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QUANTUM STRING THEORY EFFECTIVE ACTION

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We present a covariant background field method for quantum string dynamics. It is based on the effective action Γ for fields corresponding to different string modes. A formalism is developed for the calculation of Γ in the $\alpha' \to 0$ limit. It is shown that in the case of closed Bose strings Γ contains the standard kinetic terms for the scalar, external metric and the antisymmetric tensor. Our approach makes possible a consistent formulation and solution of a ground state problem (including the problem of space-time compactification) in the string theory. We suggest a solution to the old "tachyon problem" based on the generation of non-trivial vacuum values for the scalar field, metric and antisymmetric tensor. It is shown that a preferred compactification in the closed Bose string theory is to three (anti-de Sitter) space-time dimensions.

1. Introduction

String models were originally developed for the needs of the strong interaction theory. Then it was realized that a string theory when properly interpreted may play a more fundamental role, providing us with a consistent theory of all interactions including quantized gravitational ones [1, 2]. A free (closed) string can be described in terms of an infinite number of its "oscillation modes" (scalar, symmetric 2-tensor, antisymmetric 2-tensor, ...). It was observed that the zero string "size" limit ($\alpha' \to 0$) of scattering amplitudes of different string modes coincides with on-shell scattering amplitudes in a theory of fields associated with the elementary string modes [3]. It was shown [3] that the corresponding covariant action contains the Einstein gravitational term for the symmetric 2-tensor considered as a perturbation of the flat metric. In a more realistic case of supersymmetric strings in ten dimensions such an action contains an $N=2$, $D=10$ supergravity action or $N=1$, $D=10$ supergravity plus $N=1$, $D=10$ super-Yang–Mills action in the closed and open string theory cases correspondingly [2]. Given that (closed) superstring theory is likely to be finite to all orders [2] it can be considered as an interesting candidate for a fundamental theory.

There is, however, a number of conceptual as well as technical problems in a (super) string theory as formulated today. The above connection between string and corresponding field theories was previously established in a non-covariant "on-shell" way (one had first to find an $\alpha' \to 0$ on-shell scattering amplitude and then to guess a covariant action from which it could be derived). Also, expansions near a flat space–time were used in this procedure. All this made it difficult to understand how a curved space–time could be built of the "graviton" string modes and thus how a spontaneous compactification from $D=26$ or 10 to four space–time dimensions could take place. A lacking formalism is an analog of a background field method known for ordinary field theories, i.e. a covariant "off-shell" effective action Γ for the infinite number of fields corresponding to string "excitations". Had we such an effective action accumulating a string theory dynamics we could consistently formulate the ground state problem in this theory. If the solution for a ground state metric appeared to be non-trivial this would be a manifestation of a dynamical "condensation" of free string "graviton modes". If the corresponding ground state D-dimensional space–time was a product of a four-dimensional one and a compact internal space this would be a solution to the compactification problem. If this ground state was stable so that no ghosts and tachyons were present in the expansion of the effective action Γ near the vacuum values of fields this would be a solution to the unitarity problem ("tachyon problem" of the old Bose string theory).

Our aim here is to present such a "background field method" formulation of a quantum string theory starting with a covariant definition of the effective field theory action Γ in terms of a path integral over "internal" string variables (sect. 2). In this paper we mainly consider only the case of closed Bose strings. Generalization to superstrings remains an important problem for the future. We follow the covariant approach to the string theory path integral [4] (see also [5–8]), so that Γ is given by functional integrals over string coordinates and two-dimensional metrics. The action in the exponent which is averaged contains the free string term [9] as well as the infinite number of "source terms" depending on the "external" fields (the arguments of Γ) which "probe" different string "excitations".

A low-energy approximation for Γ is obtained by expanding in $\alpha' \to 0$. In sect. 3 we discuss integration over string coordinates and obtain an "intermediate" effective action W which depends on the external fields and an arbitrary two-dimensional metric.

Integration over two-dimensional metrics is studied in sect. 4. We consider the simplest case of the spherical two-dimensional topology ("tree approximation"). The resulting effective action for the "lower-lying" fields (scalar, D-dimensional metric and antisymmetric tensor) is then extremized for finding the ground state configurations. We first investigate the $D=26$ case using a simplified approach in order to illustrate some general points. Then more rigorous treatment of the general $D \leqslant 26$ case is given and it is shown that there is no ground states with a flat D-dimensional space–time. The theory prefers compactification to a three-dimensional anti-de Sitter space–time so that the ground state metric is always curved. It seems likely that more realistic four-dimensional compactification patterns may exist in the (closed) superstring case.

In concluding sect. 5 we discuss some points connected with an interpretation and extension of our approach. In particular, we present a generalization of the effective action to the case of extended objects of arbitrary dimensions (strings, membranes, etc.).

2. Effective action

Our starting point is the covariant (closed) Bose string action [9]

$$I_0=\frac{1}{2\pi\alpha'}\int d^2x\,\sqrt{g}\tfrac{1}{2}g^{\mu\nu}\partial_\mu\varphi^i\partial_\nu\varphi^i\,. \tag{1}$$

Here $[\alpha']=cm^2$, x^μ $(\mu=1,2)$ are coordinates of a 2-dimensional compact space M^2, $g_{\mu\nu}(x)$ is a metric on M^2, φ^i $(i=1,\dots,D)$ are coordinates of an external space-time M^D where strings propagate (external space metric $G_{ij}(\varphi)$ is assumed to be flat in (1)). We shall use euclidean notation in this paper. The quantum string theory is defined by the path integral [4]

$$\langle\cdots\rangle=\int[dg_{\mu\nu}]\int[d\varphi^i]\,e^{-(1/\hbar)I_0[\varphi,g]}\,. \tag{2}$$

For example, the tree amplitudes for the scattering of the "ground state" (tachyon) scalar string modes are given by [4] (see also [6, 10]; cf. [11])

$$G_N(\phi_1,\dots,\phi_N)=\left\langle\prod_{k=1}^{N}\int d^2x_k\sqrt{g(x_k)}\,\delta^{(D)}(\phi_k-\varphi(x_k))\right\rangle. \tag{3}$$

Here ϕ_k^i are the coordinates of N points in M^D and M^2 is assumed to be a closed simply connected manifold. As was found in [4, 6, 10] G_N reproduce the Virasoro–Shapiro amplitudes in the case of $D=26$. Introducing a scalar "source" field $\Phi(\varphi)$ it is easy to write down the expression for the generating functional corresponding to G_N:

$$\Gamma^{(0)}[\Phi]=\left\langle\exp\left[-\int d^2x\,\sqrt{g}\,\Phi(\varphi(x))\right]\right\rangle, \tag{4}$$

$$G_N(\phi_1,\dots,\phi_N)\sim\left.\frac{\delta^N\Gamma^{(0)}[\Phi]}{\delta\Phi(\phi_1)\cdots\delta\Phi(\phi_N)}\right|_{\Phi=0}. \tag{5}$$

The crucial point is to observe that $\Gamma^{(0)}[\Phi]$ is just a "tree" effective action for the scalar field Φ which corresponds to the "ground state" mode of the (closed) Bose string. Eq. (5) gives the amplitudes (more correctly, irreducible Green functions) on a "naive" vacuum $\Phi,\dots=0$, $G_{ij}=\delta_{ij}$. The true vacuum value of Φ (and all other fields to be introduced shortly) is to be determined by minimizing the full effective action, thus, hopefully, solving the "tachyon problem" (see sect. 4).

Now it is obvious how to generalize (4) to include fields corresponding to other closed string excitations: we are simply to add all other possible "source" terms which preserve the reparametrization invariance

$$I_{\rm source}=\int d^2x\Big[\sqrt{g}\Phi(\varphi(x))+\tfrac{1}{2}\sqrt{g}g^{\mu\nu}\partial_\mu\varphi^i\partial_\nu\varphi^j H_{ij}(\varphi(x))$$
$$+\tfrac{1}{2}i\varepsilon^{\mu\nu}\partial_\mu\varphi^i\partial_\nu\varphi^j A_{ij}(\varphi(x))+\frac{1}{4\pi}\sqrt{g}RC(\varphi(x))$$
$$+\sqrt{g}g^{\mu\nu}g^{\lambda\rho}\partial_\mu\varphi^i\partial_\nu\varphi^j\varphi_\lambda\varphi^k\partial_\rho\varphi^l B_{ijkl}(\varphi(x))+\cdots\Big]. \tag{6}$$

The symmetric tensor H_{ij} is a "source" for the "graviton" modes [12], the antisymmetric tensor A_{ij} is a "source" for the antisymmetric 2-tensor modes [13], while higher tensor fields like B_{ijkl} correspond to "spin > 2" massive modes*. C is a "source" for the "dilaton" - a massless scalar of the closed string spectrum (R is the curvature scalar of $g_{\mu\nu}$)*.

Eq. (4) is true in a "first quantized" string theory**. To account for processes with a "cubic" interaction of virtual strings (one string splitting into two and two strings recombining into one) we are to sum over all closed oriented manifolds with arbitrary number of handles n. n is thus a number of "loops" in the full "second quantized" string theory (note that n is the only topological characteristic of such 2-dimensional manifolds). As a result, we are led to the following expression for the effective field theory action corresponding to the "second quantized" string theory:

$$\Gamma[\Phi, G_{ij}, A_{ij},\ldots]=\sum_{n=0}^{\infty} e^{\sigma\chi}\int_{M_\chi^2}[dg_{\mu\nu}]\int[d\varphi^i]\,e^{-(1/\hbar)I}, \tag{7}$$

$$I=I_0+I_{\rm source}$$
$$=\int d^2x\,\sqrt{g}\Phi(\varphi)+\frac{1}{4\pi}\int d^2x\,\sqrt{g}RC(\varphi)$$
$$+\frac{1}{2\pi\alpha'}\int d^2x[\tfrac{1}{2}\sqrt{g}\,g^{\mu\nu}\partial_\mu\varphi^i\partial_\nu\varphi^j G_{ij}(\varphi)+\tfrac{1}{2}i\varepsilon^{\mu\nu}\partial_\mu\varphi^i\partial_\nu\varphi^j A_{ij}(\varphi)]$$
$$+\int d^2x\,\sqrt{g}\,g^{\mu\nu}g^{\lambda\rho}\partial_\mu\varphi^i\partial_\nu\varphi^j\partial_\lambda\varphi^k\partial_\rho\varphi^l B_{ijkl}(\varphi)+\cdots,\quad (A_{ij}\to 2\pi\alpha' A_{ij}). \tag{8}$$

Here $\chi=2-2n$ is the Euler number of M_χ^2 and $G_{ij}\equiv\delta_{ij}+2\pi\alpha' H_{ij}$ is an a priori arbitrary metric of the external space-time M^D. A dimensionless constant σ plays

* To have correspondence with the standard dilaton emission vertex [14, 15] ($\sim\partial\varphi^i\partial\varphi^i e^{ip\cdot\varphi}$) we are also to redefine H_{ij} or the metric G_{ij} in eq. (8): $G_{ij}\to G'_{ij}\exp(4C/(D-2))$, where G'_{ij} has a flat-space limit. We shall not discuss the dilaton contribution in the effective action in the main body of this paper, assuming $C=C_0=$ const in the vacuum and absorbing C_0 in σ in eq. (7) (see, however, the end of sect. 4).

