# Quantum Properties of Higher-Dimensional and Dimensionally Reduced Supersymmetric Theories

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Quantum properties of $d$-dimensional and $d=4$ reduced theories are studied with emphasis on the use of a regularization accounting for power divergences. It is shown that the vanishing of the $N=4$, $d=4$ super Yang–Mills ($SYM_4^4$) one-loop $\beta$-function and of conformal anomalies in $N=8$, $d=4$ supergravity ($SG_4^8$) are consequences of the absence of certain power divergences in corresponding $d=10$ ($SYM_{10}^1$) and $d=11$ ($SG_{11}^1$) theories.

## 1. INTRODUCTION

It is by now well known that *classical* actions of several interesting supersymmetric theories (e.g. $SYM_4^4$ [1] and $SG_4^8$ [2]) can be obtained by the simple dimensional reduction of the actions of higher-dimensional theories ($SYM_{10}^1$ and $SG_{11}^1$). We address the question of possible connection of *quantum* properties of $d$-dimensional and dimensionally reduced theories. In particular, we are going to present some new results concerning the one-loop infinities in higher-dimensional SYM and SG and trace their relation to the corresponding ones in $d=4$ reduced theories. Also we shall illustrate the efficiency of the $d$-dimensional approach with the example of one-loop effective action in $SYM_4^4$. Only final expressions will be given; the expanded version of this work[3] will be published elsewhere.

There are two possible points of view on the relation between $d>4$ and $d=4$ theories:

(a) Our world actually is described by a complete $d>4$ theory, while its apparent 4-dimensionality is only a low energy phenomenon and can be attributed to a particular structure of $d$-dimensional vacuum ($M^4 \times N^{d-4}$, $N$ is a compact space of a "scale" $r \sim m_P^{-1}$).

(b) Dimensional reduction is only a technical tool, useful in analysis of 4-dimensional theories (e.g., in construction of different versions of supergravities).

The first point of view, though quite satisfactory from the classical point of view, is inconsistent at the quantum level. The point is that we cannot correctly define the effective $d=4$ theory of light particles, after integrating over the heavy ones and using the methodology of "effective theories," in view of formal nonrenormalizability of the initial $d$-dimensional theory. One may only hope that the theory will be finite on the mass shell or pass to the theory of strings in $d$-dimensions, which can be described by a $d$-dimensional field theory in the $\alpha' \to 0$ limit. In what follows we shall ignore all massive states, using mainly the second point of view.

## 2. CORRESPONDENCE OF DIVERGENCES IN $d$-DIMENSIONAL AND REDUCED THEORIES

In order to establish the *correct* correspondence of quantum properties of higher-dimensional and dimensionally reduced theories, one is to use a regularization accounting for *power divergences*. A natural one in the one-loop approximation is based on the proper-time cutoff (see e.g. Reference 4); higher-loop generalizations are discussed in Reference 3. The one-loop effective action reads (we use Euclidean notations throughout)

$$\Gamma = \tfrac{1}{2} \log \det \Delta_2^{(d)},$$

$$\Delta_2^{(d)} = -\frac{1}{\sqrt{g}} \mathscr{D}_M g^{MN} \sqrt{g}\, \mathscr{D}_N + X, \tag{1}$$

where $M, N = 1, \ldots d$, $\mathscr{D}_M = \partial_M + A_M$. Using the well-known expansion[5–7] (omitting boundary terms)

$$\operatorname{tr} e^{-s\Delta^{(d)}_2} \overset{(s\to 0)}{\simeq} \sum_{p=0}^{d} s^{(p-d)/2} \frac{1}{(4\pi)^{d/2}} \int d^d x \sqrt{g}\, b_p(x), \tag{2}$$

we get its infinite part in the form ($\Gamma_\infty = -\frac{1}{2}\int_\varepsilon^\infty \frac{ds}{s} \mathrm{tr}\; e^{-s\Delta_2(d)} \equiv \int \mathscr{L}_\infty \sqrt{g}\, d^d x$)

$$\mathscr{L}_\infty = -\frac{1}{(4\pi)^{d/2}}\left\{\frac{1}{d} L^d b_0 + \cdots + \frac{1}{d-2K} L^{d-2K} b_{2K} + \cdots + \tfrac{1}{2} b_d \log L^2/\mu^2\right\}, \qquad L = \varepsilon^{-1/2} \to \infty.$$

