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NON-LINEAR ELECTRODYNAMICS FROM QUANTIZED STRINGS

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We compute the effective action for an abelian vector field coupled to the virtual open Bose string. The problem is exactly solved (in the "tree" and "one-loop" approximation for the string theory) for the case of a constant field strength and the number of space–time dimensions $D = 26$. The resulting tree-level effective lagrangian is shown to coincide with the Born–Infeld lagrangian, $[\det(\delta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})]^{1/2}$.

1. Introduction. In view of the present interest in string theories as candidates for a consistent unification of all fundamental interactions (see e.g. refs. [1–4]) it is important to gain deeper understanding of their properties, in particular of the non-perturbative ones. Within the standard approach to (super) string theories based on on-shell amplitudes on a flat background (or on a non-covariant action for string functionals) it is difficult to formulate and to solve the ground-state (compactification) problem and to establish connection with an effective field theory approximating string dynamics. Recently, we have proposed a new formulation of a string theory which is based on an off-shell covariant effective action Γ for an infinite number of fields corresponding to excitation modes of a first-quantized string [5,6]. Γ is simply a generating functional for all possible (off-shell) amplitudes on an arbitrary background. Extremizing Γ with respect to the background fields one can establish the true ground state of the theory. Γ computed in a proper approximation is directly the action of an effective "low-energy" field theory.

As was shown in ref. [5] the expansion of Γ in powers of derivatives of the fields (in the case of the closed Bose string theory) starts with the Einstein term ($\mathscr{R}$) for the metric $G_{\mu\nu}$ and the standard kinetic terms for the "dilaton" φ and the antisymmetric tensor $A_{\mu\nu}$. We found also a non-polynomial coupling of φ to $F^2_{\mu\nu\rho}$ which (for $D = 10$) is exactly the same as present in $D = 10$ supergravity [5,6]. To establish such results within the standard S-matrix approach it would be necessary to compute an infinite number of amplitudes with arbitrary numbers of external gravitons and dilatons (only the three-point amplitudes were previously computed [7]). Our approach thus makes possible to do calculations which are non-perturbative in a number of external fields.

To provide a consistent solution to a ground-state problem it is necessary also not to expand in a number of *derivatives* of fields (e.g. not to assume $\alpha'\mathscr{R} \ll 1$). It is likely that employing σ-model technique (cf. ref. [8]) and using some particular Ansätze for the background fields (e.g. $\mathscr{R} =$ const, $F_{\mu\nu\rho} \sim \varepsilon_{\mu\nu\rho}$, etc.) one can find expressions for Γ which are non-polynomial in the curvature, $F_{\mu\nu\rho}$, etc.

Here we are going to demonstrate how non-perturbative (in the number of fields *and* derivatives) results for the effective action can be derived in the case of the *open* Bose string theory. We shall study the dependence of Γ on the vector field A_μ of the open string spectrum assuming its strength $F_{\mu\nu}$ to be constant [‡1]. Both the "tree" and "one-loop" contributions to $\Gamma(F)$ will be computed in an *exact* way (never using $\alpha' \to 0$ approximation). In the "tree" approximation $\Gamma(F)$ coincides with the Born–Infeld action [9]. The computation of $\Gamma(F)$ is a string-theory analog of that of Schwinger [10] for a particle theory

[‡1] For simplicity we shall consider oriented U(1) open strings in $D = 26$.

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case. We note again that to establish our results within the S-matrix approach one is to sum contributions of an infinite number of amplitudes (only three-point amplitudes were previously computed for $G=U(n)$ [11,7]).

2. *General relations.* The (euclidean) effective action corresponding to the theory of interacting open and closed (oriented) strings is defined as follows [5,6]

$$\Gamma\left[\phi, A_\mu, B_{\mu\nu,\dots}\right]=\sum_{\chi=1,0,\dots} \mathrm{e}^{\sigma\chi}\int[\mathrm{d}g_{ab}][\mathrm{d}x^\mu]\,\mathrm{e}^{-I_2}\,\mathrm{tr}\left(P\,\mathrm{e}^{-I_1}\right), \tag{1}$$

$$I_2=\frac{1}{4\pi\alpha'}\int_M \mathrm{d}^2z\,\sqrt{g}\,g^{ab}\partial_a x^\mu\partial_b x^\mu, \tag{2}$$

$$I_1=\int_{\partial M}\mathrm{d}t\left[e\phi(x(t))+\mathrm{i}\dot{x}^\mu A_\mu(x(t))+e^{-1}\dot{x}^\mu\dot{x}^\nu B_{\mu\nu}(x(t))+\cdots\right]. \tag{3}$$