** A closed simply connected surface M^2 can be considered as a world sheet of a virtual string which appears at some point, propagates and then disappears at another point.

the role of a coupling constant in the theory*. The choice $e^{\sigma\chi}$ for the weight with which different topologies are summed seems to be unique and is distinguished by the fact that $\sigma\chi$ can be represented as a local addition to the action (8). In fact,

$$\chi = \frac{1}{4\pi} \int d^2x \sqrt{g} R . \tag{9}$$

Here $R = R^{\lambda}{}_{\mu\lambda\nu} g^{\mu\nu}$ and $R^{\lambda}{}_{\mu\nu\rho} = \partial_\nu \Gamma^{\lambda}_{\mu\rho} - \cdots$ is the curvature of M^2_χ.

It is important to stress that the structure of (6) or (8) is quite simple: we have written down all possible local "source" terms that respect covariance in two dimensions. The *only* "external" gauge invariances of (8) (and hence of Γ) are the general covariance in D dimensions (which follows from 2-dimensional covariance) and the abelian gauge symmetry for the antisymmetric tensor, $\delta A_{ij} = \partial_i \lambda_j - \partial_j \lambda_i$, $\partial_i = \partial/\partial\varphi^i$. Hence the *only* fields which can be "massless" in Γ are the metric G_{ij} and the antisymmetric tensor A_{ij}. "Higher-spin" tensor fields $B_{ijkl}, \ldots$ must be massive because of the absence of corresponding gauge symmetries necessary to provide their masslessness in Γ. In this way we deduce the structure of the free closed string spectrum without use of any a priori knowledge about it.

To determine Γ as defined by (7) we are thus to compute the partition function for quantized strings propagating on an arbitrarily curved space-time M^D and interacting (in addition to gravity G_{ij}) with the infinite number of local fields: Φ, A_{ij}, $B_{ijkl}, \ldots$. The important property of (7) is that it can be represented as an integral over the space-time M^D. The reason is that the free string theory is insensitive to a position of a string "centre", i.e. the action (1) is invariant under $\varphi^i(x) \to \varphi^i(x) + a^i$, $a^i = \text{const}$. Hence a free string partition function contains the corresponding zero-mode contribution (the volume of $\mathbb{R}^D$) as a factor. This translational invariance is broken by the "external" fields in (8) so that a D-dimensional zero mode integral is no longer a trivial one. It is useful to extract this integral over a "string centre" collective coordinate from the very beginning by splitting φ^i into constant and nonconstant parts:

$$\varphi^i(x) = \phi^i + \eta^i(x) , \qquad \phi^i = \text{const} ,$$

$$\int d\varphi \, F[\varphi] = \int d^D\phi \int [d\eta^i] F[\phi + \eta] ,$$

$$[d\eta] = d\eta \, \delta^{(D)}(P^i[\phi, \eta]) Q[\phi, \eta] ,$$

$$Q = \det \frac{\partial P^i[\eta + a]}{\partial a^j} \bigg|_{a=0} . \tag{10}$$

Here $P^i = 0$ is a "gauge condition" breaking the invariance under $\eta \to \eta + \text{const}$ to avoid overcounting and Q is the corresponding "ghost" determinant. Using (10)

* As we will see in sect. 4 σ is not subject to a renormalization, i.e. it has a fixed value (note that σ should be positive for convergence of the sum in (7)).

we get

$$\Gamma = \int \mathrm{d}^D\phi \, \sqrt{G(\phi)}\mathscr{L}(\Phi(\phi), \mathscr{D}_i\Phi(\phi), \mathscr{D}_i\mathscr{D}_j\Phi(\phi), \ldots ; \\ G_{ij}(\phi), \mathscr{R}^k{}_{ijm}(\phi), \ldots ; F_{ijk}(\phi), \ldots ; \ldots) , \tag{11}$$

where $\mathscr{L}$ depends on all powers of derivatives of fields (multiplied by powers of α'). The gauge invariances imply that the derivatives of G_{ij} and A_{ij} combine into the curvature tensors $\mathscr{R}^k{}_{ijm} = \partial_j \Gamma^k_{im} + \Gamma^k_{nj}\Gamma^n_{im} - (j \leftrightarrow m)$ and $F_{ijk} = 3\partial_{[i}A_{jk]}$, and that all derivatives are covariant with respect to the Christoffel connection $\Gamma^i_{jk}(G)$. The factor $\sqrt{G(\phi)}$ comes from the expansion of the covariant measure in (7): $\prod_x \mathrm{d}\varphi^i(x)\sqrt{G(\varphi(x))}$, $G = \det G_{ij}$*. The representation (11) shows that it is Γ defined by (7) (and not, for example, $\ln \Gamma$) that is the effective action.

The lagrangian in (11)

$$\mathscr{L} = \sum_\chi \mathrm{e}^{\sigma\chi} \int_{\mathrm{M}^2_\chi} [\mathrm{d}g_{\mu\nu}] \int [\mathrm{d}\eta^i] \exp\left\{-\frac{1}{\hbar} I[\phi+\eta, g_{\mu\nu}, \Phi(\phi+\eta), \ldots]\right\} \tag{12}$$

is expressed in terms of the path integrals over the "internal" string degrees of freedom so that $\mathscr{L}$ is effectively "non-local" in ϕ (this "smearing" may be responsible for the finiteness of $\mathscr{L}$)**. The fields Φ, G_{ij}, $A_{ij}, \ldots$ can be considered as some "bound state" excitations of the string degrees of freedom. Thus we get an unusual type of an induced (gravity, ...) theory where the effective fields (metric, ...) are not to be further quantized. The point is that loop effects of an approximate field theory (valid at energies $E \ll (\alpha')^{-1/2}$) are automatically accounted for by the string theory loop corrections***.

3. Integration over coordinates

Our aim is to determine the leading terms in a "low-energy" ($\alpha' \to 0$) expansion of Γ in (7), (8). According to (12), we are first to expand the fields near $\phi^i = \mathrm{const}$ and to integrate over "fluctuations" η^i. Integration over 2-dimensional metrics will be carried out in sect. 4. At this first stage the 2-metric $g_{\mu\nu}$ is considered as an arbitrary background field (in this section we may not also specialize the value of the Euler number of M^2_χ). The corresponding "first-stage" effective action is given by

$$\mathrm{e}^{-(1/\hbar)W[g,\Phi,G,\ldots]} = \int [\mathrm{d}\eta]\, \mathrm{e}^{-(1/\hbar)I[\phi+\eta, g_{\mu\nu}, \Phi(\phi+\eta),\ldots]} . \tag{13}$$

* The measure $[\mathrm{d}g_{\mu\nu}]$ is defined as in [4, 8]. In particular we divide by the volume of the full diffeomorphism group of M^2 (see also sect. 4).

** As is clear from (12) all non-polynomialities in $\mathscr{L}$ arise as a result of "virtual contact exchanges of the infinite number of the string modes", cf. [16].

*** The infinite set of modes propagating in the string theory loops is in correspondence with the set of "external" fields, Φ, G_{ij}, $A_{ij}, \ldots$.

It is useful to start the analysis with the formal case when all the "external" fields except the metric G_{ij} are set equal to zero. Then (8), i.e. the action for a string propagating in a curved space–time:

$$I_G = \frac{1}{2\pi\alpha'} \int d^2x \sqrt{g} \tfrac{1}{2} g^{\mu\nu} \partial_\mu \varphi^i \partial_\nu \varphi^j G_{ij}(\varphi) , \tag{14}$$

exactly coincides with the action for a generalized σ-model defined on a curved 2-dimensional space M^2 (the "internal" space of the σ-model is M^D with the metric G_{ij}; note that the roles of M^D and M^2 are reversed on going from a string picture to a σ-model picture). Hence (13) is the partition function for the σ-model on a curved background $g_{\mu\nu}$. Being interested in the $\alpha' \to 0$ expansion of Γ we may compute W perturbatively in $\hbar$ ($\alpha'\hbar$ counts the loop order in (13)).

To make perturbation theory manifestly covariant in D dimensions it is convenient to do a local change of the quantum variable $\eta^i \to \xi^i(\eta, \phi)$, introducing the geodesic normal coordinates with the origin at the point ϕ^i [17, 18]:

$$\eta^i \equiv \varphi^i - \phi^i - \xi^i - \tfrac{1}{2}\Gamma^i_{jk}(\phi)\xi^j\xi^k + \cdots . \tag{15}$$

Making ξ^i dimensionless by the rescaling $\xi^i \to (2\pi\alpha')^{1/2}\xi^i$, we obtain for (14) (see e.g. [19])

$$I_G = \int d^2x \sqrt{g} g^{\mu\nu} \{ \tfrac{1}{2}\partial_\mu \xi^i \partial_\nu \xi^j G_{ij}(\phi) - \tfrac{1}{6}(2\pi\alpha')\mathcal{R}_{ikjl}(\phi) \times \xi^k\xi^l \partial_\mu\xi^i\partial_\nu\xi^j - \tfrac{1}{12}(2\pi\alpha')^{3/2}\mathcal{D}_k\mathcal{R}_{iljm}(\phi) \times \xi^k\xi^l\xi^m \partial_\mu\xi^i\partial_\nu\xi^j + O(\alpha'^2\mathcal{R}^2)\} . \tag{16}$$

Higher-order terms in (16) have the following structure:

$$\int d^2x \sqrt{g} g^{\mu\nu} (\alpha'\xi^2)^{p+\frac{1}{2}q} (\mathcal{D}^q\mathcal{R}^p)_\phi \partial_\mu\xi\partial_\nu\xi , \tag{17}$$

so that (16) corresponds to a theory with the infinite number of dimensionless coupling constants (which are powers of the curvature of G_{ij} and its covariant derivatives taken at the point ϕ and rescaled by powers of α'). Observing that ξ^i transforms as a vector under the point transformations of ϕ^i we can go to the orthonormal basis introducing $\xi^a(x) = e^a_i(\phi)\xi^i(x)$, $G_{ij} = e^a_i e^a_j$, $a = 1, \ldots, D$. Then (16) takes the form

$$I_G = \int d^2x \sqrt{g} g^{\mu\nu} \{ \tfrac{1}{2}\partial_\mu\xi^a\partial_\nu\xi^a - \tfrac{1}{6}\bar{\mathcal{R}}_{abcd}\xi^a\xi^c\partial_\mu\xi^b\partial_\nu\xi^d + O(\alpha'^{\frac{3}{2}})\}$$

$$(\bar{\mathcal{R}}\ldots \equiv 2\pi\alpha'\mathcal{R}\ldots) . \tag{18}$$

A convenient covariant choice of the "gauge" in (10) is [18]: $P^a = \int d^2x \sqrt{g}\xi^a(x) = 0$ (it eliminates the constant zero mode of the scalar laplacian on a compact space

M^2_χ). Throughout this paper we ignore various local factors ($\sim\delta^{(2)}(0)$) like $\det(\partial\eta^i/\partial\xi^j)$ or Q (as well as various quadratic infinities) which are cancelled by a proper choice of the functional measure (being quadratically divergent they in any case vanish in dimensional regularization). For example, the quadratic infinities corresponding to the theory (18) are cancelled by the term ($\sim\Lambda^2\int d^2x \ln\det G(\phi+\eta(\xi,\phi))$) coming from the covariant measure $[d\varphi^i]$ ($\Lambda\to\infty$ is a cutoff).