Note that only logarithmic divergences are a priori absent if $d$ is odd ($b_{2K+1} = 0$). At present there exists[7] an explicit algorithm for computing $b_{2K}$ only for $K \leqslant 3$. Being universal (valid for any $d$), it provides the possibility to establish the leading one-loop $L^d, \ldots, L^{d-6}$ divergences in any $d$-dimensional theory. Assuming the simple reduction ($\partial_i(\phi, g_{MN}, X) = 0$) and splitting the background fields

$$g_{MN} = \begin{bmatrix} \bar{g}_{\mu\nu} + \bar{\varphi}_{ij} B^i_\mu B^j_\nu & \bar{\varphi}_{ij} B^i_\mu \\ \bar{\varphi}_{ij} B^j_\nu & \bar{\varphi}_{ij} \end{bmatrix}, \tag{4}$$

$$A_M = \{A_\mu, A_i\}, \qquad \mu, \nu = 1, \ldots, 4,$$

we can quantize the resulting 4-dimensional theory

$$\Delta_2^{(4)} = -\frac{1}{\sqrt{g}} \tilde{\mathscr{D}}_\mu \sqrt{g}\, g^{\mu\nu} \tilde{\mathscr{D}}_\nu + \tilde{X},$$

$$\tilde{\mathscr{D}}_\mu = \partial_\mu + A_\mu - B^i_\mu A_i, \qquad X = \lambda^{-1/2} X - \varphi^{ij} A_i A_j, \tag{5}$$

$$\lambda = \det \varphi_{ij}, \qquad g_{\mu\nu} = \lambda^{1/2} \bar{g}_{\mu\nu},$$

$$\bar{\varphi}_{ij} = \lambda^{-1/2} \varphi_{ij}, \qquad g = \det g_{\mu\nu},$$

and compare the expression for divergences in $d$ and four dimensions.

According to (3) the general rule of correspondence is

$$\{L^p, \log L^2\}_4 \sim \{L^{d-4+p}, L^{d-4}\}_d, \tag{6}$$

i.e.

$$(b_{2K})_4 \sim (b_{2K})_d.$$

In view of the explicit breaking of $d$-dimensional space–time symmetries by the reduction, counterterms (i.e., $b_{2K}$) of the reduced theory cannot be written in terms of only $d$-dimensional covariant objects, and, in general, $b_{2K}(\Delta_2^{(d)}) \neq b_{2K}(\Delta_2^{(4)})$.

At the same time it is easy to prove the following "lemma": If

$$A_M = \{A_\mu, 0\}, \qquad g_{MN} = \begin{bmatrix} g_{\mu\nu} & 0 \\ 0 & \delta_{ij} \end{bmatrix}, \tag{7}$$

then $b_{2K}(\Delta_2^{(d)}) = b_{2K}(\Delta_2^{(4)})$. Thus, in order to compute divergences of the reduced theory for the special background (7), one may first establish the corresponding ones in $d$ dimensions and then take into account the restriction (7). It should be stressed that this statement is valid only for the simple reduction (with $S^1 \times \cdots \times S^1$ as internal space); different reductions, for instance, the "coset space" one (see e.g. Reference 8), lead to quite different quantum theories with counterterms having no natural connection with $d$-dimensional ones.