For simplicity we ignore non-trivial background values of the fields of the closed string sector ($G_{\mu\nu}=\delta_{\mu\nu}, \varphi_{\dots}=0$). The summation in (1) goes over the Euler number χ of a virtual string world sheet M^2 which is an oriented compact two-manifold with boundary (M^2 has the topology of a disc – in a "tree" approximation, of an annulus – in a "one-loop" approximation, etc.). In eq. (3) $x^\mu(t)\equiv x^\mu(z(t))$, $\mu=1,\dots,D$, where $z^a(t)$, $a=1,2$, parametrizes the boundary ∂M. The metric on ∂M is $e^2(t)=g_{ab}(z(t))\dot{z}^a\dot{z}^b$. The fields $\phi, A_\mu, B_{\mu\nu,\dots}$ correspond to the modes of the open string spectrum (i.e. to the scalar "tachyon", the gauge vector, the symmetric tensor, ...) and take values in the algebra of the internal symmetry group $G=U(n)$ (the fields are taken to be hermitean [‡2]. P in (1) indicates the ordering along the boundary. Varying Γ with respect to the fields (and putting them equal to zero) it is easy to check that the resulting amplitudes contain the standard Chan–Paton factors. σ in (1) is related to the dimensionless coupling constant of the open strings $g_o=\mathrm{e}^{-\sigma/2}$. For correspondence with the standard normalization of the amplitudes one is also to multiply the fields in (3) by g_o. Note that all terms in (3) [in (2)] which contain an even number of e [an odd number of ε^{ab}] are to be taken with "i" in order to get a *real* total string action in Minkowski-space formulation.

Now let us take $G=U(1)$, $A_\mu\neq 0$, $\phi, B_{\mu\nu}=0$ and $D=26$. For $D=26$ the integral over two-metrics in (1) reduces [‡3] to a finite-dimensional integral over Teichmüller parameters $\lambda_1,\dots,\lambda_N$ [14,15] ($\chi=1$: $N=0$; $\chi=0$: $N=1$; $\chi<0$: $N=-3\chi$). Thus we get

$$\Gamma=\sum_\chi \mathrm{e}^{\sigma\chi}\int \mathrm{d}\mu(\lambda)\int \mathrm{e}^{-I_2-I_1}[\mathrm{d}x^\mu],\quad \mathrm{d}\mu(\lambda)\equiv \mathrm{d}\lambda_1\dots\mathrm{d}\lambda_N\mu(\lambda_1,\dots,\lambda_N), \tag{4}$$

where μ includes the contribution of the ghost determinant, corresponding to the gauge $g_{ab}=\hat{g}_{ab}$ ($\hat{g}_{ab}(\lambda_1,\dots,\lambda_N)$ a "standard" metric on M^2 defined to have a constant two-curvature and zero geodesic

‡2 In the case of non-oriented strings one is to integrate over all orientable and non-orientable two-manifolds. Slicing such manifolds by planes one finds that there are exactly *seven* types of local interactions of non-oriented open and closed strings [12,6]. Note also that in this case the fields in (3) belong to different (symmetric and antisymmetric) representations of $SO(n)$ or $USp(n)$ [1].

‡3 This reduction is strictly true for a free partition function $\Gamma[0]$. For non-vanishing background fields the integral over the conformal factor ρ ($g_{ab}=\mathrm{e}^{2\rho}\hat{g}_{ab}$) does not decouple even for $D=26$. A prescription one uses to get rid of the dependence on ρ defines an off-shell extension of the amplitudes (cf. refs. [5,13]). When only $A_\mu\neq 0$ in (3) the "$\rho=1$ gauge" is the most natural prescription (when only $A_\mu\neq 0$ the integral over the constant part of ρ, i.e. the area of M^2, automatically decouples). Furthermore, when A_μ satisfies an "on-shell" condition (e.g. $F_{\mu\nu}=$ const.) the results should not depend on a prescription used.