The classical action (18) is invariant under the Weyl transformations $g_{\mu\nu}\to\lambda^2(x)g_{\mu\nu}$ as well as under the coordinate transformations of x^μ. To preserve the latter invariance we use dimensional regularization which breaks down the Weyl invariance ((18) is Weyl invariant only at $d=2$). Covariance and Weyl anomaly considerations (see [20, 21]) suggest that W has the following general structure:

$$W[g,G]=\frac{1}{\varepsilon}\beta\int R\sqrt{g}\,d^2x+\gamma\int(R\sqrt{g})_x\Box^{-1}_{xx'}(R\sqrt{g})_{x'}\,d^2x\,d^2x'\,, \qquad (19)$$

$$\beta=4a\,,\qquad \gamma=a_1+a_2\hbar\alpha'\mathcal{R}+\gamma_3\hbar\alpha'^2(\mathcal{R}\ldots)^2+\cdots. \qquad (20)$$

Here R and $\mathcal{R}=\mathcal{R}^a_{\ bab}$ are the scalar curvatures of $g_{\mu\nu}$ and G_{ij} respectively, $\varepsilon=d-2\to 0^-$ and

$$\Box_x\Box^{-1}_{xx'}=\delta^{(2)}(x-x')\,,\qquad \Box\equiv\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu)\,. \qquad (21)$$

The first term in (19) is the "topological" ultraviolet infinity (infrared infinities are absent because of compactness of M^2_χ). The second term in (19) corresponds to the Weyl anomaly ($T^\mu_\mu=(2/\sqrt{g})g_{\mu\nu}(\delta W/\delta g_{\mu\nu})=-4\gamma R$). In the one-loop approximation for the two-dimensional theory (13), (18) W in (19) is simply the effective action for D free scalar fields on a curved 2-dimensional background. Hence $\gamma=D/96\pi$ (see e.g. [21, 4]). To determine the two-loop coefficient in (20) one may expand the metric near the flat background $g_{\mu\nu}=\delta_{\mu\nu}+h_{\mu\nu}$ and compute the "self-energy" graphs for $h_{\mu\nu}$ (fig. 1) using momentum space representation and the infrared cutoff by the explicit mass term (the contributions of the first two diagrams of fig. 1 mutually cancel).

No $1/\varepsilon^2$ or non-local $1/\varepsilon$ infinities appear in $W=\int[d^2p/(2\pi)^2]\times h_{\mu\nu}(p)K_{\mu\nu\rho\sigma}(p)h_{\rho\sigma}(-p)$ (no counterterms are to be accounted for in computing W; their use would be equivalent to a non-trivial renormalization of $G_{ij}(\varphi)$ in (14) making the effective action Γ ambiguous)*. As a result we reproduce the $h^2_{\mu\nu}$ term in the expansion of (19) with $\gamma^{(2)}=-\bar{\mathcal{R}}/128\pi^2$. Thus in the

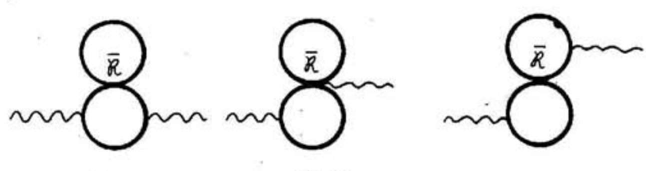

Fig. 1.

* In order to detect the infinity in (19) one assumes the background field ϕ to be non-constant.

two-loop approximation

$$\gamma=\frac{D}{96\pi}-\frac{\hbar\alpha'}{64\pi}\mathcal{R}(\phi)+\mathrm{O}(\alpha'^2\hbar^2\mathcal{R}^2)\,. \tag{22}$$

(This result being dependent on $\alpha'\mathcal{R}$ (i.e. on the square of ratio of a string "radius" to a space-time "radius") disagrees with the expression found in ref. [22] where a particular case of $\mathrm{M}^D=\mathrm{S}^D$ was considered.)

Having found $W[g, G]$ we are to integrate $\mathrm{e}^{-W/\hbar}$ over the metric $g_{\mu\nu}$. This will be done in the next section using the methods of refs. [8, 23, 24]. As a result the $\alpha'\mathcal{R}$ term in (22) will produce the Einstein-like term in the effective action Γ. Now let us include in W the contribution of the antisymmetric tensor A_{ij} (the contribution of the scalar field Φ will be discussed in sect. 4). The relevant piece of the action (8)

$$I_{G,A}=\frac{1}{2\pi\alpha'}\int \mathrm{d}^2x\{\tfrac{1}{2}\sqrt{g}g^{\mu\nu}\partial_\mu\varphi^i\partial_\nu\varphi^j G_{ij}(\varphi)$$
$$+\tfrac{1}{2}i\varepsilon^{\mu\nu}\partial_\mu\varphi^i\partial_\nu\varphi^j A_{ij}(\varphi)\} \tag{23}$$

is exactly the action for the generalized σ-model with a generalized "Wess–Zumino" term. In fact the second Kalb–Ramond [13] term in (23) coincides with the so-called Wess–Zumino term [26, 27] for the special choices of M^D and $A_{ij}(\varphi)$ corresponding to the "standard" σ-models. Expanding φ^i near ϕ^i according to (10), (15) we find that in addition to (18) the action (23) contains also the terms depending on $F_{ijk}=3\partial_{[i}A_{jk]}$, $\mathcal{R}^i_{jkl}$ and their covariant derivatives. The leading contribution in (20) comes from the following term in (23):

$$\tfrac{1}{6}i\int \mathrm{d}^2x\,\varepsilon^{\mu\nu}\bar{F}_{abc}(\phi)\xi^a\partial_\mu\xi^b\partial_\nu\xi^c\,,$$
$$\bar{F}_{abc}\equiv(2\pi\alpha')^{1/2}F_{abc}\,. \tag{24}$$

Computing the two-loop $h_{\mu\nu}$ "self-energy" diagrams with two $\alpha'^{1/2}F$ vertices (fig. 2) we conclude that the total result corresponds to (22) with the following substitution:

$$\mathcal{R}\to\mathcal{R}-\tfrac{1}{12}F_{ijk}F^{ijk}\,. \tag{25}$$

(Note a connection of this result with that found previously (on a linearized level) in the context of the old dual string model [14], and also with recent work [28, 29] on the generalized σ-model (23).)

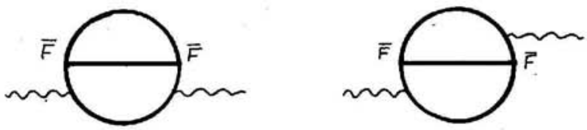

Fig. 2.

One can give an alternative derivation of eqs. (19), (22), (25) using the fact that the Weyl invariance in two dimensions makes it possible to establish the exact expression for the free scalar Green function [4, 23]. Introducing the conformal coordinates in which $g_{\mu\nu} = e^{2\rho}\delta_{\mu\nu}$ (ρ is not regular on M^2_χ) we have (projecting out the zero-mode contribution and taking $\chi = 2$)

$$\Box^{-1}_{xx'} = \frac{1}{4\pi} \ln\left[(x-x')^2 + \frac{1}{\Lambda^2} e^{-\rho(x)-\rho(x')}\right], \tag{26}$$

Here $\Lambda \to \infty$ is a covariant cutoff. As follows from (18) the two-loop contribution in W is given by

$$W^{(2)} = -\tfrac{1}{6}\bar{\mathcal{R}} \int d^2x \sqrt{g} g^{\mu\nu}$$
$$\times \lim_{x\to x'} (\Box^{-1}_{xx'} \nabla^x_\mu \nabla^{x'}_\nu \Box^{-1}_{xx'} - \nabla^x_\mu \Box^{-1}_{xx'} \nabla^x_\nu \Box^{-1}_{xx'}) . \tag{27}$$

The result of (27) is the same as in (19), (22) ($R = -2\,e^{-2\rho}\Box\rho$).

The analysis of this section can be generalized to the case of the Fermi string theory with the classical action invariant under the $N=1$ two-dimensional local supersymmetry [9, 30, 31]. Coupling a free Fermi string to the external metric G_{ij} amounts to constructing a locally supersymmetric generalization of the $N=1$ supersymmetric generalized σ-model action (see e.g. [32]). TThe result (cf. ref. [33] for the $N=2$ case) appears to be a straightforward combination of the actions of refs. [9, 30] and [32]. It is possible also to include the coupling to A_{ij} using the $N=1$ supersymmetric extension of the action (23) constructed in [28] (see also [29]). Less clear is how the Fermi string theory (apparently lacking D-dimensional supersymmetry) can be coupled to a sort of ten-dimensional supergravity anticipated as its $\alpha' \to 0$ limit (cf. [34]). Such coupling is perfectly possible for the covariant superstring action of ref. [35]. Thus generalization of the above treatment to the superstring case remains an interesting problem for the future (in particular, it is important to understand whether a superstring action with "sources" can be transformed into a sort of supersymmetric σ-model action).

4. Ground state problem (integration over 2-metrics; solution of effective equations)

Given the effective field action (7) we can study the problem of a ground state in the closed Bose string theory. Namely, we first have to compute $\Gamma[\Phi, G_{ij}, A_{ij}, B_{ijkl} \ldots]$ (B stands for all higher-spin tensor fields) for arbitrary arguments and then find the effective mean values of fields, i.e. the solutions of the effective field equations

$$\frac{\delta\Gamma}{\delta\Phi} = 0, \quad \frac{\delta\Gamma}{\delta G} = 0, \quad \frac{\delta\Gamma}{\delta A} = 0, \quad \frac{\delta\Gamma}{\delta B} = 0. \tag{28}$$

Assuming that the ground state space–time manifold should possess some global isometries (e.g. Poincaré or de Sitter) we have to look for solutions of (28) with the tensor fields $B_{ijkL\ldots}$ equal to zero. As for A_{ij}, it may be non-vanishing in the vacuum with $F_{abc} \sim \varepsilon_{abc}$ $(a, b, c = 1, 2, 3)$ thus a priori distinguishing "compactification" to 3 dimensions [36]. Hence the ground state is determined by the expectation values of the three "lowest mass" fields Φ, G_{ij} and A_{ij}.

It is easy to guess the general structure of the first several terms in the expansion of Γ in (7):

$$\Gamma[\Phi, G, A] = \int \mathrm{d}^D\phi\, \sqrt{G}\{V(\Phi) + \alpha' f_1(\Phi)(\partial_i\Phi)^2 + \alpha' f_2(\Phi)\mathcal{R} + \alpha' f_3(\Phi) F_{ijk}F^{ijk} + \mathrm{O}(\alpha'^2)\}\,. \quad (29)$$

All the indices are contracted with the help of $G_{ij}(\phi)$. To establish the ground state values Φ_0, G_0, A_0 one has thus to compute the functions V and f_n. A check of the absence of ghosts in the theory will be the physical relative signs of the "kinetic terms" in (29), while the stability of the ground state under small fluctuations will be equivalent to the absence of tachyons in their spectrum.

