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CONFORMAL SUPERGRAVITY

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Abstract:

We give a review of conformal supergravity considered as a candidate for a fundamental theory describing physics at Planck distances. Conformal supergravities are supersymmetric extensions of conformal invariant, higher derivative, power counting renormalizable Weyl theory of gravity. They are gauge theories of the superconformal group SU(2, 2|N) and naturally unify Weyl gravity with SU(N) gauge fields and matter fields. After reviewing the classical actions, transformation laws, degrees of freedom counting, etc., we present a detailed study of the quantum theory. In particular, we establish expressions for the one-loop effective actions and for the β-functions for pure conformal supergravities and for conformal supergravities coupled to matter. We discuss how to extend these results to all loops with the help of the non-renormalization theorem, or using the properties of the effective action, computed on instanton backgrounds. It is stressed that, in order to be consistent at the quantum level, superconformal theories are to be anomaly free and hence ultraviolet finite. Two candidates are found for a finite power counting renormalizable superconformal theory: a "non-minimal" $N=4$ conformal supergravity and the "minimal" $N=4$ conformal supergravity interacting with SU(2) × U(1) $N=4$ super Yang–Mills. We consider several possibilities of how to establish a low-energy correspondence between such theories and "phenomenological" unified models based on $N=1$ Poincaré supergravity coupled to matter multiplets.

0. Introduction

A central problem of modern high-energy physics is the unification of gravity with all the other fundamental interactions which would be consistent at the quantum level. A unified theory which would incorporate quantized gravity would be fundamental in the sense that it would be valid on all scales, including the sub-Planck ones. Such a theory is likely to be unique or sufficiently "constrained" to predict (or explain) the details of the standard SU(3) × SU(2)$_L$ × U(1) (QCD plus Glashow–Weinberg–Salam) model or of a GUT model.

There are various theoretical indications that a fundamental theory should be supersymmetrical. Supersymmetry (see e.g. the reviews [75, 211, 261, 129]) imposes stringent restrictions on a field theory, reducing an arbitrariness in its construction, and drastically softening ultraviolet infinities. It also has the unique property of providing a non-trivial unification of space–time and internal symmetries. The Einstein gravity interacting with a renormalizable matter model

$$\mathcal{L} = \frac{1}{k^2} R\sqrt{g} + \mathcal{L}_{\text{matter}} \tag{1}$$

is a theory with a high degree of arbitrariness. Moreover, it is meaningless at the quantum level, being perturbatively non-renormalizable (non-finite) already at the one-loop level [241]. Though $N \leq 4$ supersymmetric extensions of (1), i.e. $N \leq 4$ ordinary (Poincaré or De Sitter) supergravity interacting with $N \leq 4$ matter supermultiplets also suffer from non-renormalizable infinities (see e.g. [254]), the $5 \leq N \leq 8$ supergravities are in a certain sense unique and are finite at the first and the second loop levels. At the same time, it presently appears very unlikely that a power counting *non-renormalizable* four-dimensional field theory (e.g., $N = 8$ supergravity) can be finite to all orders in perturbation theory. To provide the vanishing of the coefficients of an infinite number of admissible on-shell counterterms (known to exist in $N \leq 8$ supergravities, see e.g. [160] and references therein), one probably needs an *infinite* number of "sum rules". However, this seems impossible in a field theory containing a finite number of fields of lower ($s \leq 2$) spins.

In addition to this non-finiteness problem, ordinary extended supergravities have a number of well-known defects from a "phenomenological" point of view: (i) even in $N = 8$ supergravity there is no place for all the particles of the standard model; (ii) one cannot gauge the rigid O(N) symmetry group (which is necessary for obtaining gauge vectors) without producing at the same time the enormous cosmological term $\Lambda \sim -M_P^2$ (M_P is the Planck mass); (iii) all fermions in the spectrum are in (real) representations of O(N) and hence cannot be identified with the SU(2)-chiral particles of the standard model; etc. The only phenomenologically acceptable supergravity models that are known at present are based on $N=1$ Poincaré supergravity (PSG) interacting with a suitable combination of $N=1$ vector and scalar multiplets (see e.g. the reviews [196, 201]). These "phenomenological $N=1$ supergravity" schemes, though having some important conceptual as well as technical advantages over the standard, or GUT, models, are ultra-violet divergent and hence can at best be considered as low-energy ($E < M_P$) effective theories which should follow from a fundamental renormalizable or finite theory describing the high-energy limit, as long-distance approximations.

Given that the ordinary extended supergravity is unlikely to be a viable candidate for a fundamental theory[1] we are left with two general options: (i) such a theory may have a fundamental mass parameter (i.e. can be a special kind of theory with a "fundamental length") but still be finite to all orders; (ii) a fundamental theory may be free of mass parameters in the high-energy limit and, as a consequence, be scale invariant and power counting renormalizable. The first possibility (that seems impossible to realize in a theory with a finite number of fields) can probably be implemented in a currently popular theory of extended objects – closed superstrings in ten dimensions [213]. The mass parameter here is related to the "size" of the strings (the square root of the slope α'). The naive $\alpha' \to 0$ limit of this theory corresponds to a variant of $N=2$ supergravity in $d=10$. Thus to make contact with a "phenomenological $N=1$ PSG" model, one is lead to assume that the six extra dimensions are compactified. This is equivalent to introducing in the theory additional mass parameters (the "radii" of the compact dimensions), as well as making the derivation of its low-energy consequences rather difficult and indirect. To this we can add that the superstring theory is not sufficiently developed at present and one cannot be sure that it could be considered as a real candidate for an ultimate theory of matter.

Before going to somewhat "exotic" theories of strings it would be desirable to first exhaust all the possibilities which exist to construct a consistent high-energy theory in the framework of quantum *field* theory (i.e. a theory of particles). Thus we are led to the second option mentioned above. It seems very natural to suppose that a local field theory, meaningful in the high-energy limit, should contain only dimensionless coupling constants (all mass parameters become irrelevant in that limit). An attempt to construct such a theory is to add to (1) the higher derivative curvature terms

$$\mathcal{L}' = (aC^2 + bR^2)\sqrt{g} \tag{2}$$

(C^2 is the square of the Weyl tensor). The resulting model (or its supersymmetric extension) is a power counting renormalizable theory [251, 231] governed by dimensionless constants ($a, b, \ldots$) in the high-energy limit. However, it suffers from an apparent problem of ghosts (which may or may not have a non-perturbative solution). What seems even more important is that such a "hybrid" model contains such a large amount of arbitrariness that it can hardly pretend at having a role of the ultimate theory. The latter is supposed to have only one dimensionless coupling constant and to provide a non-trivial unification of various fields (and hence of the corresponding terms in the action). To reduce a degree of freedom in the structure of a theory it is useful to impose additional symmetry requirements. An attractive choice for such a bosonic symmetry is the Weyl symmetry, corresponding to local scale transformations of the metric and matter fields, $g'_{\mu\nu} = e^{-2\lambda(x)} g_{\mu\nu}$, $\phi' = e^{\lambda}\phi$, etc. There are good reasons to anticipate the Weyl symmetry (as well as the general covariance) as a symmetry of a fundamental theory. In fact, the Weyl transformations are the generalization (to a generally covariant theory) of

[1] Going to higher dimensions makes the ultra-violet problem even worse. For example, in 11-dimensional supergravity, there is an admissible on-shell counter-term already at the two-loop level.

conformal transformations in flat space, while the high-energy limit of the renormalizable matter models is known to be invariant under the conformal group. The condition of Weyl invariance excludes the "naked" Einstein term from (1) and the R^2-term from (2). Next one is to invoke supersymmetry in order to unify gravity with matter fields.