curvature of ∂M [15]). We shall split x^μ in (4) on a constant and non-constant part, $x^\mu = y^\mu + \xi^\mu$. Then

$$\int [\mathrm{d}x^\mu] \mathrm{e}^{-I_2 - I_1} = \int \mathrm{d}^D y \int [\mathrm{d}\xi^\mu] \exp\left\{-\frac{1}{4\pi\alpha'}\int \mathrm{d}^2 z \sqrt{\hat{g}}\,(\partial\xi^\mu)^2 - \mathrm{i}\int \mathrm{d}t\, \dot{\xi}^\mu A_\mu(y+\xi)\right\}, \tag{5}$$

where $\xi^\mu(z)$ is a non-constant function on M^2, satisfying the Neumann boundary condition. Now let us integrate over $\xi^\mu(z)$ in all internal points of M^2, i.e. reduce (5) to a path integral over the boundary. To this end we introduce a set of (non-constant) fields $\eta^\mu_A(t_A)$, $A = 1, \ldots, p$, defined on simply connected components C_A of the boundary, $\partial\mathrm{M} = \cup_{A=1}^p \mathrm{C}_A$ ($\int_{\partial\mathrm{M}} = \Sigma_A \int_{\mathrm{C}_A}$) and insert in (5) $1 = \Pi_A \int [\mathrm{d}\eta^\mu_A]\delta(\xi^\mu|_{\mathrm{C}_A} - \eta^\mu_A)$. Representing δ-functions in terms of path integrals over $\nu^\mu_A(t_A)$ and carrying out gaussian integrations (first over $\xi^\mu(z)$ and then over ν^μ_A) we finish with the result (we rescale ξ and η by $\sqrt{2\pi\alpha'}$)

$$\int [\mathrm{d}\xi^\mu] \mathrm{e}^{-I_2 - I_1[\xi|_{\partial\mathrm{M}}]} \sim \int [\mathrm{d}\eta^\mu] \mathrm{e}^{-\frac{1}{2}\eta G^{-1}\eta - I_1[\sqrt{2\pi\alpha'}\eta]},$$

$$\eta G^{-1}\eta = \sum_{A,B} \int \mathrm{d}t_A\, \mathrm{d}t'_B\, \eta^\mu_A(t_A) G^{-1}_{AB}(t_A, t'_B) \eta^\mu_B(t'_B), \tag{6}$$

where G^{-1} is defined as follows. One first finds the Neumann function $N(z, z')$ for the Laplace operator $\Box = \partial_a(\sqrt{\hat{g}}\,\hat{g}^{ab}\partial_B)$, $-\Box N = \delta(z - z')$, then computes the matrix of its restrictions on the components of ∂M, $G_{AB}(t_A, t'_B) = \{N(z(t_A), z'(t'_B))\}$ and then finds the inverse, $G^{-1}G = \mathbf{1}$, $\mathbf{1} = \{\delta_{AB}\delta(t_A - t'_B)\}$.

Expanding A_μ in (5) is powers of ξ we have

$$\int \mathrm{d}t\, \dot{\xi}^\mu A_\mu(y+\xi) = \tfrac{1}{2} F_{\nu\mu}(y) \int \mathrm{d}t\, \dot{\xi}^\mu \xi^\nu + \tfrac{1}{3}\partial_\lambda F_{\nu\mu}(y) \int \mathrm{d}t\, \dot{\xi}^\mu \xi^\nu \xi^\lambda + \cdots. \tag{7}$$

Thus in general $\Gamma = \int \mathrm{d}^D y \mathscr{L}(\mathrm{y})$ where $\mathscr{L}$ depends on $F_{\mu\nu}$ and all powers of its derivatives at y. Now let us make our central assumption that $F_{\mu\nu} = \mathrm{const}$, i.e. let us concentrate only on the dependence of Γ on $F_{\mu\nu}$. Then the relevant path integral becomes a gaussian one

$$Z(F) \sim \int [\mathrm{d}\eta^\mu] \exp\left(-\tfrac{1}{2}\eta G^{-1}\eta + \frac{\mathrm{i}}{2}\bar{F}_{\mu\nu}\int \mathrm{d}t\, \dot{\eta}^\mu \eta^\nu\right), \quad \bar{F}_{\mu\nu} = 2\pi\alpha' F_{\mu\nu}. \tag{8}$$