Thus we get a consistent field-theoretic formulation of the "tachyon problem" in the Bose string theory. It was anticipated previously that the presence of the tachyon in the free string spectrum indicates that the expansion goes near a "wrong" vacuum and hence a sort of "spontaneous symmetry breaking" (generation of condensates of some scalar modes) is needed to establish the "true" one (see e.g. [37]). However, an off-shell method for the solution of the ground state problem (analogous to the effective action method for the study of dynamical symmetry breaking in a field theory) was not developed. A circumstance that was also ignored in the previous approaches is that the flat metric $G_{ij} = \delta_{ij}$ and $A_{ij} = 0$ may not correspond to the true vacuum of the theory. It is of course necessary to have a flat space vacuum if one wishes to apply the theory to hadrons but given that the free spectrum contains the massless "graviton" and antisymmetric tensor the question about the ground state metric and A_{ij} should be solved dynamically without prejudices about possible relevance of the theory to *flat four* dimensions.

Recalling that the theory has dimensional parameter α' it is natural to anticipate that being computed for a ground state value of Φ, Γ in (29) should contain a cosmological term (which is not ruled out by the symmetries of this theory) of natural value $\Lambda \sim \alpha'^{-1}$.

Hence the vacuum space–time (or at least a "factor" of it) is likely to be an (anti) de Sitter space with a characteristic scale of the Planck order. We stress that it is *unnatural* (in the absence of supersymmetry) to hope that the ground state metric may appear to be flat. Given that stability criteria in a curved space are in general very different from those in the flat space we conclude that a resolution of the "tachyon problem" is essentially connected with the question of the background space–time geometry.

It seems likely that the ground state problem should have a definite solution already in the "tree" (i.e. first quantized string) approximation. Hence we shall compute Γ for the case when the integration in (7) goes over closed 2-dimensional riemannian manifolds without handles M_2^2, i.e. with the topology of the sphere $S^2(\chi(M^2)=2)$. Even the treatment of this simplest case is full of technical details which we shall partly omit here (some basic material about computation of a free partition function, i.e. $\Gamma[0]$, can be found in refs. [23, 24, 8]). Separating the constant part ϕ^i of φ^i as in (10), (15) we get (cf. (12)–(15))

$$\Gamma[\Phi, G, A] \sim \int d^D\phi \int [dg_{\mu\nu}] \int [d\xi^i] \exp\left\{-\int d^2x\sqrt{g}\,\Phi(\phi,\xi) - \frac{1}{2\pi\alpha'}\int d^2x[\tfrac{1}{2}\sqrt{g}g^{\mu\nu}\partial_\mu\xi^i\partial_\nu\xi^j G_{ij}(\phi)+\ldots]\right\}. \quad (30)$$

The integral over ξ^i goes over the regular *non-constant* functions on M_2^2 (satisfying $\int d^2x\sqrt{g}\,\xi^i(x)=0$) and

$$\Phi(\phi,\xi) \equiv \Phi(\phi+\eta(\phi,\xi)) = \Phi(\phi)+(\partial_i\Phi)_\phi\xi^i + \sum_{n=2}^{\infty}\frac{1}{n!}(\mathscr{D}_{i_1}\cdots\mathscr{D}_{i_n}\Phi)_\phi\xi^{i_1}\cdots\xi^{i_n}. \quad (31)$$

($\mathscr{D}_i$ are covariant derivatives corresponding to G_{ij}). Integrating over ξ^i, we find the "effective action" W (defined in (13)), which depends on $\Phi(\phi), \partial_i\Phi(\phi), \ldots, G_{ij}(\phi)$, $\mathscr{R}^i{}_{jkl}(\phi), \ldots, F_{ijk}(\phi), \ldots$ ($W=\Phi(\phi)\int d^2x\sqrt{g}+\ldots$). Simple power counting reveals that W contains ultraviolet infinities, i.e. depends on a cutoff Λ of the field theory defined on a fixed M_2^2. For example, ignoring various mixing terms, one finds

$$W_\infty(\Phi) = -\sum_{n=1}^{\infty}\frac{1}{\varepsilon^n}\frac{\alpha'^n}{(2n)!!}[(\mathscr{D}^2)^n\Phi]_\phi\int d^2x\sqrt{g}+\ldots, \quad (32)$$

where $1/\varepsilon = -\ln\Lambda$. Hence W is not "unambiguously calculable" in the Bose string case we consider*. It appears likely that W will be calculable in the superstring case where the "tachyon field" Φ is absent and infinities in other sectors have a better chance of cancelling.

The apparent cutoff dependence of W may look like a serious problem of the Bose string theory. There are two possibilities of overcoming this difficulty. A naive

* Power counting indicates the presence of infinities also in other field sectors. For example, in the case of a "higher-spin" field B_{ijkl} we have

$$I=\frac{1}{2\pi\alpha'}\int d^2x\sqrt{g}\tfrac{1}{2}(\partial\xi)^2+\int d^2x\sqrt{g}[B_{ijkl}(\phi) + \mathscr{D}_n B_{ijkl}\xi^n+\cdots]\partial_\mu\xi^i\partial_\mu\xi^j\partial_\rho\xi^k\partial_\rho\xi^l+\cdots,$$

and hence

$$W_\infty(B)\sim\frac{\alpha'^2}{\varepsilon^2}B_{iijj}\int R^2\sqrt{g}\,d^2x+\cdots.$$

one is to "subtract" the infinities by redefining the field ($\Phi \to \Phi - \alpha' \ln \Lambda \mathscr{D}^2 \Phi + \cdots$) in each order of expansion in α' (a redefinition $\Phi \to \Phi + c\Lambda^2$ is in any case needed to absorb quadratic infinities). This procedure makes W finite but dependent on an infinite number of arbitrary ("subtraction-dependent") constants. Another more appealing possibility is connected with the existence of some natural "built-in" mechanism leading to a well-defined Γ as a functional of fields rescaled by powers of the cutoff so as to make them dimensionless ($\Phi \to \Lambda^2 \tilde{\Phi}, \ldots$). The basic observation is that the rescaling of the cutoff should be equivalent to a rescaling of a two-dimensional metric $g_{\mu\nu}$. But $g_{\mu\nu}$ itself is an integration variable in (7). Hence all the cutoff dependence (left after the rescaling of the fields) should be absorbable in a formal redefinition of the integration variable, and hence should be absent in Γ. There of course may be additional infinities coming from the integral over metrics itself. For example, the integral over a "scale" of $g_{\mu\nu}$ may diverge at lower limit. However, this integral should necessarily be cutoff at $1/\Lambda$ because a (covariant) short-distance cutoff $1/\Lambda \to 0$ was already introduced in the theory defined on M_2^2 with a fixed metric. As a result, this integral will be automatically convergent after the rescaling of the metric discussed above. That this second approach is sensible can be seen on the example of a free partition function or a "naive effective potential" $\Gamma[\Phi = \mu_0^2 = \text{const}, G_{ij} = \delta_{ij}, A_{ij}, \ldots = 0] = \langle \exp(-\mu_0^2 \int d^2x \sqrt{g}) \rangle$, which as will become clear (see also [24, 8]), is a well-defined function of $\tilde{\Phi} = \Phi \Lambda^{-2}$. It is interesting to note that if this mechanism is indeed operative (as we shall see) no actual renormalization of σ in (7) is needed at all so that σ is a fixed coupling constant of the string theory (cf. [8]).

Being interested in the lowest-order terms in (29) we can ignore higher-derivative couplings in (31) and carry out explicit (one-loop) integration over ξ:

$$W = \Phi(\phi) \int d^2x \sqrt{g} + \tfrac{1}{2} \ln \det(\delta_{ij} \Delta + 2\pi\alpha' (\mathscr{D}_i \mathscr{D}_j \Phi)_\phi) + \cdots,$$

$$\Delta = -\frac{1}{\sqrt{g}} \partial_\mu (g^{\mu\nu} \sqrt{g} \partial_\nu) \equiv -\frac{1}{\sqrt{g}} \Box . \tag{33}$$

There is no contribution from the linear term because $\int d^2x \sqrt{g} \xi^i$ vanishes for a class of functions on which the ξ^i integral in (30) is defined, namely for the regular functions expandable in eigenfunctions (with non-zero eigenvalues) of the Laplace operator on M_2^2*. Thus, to the lowest order in α'

$$W = \Phi(\phi) \int d^2x \sqrt{g} - \pi\alpha' (\mathscr{D}^2 \Phi)_\phi \int d^2x \sqrt{g_x} \Box^{-1}_{xx} + \cdots, \tag{34}$$

* This in fact is a subtle point. Assuming first that $(\partial_i \Phi)_\phi = J_i$ is non-constant on M_2^2, integrating over ξ and *then* taking the limit $J \to \text{const}$ one finds the additional term in (33): $\pi\alpha' (\partial_i \Phi)^2 \int d^2x\, d^2x' \sqrt{g} \Box^{-1}_{xx'} \sqrt{g_{x'}}$. However, the function $\xi_0 \sim \Box^{-1} J$ needed to compute the gaussian integral by a shift is not regular on M_2^2. For regularity of the limit one has to take $\Box^{-1}$ with projectors on non-constant functions so that the final result vanishes.

where $\Box_x \Box^{-1}_{xx'} = \delta^{(2)}(x-x')$. Introducing a proper-time cutoff $1/\Lambda^2 \to 0$ we get (cf. [4] and (26))

$$\int d^2x\,\sqrt{g_x}\Box^{-1}_{xx} = -\frac{1}{4\pi}(\ln \Lambda^2)\int d^2x\,\sqrt{g} \;+\frac{1}{4\pi}\int d^2x\, d^2x'\,\sqrt{g_x}\Box^{-1}_{xx'}(R\sqrt{g})_{x'}\,. \quad (35)$$

Hence,

$$\begin{aligned} W = {} & W_\infty(0) + \Phi \int d^2x\,\sqrt{g} + \tfrac{1}{4}\alpha' (\ln \Lambda^2)\mathscr{D}^2\Phi \int d^2x\,\sqrt{g} \\ & -\tfrac{1}{4}\alpha'\mathscr{D}^2\Phi \int d^2x\, d^2x'\,\sqrt{g_x}\Box^{-1}_{xx'}(R\sqrt{g})_{x'} + \tfrac{1}{4}\alpha'(\ln \Lambda^2)(\mathscr{R} - \tfrac{1}{12}F^2) \\ & + \left[\gamma_0 - \frac{\alpha'}{64\pi}(\mathscr{R} - \tfrac{1}{12}F^2)\right]\int d^2x\, d^2x'\,(R\sqrt{g})_x\Box^{-1}_{xx'}(R\sqrt{g})_{x'} \\ & + O(\alpha'^2)\,, \qquad \gamma_0 \equiv (D-26)/96\pi\,. \end{aligned} \quad (36)$$