The resulting theory is invariant under superconformal transformations which include general coordinate transformations, Weyl transformations, "ordinary" supersymmetry and special conformal supersymmetry transformations and the U(N) (SU(N) for $N = 4$) chiral gauge transformations. The action contains in general two groups of terms: 1) the super-extension of the Weyl C^2-action, or the action of "pure" *conformal supergravity* (CSG) [172, 8], which has the following symbolic form

$$I^{(N)}_{\mathrm{CSG}} \sim \frac{1}{\alpha^2}\int \mathrm{d}^4x\, e\{C^2 + F^2_{\mu\nu}(V) + F^2_{\mu\nu}(A) + \bar{\psi}_\mu \mathscr{D}^3 \psi_\mu + \bar{\chi}\mathscr{D}\chi + (\partial E)^2 + (\partial T_{\mu\nu})^2 + \bar{\Lambda}\mathscr{D}^3\Lambda + (\Box\varphi)^2 + \cdots\} \tag{3}$$

($N \leq 4$ is the number of supersymmetry generators, V_μ and A_μ are U(N) gauge vectors, ψ^i_μ are "conformal" gravitinos and $T_{\mu\nu}$, χ, Λ, E, φ are "matter" fields); 2) the action for superconformal matter multiplets, e.g. scalar and vector multiplets for $N = 1$ and vector multiplets for $N = 4$. Though a "naturally" unified superconformal theory (to which one cannot add arbitrary matter multiplets) is probably possible only starting with $N = 5$ and is presently unknown, it is the central requirement of quantum consistency that helps to make the choice of an acceptable $N \leq 4$ superconformal theory practically unique. In fact, in order to maintain superformal invariance at the quantum level, one has to obtain a cancellation of the superconformal anomalies. This cancellation in turn implies that the resulting anomaly-free theory will be *finite*. Hence the important conceptual advantage of the requirement of (super)conformal symmetry is that it presupposes the absence of ultraviolet divergences in the corresponding quantum theory.

It is thus proposed that a fundamental theory that describes physics at sub-Planck distances is a finite theory possessing maximal superconformal symmetry at the quantum level and non-trivially unifying gravity gauge and "matter" fields. It appears [100, 101, 106] that there are at least two candidates for such a theory. The first one is what we call a "non-minimal" version of $N = 4$ conformal supergravity and the second one is a "minimal" version of $N = 4$ CSG coupled to $N = 4$ super Yang–Mills theory with SU(2) × U(1) as a gauge group. These finite theories (which are the first examples of power counting renormalizable generally covariant finite theories) provide natural locally superconformal analogs of the finite $N = 4$ super-Yang–Mills theory [140, 21, 20, 189, 160] known to be invariant under the rigid $N = 4$ superconformal group. There is a deep connection between $N = 4$ conformal supergravity and $N = 4$ super-Yang–Mills theory. For example, gauging the rigid superconformal group of the latter theory one obtains $N = 4$ CSG [8]. Also, the conformal supergravity action can be found, e.g., as a counter-term for an $N = 4$ super-Yang–Mills theory interacting with an external $N = 4$ CSG.

Comparing the conformal supergravity theories with the ordinary Poincaré or De Sitter ones, we see that the former theories have a number of "phenomenological" advantages which follow from the structure of the superconformal gauge group. Extending the rigid conformal algebra by adding the fermionic (ordinary and special conformal) supersymmetry generators one finds [150] that, in order to close the algebra, it is necessary to introduce also the generators of an internal U(N) symmetry. This symmetry is chiral in the sense that the chiral projections of the Majorana spinors, corresponding to the supersymmetry generators, transform according to the N and $\bar{N}$ representations of U(N). As a consequence, the fermions of conformal supergravities are in *chiral* (complex) representations of the gauge group U(N). Another consequence of the structure of the superconformal algebra is that the gauge U(N) vectors of the conformal supergravities are not accompanied by the enormous cosmological term present in gauged O(N) supergravities.[2]

To see whether these welcome properties help to establish a low-energy correspondence with "phenomenological $N = 1$ PSG" models, it is first necessary to establish the presence of the Einstein (or $N = 1$ PSG) term in the low-energy effective action. To give to a finite conformal supergravity the status of a physically acceptable theory, one has also to prove that, in spite of higher derivatives in the fundamental action (cf. eq. (3)), the low-energy effective action contains no tree-level ghosts (i.e. corresponds to a unitary theory). This at present remains only a program. In order to provide the answer as to whether or not it can be realized, one has to understand the quantum dynamics of finite superconformal theories. This may not be asking for the impossible because the preservation of superconformal symmetry at the quantum level may help to find an exact solution to the quantum problem using a "super-extension" of the methods developed in conformal quantum field theory (see e.g. [96, 242] and in particular [97] for the case of gauge theory).

It appears likely that for a "natural" value of the dimensionless coupling $\alpha \sim 1$ in (3), the theory is in a strong coupling phase and that all the fundamental fields (in particular the multipole ghosts) are "confined". The scale of confinement (and hence a dimensional parameter) taken from physical considerations to be of the order of the Planck scale is supposed to be generated by the *infrared* instability of the initially scale invariant theory. This instability may also be a reason for a dynamical supersymmetry breakdown from $N = 4$ to $N = 1$ (cf. [122]). The bound states built from the fields in the fundamental superconformal action can then be identified with observable particles (graviton, quarks, leptons, etc.) described by a low-energy "phenomenological $N = 1$ Poincaré supergravity" model. A plausibility of this scenario is supported by the first attempts [168, 246, 197] to study the non-perturbative properties of the higher derivative Weyl theory ($N = 0$ conformal supergravity).

The aim of the present article is to give a review of the basic facts concerning classical and quantum conformal supergravity. We do not try to give an exhaustive exposition of the general topic of conformal invariance in supergravity which was already discussed in a number of reviews [254, 48–50, 56, 7, 183]. The accent in these reviews was put on the advantages of superconformal approach for the understanding the off-shell structure and for constructing the tensor calculus of the ordinary ($N = 1$ and $N = 2$) supergravity. We instead concentrate on conformal supergravity theories considered as candidates for a fundamental theory.