Now we can make a O(D) rotation to put $F_{\mu\nu}$ in a standard block-diagonal form:

$$\bar{F}_{\mu\nu} = \begin{pmatrix} 0 & \bar{f}_1 & & & \\ -\bar{f}_1 & 0 & & 0 & \\ & & \ddots & & \\ & 0 & & 0 & \bar{f}_n \\ & & & -\bar{f}_n & 0 \end{pmatrix}, \quad n = D/2, \quad \bar{f}_k = 2\pi\alpha' f_k, \tag{9}$$

and consider each block of (9) separately in (8). For example, for the first block we have: $\frac{1}{2}\eta^1 G^{-1}\eta^1 + \frac{1}{2}\eta^2 G^{-1}\eta^2 + \mathrm{i}\bar{f}_1 \int \mathrm{d}t\, \dot{\eta}^1 \eta^2$ and thus may integrate over η^2. As a result

$$Z(F) \sim \prod_{k=1}^{D/2} \int [\mathrm{d}\eta] \exp\left(-\tfrac{1}{2}\eta G^{-1}\eta - \tfrac{1}{2}\bar{f}_k^2 \dot{\eta} G \dot{\eta}\right), \tag{10}$$

where now η does not carry a spacetime index. The final expression for Γ is:

$$\Gamma(F) = \sum_\chi \mathrm{e}^{\sigma\chi} \int \mathrm{d}\mu(\lambda)\, Z(0) \bar{Z}(F), \quad Z(0) = \int [\mathrm{d}x^\mu] \mathrm{e}^{-I_2}, \tag{11}$$

$$\bar{Z}(F) = \prod_{k=1}^{D/2} \int [\mathrm{d}\tilde{\eta}] \mathrm{e}^{-\frac{1}{2}\tilde{\eta}\Delta_k\tilde{\eta}} = \prod_{k=1}^{D/2} (\det \Delta_k)^{-1/2}, \quad \Delta_k = 1 + \bar{f}_k^2 \ddot{G} \cdot G, \tag{12}$$

where $\ddot{G} = (\mathrm{d}^2/\mathrm{d}t\,\mathrm{d}t') G(t, t')$ (G and Δ are defined on non-constant functions).

3. Tree approximation. For $\chi = 1$ the world sheet M^2 can be taken to be a unit disc on a complex plane. The corresponding Neumann function is (see e.g. ref. [16])

$$N(z,z') = -(1/2\pi)\ln|z-z'||z-\bar{z}'^{-1}|, \tag{13}$$

and thus

$$G(\theta,\theta') = N(e^{i\theta}, e^{i\theta'}) = -(1/2\pi)\ln(2-2\cos\zeta), \quad \zeta=\theta-\theta', \quad 0\leqslant\theta\leqslant 2\pi. \tag{14}$$

To compute $\det\Delta_k$ in (12) we use Fourier expansions on the boundary circle, $\eta(\theta) = \sum_{m=1}^{\infty}(1/\sqrt{\pi})(a_m\cos m\theta + b_n \sin m\theta)$. In view of the important formula [17]

$$\ln(1+b^2-2b\cos\zeta) = -2\sum_{m=1}^{\infty}\frac{b^m}{m}\cos m\zeta, \quad b\leqslant 1, \tag{15}$$

we find

$$G = \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\cos m\zeta, \quad \ddot{G} = G^{-1} = \frac{1}{\pi}\sum_{m=1}^{\infty} m\cos m\zeta,$$

$$\ddot{G}\cdot G = \frac{1}{\pi}\sum_{m=1}^{\infty}\cos m\zeta \equiv \bar{\delta}(\theta-\theta'), \; \Delta_k = (1+\bar{f}_k^2)\bar{\delta}(\zeta), \tag{16}$$

where $\bar{\delta}$ is a function defined on non-constant functions. Hence $\bar{Z}(F)$ in (12) is:

$$\bar{Z}(F) = \prod_{k=1}^{D/2}\prod_{m=1}^{\infty}(1+\bar{f}_k^2)^{-1} = \prod_{k=1}^{D/2}(1+\bar{f}_k^2)^{1/2}. \tag{17}$$

Here we employed the standard prescription for the definition of one-dimensional path integrals based on the use of the Riemann ζ-function (see e.g. ref. [18]):

$$\int[d\eta]\,e^{-c\eta^2} \sim \int\prod_{m=1}^{\infty}da_m\,db_m\,e^{-c(a_m^2+b_m^2)} \sim \prod_{m=1}^{\infty}c^{-1} = e^{-\zeta(0)\ln c} = c^{1/2}. \tag{18}$$