## 3. DIVERGENCES IN $d>4$ SUPER YANG–MILLS THEORY

Let us now illustrate Lemma (7) on $\mathrm{SYM}_{10}^1$ and its reductions. Quantizing the $d=10$ theory

$$\mathscr{L}_{10} = \frac{1}{4g_{(10)}^2}(F_{MN}^a)^2 + i\bar{\psi}^a \hat{\mathscr{D}} \psi^a \tag{8}$$

(here $a = 1, \ldots, \dim G$, $\hat{\mathscr{D}} = \gamma_M \mathscr{D}_M$, $\gamma_M$ are $\nu \times \nu$ Dirac matrices, $\nu = 2^{[d/2]}$,

$\psi$ is a Majorana–Weyl spinor) in the standard background gauge $\mathscr{D}_M A_M = \xi(x)$ in the gauge field sector, we get

$$Z = \frac{\det \Delta_0}{\sqrt{\det \Delta_1}} [\sqrt{\det \Delta_{1/2}}]^{1/4}, \tag{9}$$

$$\Delta_0 = -\mathscr{D}^2, \quad \Delta_{1MN} = -g_{MN}\mathscr{D}^2 - 2F_{MN},$$

$$\Delta_{1/2} = -\mathscr{D}^2 - \tfrac{1}{2}\gamma_{MN}F_{MN}.$$

Introducing the notations

$$b_0 = \beta_0, \qquad b_4 = \beta_1 \frac{1}{12} \operatorname{tr} F^2_{MN}, \tag{10}$$

$$b_6 = -\beta_2 \frac{1}{60} \operatorname{tr} (\mathscr{D}_M F_{MN})^2 - \beta_3 \frac{1}{72} \operatorname{tr} (F_{MN}F_{NK}F_{KM}) \tag{11}$$

and employing the algorithm of Reference 7, we find

| | $\beta_0$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
|---|---|---|---|---|
| $\Delta_0$: | 1 | 1 | 1 | 1 |
| $\Delta_1$: | $d$ | $d-24$ | $d-40$ | $d$ |
| $\Delta_{1/2}$: | $-\nu$ | $-2\nu$ | $-4\nu$ | $-\nu$ . |

(12)

Finally, we obtain

$$b_0 = b_2 = b_4 = b_6 = 0 \tag{13}$$

as a property of $\mathrm{SYM}^1_{10}$ theory. Applying Lemma (7), we establish the validity of (13) for all $d < 10$ reductions of this theory. For example, now we understand the known one-loop finiteness of $\mathrm{SYM}^4_4$ ($b_4 = 0$) as a consequence of the absence of $F^2_{MN} \times L^6$ one-loop infinities in $\mathrm{SYM}^1_{10}$ (in general, it is the absence of $F^2_{MN}$ infinitities in $d = 10$ theory in any loop order that is responsible for the finiteness of $\mathrm{SYM}^4_4$; cf. Reference 9).

Note that in view of the irreducible supersymmetry it is sufficient to prove finiteness in the gauge field sector. Also, we conclude that off-shell one-loop finiteness takes place for power-counting nonrenormalizable $d \leqslant 7$ maximal SYM theories e.g. $\mathrm{SYM}_6^2$ (this result agrees with predictions of various super-power-counting rules[10]). When viewed from four dimensions, $\mathrm{SYM}_6^2$ it contains $\mathrm{SYM}_4^4$ as well as the infinite tower of its "massive analogs." We see that the account of all massive states does not spoil the one-loop finiteness of $\mathrm{SYM}_4^4$.

## 4. EFFECTIVE LAGRANGIAN AND EFFECTIVE POTENTIAL IN *N*=4 SUPER YANG–MILLS

Now we turn to the finite part of the effective action in (9). First let us present the result for effective Lagrangian for the $d=4$ constant abelian gauge field background ($G=\mathrm{SU}_2$):

$$A_\mu^a = -\tfrac{1}{2}F_{\mu\nu}x_\nu n^a, \qquad (n^a)^2=1,\, A_i=0. \tag{14}$$