1. Preliminary considerations

1.1. *Weyl symmetry*

Our aim is to construct a unified four-dimensional supersymmetric theory of all particles and interactions which initially does not contain any dimensional parameters (like masses, gravitational and cosmological constants etc.). Such a theory (supposed to be a fundamental theory in the high-energy

[2] It is useful to mention that it is conformal supergravity that provides a true unification of gravity and gauge fields (gauged supergravity has *two* coupling constants, one for the graviton and one for the gauged vector). Also, incorporating the Yang–Mills term, the conformal supergravity is "tied" to four dimensions, while the ordinary supergravity can be defined (and has the same structure of action) in various numbers ($d \leq 11$) of dimensions.

limit) will be scale invariant when expanded near flat space, i.e. invariant under the scale transformations[1]

$$x'^{\mu} = e^{-\lambda} x^{\mu}, \qquad \Phi'(x') = e^{k\lambda}\, \Phi(x), \tag{1.1}$$

where $\lambda = \text{const.}$ and k is the canonical (mass) dimension of the fields (e.g., $k = +1$ for scalars and vectors). A natural generalization of (1.1) in curved space is the local Weyl transformations

$$g'_{\mu\nu} = e^{-2\lambda(x)} g_{\mu\nu}, \qquad \Phi' = e^{w\lambda(x)}\Phi. \tag{1.2}$$

Here $g_{\mu\nu}$ is the metric tensor and w is the Weyl weight of Φ (thus for the "graviton" $w = -2$).

The condition of Weyl invariance (in addition to that of general covariance) is central for the construction of the theory.[2] It is important to stress that we do not want to introduce a special gauge field b_μ, which "compensates for" (1.2) (i.e., $\delta b_\mu = \partial_\mu \lambda$) as a physical field present in the action. This is the point of departure from the original Weyl's suggestion [266] which was to identify b_μ with the electromagnetic field. The absence of b_μ provides a maximal "irreducibility" of the theory and essentially restricts the form of its action. As for the electromagnetic field it can be non-trivially unified with gravity only in the context of supersymmetric theory, and thus its presence (among all other necessary gauge fields) is to be dictated independently by supersymmetry.

The symmetry (1.2) which lacks the corresponding gauge fields is assumed to be an "accidental" or "hidden" symmetry of our action. A unique Weyl-invariant $d = 4$ Lagrangian built only from the metric is proportional to the square of the Weyl tensor (see appendix B)[3]

$$\mathscr{L}_{\mathrm{W}} = -\frac{1}{2\alpha^2} C^{\lambda}{}_{\mu\nu\rho} C^{\mu}{}_{\lambda\alpha\beta}\, g^{\nu\alpha} g^{\rho\beta} \sqrt{g} \tag{1.3}$$

(here α is a dimensionless coupling constant). Hence we are to look for supersymmetric extensions of the Weyl Lagrangian or "conformal supergravities". Supersymmetry dictates that the "graviton" $g_{\mu\nu}$ is to be accompanied by a fermionic partner, the simplest choice for which is the spin 3/2 "gravitino" ψ_μ. The corresponding analog of the general coordinate (or "X") transformation is the "ordinary" (or "Q") supersymmetry transformation

$$\delta\psi_\mu = \partial_\mu \varepsilon + \cdots, \tag{1.4}$$

while the analog of the Weyl (or local dilational "D") transformation is the special conformal (or "S") supersymmetry transformation

$$\delta\psi_\mu = -\gamma_\mu \eta + \cdots, \qquad \delta e^a_\mu = 0. \tag{1.5}$$

Here ε and η are Majorana spinor parameters, and dealing with spinors, we have substituted the metric

[1] For reviews of scale (and conformal) invariance in flat space, see e.g. [260, 123, 5].

[2] For discussions of Weyl invariant gravitational theories see also [266, 267, 2, 44, 120].

[3] Note that if b_μ were present we could use the Weyl covariant derivative $\hat\partial_\mu = \partial_\mu + 2b_\mu$ to construct two additional invariants: $\hat R^2\sqrt{g}$ and $\hat R^2_{\mu\nu}\sqrt{g}$ (along with $F^2_{\mu\nu}(b)\sqrt{g}$), where $\hat R_{\lambda\mu\nu\rho}$ is obtained from the ordinary curvature by substituting $\partial g \to \hat\partial g$.

for the vierbein e^a_μ. Again we do not want to introduce an independent field to gauge the S-supersymmetry.

1.2. Weyl invariant and generally covariant actions

Let us now discuss the construction of D (and X)-invariant bilinear actions on the gravitational background, for various fields which may occur in conformal supergravities. Suppose that the flat space action for a generic field $\Phi_{\mu_1\cdots\mu_n}$ is given by

$$I = \int d^4x\, (\Phi_{\mu_1\cdots\mu_n}\partial_{\lambda_1}\cdots\partial_{\lambda_{2p}}\Phi_{\nu_1\cdots\nu_n} + \cdots) \tag{1.6}$$

(p is half-integer for fermions). By assumption, we consider only actions without dimensional parameters. Thus the canonical (mass) dimension of Φ is equal to

$$k = 2 - p, \qquad [\Phi] = k\,. \tag{1.7}$$

If Φ does not carry world indices $n = 0$ (e.g. it is a scalar or a spinor), then its Weyl weight w is equal to its canonical weight. This is sufficient for the invariance of the curved space analog of (1.6) under the rigid D-transformations ((1.2) with $\lambda = \text{const.}$). In fact, the $2p$ derivatives in (1.6) are multiplied by a tensor $e^{\lambda_1}_{a_1}\cdots e^{\lambda_{2p}}_{a_{2p}}e$ ($e = \det e^a_\mu$), which scales under (1.2) ($w(e^\lambda_a) = 1$) as $\exp[(2p-4)\lambda]$. If $n \neq 0$, then we can define a new field $\tilde{\Phi}_{a_1\cdots a_n} = \Phi_{\mu_1\cdots\mu_n}e^{\mu_1}_{a_1}\cdots e^{\mu_n}_{a_n}$ with $\tilde{n} = 0$ and $\tilde{w} = k$. As a result, we find

$$w(\Phi_{\mu_1\cdots\mu_n}) = k(\Phi_{\mu_1\cdots\mu_n}) - n = 2 - p - n \tag{1.8}$$

(if Φ has upper world indices their number is to be subtracted from (1.8)).

As an illustration, let us now write down the free Lagrangians for a number of the simplest relevant cases

$$\begin{aligned} &\mathcal{L}_0 \sim \phi\Box\phi, \qquad \mathcal{L}_{1/2} \sim \bar{\chi}\partial\!\!\!/\chi, \qquad \mathcal{L}_1 \sim A_\mu\Box A_\mu + \cdots, \\ &\mathcal{L}'_0 \sim \varphi\Box^2\varphi, \qquad \mathcal{L}'_{1/2} \sim \bar{\Lambda}\partial\!\!\!/^3\Lambda, \qquad \mathcal{L}'_1 \sim T_{\mu\nu}\Box T_{\mu\nu} + \cdots, \end{aligned} \tag{1.9}$$

$$\mathcal{L}_2 \sim h_{\mu\nu}\Box^2 h_{\mu\nu} + \cdots, \qquad \mathcal{L}_{3/2} \sim \bar{\psi}_\mu\partial\!\!\!/^3\psi_\mu + \cdots \tag{1.10}$$

(here $T_{\mu\nu} = T_{[\mu\nu]}$, $h_{\mu\nu} = g_{\mu\nu} - \delta_{\mu\nu}$).