Recalling (5) and (9) we can put the "tree" contribution in (11) in the following form

$$\Gamma(F)_{\rm tree} = Z_0 g_0^{-2}\alpha'^{-D/2}\int d^D y\left[\det(\delta_{\mu\nu}+2\pi\alpha' F_{\mu\nu})\right]^{1/2}, \tag{19}$$

where $Z_0 = $ const. is the free partition function for a unit disc and $g_0 = e^{-\sigma/2}$. Eq. (19) coincides with the D-dimensional Born–Infeld action [9] [‡4]. Expanding (19) in powers of $F_{\mu\nu}$

$$\Gamma(F)_{\rm tree} = \int d^D y\left[\text{const.} + (1/4g^2)F_{\mu\nu}^2 + O(\alpha'^2F^4/g^2)\right], \quad g \sim g_0(\alpha')^{(D-4)/4}, \tag{20}$$

we find agreement with the result of the $\alpha'\to 0$ expansion of the amplitudes [11,7] (F^3-term is absent in the abelian case).

It is instructive to compare (19) with the corresponding result for the case of particle dynamics [10]. Consider a charged scalar loop in an external electromagnetic field. Then

$$\ln\det(-D^2(A)) = -\tfrac{1}{2}\int_{\epsilon}^{\infty}\frac{dT}{T}Z_p[A], \quad Z_p[A] = \int[dx^\mu]\exp\left(-\frac{\pi}{2T^2}\int_0^{2\pi}d\theta\,\dot{x}^2 - i\int_0^{2\pi}d\theta\,\dot{x}^\mu A_\mu\right). \tag{21}$$

‡4 In $D=4[\det(\delta_{\mu\nu}+\bar{F}_{\mu\nu})]^{1/2} = [1+\frac{1}{2}\bar{F}_{\mu\nu}^2 + \frac{1}{16}(\bar{F}_{\mu\nu}\bar{F}_{\mu\nu}^*)^2]^{1/2}$. This lagrangian for non-linear electrodynamics was singled out in ref. [9] because of its geometrical appeal (for its connection with unified theories see ref. [19]).

where T is the length of a closed path ($e \sim T^2$ in the proper-time gauge). Assuming $A_\mu = -\frac{1}{2}F_{\mu\nu}x^\nu$ we can put Z_p in the form (8) with

$$G^{-1} = -\frac{\pi}{T^2}\frac{d^2}{d\theta^2}\delta(\theta-\theta') = \frac{1}{T^2}\sum_{m=1}^{\infty} m^2\cos m\zeta, \quad \zeta = \theta - \theta'$$

[cf. (16)]. We conclude that in contrast with the string case here G^{-1} is *not* scale-invariant (this is the reason why the integral over the length of the path does not decouple in the particle case). Computing (12) we get the well-known result (cf. refs. [17,19])

$$Z_p(F) \sim \prod_{k=1}^{D/2}\frac{f_k}{\sinh(f_k T^2)}. \tag{22}$$

4. *One-loop approximation.* The world surface for $\chi = 0$ can be taken to be a flat annulus with the radius of the internal circle equal to $a < 1$ and the radius of the external circle equal to 1 [the corresponding Teichmüller parameter λ in (4) is $\ln a$]. The Neumann function for the annulus is [16]

$$N(z-z') = -\frac{1}{2\pi}\left(\ln|z-z'||z-\bar{z}'^{-1}| + \sum_{n=1}^{\infty}\ln\left[|1-a^{2n}z/z'||1-a^{2n}z'/z||1-a^{2n}z\bar{z}'||1-a^{2n}/z\bar{z}'|\right]\right). \tag{23}$$

The matrix of boundary functions G in (6) is obtained by setting z and z' equal to $e^{i\theta}$ or $a\,e^{i\theta}$ (here we can use the same angle θ to parametrize both boundaries, $\zeta = \theta - \theta'$):

$$G_{11} = -\frac{1}{2\pi}\left(Q(1) + 2\sum_{n=1}^{\infty}Q(a^{2n})\right), \quad Q(b) \equiv \ln(1 + b^2 - 2b\cos\zeta),$$

$$G_{12} = -\frac{1}{2\pi}\left(Q(a) - \ln a + \sum_{n=1}^{\infty}\left[Q(a^{2n+1}) + Q(a^{2n-1})\right]\right), \quad G_{21} = G_{12} - \frac{1}{2\pi}\ln a,$$

$$G_{22} = -\frac{1}{4\pi}\left(Q(1) + Q(a^2) + \sum_{n=1}^{\infty}\left[2Q(a^{2n}) + Q(a^{2n+2}) + Q(a^{2n-2})\right]\right) = G_{11}. \tag{24}$$