The corresponding expressions for $\mathrm{SYM}_{10}^1$ and $\mathrm{SYM}_4^4$ theories are (cf. (7))

$$\mathscr{L}_{10}^{(1)} = -\frac{\hbar}{2(4\pi)^5}\times\int_0^\infty \frac{\mathrm{d}s}{s^6}\,\Phi,$$

$$\mathscr{L}_4^{(1)} = -\frac{\hbar}{2(4\pi)^2}\times\int_0^\infty \frac{\mathrm{d}s}{s^3}\,\Phi, \tag{15}$$

where $\Phi$ is given by (for related previous $d=4$ results see, e.g. Reference 11)

$$\Phi = 8\cdot\frac{sF_1}{\sinh sF_1}\cdot\frac{sF_2}{\sinh sF_2}(\cosh sF_1 - \cosh sF_2)^2, \tag{16}$$

where

$$F_{1,2}^2 = J_1 \pm \sqrt{J_1^2 - J_2^2},$$

$$J_1 = \tfrac{1}{4}F_{\mu\nu}F_{\mu\nu},\; J_2 = \tfrac{1}{4}F_{\mu\nu}\overset{*}{F}_{\mu\nu}.$$

Observing that $\Phi|_{s\to 0}\simeq\sum_{K=0}^{\infty}s^K b_{2K}$, we get $b_8=8(J_1^2-J_2^2)\neq 0$ (and also $b_{4K+2}=0$, $b_{4K}\neq 0$, $K\geqslant 4$). This explicitly demonstrates that $\mathrm{SYM}_{10}^1$ is not finite in one loop; e.g., $\mathscr{L}_{10}^{(1)}$ is quadratically divergent in contrast with the manifest UV finiteness of $\mathscr{L}_4^{(1)}$. However, $\mathscr{L}_4^{(1)}$ (and $\mathscr{L}_{10}^{(1)}$) is IR ($s\to\infty$) divergent if $F\neq\pm F^*$. This instability originates from the gauge field contribution, cf. Reference 12 (supersymmetry does not rule out possible negative modes). This probably indicates the analogy in IR behavior of $\mathrm{YM}_4$ and $\mathrm{SYM}_4^4$ theories (in spite of the zero $\beta$-function in the latter).

Next we are going to consider the scalar effective potential, corresponding to the following choice of background in (9): $A_\mu=0$, $A_i=\mathrm{const}$. The $\mathrm{SYM}_{10}^1$ effective action calculation is then analogous to that in $\mathrm{YM}_4$ theory for a constant nonabelian gauge potential background. In order to extract the $\mathrm{SYM}_4^4$ result, we are to drop all terms with $\partial_i$-derivatives in the corresponding one for $\mathrm{SYM}_{10}^1$. This yields

$$V=V_0+V_1,\qquad V_0=\frac{1}{4g^2}(F_{ij}^a)^2,$$

$$F_{ij}^a=f^{abc}A_i^bA_j^c,$$

$$V_1=\mathscr{L}_4^{(1)}$$

$$=\frac{\hbar}{64\pi^2}\operatorname{tr}(M_0^4\log M_0^2+2M^4\log M^2-\tfrac{1}{4}M_{1/2}^4\log M_{1/2}^2),\qquad(17)$$

$$M_{ab}^2=f_{aec}f_{bed}A_i^cA_i^d,\qquad M_{1/2}^2=M^2-\tfrac{1}{2}\gamma_{ij}F_{ij},$$

$$M_{0ij}^2=\delta_{ij}M^2-2F_{ij}.$$

This (UV finite) expression is *scale invariant*, and, hence, the potential, calculated on its extrema is always equal to zero (no supersymmetry breaking). This connection of UV finiteness and the absence of supersymmetry breaking can in fact be generalized to all loops: if the $\beta$-function is zero, then $T_\mu^\mu=0$, and thus $V=\frac{1}{4}\langle T_\mu^\mu\rangle=0$. Moreover, it is easy to prove that the classical minima $F_{ij}=0$ are the only solutions of the effective equations. Note that, in general, the off-shell expression for $V_1$ is not real because of negative modes of $M_0^2$ (the scalar operator appears as a part of the $d=10$ gauge field one, and thus the origin of this

IR instability can be traced to the "anomalous magnetic moment" term in $\Delta_1$ in (9); cf. also Reference 13). Thus we see that one cannot solve the problem of masses and vacua degeneracies within perturbation theory for pure $\mathrm{SYM}_4^4$ theory.