Here $F^2 = F_{ijk}F^{ijk}$ and $W_\infty(0) = -\frac{1}{6}(D-26)\ln \Lambda^2$ (note that $\int R\sqrt{g}\, d^2x = 8\pi$)*. In (36) we accounted for the leading gravitational and antisymmetric tensor contributions already found in (22), (25). We also included the contribution of the ghost operator corresponding to the gauge $g_{\mu\nu} = e^{2\rho}\hat{g}_{\mu\nu}$ (with $\hat{g}_{\mu\nu}$ being a metric on S^2). For $g_{\mu\nu} = e^{2\rho}\hat{g}_{\mu\nu}$**

$$\begin{aligned} & -\tfrac{1}{4}\alpha'\mathscr{D}^2\Phi \int d^2x\, d^2x'\,\sqrt{g_x}\Box^{-1}_{xx'}(R\sqrt{g})_{x'} + \tfrac{1}{4}\alpha'(\ln \Lambda^2)(\mathscr{R} - \tfrac{1}{12}F^2) \\ & + \left[\gamma_0 - \frac{\alpha'}{64\pi}(\mathscr{R} - \tfrac{1}{12}F^2)\right]\int d^2x\, d^2x'\,(R\sqrt{g})_x\Box^{-1}_{xx'}(R\sqrt{g})_{x'} \end{aligned}$$

$$\hat{\Box} = \Box(\hat{g})\,, \qquad \hat{R} = R(\hat{g})\,. \quad (37)$$

All the cutoff dependence in (36) can be absorbed in rescaling of the metric ($g_{\mu\nu} = \Lambda^{-2}\tilde{g}_{\mu\nu}$) and the scalar field ($\Phi = \Lambda^2\tilde{\Phi}$):

$$W = \tilde{\Phi}\int d^2x\,\sqrt{\tilde{g}} - \tfrac{1}{4}\alpha'\mathscr{D}^2\tilde{\Phi}\int \tilde{\Box}^{-1}\tilde{R} + \left[\gamma_0 - \frac{\alpha'}{64\pi}(\mathscr{R} - \tfrac{1}{12}F^2)\right]\int \tilde{R}\tilde{\Box}^{-1}\tilde{R}\,. \quad (36')$$

* To obtain this value of $W_\infty(0)$ one has to correctly subtract the contribution of the six zero modes (corresponding to the conformal Killing vectors) of the ghost operator $-\nabla^2_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ and to define $[dg_{\mu\nu}]$ in (30) by dividing the formal measure by the volume of the diffeomorphism group (see [8]).

** It may seem that $\int R\Box^{-1}R$ should not change under the constant scaling of the metric. However, this symbol in (19) should be understood precisely according to (37).

Expanding e^{-W} in powers of α' we obtain

$$\Gamma[\Phi, G, A] \sim \mathscr{V}^{-1} \int d^D\phi \sqrt{G(\phi)} \int [d\rho]\, e^{-W_0}$$

$$\times \left\{ 1 - a\alpha' \left[\mathscr{D}^2\Phi \int d^2x \sqrt{g} + (\mathscr{R} - \tfrac{1}{12}F^2) \right] \right.$$

$$\left. + \tfrac{1}{4}\alpha' \mathscr{D}^2\Phi \int \Box^{-1}R + \frac{\alpha'}{64\pi}(\mathscr{R} - \tfrac{1}{12}F^2) \int R\Box^{-1}R + O(\alpha'^2) \right\}, \tag{38}$$

$$W_0 = W_\infty(0) + \Phi \int d^2x \sqrt{g} + \gamma_0 \int R\Box^{-1}R, \qquad a \equiv \tfrac{1}{4}\ln \Lambda^2 .$$

Here

$$g_{\mu\nu} = e^{2\rho}\hat{g}_{\mu\nu}, \qquad R\sqrt{g} = \hat{R}\sqrt{\hat{g}} - 2\hat{\Box}\rho ,$$

$$\hat{g}_{\mu\nu} = \frac{\delta_{\mu\nu}}{(1 + x^2/4r^2)^2},$$

$$\int R\Box^{-1}R \equiv \int d^2x\, d^2x' \sqrt{g_x} R_x \Box^{-1}_{xx'} \sqrt{g_{x'}} R_{x'} ,$$

$$\int \Box^{-1}R \equiv \int d^2x\, d^2x' \sqrt{g_x} \Box^{-1}_{xx'} R_x \sqrt{g_{x'}} .$$

ρ-integration goes over regular functions on S^2. Note that (38) does not contain additional integrations over parameters of "Teichmuller deformations" of the metric [8] because any traceless deformation of a metric on M_2^2 can be generated by a local diffeomorphism. The integrand in (38) is invariant under the diffeomorphisms that preserve the gauge condition $g_{\mu\nu} = e^{2\rho}\hat{g}_{\mu\nu}$. They are generated by the six conformal Killing vectors (three SO(3) rotations and three proper conformal boosts). The corresponding global transformations form the conformal group SO(2, 2). $\mathscr{V}$ is the (infinite) volume of this group which cancels the analogous factor coming from the integral over ρ (ρ is invariant under SO(3) but is "shifted" by the conformal boosts).

Let us first investigate the case of $\gamma_0 = 0$ or $D = 26$. The free theory ($\Phi = 0$, $G_{ij} = \delta_{ij}$, $A_{ij}, \ldots = 0$) is then Weyl invariant, i.e. ρ is a gauge degree of freedom. The corresponding partition function $\Gamma[0]$ then diverges and is to be regulated by introducing a gauge. In analogy with philosophy adopted in ordinary gauge theories it may seem natural to insert the Weyl gauge ($\rho = 0$) also in the presence of "sources" ("external" fields) even though they formally break the Weyl symmetry. In any case, the integral over ρ is not a well-defined one for $\gamma_0 = 0$ and hence some prescription of how it is to be evaluated is to be adopted. That the gauge-fixing prescription is a reasonable one is seen from the fact that using it one can prove that the amplitude

(2) reproduces the Virasaro-Shapiro (VS) amplitude for $D=26^\star$. Putting ρ equal to zero in (38) and computing all the integral with the S^2 metric $(=\hat{g}_{\mu\nu})$ we finish with the result

$$\Gamma^{(26)}[\Phi, G, A] \simeq -M^D \int \mathrm{d}^D\phi \sqrt{G}\, \mathrm{e}^{-\Phi}[1-\tfrac{1}{4}a_1\alpha'\mathcal{D}^2\bar{\Phi} + \alpha' b_1(\mathcal{R}-\tfrac{1}{12}F^2)+\cdots], \tag{39}$$

where a_1 and b_1 are constants, $\bar{\Phi}=\Phi\int\sqrt{\hat{g}}\,d^2x$ and M is a normalization mass that can be taken equal to $(2\pi\alpha')^{-1/2}$. Having completely "omitted" the integral over the metrics, we cannot apply the natural "mechanism" for elimination of infinities described above. Thus in (39) we assume that a_1 is a finite but "subtraction-dependent" constant.

It may appear that $\rho=0$ is a too stringent condition. In fact, it may be desirable to preserve the "small" conformal group (see previous footnote) under which $\delta\rho=\varepsilon_{(\alpha)}\nabla_\mu\zeta^{(\alpha)}_\mu$ ($\zeta^{(\alpha)}$ are the proper conformal Killing vectors), i.e. to fix the Weyl gauge only for the "transverse" directions. Then the (finite-dimensional) integral over the "pure conformal gauge" ρ's will remain and our "mechanism" will be applicable. We shall discuss this more rigorous approach later but it is instructive first to analyze the ground state problem for the "model action" (39). Introducing the new dimensionless scalar field $\Omega=\exp(-\tfrac{1}{2}\bar{\Phi})$ we can rewrite (39) as

$$\Gamma^{(26)}[\Phi, G, A] \simeq -M^D \int \mathrm{d}^D\phi \sqrt{G}[\Omega^2-\alpha' a_1(\partial_i\Omega)^2 + \alpha' b_1(\mathcal{R}-\tfrac{1}{12}F^2)\Omega^2+\mathrm{O}(\alpha'^2)]. \tag{40}$$

Hence if we ignore the metric and the antisymmetric tensor backgrounds we get a free scalar field which is a ghost for $a_1<0$ and a tachyon for $a_1>0$. If $a_1>0$ (as we shall assume) all the three "kinetic terms" in (40) have the physical signs.

Correspondence with the free string spectrum** suggests that $a_1=\tfrac{1}{4}$, though the choice of this particular value will not be important for the subsequent discussion.

A naive flat-space vacuum $\Omega=0$ is unstable due to the tachyonic nature of the fluctuation mode. Let us see if the full action (40) admits a stable ground state

* In the momentum representation $G_N(p_1,\ldots,p_N)=\langle\prod_{k=1}^N\int \mathrm{d}^2x_k\sqrt{g_{x_k}}\,\mathrm{e}^{i\varphi(x_k)\cdot p_k}\rangle$. Integration over ϕ produces $\delta(\Sigma p)$. Inserting $\prod_x\delta(\rho(x))$ we get just the VS amplitude but without the mass-shell condition $\alpha' p_k^2=4$. This condition is recovered under additional requirement of conformal invariance (duality). It was noted in [10] that using a different prescription for integration over $\rho(x_k)$ one can obtain in G_N the pole factors $\prod_k(4-\alpha' p_k^2)^{-1}$ in front of the VS amplitude. However, this prescription is completely ad hoc. The only important fact is that the G_N with $\rho(x_k)$ integrations included is invariant under the conformal SO(2, 2) transformations for arbitrary $p_k^2=-m^2$, while the VS amplitude is conformal invariant only at the point $p_k^2=4/\alpha'$. Hence the only effect of the conformally invariant removing of the ρ-integrations should be restriction of the VS amplitude on the mass shell. This justifies the use of the conformal invariance requirement in addition to the gauge-fixing prescription.

** This correspondence is not obvious in fact. The naive vacuum seems to be $\Phi=0$ and not $\Omega=0$ (cf. (2), (14)). Note, however, that Ω depends on a "subtracted" value of Φ and hence a direct interpretation is complicated due to the dependence on the cutoff.

configuration. Suppose we look for maximally symmetric vacua, so that the vacuum value of Ω is $\Omega_0 = \text{const} \neq 0$. Then the classical field equations corresponding to (40) take the form

$$1 + \alpha' b_1(\mathcal{R} - \tfrac{1}{12}F^2) = 0\,,$$
$$\mathcal{R}_{ij} - \tfrac{1}{4}F_{imn}F_j^{\ mn} = 0\,, \qquad \mathcal{D}_i F^{ijk} = 0\,. \tag{41}$$

The value of Ω_0 is thus arbitrary. It is easy to see that there are no solutions with $F_{ijk} = 0$. No solutions exist also if G_{ij} has the euclidean signature (so that $F^2 > 0$). Hence we *deduce* that the D-dimensional space M^D, where strings are defined, must have at least one "time" direction. Only one "time" is consistent with the absence of ghosts. Solutions with a maximal symmetry are obtained in the case when $F_{ijk} \sim \varepsilon_{ijk}$ in some two space and one time dimensions. Thus $\mathrm{M}^D = \tilde{\mathrm{S}}^3 \times \mathrm{M}^{D-3}$ where $\tilde{\mathrm{S}}^3$ is an anti-de Sitter three-dimensional "space-time" (with signature $-++$), i.e.

$$\mathcal{R}_{ab} = -\tfrac{1}{2}f_0^2 G_{ab}\,, \qquad F_{abc} = f_0 \varepsilon_{abc}\,, \tag{42}$$
$$f_0^2 = 1/\alpha' b_1\,, \qquad a, b, c = 0, 1, 2\,.$$

If F_{ijk} has all other components equal to zero, M^{D-3} is an arbitrary Einstein space with a zero cosmological constant, i.e.

$$\mathcal{R}_{mn} = 0\,, \qquad m, n = 4, \ldots, D \tag{43}$$

(We assume that G_{ij} is block diagonal). Hence the maximally symmetric case is $\mathrm{M}^{D-3} = \mathbb{R}^{D-3}$. Less symmetric solutions are found if $F_{ijk} \sim \varepsilon_{ijk}$ for some 3-dimensional subspaces of M^{D-3}. Then M^{D-3} splits on a product of compact S^3 factors:

$$\mathcal{R}_{\alpha_n\beta_n} = \tfrac{1}{2}f_n^2 G_{\alpha_n\beta_n}\,, \qquad F_{\alpha_n\beta_n\gamma_n} = f_n \varepsilon_{\alpha_n\beta_n\gamma_n}\,,$$
$$f_0^2 - \sum_n f_n^2 = 1/b_1\alpha' > 0\,. \tag{44}$$

The general solution corresponds to M^D equal to a product of $\tilde{\mathrm{S}}^3$, of N S^3 factors and of some ($\dim = D - 3(N+1)$) Einstein manifold satisfying (43)*.