Making use of formula (15) and $\sum_{n=1}^{\infty} b^n = b/(1-b)$ we can easily compute the Fourier expansions (recall that G is defined on non-constant functions)

$$G(\theta,\theta') = \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\Omega_m\cos m\zeta, \quad \Omega_m = \begin{pmatrix} A_m & B_m \\ B_m & A_m \end{pmatrix}, \quad A_m = \frac{1+a^{2m}}{1-a^{2m}}, \quad B_m = \frac{2a^m}{1-a^{2m}}. \tag{25}$$

Then

$$\tilde{G} = \frac{1}{\pi}\sum_{m=1}^{\infty} m\Omega_m\cos m\zeta, \quad \Delta_k(\theta,\theta') = \frac{1}{\pi}\sum_{m=1}^{\infty}\Lambda_m^{(k)}\cos m\zeta, \tag{26}$$

$$\Lambda_m^{(k)} = \begin{pmatrix} 1 + \bar{f}_k^2(A_m^2 + B_m^2) & 2\bar{f}_k^2 A_m B_m \\ 2\bar{f}_k^2 A_m B_m & 1 + \bar{f}_k^2(A_m^2 + B_m^2) \end{pmatrix}. \tag{27}$$

Hence $\bar{Z}(F)$ in (12) is given by

$$\bar{Z}(F) = \prod_{k=1}^{D/2}\prod_{m=1}^{\infty}\left[\det\Lambda_m^{(k)}\right]^{-1} = \prod_{k=1}^{D/2}\prod_{m=1}^{\infty}\left[(1 + p_m^2\bar{f}_k^2)(1 + p_m^{-2}\bar{f}_k^2)\right]^{-1}, \tag{28}$$

$$\bar{Z}(F) = \prod_{m=1}^{\infty} \left[\det\left(\delta_{\mu\nu} + p_m \bar{F}_{\mu\nu}\right) \det\left(\delta_{\mu\nu} + p_m^{-1} \bar{F}_{\mu\nu}\right)\right]^{-1}, \quad p_m \equiv \frac{1+a^m}{1-a^m}. \tag{29}$$

One can also rewrite (28) as $[\gamma_k \equiv (\bar{f}_k^2 - 1)/(\bar{f}_k^2 + 1)]$

$$\bar{Z}(F) = \prod_{k=1}^{D/2} \left[(1 + \bar{f}_k^2) \prod_{m=1}^{\infty} \left(1 - \frac{\gamma_k^2}{\cosh^2(m \ln a)}\right)^{-1} \right] \left(\prod_{m=1}^{\infty} \tanh(m \ln a) \right)^{D}. \tag{30}$$

The partition function on the annulus with Neumann boundary conditions is [20,21]

$$Z(0) \sim [P(a)]^{D}, \quad P(a) = a^{-1/12} \prod_{m=1}^{\infty} (1 - a^{2m})^{-1} = \prod_{m=1}^{\infty} [2 \sinh(m \ln a)]^{-1},$$
$$[\zeta(-1) = -1/12]. \tag{31}$$

The measure μ in (4), (11) contains the contribution of the ghost determinant (computed with "mixed" boundary conditions [13])

$$Z_{\mathrm{gh}} = \det{}' \Box \sim [P(a)]^{-2}. \tag{32}$$

Observing that the natural measure of integration over conformally inequivalent domains is $\mathrm{d}\lambda = \mathrm{d}a/a$ [20] and combining all the factors we find for the one-loop contribution in (11) ($D = 26$)

$$\Gamma_{\mathrm{loop}} = V \int_0^1 \frac{\mathrm{d}a}{a^3} \prod_{m=1}^{\infty} (1 - a^{2m})^{-24} \bar{Z}(F) \tag{33}$$

$$= V \int_0^1 \frac{\mathrm{d}a}{a^3} \prod_{m=1}^{\infty} \left[(1 - a^{2m})^2 (1 + a^{2m})^{-26}\right] \prod_{k=1}^{13} \left[(1 + \bar{f}_k^2) \prod_{m=1}^{\infty} \left(1 - \frac{\gamma_k^2}{\cosh^2(m \ln a)}\right) \right], \tag{34}$$

where $V \sim (\alpha')^{-D/2} \int \mathrm{d}^D y$. The equivalent form of (34) is

$$\Gamma_{\mathrm{loop}} = -\tfrac{1}{2} V \int_0^{\infty} \mathrm{d}\lambda \prod_{m=1}^{\infty} \sinh^2(m\lambda) \prod_{k=1}^{13} \left\{ (1 + \bar{f}_k^2) \prod_{m=1}^{\infty} \left[\cosh^2(m\lambda) - \gamma_k^2\right]^{-1} \right\}. \tag{35}$$