One way to improve the situation may be to couple $\mathrm{SYM}_4^4$ to $N=4$ (conformal or Poincaré) supergravity. This coupling can probably be simulated by adding some scalar and spinor mass terms and thus softly breaking supersymmetry at the tree level:

$$\Delta\mathscr{L}=\frac{1}{2g^2}\mu_0{}^2_{ij}A_i^a A_j^a+\mu_{1/2\alpha\beta}\bar{\psi}_\alpha^a\psi_\beta^a. \tag{18}$$

Then the gauge coupling $\beta$-function will still be zero, and no field-dependent quadratic divergences will be induced (cf. Reference 14) (one can even cancel the logarithmic mass renormalizations relating $\mu_0$ and $\mu_{1/2}$). Now it is possible to avoid negative modes of $M_0^2$ and, thus, to obtain a well-defined effective potential that is not forbidden to have nontrivial minima. Therefore we can study the question of dynamical gauge symmetry breaking, and the possibility of solving the gauge hierarchy problem and hence constructing *a realistic unified model, starting with* $\mathrm{SYM}_4^4$. This program seems superior to those of Reference 8, where a coset reduction of $\mathrm{SYM}_{10}^1$ was used to generate some $d=4$ theory with broken supersymmetry but with, very probably, bad UV behavior. A preliminary analysis made in Reference 3 indicates that one cannot generate an exponential ($\sim\exp c/g^2$) hierarchy without dropping the relation between $\mu_0$ and $\mu_{1/2}$, necessary for the manifest UV finiteness of effective potential.

## 5. ONE-LOOP DIVERGENCES IN $d\leqslant 11$ SUPERGRAVITIES

Let us now study the one-loop infinities in higher-dimensional supergravities, calculating the $b_{2K}$, $K\leqslant 3$, coefficients in (3) for the gravitational background. We shall use the following notations:

$$b_0=N,\quad b_2=\rho R,\quad b_4=\alpha_1 R^2_{MNPQ}+\alpha_2 R^{2MN}+\alpha_3 R^2+\alpha_4\mathscr{D}^2 R, \tag{19}$$

$$b_6=\sigma_1 I_1+\sigma_2 E,\qquad I_1=R^{MN}_{PQ}R^{PQ}_{KS}R^{KS}_{MN},$$

$$I_2=R^{MN}_{PQ}R^{QS}_{NK}R^{KM}_{SP},\qquad E=I_1-2I_2, \tag{20}$$

where in (20) we assume $R_{MN}=0$ (we shall compute $b_6$ only on the mass shell) and omit all total derivative terms (note also that $E$ is the integrand of the Euler number in $d=6$). Supergravities are known to contain gravitons and antisymmetric gauge tensors along with gravitinos and spinors. That is why we first need to establish the contributions of all these fields in (19), (20) *in d dimensions*.

## (a) Contribution of gravity

Within the background field method, one-loop quantization of gravity in $d$ dimensions is done straightforwardly in the standard gauge $\mathscr{D}_M(h_{MN}-\frac{1}{2}g_{MN}h)=\xi_N(x)$,

$$Z=\frac{\det\Delta_g}{\sqrt{\det\Delta_h}},\qquad \Delta_{g_{MN}}=-g_{MN}\mathscr{D}^2-R_{MN},$$

$$\begin{aligned}(\Delta_h)^{MN}_{PQ}=&-(\delta^M_{(P}\delta^N_{Q)}-\tfrac{1}{2}g^{MN}g_{PQ})(\mathscr{D}^2-R)\\&+2R^P{}_{(MN)}{}^Q-2\delta^M_{(P}R^N_{Q)}\\&+g_{MN}R^{PQ}+g^{PQ}R_{MN}.\end{aligned}\tag{21}$$