We see that the Bose string model prefers the *three* space–time dimensions with the anti-de Sitter metric and the non-vanishing antisymmetric tensor field strength. Direct analysis shows that there are no tachyonic modes in the spectrum of fluctuations of the fields near their vacuum values. Thus in contrast to the "naive" $\Omega = 0$ vacuum the true vacuum is *stable*. Note that the non-zero vacuum value of Ω implies the non-zero constant vacuum value for $\Phi \sim -\ln \Omega$**. Thus our conclusion is that generation of non-trivial background values for Φ, G_{ij} and A_{ij} formally solves the "tachyon problem".

* In contrast to the Freund–Rubin mechanism [36], the presence of Ω makes it possible to have flat space-like subspaces.

** Note also that the vacuum value of the action (40) is equal to zero and that the value of Ω determines the value of the gravitational constant.

This result (if true also for $D<26$) seems to rule out a direct application of the closed Bose string theory to a description of glueballs but is quite natural from the "fundamental" point of view on a string theory as a candidate for a theory of all interactions. In a more realistic superstring case other antisymmetric tensors will be present in (40) (while Ω will be absent) and hence more realistic patterns of a ground state compactification may exist. Here we would like to note the following problem that must be carefully analyzed in the approach to compactification based on the effective action like (30), (40). In the derivation of the action (40) we used the $\alpha' \to 0$ approximation (and neglected higher tensor fields). However, the resulting ground state space appears to have a characteristic scale of the order of $(\alpha')^{1/2}$. Hence we have to study whether the higher-order terms in (40) are in fact irrelevant. It seems likely that it is incorrect to study compactification in the superstring theory starting only with a D = 10 supergravity action. Maybe it is sufficient to include a number of other terms (like $\alpha'^2 \mathcal{R}^2 + \cdots$) but maybe one has to invent a new approximation scheme for Γ without expanding it in powers of α'.

Let us now consider the computation of Γ for the general case $D \leqslant 26$ (for $D>26$ ρ is a ghost). It is useful to isolate first the integral over a constant scale of the metric and then to do a loop expansion for the remaining degrees of freedom. This is done systematically by inserting $1=\int_0^\infty \mathrm{d}A\, \delta(\int \sqrt{g}\, \mathrm{d}^2x - A)$ in eq. (36), i.e. by extracting the integral over the surface area A [23] (see also [24]). Starting with eq. (38) and using (37) we get

$$\begin{aligned}
\Gamma[\Phi, G, A] \sim \mathcal{V}^{-1} \int \mathrm{d}^D\phi \sqrt{G} \int_0^\infty \frac{\mathrm{d}A}{A} \mathrm{e}^{-\Phi A} \int \mathrm{d}\tilde{\rho}\, \delta\Big(\int \mathrm{d}^2x \sqrt{\hat{g}}(\mathrm{e}^{2\tilde{\rho}}-1)\Big) \\
\times \mathrm{e}^{-S}\Big\{1 - \tfrac{1}{2}\alpha' \mathcal{D}^2(\Phi A) \hat{A}^{-1} \int \mathrm{d}^2x \sqrt{\hat{g}}\, \mathrm{e}^{2\tilde{\rho}} [\tfrac{1}{2} \ln(\Lambda^2 A) + \tilde{\rho} - \tfrac{1}{2}\hat{\Box}^{-1}\hat{R}] \\
+ \tfrac{1}{4}\alpha'(\mathcal{R} - \tfrac{1}{12}F^2)\Big[1 + \ln(\Lambda^2 A) + \frac{1}{4\pi}\int \mathrm{d}^2x\, \tilde{\rho} \Box_0 \tilde{\rho} \\
- 2\hat{A}^{-1} \int \mathrm{d}^2x \sqrt{\hat{g}} \tilde{\rho}\Big] + \mathrm{O}(\alpha'^2)\Big\}, \\
S \equiv \kappa - \kappa \ln(\Lambda^2 A) + \frac{\kappa}{4\pi}\Big[\int \mathrm{d}^2x\, \tilde{\rho}\Box_0\tilde{\rho} + \tfrac{1}{2}R_0 \int \mathrm{d}^2x \sqrt{g_0}(\mathrm{e}^{2\tilde{\rho}} - 2\tilde{\rho} - 1)\Big]. \qquad (45)
\end{aligned}$$

Here $\kappa = \frac{1}{6}(D-26)$ and $\hat{A} = \int \mathrm{d}^2x \sqrt{\hat{g}}$ ($\hat{g}_{\mu\nu}$ is the same as in (38)). The metric $g_{0\mu\nu} = (A/\hat{A})\hat{g}_{\mu\nu}$ (R_0 is the curvature scalar of g_0, $R_0 = 8\pi/A$, $\Box_0 = \Box(g_0)$) is the stationary point of the action $\sim \int R\Box^{-1}R$ under the constraint $\int \mathrm{d}^2x \sqrt{g} = A$. The integration variable $\tilde{\rho}$ (a regular function on S^2) has a vanishing vacuum value (the total metric is now $g = \mathrm{e}^{2\tilde{\rho}} g_0$, cf. (38)). The Liouville-type action for $\tilde{\rho}$ in (45) is positive under the constraint $\int \mathrm{d}^2x \sqrt{\hat{g}}(\mathrm{e}^{2\tilde{\rho}} - 1) = 0$ [38].

All the A-dependences in (45) are shown explicitly except that coming from the dependence on $g_{0\mu\nu} \sim A\hat{g}_{\mu\nu}$. In view of the Weyl invariance of $\Box_0$ and $R_0\sqrt{g_0}$ the latter dependence can be only of the "anomalous" type, $f(\Lambda^2 A)$. Thus we see (in agreement with the suggestion made above) that all the cutoff dependence in (45) can be eliminated by introducing the new variables $\tilde{A}=\Lambda^2 A$ and $\tilde{\Phi}=\Lambda^{-2}\Phi$ so that *Γ expressed in terms of $\tilde{\Phi}$, G_{ij} and A_{ij} is independent of the cutoff, i.e. is an unambiguously calculable functional.* This conclusion is unchanged if the integral over A formally diverges at $A=0$ because in fact it must be cut off at $A_{\min}\sim 1/\Lambda^2 \to 0$.

We shall evaluate Γ in the simplest possible approximation, including the one-loop contribution from the $\tilde{\rho}$ action but ignoring the $\tilde{\rho}$ dependence of the pre-exponential terms. The integral over $\tilde{\rho}$ then is

$$\int \mathrm{d}\tilde{\rho}\,\delta\left(\int \sqrt{\hat{g}}\,\tilde{\rho}\,\mathrm{d}^2x\right)\exp\left(\frac{\kappa}{4\pi}\int \mathrm{d}^2x\,\sqrt{g_0}\,\tilde{\rho}\tilde{\Delta}_0\tilde{\rho}\right)\{1+\cdots\},$$

$$\tilde{\Delta}_0 \equiv \Delta_0 - R_0\,,\qquad \Delta_0 = -\frac{1}{\sqrt{g_0}}\partial_\mu(g_0^{\mu\nu}\sqrt{g_0}\,\partial_\nu)\,. \tag{46}$$

The operator $\tilde{\Delta}_0$ defined on S^2 has the spectrum $\lambda_n=(4\pi/A)[n(n+1)-2]$, i.e. has one negative mode (with eigenfunction $\tilde{\rho}=\text{const}$) and three zero modes (corresponding to the invariance of the integrand of (45) under the (proper) conformal transformations). The negative mode is projected out by the δ-function in (46) while the infinite integral over the (normalized) zero modes is cancelled out by the $\mathcal{V}^{-1}$ factor in (45). The contribution of (46) is thus proportional to

$$(\Lambda^2 A)^{\frac{1}{2}\times 4}(\det{}'\,\tilde{\Delta}_0)^{-1/2}\,,$$

$$\ln\det{}'\,\tilde{\Delta}_0 = -(B_2-4)\ln(\Lambda^2 A)+\text{const}\,, \tag{47}$$

where the first factor comes from the normalization of the negative and zero modes, $B_2=(1/4\pi)\int \mathrm{d}^2x\,\sqrt{g_0}(\frac{1}{6}R_0+R_0)=\frac{7}{3}$ is the corresponding DeWitt–Seeley coefficient for $\tilde{\Delta}_0$ and the prime and (-4) indicate that the first four modes of $\tilde{\Delta}_0$ are not included. Introducing $\tilde{A}=\Lambda^2 A$ and $\tilde{\Phi}=\Lambda^{-2}\Phi$ we can then put the result for Γ in the form*

$$\Gamma \sim \int \mathrm{d}^D\phi\,\sqrt{G}\int \mathrm{d}\tilde{A}\,\mathrm{e}^{-\tilde{\Phi}\tilde{A}+\nu\ln\tilde{A}}\{1+\tfrac{1}{2}\alpha'\tilde{A}\mathcal{D}^2\tilde{\Phi}$$

$$\times(1-\tfrac{1}{2}\ln\tilde{A})+\tfrac{1}{4}\alpha'(\mathcal{R}-\tfrac{1}{12}F^2)(1-\ln\tilde{A})+\cdots\}\,. \tag{48}$$

Here $\nu=\frac{1}{6}(D-25)$ if $D<26$, and $\nu=1$ if $D=26$ (the contribution of $\det{}'\,\tilde{\Delta}_0$ in (47) is absent in the latter case, while the integral over $\tilde{\rho}$'s expandable in higher $n>1$ eigenfunctions of $\tilde{\Delta}_0$ is regulated by inserting the Weyl gauge). For the free case ($\Phi=\text{const}$, $G_{ij}=\delta_{ij}$, $A_{ij}=0$) eq. (48) agrees with the result of ref. [24].

* Note that to absorb the quadratic infinities (which we did not indicate explicitly above) one apparently has to redefine $\Phi\to\Phi+\text{const}\,\Lambda^2$. As a result the constant part of $\tilde{\Phi}$ appears to be ambiguous. Here it may be useful to recall once again that one can get free of this ambiguity by using a dimensionless cutoff or by properly defining the measure of the path integrals involved.