The corresponding dimensions and weights are presented in table 1.1.

Table 1.1
Weyl weights and dimensions of the fields

	ϕ	χ	A_μ	φ	Λ	$T_{\mu\nu}$	$h_{\mu\nu}$	ψ_μ
p	1	1/2	1	2	3/2	1	2	3/2
k	1	3/2	1	0	1/2	1	0	1/2
w	1	3/2	0	0	1/2	−1	−2	−1/2

It is important to note that we have omitted two a priori possible cases: $h_{\mu\nu}\Box h_{\mu\nu}$ and $\bar{\psi}_\mu \not{\partial} \psi_\mu$. The reason is that these terms do not possess D-invariant extensions, which would be consistent with the corresponding gauge symmetries ($\delta h_{\mu\nu} = \mathscr{D}_\mu \xi_\nu + \mathscr{D}_\nu \xi_\mu$, $\delta\psi_\mu = \mathscr{D}_\mu \varepsilon$. Here $\mathscr{D}_\mu$ is the covariant derivative with respect to the background metric). This could be already anticipated from the fact that the well-known Einstein and Rarita–Schwinger actions are not invariant under (1.2) for any Weyl weight assignment. This remark is also related to the fact that the D-invariant extension of $\mathscr{L}'_1$ will not coincide with the gauge antisymmetric tensor Lagrangian (i.e. Weyl and gauge invariances cannot be realized simultaneously for a field with $w = 1$).

The next step is to establish the covariant extensions of (1.9), (1.10) that are invariant under the *local* D-transformations (1.2). The idea is to compensate for the D-non-invariance of $\partial\Phi$ by proper insertions of terms involving the curvature tensor of the background metric. Substituting the derivatives for the background covariant ones, and adding all possible (dimensionless) curvature-dependent and derivative terms, we can then fix the unknown coefficients by imposing the condition of D-invariance on the action. For example, in the ordinary scalar case, we have

$$e^{-1}\mathscr{L}_0 = -\tfrac{1}{2}\phi(-\mathscr{D}^2 + aR)\phi, \qquad e = \sqrt{g}\,. \tag{1.11}$$

The simplest way to determine the constant a is to take the metric to be conformally flat $g_{\mu\nu} = \mathrm{e}^{-2\lambda(x)}\delta_{\mu\nu}$, to substitute ϕ for $\mathrm{e}^{\lambda}\phi$ and then to find the value of a for which (1.11) is independent of λ. Using the formulas of appendix B, we easily obtain an equation for a

$$-\mathrm{e}^{\lambda}\partial_\mu(\mathrm{e}^{-2\lambda}\partial_\mu\,\mathrm{e}^{\lambda}) + 6a(\partial_\rho\xi_\rho - \xi_\rho\xi_\rho) = 0\,, \qquad \xi_\rho = \partial_\rho\lambda \tag{1.12}$$

which is identically satisfied for $a = \frac{1}{6}$. Analogously, we find that D-invariant Lagrangians are unique for ordinary spinors and vectors and have the standard form

$$e^{-1}\mathscr{L}_{1/2} = \bar{\chi} e^{\mu}_{a}\gamma^a \mathscr{D}_\mu \chi, \qquad \mathscr{D}_\mu = \partial_\mu + \tfrac{1}{2}\sigma_{ab}\,\omega^{ab}_{\mu}(e) \tag{1.13}$$

$$\mathscr{L}_1 = -\tfrac{1}{4}F_{\mu\nu}F_{\alpha\beta}g^{\mu\alpha}g^{\nu\beta}\sqrt{g}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{1.14}$$

Note that the gauge invariance property of the D-invariant action (1.14) is essentially due to the fact that, in $d = 4$, $w(A_\mu) = 0$ (gauge- and D-invariances for A_μ are incompatible for $d \neq 4$, cf. [45]).

More work is needed in the antisymmetric tensor case. Starting with

$$\mathscr{L}'_1 = (\mathscr{D}_\mu T_{\mu\nu}\,\mathscr{D}_\rho T_{\rho\nu} + a_1\,\mathscr{D}_\lambda T_{\mu\nu}\,\mathscr{D}_\lambda T_{\mu\nu} + a_2 R_{\mu\nu}T_{\mu\alpha}T_{\nu\alpha} + a_3 R T_{\mu\nu}T_{\mu\nu} + a_4 R_{\mu\alpha\nu\beta}T_{\mu\nu}T_{\alpha\beta})\sqrt{g} \tag{1.15}$$

and substituting $T_{\mu\nu} \to \mathrm{e}^{-\lambda}T_{\mu\nu}$, $g_{\mu\nu} \to \mathrm{e}^{-2\lambda}\delta_{\mu\nu}$, we find that all λ-dependent terms cancel if $a_1 = -\frac{1}{4}$, $a_2 = -1$, $a_3 = \frac{1}{8}$, $a_4 = -\frac{1}{2}$. The resulting Lagrangian can be rewritten as follows

$$e^{-1}\mathscr{L}'_1 = (\mathscr{D}_\mu T_{\mu\nu})^2 - \tfrac{1}{4}(\mathscr{D}_\mu T_{\rho\sigma})^2 - R_{\mu\nu}T_{\mu\lambda}T_{\nu\lambda} + \tfrac{1}{8}RT_{\mu\nu}T_{\mu\nu} + \tfrac{1}{2}R_{\mu\alpha\nu\beta}T_{\mu\nu}T_{\alpha\beta} \tag{1.16}$$

$$\equiv 2\,\mathscr{D}_\mu T^{+}_{\mu\nu}\,\mathscr{D}_\rho T^{-}_{\rho\nu} - R_{\mu\rho}T^{+}_{\mu\alpha}T^{-}_{\rho\alpha}\,, \tag{1.17}$$

where $T^{\pm}_{\mu\nu} = \frac{1}{2}(T_{\mu\nu} \pm T^{*}_{\mu\nu})$, $T^{*}_{\mu\nu} = \frac{1}{2}e\varepsilon_{\mu\nu\lambda\rho}T^{\lambda\rho}$. This Lagrangian seems first to appear in refs. [59, 8] (see also [100, 49]). Note that (1.17) is real *in Minkowski space*, because $T^{+}_{\mu\nu} \leftrightarrow T^{-}_{\mu\nu}$ under complex conjugation.

It is straightforward also to find the extensions of $\mathscr{L}'_{1/2}$ and $\mathscr{L}'_0$ in (1.9) [100]

$$
\begin{aligned}
e^{-1}\mathscr{L}'_{1/2} &= \bar{\Lambda}[\mathscr{D}\!\!\!/^3 + (R_{\mu\nu} - \tfrac{1}{6}g_{\mu\nu}R)\gamma_\mu \mathscr{D}_\nu]\Lambda\,,\\
e^{-1}\mathscr{L}'_0 &= \mathscr{D}^2\varphi\,\mathscr{D}^2\varphi - 2(R_{\mu\nu} - \tfrac{1}{3}g_{\mu\nu}R)\,\mathscr{D}_\mu\varphi\,\mathscr{D}_\nu\varphi + f(\varphi)\,C_{\lambda\mu\nu\rho}C^{\lambda\mu\nu\rho}\,.
\end{aligned}
\tag{1.18}
$$

We observe that (1.18) is not unique: f can be taken to be an arbitrary function because φ remains unchanged under transformation (1.2).