Using the algorithms of Reference 7 we get the following results for $b_p^{(h)}=b_p(\Delta_h)-2b_p(\Delta_g)$:

$$N=\tfrac{1}{2}d(d-3),\qquad \rho=\tfrac{1}{12}(-5d^2+9d-48),$$

$$\alpha_1=\tfrac{1}{180}N-\tfrac{1}{12}(d-18),\qquad \alpha_2=\tfrac{1}{360}(-d^2+543d-3600),\tag{22}$$

$$\alpha_3=\tfrac{1}{144}(43d^2-303d+696),\qquad \alpha_4=\tfrac{1}{60}(-4d^2+7d-40),$$

$$\sigma_1=\tfrac{1}{15120}N,\qquad \sigma_2=\tfrac{1}{3240}N-\frac{\mathrm{d}+30}{180}.\tag{23}$$

They generalize the previously known ones for $d=4$ ($b_2$[15,4], $b_4$[16,17]) and $d=6$ ($\sigma_1$ in $b_6$[8,17]). We conclude that *all gravities in* $\mathrm{d}>4$ *have at least* $\mathrm{L}^{d-4}$ *and* $\mathrm{L}^{d-6}$ (*log* $\mathrm{L}^2$ *for* $\mathrm{d}=6$) *divergences on the mass shell.*

### (b) Contribution of antisymmetric tensors

The correct recipe for the quantization antisymmetric tensor gauge field $A_{M_1\dots M_n}$ on the $d$-dimensional gravitational background is (cf. e.g. References 19 and 20)

$$Z^{(n)} = \prod_{K=0}^{n} [\det \Delta_{n-K}]^{-(-1)^K(K+1)/2},$$

$$b_p^{(n)} = \sum_{K=0}^{n} b_p(\Delta_{n-K}) C^K_{-2}, \tag{24}$$

where $\Delta_p$ are Hodge–de Rham operators:

$$(\Delta_p)^{N_1\dots N_p}_{M_1\dots M_p} = -\delta^{N_1}_{[M_1}\cdots\delta^{N_p}_{M_p]}\mathscr{D}^2 + \sum_{i=1}^{p} R^{N_i}_{[M_i}\delta^{N_1}_{M_1}\dots\delta^{M_p}_{M_p]}$$
$$-\sum_{j>i=1}^{p} R^{N_iN_j}_{[M_iM_j}\delta^{N_1}_{M_1}\dots\delta^{N_p}_{M_p]},$$

and $C^n_r = r(r-1)\cdots(r-n+1)/n!$ are binomial coefficients. Again employing the formulas of Reference 7 we obtain

$$N = C^n_{d-2}, \qquad \rho = \tfrac{1}{6}C^n_{d-2} - C^{n-1}_{d-4}, \tag{25}$$

$$\alpha_1 = \tfrac{1}{180}C^n_{d-2} - \tfrac{1}{12}C^{n-1}_{d-4} + \tfrac{1}{2}C^{n-2}_{d-6},$$

$$\alpha_1 = -\tfrac{1}{180}C^n_{d-2} + \tfrac{1}{2}C^{n-1}_{d-4} - 2C^{n-2}_{d-6},$$

$$\alpha_3 = \tfrac{1}{72}C^n_{d-2} - \tfrac{1}{6}C^{n-1}_{d-4} + \tfrac{1}{2}C^{n-2}_{d-6},$$

$$\alpha_4 = \tfrac{1}{30}C^n_{d-2} - \tfrac{1}{6}C^{n-1}_{d-4}, \tag{26}$$

$$\sigma_1 = \tfrac{1}{15120}C^n_{d-2},$$

$$\sigma_2 = \tfrac{1}{3240}C^n_{d-2} - \tfrac{1}{180}C^{n-1}_{d-4} + \tfrac{1}{6}C^{n-2}_{d-6} - \tfrac{2}{3}C^{n-3}_{d-8}. \tag{27}$$