The integral over $\tilde{A}$ is also to be evaluated in a "saddle-point" approximation. The "saddle point" for the "action" in (48) is

$$\tilde{A}_0 = \nu / \tilde{\bar{\Phi}} . \tag{49}$$

Expanding $\tilde{A}$ near $\tilde{A}_0$ and integrating over $\tilde{A} - \tilde{A}_0$ in the gaussian approximation for the "action" (putting $\tilde{A} = \tilde{A}_0$ in the pre-exponential terms) we find from (48)

$$\Gamma \simeq cM^D \int \mathrm{d}^D\phi \sqrt{G} (\tilde{\bar{\Phi}}/\nu)^\delta \times \left\{ 1 - \tfrac{1}{2}\alpha' \nu \frac{\mathscr{D}^2 \tilde{\bar{\Phi}}}{\tilde{\bar{\Phi}}} (1 - \tfrac{1}{2} \ln (\tilde{\bar{\Phi}}/\nu)) + \tfrac{1}{2}\alpha' (\mathscr{R} - \tfrac{1}{12}F^2)(1 + \ln (\tilde{\bar{\Phi}}/\nu)) + \mathrm{O}(\alpha'^2) \right\} , \tag{50}$$

where $\delta = -1 - \nu$ (i.e. $\delta = \frac{1}{6}(19 - D)$ for $D < 26$ and $\delta = -2$ for $D = 26$) and $c = \text{const}$. Introducing the dimensionless scalar field $\Omega = (\tilde{\bar{\Phi}}/\nu)^{\delta/2}$ we can rewrite (50) as

$$\Gamma \simeq cM^D \int \mathrm{d}^D\phi \sqrt{G} [\Omega^2 - \alpha' (\partial_i \Omega)^2 (a_1 + a_2 \ln \Omega) + \tfrac{1}{4}\alpha' (\mathscr{R} - \tfrac{1}{12}F^2) \Omega^2 \left(1 + \frac{2}{\delta} \ln \Omega \right) + \mathrm{O}(\alpha'^2)] , \tag{51}$$

where $a_1 = -\delta^{-2}(1 + \delta)(3 - 2\delta)$, $a_2 = -2\delta^{-3}(1 + \delta)$. For $\delta = \infty$ (51) has the same form as the "model action" (40). This suggests that the ground-state structure should be similar for both actions. Assuming that the ground state value of Ω is $\Omega_0 = \text{const} \neq 0$ and ignoring $\ln \Omega$-terms we find that the classical field equations corresponding to (51) coincide with (41):

$$\mathscr{R} - \tfrac{1}{12}F^2 = -4\alpha'^{-1} , \qquad \mathscr{R}_{ij} - \tfrac{1}{4} F_{imn} F_j{}^{mn} = 0 , \tag{52}$$

Again there are no solutions corresponding to the flat D-dimensional space-time, while the maximally symmetric solution is (anti-de Sitter)$_3 \times \mathbf{R}^{D-3}$ with $F_{ijk} \sim \varepsilon_{ijk}$*. The stability of the solutions depends on the values of a_1 and a_2 and deserves special study.

Our basic conclusion is that (either for $D = 26$ or for $D < 26$) it is necessary to *account for the "condensates" of the D-dimensional metric and the antisymmetric tensor* (as well as for the "condensate" of the "ground state" scalar) in order to determine *the true ground state* of the closed Bose string theory. It is inconsistent (from the effective field theory point of view) to take D-dimensional space-time to be flat.

A non-trivial question is whether a solution of (52) approximates a solution of the full effective equations (28). This may not be the case because (52) imply that $\alpha' \mathscr{R} = -24$, $\alpha' F^2 = -96$ and hence all higher order ($\sim \alpha'^2 \mathscr{R}^2 + \cdots$) terms in (51) a priori cannot be neglected. However, it is possible to give an indirect argument that

* An interesting property of this solution is that it "predicts" not only the dimension of the "effective" space-time (three) but also its *non-euclidean signature*. This raises a hope that both the dimension and the signature of the physical space-time may be predicted in a similar fashion in a more realistic superstring context.

$(\mathrm{adS})_3 \times \mathbb{R}^{D-3}$ with $\mathcal{R} - \frac{1}{4}F^2 = 0$ may still be an exact (tree level) solution of string theory ground state equations (28). Eqs. (28) imply vanishing of expectation values of some string-theory operators constructed of φ^i and $g_{\mu\nu}$. Particular vacuum backgrounds may be distinguished by some symmetry properties implying the vanishing of these expectation values. For example, if the generalized string action (8) corresponds to a conformal invariant two-dimensional field theory then (at least in the tree approximation and for the critical number of dimensions) the expectation value of any local operator of non-negative dimension vanishes and thus the corresponding background fields solve eqs. (28) (cf. [43]). The only known conformally invariant model (with vanishing β-function) of the type (8) is the principal σ-model with the Wess–Zumino term having a properly chosen coefficient [26] (supersymmetry offers additional possibilities [19, 28]). The simplest example is (8) with G_{ij} being a metric on S^3, A_{ij} having $F_{ijk} = f\varepsilon_{ijk}$ (f = const) on S^3, Φ = const and all other fields vanishing. If the scalar curvature of G_{ij} is $\mathcal{R} = 6/r^2$ (r is the radius of S^3) then the value of f ensuring the vanishing of the β-function is $f = \pm 2/r$ so that $\mathcal{R} - \frac{1}{4}F_{ijk}F^{ijk} = 0$ [26, 28]. Note that not only S^3 but also its "analytic continuation" obtained by changing the signature from $(+++)$ to $(-++)$ without violating the relation $\mathcal{R} - \frac{1}{4}F^2 = 0$ (i.e. $(\mathrm{adS})_3$ with $F_{ijk} \sim \varepsilon_{ijk}$) is admissible as providing the vanishing β-function. Thus in spite of the fact that higher-order terms in (51) are not naturally suppressed for the above $(\mathrm{adS})_3 \times \mathbb{R}^{D-3}$ solution it is likely to coincide with an exact (tree-level) ground state solution of the Bose string theory.

The action analogous to (51) is found if we include the contribution of the "dilaton" field C in (8) (see ref. [42]). Taking for simplicity Φ = const we get in the tree approximation (at the critical dimension)

$$\Gamma \sim \int \mathrm{d}^D\phi \, \sqrt{G} \, \mathrm{e}^{-2C}\{1 + \tfrac{1}{4}\alpha'[\mathcal{R} + 4(\partial_i C)^2 - \tfrac{1}{12}F_{ijk}^2] + \mathrm{O}(\alpha'^2)\}\,, \tag{53}$$

or after the Weyl rescaling $G_{ij} \to G_{ij} \exp(4C/(D-2))$

$$\begin{aligned}\Gamma \simeq &-\frac{4}{\kappa^2\alpha'}\int \mathrm{d}^D\phi \sqrt{G} \exp\left(\frac{4C}{D-2}\right) \\ &-\frac{1}{\kappa^2}\int \mathrm{d}^D\phi \, \sqrt{G}\left\{\mathcal{R} - \frac{4}{D-2}(\partial_i C)^2 - \tfrac{1}{12}F_{ijk}^2 \, \mathrm{e}^{-8C/(D-2)} + \mathrm{O}(\alpha')\right\}, \\ &\kappa \sim g(\alpha')^{(D-2)/4}, \qquad g = \mathrm{e}^{-\sigma}. \end{aligned} \tag{53'}$$

The second term here has the same form as the bosonic part of the action of $D = 10$ supergravity. The important advantage of our approach over the previous ones [3, 14] (based on S-matrix considerations) is that we are able to establish the full *non-polynomial* structure of dilaton couplings. Assuming C = const in the vacuum, the effective equation $\delta\Gamma/\delta C = 0$ implies the vanishing of the "lagrangian" in the curly brackets in (53) (and thus, incidentally, the vanishing of the vacuum value of cosmological constant). Now it is easy to prove the observation made in ref. [43]

that (in the tree approximation) type I superstring theory always admits a vacuum solution $M^{10} = M^4 \times \Sigma^6$, where M^4 is flat. In fact, because of $\delta\Gamma/\delta C = 0$ the equation $\delta\Gamma/\delta G_{ij} = 0$ looks like (cf. (52); e^{-2C} in (53) plays the role of Ω^2 in (51))

$$\mathcal{R}_{ij} \sim F_{imn}F_j{}^{mn} + \mathcal{R}_{i\dots}\mathcal{R}_j{}^{\dots} .$$

If M^4 is maximally symmetric, all "matter" fields should have vanishing 4-dimensional components (the argument breaks down for type IIA theory where one may have $F_{ijkl} \sim \varepsilon_{ijkl}$). Then $\mathcal{R}_{i'j'k'l'} = 0$, i', $j'_{,\dots} = 0, \dots, 3$, is always a solution of $\delta\Gamma/\delta G_{ij} = 0$.

One may question how the tachyon problem can be solved in an open Bose string theory. The generalization of (7) to the (oriented) open-string theory (more properly, to the theory of interacting open and closed strings) is

$$\Gamma[\Phi, G_{ij}, A_{ij}, C_{,\dots}|X, A_{i,\dots}] = \sum_{\chi} e^{\sigma\chi} \int [dg_{\mu\nu}][d\varphi^i]\, e^{-I}\, \mathrm{tr}\,(P\, e^{-I_{\partial M}}) , \tag{54}$$

where the summation goes over compact (oriented) 2-manifolds with the topology of a disk with holes (k) and handles (n) ($\chi = 1 - k - 2n$); I is given by (8) and

$$I_{\partial M} = \int_{\partial M^2} dt\, \{eX(\varphi(t)) + i\dot\varphi^j A_j(\varphi(t)) + \cdots\} , \qquad \dot\varphi = \frac{d\varphi}{dt} ,$$

$\varphi^i(t) \equiv \varphi^i(x(t))$, $x^\mu(t)$ parametrizes the boundary, $e^2(t) = (g_{\mu\nu}\dot x^\mu \dot x^\nu)_{\partial M}$. The fields X, $A_i, \dots$ correspond to the modes of the open-string spectrum (scalar tachyon, massless vector, ...) and are assigned to representations of an internal symmetry group G (in the oriented string case $G = U(N)$ and fields belong to the adjoint representation). P in (54) is the ordering along ∂M^2. Ignoring non-trivial backgrounds of the fields of the closed string sector ($G_{ij} = \delta_{ij}$, $\Phi_{,\dots} = 0$) we get for $G = U(1)$ in a tree approximation ($M^2 = \text{disc}$)

$$\Gamma \sim \int d^D\phi \int_0^\infty d\tilde L \tilde L^\gamma\, e^{-\tilde X \tilde L}\{1 + \pi\alpha' k_1(\tilde L)\Box\tilde X + \tfrac{1}{2}\pi^2\alpha'^2 k_2 F_{ij}F^{ij} + \cdots\}$$

$$\sim \int d^D\phi\, f(\tilde X)\{1 + b(\tilde X)\Box\tilde X + \pi^2\alpha'^2 F_{ij}F^{ij} + \cdots\} ,$$

$$\tilde L = \Lambda L , \qquad \tilde X = \Lambda^{-1} X , \qquad \Lambda \to \infty ,$$

$$L = (\text{the length of } \partial M^2) = 2\pi a , \qquad F_{ij} = \partial_i A_j - \partial_j A_i ,$$

$$k_1 = \tilde L \oint d\theta\, K(\theta, \theta) = -\tilde L \ln \tilde L + \text{const} , \qquad 0 \leq \theta \leq 2\pi ,$$

$$k_2 = \oint d\theta \oint d\theta' \left[\frac{\partial}{\partial\theta} K(\theta, \theta') \frac{\partial}{\partial\theta'} K(\theta, \theta') - K(\theta, \theta') \frac{\partial^2}{\partial\theta\partial\theta'} K(\theta, \theta')\right] = 2 ,$$

where $K(\theta, \theta') = K(z, z')|_{\partial M}$ is the boundary value of the Neumann function on a standard disc with radius a,

$$K(z,z')=\frac{1}{2\pi}\ln|z-z'||z-\bar{z}'^{-1}|,\qquad z|_{\partial M}=a\,e^{i\theta}.$$

A stable vacuum may correspond to $X=\text{const}$, $F_{ij}\sim\varepsilon_{ij}$, i.e. to a breakdown of D-dimensional Lorentz symmetry to $O(2)\times O(D-1,1)$. It may happen, however, that a stable ground state is possible only after the account of loop corrections, and thus of "condensates" of G_{ij} and A_{ij}.