Following the same pattern we can check the Weyl invariance of

$$
\begin{aligned}
\mathscr{L}_2 = \frac{\sqrt{g}}{2\alpha^2}\{&\tfrac{1}{2}(\mathscr{D}^2\bar{h}_{\mu\nu})^2 - \tfrac{1}{2}(\mathscr{D}_\mu\mathscr{D}_\lambda\bar{h}_{\lambda\nu} - \mathscr{D}_\nu\mathscr{D}_\lambda\bar{h}_{\lambda\mu})^2 - \tfrac{2}{3}(\mathscr{D}_\lambda\mathscr{D}_\mu\bar{h}_{\lambda\mu})^2 - 2C_{\alpha\mu\nu\beta}\bar{h}_{\alpha\beta}\,\mathscr{D}^2\bar{h}_{\mu\nu}\\
&- R_{\mu\nu}(2\bar{h}_{\mu\lambda}\,\mathscr{D}^2\bar{h}_{\nu\lambda} + 4\bar{h}_{\mu\lambda}\,\mathscr{D}_{(\nu}\mathscr{D}_{\rho)}\bar{h}_{\rho\lambda} - \bar{h}_{\alpha\beta}\,\mathscr{D}_\mu\mathscr{D}_\nu\bar{h}_{\alpha\beta}) - \tfrac{1}{3}R\,(\tfrac{3}{4}\bar{h}_{\mu\nu}\,\mathscr{D}^2\bar{h}_{\mu\nu} + \bar{h}_{\mu\lambda}\,\mathscr{D}_{(\mu}\mathscr{D}_{\nu)}\bar{h}_{\nu\lambda})\\
&+ \tfrac{1}{3}(R_{\mu\nu}\bar{h}_{\mu\nu})^2 + 4\bar{R}_{\mu\nu}\bar{R}_{\nu\lambda}\bar{h}_{\lambda\rho}\bar{h}_{\rho\mu} + 2\bar{R}_{\mu\nu}\bar{R}_{\lambda\rho}\bar{h}_{\nu\lambda}\bar{h}_{\rho\mu} - \tfrac{1}{2}(\bar{R}_{\mu\nu})^2\bar{h}^2_{\alpha\beta} + \tfrac{1}{12}R^2\bar{h}^2_{\mu\nu} + 2C_{\alpha\mu\nu\beta}C_{\lambda\mu\nu\rho}\bar{h}^{\alpha\beta}\bar{h}^{\lambda\rho}\}\\
\bar{R}_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{4}&g_{\mu\nu}R, \qquad \bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}h^\lambda_{\ \lambda}\,.
\end{aligned}
\tag{1.19}
$$

Here for simplicity we assumed that the background curvature is covariantly constant (i.e. omitted are terms involving $\mathscr{D}_\mu R$ which in fact are not essential for the D-invariance of (1.19)). It goes without saying that the easiest way to derive this expression is to take the bilinear term in the $h_{\mu\nu}$-expansion $(g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu})$ of the Weyl action (1.3). The Lagrangian (1.19) is also invariant under the background gauge transformations $\delta h_{\mu\nu} = \mathscr{D}_\mu\xi_\nu + \mathscr{D}_\nu\xi_\mu$ (under the condition that $g_{\mu\nu}$ satisfies the classical equations following from (1.3)) and also under $\delta h_{\mu\nu} = 2\lambda g_{\mu\nu}$. It can be thus proven that the flat space limit of (1.19) describes off-shell the pure spin 2 states (see [231]).

Finally, the Weyl invariant extension of $\mathscr{L}_{3/2}$ is given by

$$
\begin{aligned}
e^{-1}\mathscr{L}_{3/2} = &- 4e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\phi}_\rho\gamma_5\gamma_\sigma\,\mathscr{D}_\mu\phi_\nu - R_{\mu\nu}[2\bar{\psi}_\lambda\sigma_{\lambda\nu}\phi_\mu - 2\bar{\psi}_\mu\sigma_{\lambda\nu}\phi_\lambda + 2\bar{\psi}_\lambda\gamma_\nu(\mathscr{D}_{[\mu}\psi_{\lambda]} - \gamma_{[\mu}\phi_{\lambda]})]\\
&+ \tfrac{4}{3}R\bar{\psi}_\lambda\sigma_{\lambda\nu}\phi_\nu\,,
\end{aligned}
\tag{1.20}
$$

where

$$
\begin{aligned}
\phi_\mu &\equiv \tfrac{1}{3}\gamma^\nu(\mathscr{D}_\nu\psi_\mu - \mathscr{D}_\mu\psi_\nu + \tfrac{1}{2}\gamma_5 e\varepsilon_{\nu\mu\alpha\beta}\,\mathscr{D}_\alpha\psi_\beta)\,,\\
\mathscr{D}_\mu\psi_\nu &= (\partial_\mu + \tfrac{1}{2}\sigma_{ab}\omega_\mu^{\ ab}(e))\psi_\nu
\end{aligned}
\tag{1.21}
$$

and, as in (1.19), we omitted $\mathscr{D}R\,\bar{\psi}\psi$-terms. Direct computation reveals that (1.20) is S-invariant[1] (under (1.5)) but is not invariant under the background Q-supersymmetry transformations $(\delta\psi_\mu = \mathscr{D}_\mu\varepsilon)$. However, Q-invariance is obviously present in the flat space limit of (1.20), which thus describes off-shell the pure spin 3/2 states. To impose Q-invariance upon the interacting theory it is natural to try to couple (1.20) to the Weyl Lagrangian (1.3) (and to make the "graviton" transform under Q), i.e. to try to construct a consistent theory of conformal supergravity. As we shall see, (1.20) does in fact coincide with the corresponding part of the conformal supergravity Lagrangian.

[1] It is interesting to note that the "non-minimal" terms in (1.20) can be found also from the condition of S-invariance of the action.

1.3. "Pure spin" conformal invariant Lagrangians and count of degrees of freedom

The above remarks concerning the linearized Weyl graviton and gravitino Lagrangians (1.19) and (1.20) suggest another point of view on the structure of Lagrangians and of the spectrum of fields of conformal supergravities. Namely, the linearized Lagrangians are to describe only states of some definite ("pure") spins s and thus are to possess a *maximal* degree of gauge invariance and irreducibility even *off the mass shell.* To realize this condition for $s>1$ in four dimensions it turns out to be necessary to introduce higher derivatives (and thus multiple poles in propagators). Let us consider first the linearized Einstein and Rarita–Schwinger Lagrangians (see e.g. [253, 254])

$$e^{-1}\mathscr{L}_{\mathrm{E}}=\frac{1}{k^2}R\simeq-\frac{1}{k^2}\{-\tfrac{1}{4}h\Box(P_2-2P_0)h+\Box(\pi_{\mu\nu}h_{\mu\nu})\}\,, \tag{1.22}$$