It is possible also to prove the following general results

$$b_p(\Delta_n/d)=b_p(\Delta_{d-n}/d),\quad b_p(\Delta_n/d-2)=b_p^{(n)}(d),\qquad p<d,$$

$$b_p^{(n)}(d)-b_p^{(d-2-n)}(d)\quad\begin{cases}=0, & p<d,\\ =(-1)^n(n+1-d/2)H_p(d), & \\ & p\geqslant d,\quad d=\text{even}\\ \neq 0, & d=\text{odd},\end{cases}\tag{28}$$

where, according to Reference 7,

$$H_p(d)=\sum_{K=0}^{p}(-1)^K b_p(\Delta_K|d)$$

$$=\{0 \text{ if } p<d;$$

$$\varepsilon_d \quad \text{if } p=d\}$$

and $H_p(d)=0$ for $d=\text{odd}$ (the Euler number is $\chi=(4\pi)^{-d/2}\int\varepsilon_d$). Equation 28 is the precise statement of "quantum nonequivalence" observed for $d=4$ in Reference 20.

### (c) Gravitino contribution

Within the background field method the quantization of the gravitino Lagrangian

$$\mathscr{L}_{3/2}=\bar{\psi}_M\gamma^{MNK}\mathscr{D}_N\psi_K,\qquad \gamma_{MNK}=\gamma_{[M}\gamma_N\gamma_{K]}\tag{29}$$

does not appear to be a trivial generalization of the $d=4$ procedure (see e.g. Reference 21 and 22), contrary to the gravitational case (21). Namely, it is impossible to obtain a diagonal squared gravitino operator only by choosing the gauge $\gamma_M\psi_M=\zeta(x)$ and averaging over $\zeta$ with the help of $\hat{\mathscr{D}}=\gamma_M\mathscr{D}_M$, $\mathscr{L}_{3/2}+\mathscr{L}_{\text{gauge}}=\bar{\psi}_M\mathscr{D}^{(\xi)}_{MN}\psi_N$,

$$Z=\sqrt{\frac{\det D^{(\xi)}}{(\det\hat{\mathscr{D}})^3}},\qquad D^{(\xi)}_{MN}=(\gamma_{MKN}+\xi\gamma_M\gamma_K\gamma_N)\mathscr{D}_K. \tag{30}$$

One is also to "rotate" $D^{(\xi)}\to\tilde{D}=\Lambda D\bar{\Lambda}^T$ with the help of the algebraic operator $\Lambda_{MN}=g_{MN}+a\gamma_M\gamma_N$. The choice $a=-1/(d-2)$, $\xi=-\frac{1}{4}(d-2)$ corresponds to the "standard gauge," where the gravitino operator takes its simplest form:

$$D_{MN}=g_{MN}\hat{\mathscr{D}},$$

$$\Delta_{3/2MN}=-D^2_{MN}=-g_{MN}\mathscr{D}^2+\frac{R}{4}g_{MN}-\tfrac{1}{2}\gamma_{PQ}R^{PQ}_{MN}. \tag{31}$$

It is this operator that is suitable for off-shell calculation of divergences, and we find, for $b_p^{(3/2)}=-(1/\gamma)[b_p(\Delta_{3/2})-3b_p(\Delta_{1/2})]$ ($\gamma=1,2,4$ for Dirac, Majorana, and Majorana–Weyl cases, respectively) that

$$N=-\frac{\nu}{\gamma}(d-3),\quad \nu=2^{[d/2]},\quad \rho=-\tfrac{1}{12}N,$$

$$\alpha_1=\tfrac{1}{180}N+\frac{\nu}{\gamma}\frac{d-19}{96},\qquad \alpha_2=-\tfrac{1}{180}N,$$

$$\alpha_3=\tfrac{1}{288}N,\qquad \alpha_4=-\tfrac{1}{120}N, \tag{32}$$

$$\sigma_1=\tfrac{1}{15120}N,\qquad \sigma_2=\tfrac{1}{3240}N+\frac{\nu}{\gamma}\cdot\frac{d+5}{1440} \tag{33}$$

(while for $b_p^{(1/2)}=-1/\gamma\, b_p(\Delta_{1/2})$ we have

$$N=-\frac{\nu}{\gamma},\qquad \alpha_1=\tfrac{1}{180}N+\frac{\nu}{\gamma}\tfrac{1}{96},$$

$$\sigma_2=\tfrac{1}{3240}N+\frac{\nu}{\gamma}\frac{1}{1440}$$

with all other coefficients the same as in (33)). For $d=4$ these values are the same as in References 21 ($\alpha_1$) and 22 ($b_2, b_4$).