5. Discussion

The argument which led to the expression (7) for the effective action used a string theory motivation. At the same time, another interpretation of Γ is possible. One can forget completely about the string theory and consider (7) as a basic definition of the effective action in a fundamental theory of infinite number of fields with the "massless" ones being the metric and the antisymmetric tensor. (In the supersymmetric case the massless sector will include also other "lower-spin" Bose and Fermi fields in a number hopefully sufficient for a low-energy correspondence with the "standard model".) The "internal" variables which are quantized in this theory are the coordinates φ^i (and also the "internal" metric $g_{\mu\nu}$). The "classical" space-time coordinates ϕ^i are the mean values of $\varphi^i(x)$ (cf. (10)). All "ordinary" fields are not explicitly quantized but their quantum dynamics is implicitly accounted for due to their dependence on the fluctuating space–time coordinates φ^i (see (8)).

A reasonable point of view is that all we need to know from a fundamental "quantum gravity" theory is an *effective action* $\Gamma[G_{ij},\psi]$ (G_{ij} is the metric and ψ stands for all other fields) that should satisfy several conditions: (i) it should be mathematically well defined, i.e. finite, calculable, etc; (ii) it should possess a consistent (unitary) low-energy limit, i.e. should contain the Einstein term and kinetic terms for ψ with correct physical signs and no instability should be present (no ghosts and tachyons, i.e. positive energy); (iii) it should be a starting point for a study of fundamental problems of Planck scale physics (quantum cosmology, compactification of extra dimensions, black holes, etc).

The Γ in eq. (6) may be viewed as a candidate for such an effective action. It is free of the usual problems of effective actions in ordinary field theories: it is manifestly covariant and "gauge independent" (no guage is needed to be fixed for G_{ij}) and it is free of the ultraviolet infinities. There are also no other problems (like indefiniteness of the classical action) arising in the naive quantization of the Einstein theory itself.

It of course remains to be seen whether Γ as defined by (7) is completely free of pathologies like some special sorts of infinities or inconsistency of the "loop expansion" in (7). In fact, the prescription of summation over topologies may look ad hoc* so that it may be desirable to derive it as a perturbation expansion for the quantum field theory of string functionals $\Psi[C]$ with the action $\int \Psi \Delta \Psi + \Psi^3$ (in analogy with what was done in the light-cone gauge in [39]). A covariant formulation of such a "second quantized" theory (generalizing the Polyakov approach to the "first quantized" string) remains to be developed.

Adopting this point view on the action (7) we are led to the following natural question: why the action (8) standing in the exponent in (7) is given by a *two*-dimensional integral, i.e. corresponds to *one*-dimensional extended objects (strings)? We know that historically the string theory was first developed in the context of a strong interaction theory and only then was it observed that the presence of "graviton" in the string excitation spectrum suggests a more fundamental role of strings. Nowdays it is clear that the quark-antiquark pair connected by an open string is indeed an adequate description of mesons, following from QCD but there seems to be no general arguments distinguishing strings as basic objects for the construction of a fundamental theory like (7). In fact, it is straightforward to write down a generalization of (7) for the case of a closed "extended" object of dimension $d-1$ (particle, string, membrane, ...)**

$$\Gamma[\Phi, G_{ij}, A_{i_1\cdots i_d}, \ldots] = \sum_{\text{topologies}} \int [\mathrm{d}g_{\mu\nu}] \int [\mathrm{d}\varphi^i]\, \mathrm{e}^{-(1/\hbar)I},$$

$$I = \Lambda^d \int \mathrm{d}^d x \sqrt{g}\,\Phi(\varphi) + \Lambda^{d-2} M^2 \int \mathrm{d}^d x \sqrt{g}\tfrac{1}{2} g^{\mu\nu} \partial_\mu \varphi^i \partial_\nu \varphi^j G_{ij}(\varphi)$$

$$+ iM^d \int \mathrm{d}^d x\, \varepsilon^{\mu_1 \cdots \mu_d} \partial_{\mu_1} \varphi^{i_1} \cdots \partial_{\mu_d} \varphi^{i_d} A_{i_1 \cdots i_d}(\varphi) + \ldots \quad (55)$$

Here $\mu, \nu = 1, \ldots, d$, $i, j = 1, \ldots, D$, $M = (2\pi\alpha')^{-1/2}$, Λ is a cut off, and dots stand for "higher-spin" field terms like $\int \sqrt{g}\, g^{\mu\nu} g^{\lambda\rho} \partial_\mu \varphi^{i_1} \partial_\nu \varphi^{i_2} \partial_\lambda \varphi^{i_3} \partial_\rho \varphi^{i_4} B_{i_1 \cdots i_4}\, \mathrm{d}^d x$, etc. The action I (and hence $\Gamma = \int \mathrm{d}^D \phi \sqrt{G} \mathscr{L}(\Phi(\phi), \ldots)$, cf. (11)), possesses only two kinds of "external" gauge invariances: (i) the D-dimensional covariance which is simply a consequence of the d-dimensional covariance of I; (ii) the abelian gauge invariance for the antisymmetric tensor, $\delta A_{i_1 \cdots i_d} = \partial_{[i_1} \lambda_{i_2 \cdots i_d]}$. As a by-product, we deduce that the spectrum of excitations in a theory of closed $d-1(>0)$-dimensional objects should contain the *massless* (gauge) fields (represented by the metric G_{ij} and the antisymmetric tensor of rank d) in addition to the infinite number of the *massive* fields***. In this way we understand that any theory (either supersymmetric or not) based on extended objects of an arbitrary dimension should contain (when

* One may question why different "loop diagrams" are taken with weights exp $(\sigma\chi)$ and not with $\omega(\chi)$ where ω is some unknown function. A partial justification is provided by the observation that $\sigma\chi$ can be written as a local addition to the action (8).

** We assume that Φ contains a constant part $\sim(1-\frac{1}{2}d)$, cf. [40].

***The spectrum of a theory of open $(d-1)$-dimensional objects is a "combination" of the spectra of the theories of closed objects of dimensions $(d-1)$ and $(d-2)$.

interpreted in four dimensions) *massless* particles only of spins $s \leq 2$. The point is that "higher-spin" fields in (55) lack the corresponding gauge groups needed to ensure their masslessness in Γ. The distinguished role played by G_{ij} and $A_{i_1 \cdots i_d}$ is due to the existence of only *two* covariant objects ($g_{\mu\nu}$ and $\varepsilon^{\mu_1 \cdots \mu_d}$) on an arbitrary finite-dimensional riemannian manifold. To get massless higher-spin fields we need a new (infinite-dimensional?) geometry with higher symmetric invariants.

What is less clear is how to give meaning to the sum over topologies in (55). Lacking topological classification of $(d>2)$-dimensional closed manifolds it is difficult to classify admissible types of interactions of extended objects. At the same time, the inclusion of non-trivial topologies ("loops") is necessary in order to have the effective "duality" between the "source" (external) fields in (55) and virtual states that propagate in loops (this "duality" is absent in the case of particles, i.e. $d=1$). If this duality takes place, we can consistently truncate Γ to an effective action $\tilde{\Gamma}$ for a field theory of finite number of fields defined at energies $E \ll M$, so that "loop" corrections to Γ reproduce ordinary loop corrections to $\tilde{\Gamma}$ computed with a cutoff at higher energies. Another problem is that the kinetic part of the action I in (55) lacks Weyl invariance if $d>2$ and hence the "first stage" effective action $W[g, \Phi, \dots]$ (cf. (13)) will be essentially non-local and difficult to compute. These and other technical problems present for $d>2$ distinguish the string case $d=2$ as a first non-trivial but yet tractable one.

Leaving aside these complications it is possible to guess the low-energy ($M \to \infty$) structure of Γ in (55). As in the string case it is natural to suppose again that the higher tensor fields $B_{ijkl\dots}$ are irrelevant for a study of a ground state problem, so that the $1/M$ expansion of Γ looks like (cf. (39), (48))

$$\Gamma[\Phi, G, A] \sim \int \mathrm{d}^D\phi \sqrt{G} \left\{ V(\Phi) + \frac{1}{M^2} f_1(\Phi) \partial_i \Phi \partial_j \Phi G^{ij} + \frac{1}{M^2} f_2(\Phi) \mathscr{R} + \frac{1}{M^2} f_3(\Phi) F_{i_1 \cdots i_{d+1}} F^{i_1 \cdots i_{d+1}} + \mathrm{O}(1/M^4) \right\}. \quad (56)$$

In analogy with the string case, it appears likely that the ground state values of fields will be $\Phi = \text{const}$, $G_{ij} = \{$block diagonal metric corresponding to a D-dimensional space-time being a product space containing one or several $(d+1)$-dimensional factors$\}$ and $F_{i_1 \cdots i_{d+1}} \sim \varepsilon_{i_1 \cdots i_{d+1}}$ (for indices belonging to a $(d+1)$ subspace). Hence compactification to four dimensions is preferred in the case of membranes. We would like to note, however, that the actual compactification mechanism may be different from that of ref. [36] due to the presence of the scalar Φ (as we already saw in sect. 4). Also, higher-order ($(1/M^4)\mathscr{R}^2 + \cdots$) terms in (56) may be important.

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Note added

After this paper was written we learned about the preprint [41] where a "phenomenological" approach to the study of compactification in a Bose string theory is discussed. It seems that the "first principles" approach to a field-theoretic description of a quantum string theory developed in our work provides a consistent framework for investigation of the ground state problem (including space–time compactification) in this theory and gives answers to the principal questions raised in ref. [41].

Note added in proof

Computing Γ at the critical dimension we are to discard the integral over the conformal degree of freedom ρ using a Weyl gauge. The resulting off-shell expression for Γ a priori depends on the Weyl gauge used. A crucial consistency requirement should be gauge independence of the effective background (a solution of (28)) and of Γ computed at the stationary point. A sufficient condition for this is the Weyl invariance of the integrand (i.e. a decoupling of ρ) for the stationary values of the fields. The on-shell Weyl invariance is also crucial for consistency of the amplitudes generated by Γ[44]. It is remarkable that the effective equations corresponding to (53) do imply (to the leading order) the quantum Weyl invariance of the string action (8). As was noted in ref. [45] they also imply the (one-loop) ultraviolet finiteness of the corresponding σ-model. Our conjecture (which may turn out to be true only in the superstring case) is that the stationary points of the effective action *always* correspond to Weyl-invariant (and *thus* finite) two-dimensional theories.

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