$$e^{-1}\mathscr{L}_{\mathrm{RS}}=-e^{-1}\varepsilon^{\mu\nu\lambda\rho}\bar{\psi}_\mu\gamma_5\gamma_\nu\mathscr{D}_\lambda\psi_\rho\simeq-\bar{\psi}(P_{3/2}-2P_{1/2})\not{\partial}\psi \tag{1.23}$$

where the spin projectors P_s are given by

$$P_1{}^\mu_\nu\equiv\pi^\mu_\nu=\delta^\mu_\nu-\partial^\mu\Box^{-1}\partial_\nu,\qquad\Box=\partial_\mu\partial_\mu\,, \tag{1.24}$$

$$P_0{}^{\mu\nu}_{\alpha\beta}=\tfrac{1}{3}\pi^{\mu\nu}\pi_{\alpha\beta},\qquad P_2{}^{\mu\nu}_{\alpha\beta}=\pi^\mu_{(\alpha}\pi^\nu_{\beta)}-\tfrac{1}{3}\pi_{\alpha\beta}\pi^{\mu\nu}\,, \tag{1.25}$$

$$P_{1/2}{}^\mu_\nu=\tfrac{1}{3}\pi^\mu_\rho\pi^\sigma_\nu\gamma^\rho\gamma_\sigma,\qquad P_{3/2}{}^\mu_\nu=\pi^\mu_\nu-\tfrac{1}{3}\pi^\mu_\rho\pi^\sigma_\nu\gamma^\rho\gamma_\sigma\,. \tag{1.26}$$

The corresponding actions known to describe physical spin 2 and 3/2 particles on mass shell, contain a mixture of pure spin states *off mass shell.* This is a manifestation of the non-invariance of (1.22) and (1.23) under the "algebraic" D and S transformations (eqs. (1.2), (1.5)). Analogous facts are true also for the standard ($\Box$ or $\not{\partial}$) higher spin actions [72, 121, 6, 38, 53, 257].[1]

Now it is clear that the action which describes pure spin states should look like $\Phi_s\,\partial^{2p}P_s\Phi_s$. The condition of locality adds a restriction $p\geq s$. To represent the states of spin s we shall use the real totally symmetric tensor field $\phi_s=\phi_{\mu_1\cdots\mu_s}$ in the Bose case ($s=$ integer) and the Majorana spinor-tensor field $\psi_s=\psi_{\mu_1\cdots\mu_{s-1/2}}$ in the Fermi case ($s=$ half integer) (see e.g. [214, 38]). We always suppress the spinor index. The simplest local Lagrangian now takes the universal form

$$\mathscr{L}_s=\Phi_s\Box^sP_s\Phi_s,\qquad s>0\,, \tag{1.27}$$

where Φ_s stands either for ϕ_s or for ψ_s. The more explicit form of $\mathscr{L}_s$ in the latter case is ($\Box^{1/2}\equiv\not{\partial}$)

$$\mathscr{L}_s^{(\mathrm{F})}=\bar{\psi}_s\Box^{s-1/2}\not{\partial}P_s\psi_s\,. \tag{1.28}$$

The projectors P_s satisfy the following requirements: they are totally symmetric, "traceless"

$$\delta_{\mu_i\mu_j}P_s{}^{\mu_1\cdots\mu_i\cdots\mu_j\cdots\mu_s}_{\nu_1\cdots\nu_s}=0,\qquad\gamma_{\mu_i}P_s{}^{\mu_1\cdots\mu_i\cdots\mu_{s-1/2}}_{\nu_1\cdots\nu_{s-1/2}}=0 \tag{1.29}$$

[1] Note that it is the presence of the negative norm spin 0 particle in (1.22) that is the reason for the indefiniteness of the Einstein gravitational action [136], which is to be contrasted to the formal positivity of the Weyl (Euclidean) action.

and transverse

$$\partial_{\mu_i} P_{s\,\nu_1\cdots\nu_s}^{\;\mu_1\cdots\mu_i\cdots\mu_s} = 0, \qquad \partial_{\mu_i} P_{s\,\nu_1\cdots\nu_{s-1/2}}^{\;\mu_1\cdots\mu_i\cdots\mu_{s-1/2}} = 0 \tag{1.30}$$

(the same relations hold for lower indices). As a result, the actions (1.27) possess the following "differential" gauge symmetries

$$\delta\phi_{\mu_1\cdots\mu_s} = \partial_{(\mu_1}\xi_{\mu_2\cdots\mu_s)}, \qquad \xi_{\mu_1\cdots\mu_i\cdots\mu_j\cdots\mu_{s-1}}\delta^{\mu_i\mu_j} = 0 \tag{1.31}$$

$$\delta\psi_{\mu_1\cdots\mu_{s-1/2}} = \partial_{(\mu_1}\varepsilon_{\mu_2\cdots\mu_{s-1/2})}, \qquad \gamma^{\mu_i}\varepsilon_{\mu_1\cdots\mu_i\cdots\mu_{s-3/2}} = 0 \tag{1.32}$$

(generalizing the standard gauge invariance, general covariance and Q-supersymmetry) as well as the "algebraic" gauge symmetries

$$\delta\phi_{\mu_1\cdots\mu_s} = -\sum_{i\neq j}\delta_{\mu_i\mu_j}\lambda_{\mu_1\cdots\hat{\mu}_i\cdots\hat{\mu}_j\cdots\mu_s}, \tag{1.33}$$

$$\delta\psi_{\mu_1\cdots\mu_{s-1/2}} = -\sum_i \gamma_{\mu_i}\eta_{\mu_1\cdots\hat{\mu}_i\cdots\mu_{s-1/2}} \tag{1.34}$$

which are analogs of the Weyl and S-supersymmetry transformations (1.2), (1.5).

Using (1.7) and (1.8) we can find the dimensions and Weyl weights of Φ_s

$$\begin{aligned} &\phi_{\mu_1\cdots\mu_s}: \quad k = 2-s, \quad w = 2-2s \\ &\psi_{\mu_1\cdots\mu_{s-1/2}}: \quad k = 2-s, \quad w = \tfrac{5}{2}-2s. \end{aligned} \tag{1.35}$$

The most important property of the Lagrangians (1.27) is that they have generally covariant *and Weyl invariant extensions*, describing the couplings of Φ_s to the background metric. These extensions possess also the algebraic symmetries (1.33), (1.34) but, in general (for $s > 1$), are not invariant under covariant versions of (1.31) and (1.32) for arbitrary background metric.[2] Gauge invariance is supposed to be restored after coupling several fields Φ_s (and possibly some "matter" fields) in a locally supersymmetric fashion.