### (d) Results for supergravities

Now it is possible to compute the leading $L^{d-2K}$, $K \leqslant 3$, one-loop gravitational infinities: e.g., for $N=1$, $d=11$ supergravity[2], containing $g_{MN}$, Majorana $\psi_M$ and $A_{MNK}$. On-shell ($R_{MN}=0$) we get $b_{2K}=0$, $K \leqslant 3$—i.e., Eq. (13). In view of Lemma (7) we conclude that (13) holds also for all reductions of $\mathrm{SG}^1_{11}$—i.e., for all maximal supergravities; e.g., for $SG^8_4$. Thus we recognize the absence of anomalies ($b_4=0$) in the "reduction" version of $\mathrm{SG}^8_4$ [23] to be a consequence of the absence of $L^7$ infinities in $\mathrm{SG}^1_{11}$. A new result is $b_6(\mathrm{SG}^8_4)=0$, which is valid again only for the "reduction" version, because in $d=4$, $b_6^{(3)} \neq 0$, $b_6^{(2)} - b_6^{(0)} \neq 0$; cf. (28).

Another consequence is the one-loop finiteness of maximal SGs in $d \leqslant 7$ (i.e. $\mathrm{SG}^8_5$, $\mathrm{SG}^4_6$, $\mathrm{SG}^4_7$). They provide first examples of ($d>4$)-dimensional one-loop finite gravitational theories (recall that the pure gravity is infinite for $d>4$, and thus their finiteness is due to *cancellations* and not merely to nonexistence of nonzero on-shell invariants as in the case of $d=4$ SGs). Therefore these theories (corresponding, from the 4-dimensional point of view, to $\mathrm{SG}^8_4$ plus the infinity of its "massive copies") may serve as a basis for a *consistent* (at least at one loop) *Kaluza–Klein program.*

As for the on-shell finiteness of $\mathrm{SG}^1_{11}$ conjectured in Reference 24, it seems very doubtful in view of the probable $b_8(\mathrm{SG}^1_{11}) \neq 0$ property, which is supported by the analogy with the $\mathrm{SYM}^1_{10}$ case and also by the nonzero result for 4-particle amplitude in $\mathrm{SG}^8_4$ [10].

One can prove that (13) is valid also for the $\mathrm{SG}^1_{10}$ theory containing[1,26] $g_{MN}$, $A_{MN}$, $A$, and Majorana–Weyl fields $\psi_M$ and $\psi$ (and also for $\mathrm{SYM}^1_{10}$ in external metric). However, this *does not* imply finiteness of corresponding $d \leqslant 7$ reductions (cf. Reference 9 for $d=4$). Really, it is *irreducible* supergravity that is finite if it is finite in the gravitational sector, but the irreducibility is lost if we apply dimensional reduction to a *nonmaximal* supergravity (see e.g. References 25 and 26 for discussion of reductions of $\mathrm{SG}^1_{10}$, $\mathrm{SG}^1_5$, $\mathrm{SG}^4_5$). It is interesting to note that (13) holds also for the version of $\mathrm{SG}^1_{10}$ with $A_{M_1 \ldots M_6}$ instead of $A_{MM}$ [27]. In fact, according to (28) all $b_p$, $p<10$, are equal in both versions, and thus duality transformations in higher dimensions do not affect the infinities of reduced theories. As a final remark let us note that if the $M^4 \times S^7$ version of $\mathrm{SG}^1_{11}$ is actually connected via a reduction with the gauged $\mathrm{SG}^8_4$ [28] (as was conjectured in Reference 24) then it may turn out to be possible to understand the $b_4 = 0$ property of $SG^1_{11}$ as the origin not only of the vanishing of anomalies in the "simply reduced" version of $SG^8_4$ but also of the vanishing of the $\beta$-function in its gauged version[29].

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