The measure of integration in (33) is the same as found in the operator formalism for the one-loop amplitudes (see e.g. refs. [16,22,1]). As is clear from (34) the integral has a cut singularity at $a = 1$ ‡5 and diverges as $\int \mathrm{d}a/a^3$ at $a = 0$ (these properties are again the same as found for the tachyon scattering amplitudes). The infinite part is proportional to $\prod_{k=1}^{13}(1 + \bar{f}_k^2) = \det(\delta_{\mu\nu} + \bar{F}_{\mu\nu})$. This expression is the *square* of the tree result (19) and thus it appears as if the infinity cannot be absorbed by a renormalization of the coupling constant g_0 and α' (in contradiction with expectations based on analysis of the amplitudes [22,23]).

The reason for the above "squaring" of $[\det(\delta_{\mu\nu} + \bar{F}_{\mu\nu})]^{1/2}$ can be traced to the fact that the annulus has *two* boundaries while the disc has only one. However, the inner boundary should disappear in the limit $a \to 0$. In fact, if we introduce a short distance cut-off by substituting $\ln(|z - z'| + \varepsilon)$ for $\ln|z - z'|$ in (23) and take $a \to 0$ *before* $\varepsilon \to 0$ then G_{22} in (24) vanishes for $a = 0$. This is equivalent to putting the 22-element of Ω_m in (25) equal to A_m for $a \neq 0$ and equal to zero for $a = 0$. With this prescription the $a \to 0$ limit of $\bar{Z}(F)$ and hence of (33) is proportional to the *first* power of $[\det(\delta_{\mu\nu} + \bar{F}_{\mu\nu})]^{1/2}$. The one-loop corrected effective action will then look like ($\Lambda \to \infty$)

$$\Gamma = V\left\{ \frac{1}{g_0^2} \left[\det\left(\delta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\right)\right]^{1/2} + c\Lambda^2 \left[\det\left(\delta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\right)\right]^{1/2} + \text{finite part} \right\} \tag{36}$$

and thus will be *renormalizable* by a redefinition of g_0 ‡6.

‡5 The structure of the singularity at $a = 1$ is $\exp(\pi^2/3\varepsilon)\exp[(1/4\varepsilon)\sum_{k=1}^{13}(b_k^2 - 2\pi b_k)]$, $\varepsilon = 1 - a$, $b_k = \arccos[1 - 8\bar{f}_k^2(1 + \bar{f}_k^2)^{-1}]$. There is an interval of values of f_k's in which the singularity is absent.

‡6 We understand that the above prescription of taking the limit $a \to 0$ may seem dubious under the natural condition that the radius of the inner hole should be always larger than the short-distance cut-off on M^2. The important question of the connection between the $a \to 0$ limit and the removal of the ε cut-off on the world surface deserves further study.

5. *Concluding remarks.* We find it remarkable that the Born–Infeld lagrangian is the exact (tree-level) solution of a constant external vector field problem in the open string theory ‡7. It is interesting that the Born–Infeld-type actions were already discussed in connection with string theory in ref. [24] (it was noted there that non-linear actions may admit non-trivial vortex solutions with $\int[\det F_{\mu\nu}]^{1/2}$ giving the Nambu action). Maybe there should be a kind of a bootstrap and the effective field theory action corresponding to the fundamental string theory should have string-like classical solutions ("solitonic" strings).

There are a number of possible generalizations of our discussion. We may consider the non-abelian case Representing the path-ordered exponent in (1) as a path integral over the auxiliary Grassmann degrees of freedom (see e.g. ref. [25])

$$\operatorname{tr} P\exp\left(-\mathrm{i}\int \mathrm{d}t\,\dot{x}^{\mu}A_{\mu}\right)\sim\int \mathrm{d}\psi\,\mathrm{d}\bar{\psi}\exp\left[-\mathrm{i}\int \mathrm{d}t\left(\bar{\psi}\dot{\psi}+\dot{x}^{\mu}\bar{\psi}A_{\mu}\psi\right)\right],$$

we conclude that the problem is no longer exactly solvable and thus the best we can do is a perturbation theory both in ψ and η^{μ} [the starting point is again eq. (6) and hence the only difference from the particle case is again in the structure of the "kinetic" operator G^{-1}].