Let us illustrate the above general discussion with a number of the simplest examples: $s = \frac{1}{2}, 1, \frac{3}{2}, 2$. The corresponding X- and D-invariant actions were already given in the previous section:

$$\begin{aligned} &s = \tfrac{1}{2}: \quad \mathscr{L}_{1/2} = \bar{\psi}\partial\!\!\!/\psi \to \mathscr{L}_{1/2}\ (1.13)\ (\chi\to\psi) \\ &s = 1: \quad \mathscr{L}_1 = A_\mu(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)A_\nu \to 2\mathscr{L}_1\ (1.14) \\ &s = \tfrac{3}{2}: \quad \mathscr{L}_{3/2} = \bar{\psi}_\mu P_{3/2}{}^{\mu}_{\nu}\Box\partial\!\!\!/\psi_2 \to -\mathscr{L}_{3/2}\ (1.20) \\ &s = 2: \quad \mathscr{L}_2 = h_{\mu\nu}P_2{}^{\mu\nu}_{\alpha\beta}\Box^2 h_{\alpha\beta} \to 4\alpha^2\mathscr{L}_2\ (1.19). \end{aligned} \tag{1.36}$$

[2] Though for the $s = \frac{3}{2}$ and 2 cases, we can restore gauge invariance by imposing some non-trivial restrictions on the background metric (e.g. $R_{\mu\nu} = 0$), this seems to be impossible for $s > 2$. Thus the situation with gauge invariance here is analogous to that in the case of higher spin fields with "standard" actions (see e.g. [6]).

It is curious to note that (in $d=4$) $s=1$ is the upper value for which we can simultaneously satisfy the conditions of off-shell pure spin states, Weyl (conformal) invariance, gauge invariance of gravitational coupling, positivity of Euclidean action and physical unitarity (absence of higher derivatives). The Lagrangians (1.36) provide natural candidates for the corresponding parts of the total Lagrangian of conformal supergravity. We also conclude that the conformal invariant Lagrangians of higher ($s>2$) spin fields (which may be relevant for $N>4$ conformal supergravity) must necessarily have the structure of (1.27) (i.e. $\bar{\psi}_{\mu\nu}P_{5/2}{}^{\mu\nu}_{\alpha\beta}\Box^2\not{\partial}\psi^{\alpha\beta}$, etc.), because lower derivative Lagrangians cannot be made D-invariant without spoiling the gauge-invariance property of their flat space expressions.

Now we turn to the count of the degrees of freedom which are described by (1.27). We shall distinguish between the "off-shell" (n) and "on shell" (ν) numbers of degrees of freedom. By definition,

$$n = (\text{number of field components}) - (\text{dimension of local symmetry group}), \tag{1.37}$$

i.e., n depends on the field representation and on the gauge symmetry but not on the particular gauge invariant Lagrangian considered. On the contrary, ν, being the number of independent states propagated according to the classical fields equations, depends essentially on the dynamics (e.g., on the number of derivatives in the Lagrangian). It is useful to give also another (equivalent) definition of ν. Take the linearized (Euclidean) action and compute the corresponding "partition function" (or vacuum functional), introducing all the necessary gauge breaking terms and ghosts. The result can always be put into the form

$$Z = Z_0^{\nu}, \qquad Z_0 = [\det(-\Box)]^{-1/2} \tag{1.38}$$

where Z_0 is the "partition function" for a real scalar field. Let us now establish the values of n and ν for the fields Φ_s in (1.27). Taking into account the gauge invariance conditions (1.31)–(1.34) we find (in four dimensions)

$$n(\phi_s) = \underbrace{\binom{3+s}{s}}_{\phi_s} - \underbrace{\left[\binom{2+s}{s-1} - \binom{s}{s-3}\right]}_{\xi_{s-1}} - \underbrace{\binom{1+s}{s-2}}_{\lambda_{s-2}} = 2s+1\,, \tag{1.39}$$

$$n(\psi_s) = -4\left\{\underbrace{\binom{3+p}{p}}_{\psi_s} - \underbrace{\left[\binom{2+p}{p-1} - \binom{1+p}{p-2}\right]}_{\varepsilon_{s-1}} - \underbrace{\binom{2+p}{p-1}}_{\eta_{s-2}}\right\} = -2(2s+1), \qquad p \equiv s-\tfrac{1}{2} \tag{1.40}$$

(we have used the fact that the number of components of a Majorana spinor is 4 and took the number of Fermi degrees of freedom with the negative sign). Hence the number of *off-shell* degrees of freedom, described by (1.27), is just equal to the number of states for a *massive* particle of spin s (i.e., $n=3$ for a gauge vector, $n=5$ for "Weyl" graviton, $n=-8$ for the "Weyl" gravitino, etc.).

To compute ν we first fix the algebraic gauge freedoms (1.33), (1.34) by imposing the (ghost-free) gauges: $\delta^{\mu_i\mu_j}\phi_{\dots\mu_i\dots\mu_j\dots}=0$, $\gamma^{\mu_i}\psi_{\dots\mu_i\dots}=0$. In other words, we change the path integral variables $\phi\to(\hat{\phi},\lambda)$, $\psi\to(\hat{\psi},\eta)$ and drop the integrals over λ and η. Consider first the case of integer s and take $\partial_{\mu_1}\hat{\phi}_{\mu_1\cdots\mu_s}=\hat{\zeta}_{\mu_2\cdots\mu_s}$ as a gauge for (1.31) ("hats" denote traceless tensors). The corresponding ghost operator is defined on $\hat{\xi}_{s-1}(\Delta_{\rm gh} = \Box + \cdots)$. Next we average over $\hat{\zeta}$ with the help of a suitable operator (defined on $\hat{\zeta}_{s-1}$): $H = \Box^{s-1} + \cdots$. The order of this operator is dictated by the condition of "diagonality" of the final action with a fixed gauge

$$\hat{\phi}_s P_s \Box^s \hat{\phi}_s + (\partial \cdot \hat{\phi}_s) H (\partial \cdot \hat{\phi}_s) = \hat{\phi}_s \Delta_s \hat{\phi}_s, \qquad \Delta_s = \Box^s. \tag{1.41}$$

As a result, we find for the partition function

$$Z_\phi = \frac{\det \Delta_{\rm gh} [\det H]^{1/2}}{[\det \Delta_s]^{1/2}} = \left[\frac{\det(\Box^{s+1})_{\hat{\xi}_{s-1}}}{\det(\Box^s)_{\hat{\phi}_s}}\right]^{1/2} \tag{1.42}$$

and therefore (cf. (1.38))[3]

$$\nu(\phi_s) = sN_s - (s+1)N_{s-1} \tag{1.43}$$

where N_s is the number of components of the totally symmetric traceless tensor of rank s,

$$N_s = \binom{3+s}{s} - \binom{1+s}{s-2} = (s+1)^2. \tag{1.44}$$

Thus

$$\nu(\phi_s) = s(s+1), \qquad s > 0. \tag{1.45}$$

This expression readily reproduces the well-known result $\nu(A_\mu) = 2$ and also the result for the number of on-shell degrees of freedom of the Weyl graviton $\nu(h_{\mu\nu}) = 6$ first found in [98, 99] (see also [100, 185]).

