e^{\sigma}\xi_{\bar{t}})=0$ in (3.15). Note that geometrical Robin conditions like (C.5) were considered also in [40]. Following [12, 28] it is useful to consider a generalization of above problem:

$$\Delta_j=-\nabla_c^2-j\frac{R}{2}, \qquad \Delta_{-j-1}=-\nabla_c^2+(j+1)\frac{R}{2}, \tag{C.7}$$

$$\Delta_{jD}\colon \phi_j|_{\partial\mathscr{D}}=0, \qquad \Delta_{jR}\colon (\partial_n+jK)\,\phi_j|_{\partial\mathscr{D}}=0, \tag{C.8}$$

$$\delta\Gamma(\Delta_{jab})=-(j+1)\,\mathscr{A}_2^{(j)}[\delta\sigma]+j\mathscr{A}_2^{(-j-1)}[\delta\sigma], \tag{C.9}$$

$$\Gamma_\infty(\Delta_{jab})=-\frac{1}{2}\left(A_0^{(j)}L^2+2A_1^{(j)}L+A_2^{(j)}\log\frac{L^2}{\mu^2}\right), \tag{C.10}$$

$$\mathscr{A}_k^{(j)}\equiv\mathscr{A}_k^{(\Delta_{jD})}+\mathscr{A}_k^{(\Delta_{jR})}, \qquad k=0,1,2. \tag{C.11}$$

Here Δ_{jab} is defined on ξ^a and in our case of Δ_1, $j=1$ (i.e., $\hat{\Delta}_1=\Delta_{-2}$). Applying (B.10)–(B.12) for Δ_j with the above boundary conditions and using (C.11) we get

$$4\pi A_0^{(j)}=2\int_{\mathscr{D}}\sqrt{g}\,d^2z, \qquad A_1^{(j)}=0, \qquad A_2^{(j)}=2\cdot\frac{\chi}{6}, \tag{C.12}$$

$$4\pi\mathscr{A}_2^{(j)}[\delta\sigma]=(\tfrac{1}{3}+j)\left(\int_{\mathscr{D}}R\sqrt{g}\,\delta\sigma\,d^2z+\int_{\partial\mathscr{D}}2K\,\delta\sigma\,ds\right). \tag{C.13}$$

Using (A.2), (A.3), (A.10) and the Gauss theorem we can rewrite the expression in the bracket in (C.13) as a total variation and thus integrate over σ. This is essentially due to conformal invariance of the boundary conditions providing the "correlation" of the "R" and "$2K$"-terms in $\mathscr{A}_2^{(j)}$. The result is (cf. [12, 28])

$$\Gamma_\sigma(\Delta_{jab})=-\frac{1+6j(j+1)}{6\pi}\left(\int_{\mathscr{D}}\frac{1}{2}(\partial_a\sigma)^2\,d^2z\right.$$
$$\left.+\int_{\partial\mathscr{D}}K_T\cdot\sigma\cdot dz\right)+\text{const}, \tag{C.14}$$

coinciding for $j=1$ with (3.18). Note that if the boundary is flat (as in the case of applications to scattering amplitudes; see Section 5), $K_T=0$ and hence boundary term in (C.14) vanishes.

As a final comment let us indicate some connection of (C.7) with operators $\mathscr{L}_j=-e^{-2(j+1)\sigma}\partial e^{2j\sigma}\bar{\partial}$ on complex functions used in [12, 28].

According to (A.2), (A.3),

$$\Delta_j=-e^{-2\sigma}\partial_a e^{j\sigma}\partial_a e^{-j\sigma}, \tag{C.15}$$

or after the change of variables $(\Delta_j \phi_j = \lambda \phi_j \to \tilde{\Delta}_j \tilde{\phi}_j = \lambda \tilde{\phi}_j)$

$$\tilde{\phi}_j = e^{-j\sigma} \phi_j, \qquad \tilde{\Delta}_j = -e^{-2\sigma(j/2+1)} \partial_a e^{2\sigma \cdot (j/2)} \partial_a. \tag{C.16}$$

Note added in proof. Recently we derived the general formula for the string scattering amplitude in the context of the $N \to \infty$ QCD [41].

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