It is relatively straightforward to generalize our results to the case of fermionic strings with gauge group $G=U(1)$ or SO(2) [the new interaction term is $\int \mathrm{d}t(\psi^{\mu}\psi^{\nu}F_{\mu\nu})$ and thus the problem remains exactly solvable if $F_{\mu\nu}=$ const.]. We can also consider directly the case of superstrings taking $F_{\mu\nu}$ to correspond to an abelian subgroup of SO(n) or USp(n). The relevant interaction lagrangian was recently constructed in ref. [6]. The only terms contributing when $F_{\mu\nu}=$ const. are $\int \mathrm{d}t(\dot{x}^{\mu}A_{\mu}-\frac{1}{4}\mathrm{i}\bar{\theta}\gamma^{\mu\nu\rho}\theta F_{\mu\nu}\dot{x}_{\rho})$. In the light-cone gauge the second term reduces to $\int \mathrm{d}t\,\bar{\theta}\gamma^{ij}\gamma^{-}\theta F_{ij}$ and thus the path integral over θ becomes gaussian. We anticipate that the contributions of non-zero (non-constant) modes of x^{i} and θ cancel due to supersymmetry and thus $\Gamma(F)$ is completely given by the integral over the zero mode $\theta=$ const. Computing this integral with a proper measure we get $\Gamma\sim VF_{ij}^{2}$, i.e. the standard Maxwell action and *not* the Born–Infeld one.

To find the one-loop correction one has also to include the Möbius strip contribution. Eq. (33) should then take the form $\int \mathrm{d}a\,a^{-1}\bar{Z}(F)$ with a logarithmic divergence at $a=0$ (cf. refs. [1,12]). The infinity will probably cancel for the number of abelian components of $F_{\mu\nu}$ equal to the dimension of SO(32) [12].

An analogous problem can be studied also for the closed heterotic superstring [4]. The action corresponding to the coupling of the heterotic string to a gauge field A_{μ} belonging to an abelian subgroup of SO(32) and having $F_{\mu\nu}=$ const. is given by

$$I=\frac{1}{4\pi\alpha'}\int \mathrm{d}^{2}z\left[(\partial_{a}x^{\mu})^{2}+\mathrm{i}\bar{\psi}^{\mu}\partial\!\!\!/\psi^{\mu}+\mathrm{i}\bar{\zeta}^{I}D\!\!\!\!/\,\zeta^{I}\right],$$

$$D_{a}=\partial_{a}+\mathrm{i}B_{a},\quad B_{a}^{IJ}=\tfrac{1}{2}F_{\mu\nu}^{IJ}(x^{\nu}\partial_{a}x^{\mu}+\mathrm{i}\bar{\psi}^{\mu}\rho_{a}\psi^{\nu}),\qquad(37)$$

where ρ_{a} are the two-dimensional Dirac matrices ψ^{μ} and ζ^{I} are two-dimensional Majorana–Weyl spinors of opposite chirality. It is possible to integrate out ζ^{I} getting instead a chiral generalization of the Schwinger term ($\ln\det D\!\!\!\!/_{\mathrm{MW}}=-(1/8\pi)\int \mathrm{d}^{2}z\,\partial_{a}\varphi(\partial_{a}\varphi+\mathrm{i}\partial_{a}\phi)$, $B_{a}\equiv\partial_{a}\phi+\varepsilon_{ab}\partial_{b}\varphi$). However, the resulting expression is fourth-order in x^{μ} and ψ^{μ} and hence the remaining integrals are non-gaussian. Yet it may happen that the problem has an exact and non-trivial (non-Maxwell) solution being a problem of two-dimensional QFT.

‡7 Let us recall that the Born–Infeld-type actions provided the first examples of non-linear electrodynamics. They were followed by the Heisenberg–Euler action derived from QED. The later action was found to be a piece of the Schwinger action which may be considered as an exact consequence of particle dynamics. A historical paradox is that the "prototypical" Born–Infeld action follows itself from the string theory.

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