The case of half integer s is treated similarly: "averaging over gauges" $\partial_{\mu_1} \hat{\psi}_{\mu_1 \cdots \mu_p} = \hat{\zeta}_{\mu_2 \cdots \mu_p}$, $p \equiv s - \frac{1}{2}$, we are finally left with

$$Z_\psi = \left[\frac{\det(\Box^{s+1})_{\hat{\varepsilon}_{s-1}}}{\det(\Box^s)_{\hat{\psi}_s}}\right]^{-1/2} \tag{1.46}$$

which according to (1.38) gives

$$\nu(\psi_s) = -s\bar{N}_s + (s+1)\bar{N}_{s-1}, \qquad \bar{N}_s = N(\hat{\psi}_s). \tag{1.47}$$

In order to determine the fields Φ_s which can be combined to form off- and on-shell multiplets of global supersymmetry, we have to meet the conditions of having zero total (off- and on-shell) degrees of freedom

$$\sum_i n(\Phi_{s_i}) = 0, \tag{1.48}$$

$$\sum_i \nu(\Phi_{s_i}) = 0. \tag{1.49}$$

[3] To derive (1.42), (1.43) one has to make some trivial "rotations" of fields to put all operators in the $\Box^k$-form (e.g., $-\delta_{\mu\nu}\Box + \frac{1}{2}\partial_\mu\partial_\nu \to -\delta_{\mu\nu}\Box$, etc.).

Consider first the case of $N = 1$ supersymmetry and suppose that a multiplet contains only two fields Φ_s with adjacent spins $(s, s-\frac{1}{2})$. It is easy to check that there is no such solution which satisfies both equations in (1.48). Thus it is necessary to include *three* fields in the multiplet: $(s, s-\frac{1}{2}, s-1)$.[4] Again no solution of (1.48) exists if the highest spin s is half-integer. At the same time (1.48) is identically satisfied for an integer spin s (bosonic highest spin field)

$$\sum n = (2s+1) - 2[2(s-\tfrac{1}{2})+1] + [2(s-1)+1] \equiv 0\,, \tag{1.50}$$

$$\sum \nu = s(s+1) - 2[(s-\tfrac{1}{2})+\tfrac{1}{2}]^2 + (s-1)[(s-1)+1] \equiv 0\,. \tag{1.51}$$

The Lagrangian corresponding to this multiplet, and the global supersymmetry transformations which leave it invariant (up to a total derivative term) can be written as follows ($\{s\}_1 = (s, s-\frac{1}{2}, s-1)$)

$$\mathscr{L}_{\{s\}_1} = \phi_s \Box^s P_s \phi_s + \bar{\psi}_{s-1/2} \Box^{s-1} P_{s-1/2} \not{\partial} \psi_{s-1/2} + \phi_{s-1} \Box^{s-1} P_{s-1} \phi_{s-1}\,, \tag{1.52}$$

$$\begin{aligned} &\delta\phi_{\mu_1\cdots\mu_s} = \bar{\varepsilon}\gamma_{(\mu_1}\tilde{\psi}_{\mu_2\cdots\mu_s)}, \qquad \tilde{\psi}_{s-1/2} \equiv P_{s-1/2}\psi_{s-1/2}\,, \\ &\delta\bar{\psi}_{\mu_1\cdots\mu_{s-1}} = -2\bar{\varepsilon}\gamma_{[\rho}\gamma_{\nu]}\partial_\rho\tilde{\phi}_{\nu\mu_1\cdots\mu_{s-1}} - \bar{\varepsilon}\tilde{\phi}_{\mu_1\cdots\mu_{s-1}}\,, \\ &\delta\phi_{\mu_1\cdots\mu_s} = \bar{\varepsilon}\not{\partial}\tilde{\psi}_{\mu_1\cdots\mu_{s-1}}, \qquad \tilde{\phi} \equiv P_s\phi_s\,. \end{aligned} \tag{1.53}$$

These transformations can be made local by combining them with gauge transformations (1.31), (1.32) (this amounts to dropping the "wave" signs on the fields).

Let us now give several examples of these $N = 1$ supersymmetry multiplets $\{s\}_1$:

$$\{1\}_1 = (1, \tfrac{1}{2}, 0), \qquad \{2\}_1 = (2, \tfrac{3}{2}, 1), \qquad \{3\}_1 = (3, \tfrac{5}{2}, 2), \ldots \tag{1.54}$$

$\{1\}_1$ is recognized to be the well-known gauge vector multiplet (the $s = 0$ component is a non-propagating scalar field, cf. (1.27)). The corresponding Lagrangian (1.52) and the transformation laws (1.53) obviously coincide (after a trivial rescaling of fields) with the standard ones [88]. $\{2\}_1$ (which contains only propagating fields) represents a natural candidate for a multiplet of $N = 1$ conformal supergravity [171, 172]. The full non-linear Lagrangian of this theory can now be (straightforwardly in principle) constructed starting with the sum of covariant bilinear terms (1.19), (1.20), (1.14), and transformation laws (1.2), (1.4), (1.5), (1.53). We have then to guess (mainly from dimensional considerations) the form of the S-supersymmetry transformations for e^a_μ and A_μ and to apply the Noether procedure to complete the Lagrangian and the Q-supersymmetry transformation laws.[5] Even before starting such a construction we already know much about this theory (for example, it is

[4] We assume that the supersymmetry parameter is a spin 1/2 Majorana spinor. If we had tried to use instead a gauge parameter $\varepsilon_{\mu_1\cdots\mu_{s-3/2}}$ in (1.32), then the "multiplet" with the "middle" (Fermi) spin s would have contained $(2s-1, s, 1)$. Imposing (1.48) would leave us with only two possible values: $s = \frac{1}{2}$ and $\frac{3}{2}$, which are already covered by our initial assumption. This remark suggests the absence of a special kind of higher ($s > 2$) spin theory in which Fermi gauge transformations (1.32) are treated as a kind of supersymmetry. To construct a consistent higher spin theory one has probably to distinguish (1.32) from an "ordinary" (spin $\frac{1}{2}$) $N > 1$ supersymmetry, which is to be introduced independently.

[5] We note that the presence of linearly linearized S-supersymmetry (and the dimensionless nature of the coupling constant) essentially simplifies the construction. For example, the condition of S-supersymmetry is sufficient to fix the coupling of ψ_μ to A_μ (including non-minimal $\bar{\psi}\mathscr{D}\psi F$-terms, see below).

interesting to note that the bilinear Lagrangians (1.20), (1.19) and (1.14) are sufficient to compute the one-loop β-function in $N=1$ conformal supergravity [100, 101]). However, we are still to answer the question of the existence of the full supersymmetric theory. This question is in fact difficult to answer using the inductive method of the Noether procedure. Thus we need a more effective method to establish the Lagrangian of $N=1$ conformal supergravity.

Turning to $N>1$ supersymmetry, we conclude that the above reasoning fails to predict the content of $N=2,\dots$ supersymmetry multiplets. The point is that the fields Φ_s in (1.27) are essentially the "gauge fields", while $N>1$ multiplets must contain also "matter" fields (like those present in (1.11), (1.16), (1.18), (1.19)). For example, $\{1\}_2=(1, 2\times\frac{1}{2}, 2\times 0$, auxiliary fields). Hence we have also to learn how to determine the spectra of $N>1$ theories. Trying to work out a more systematic understanding of conformal supergravities it is useful to appreciate the relation between the local symmetries (X, D, Q, S) which we are imposing on these theories and the rigid (super)conformal symmetry of the flat space theories.

1.4. Rigid superconformal algebra and its gauging

Let $I[g_{\mu\nu}, \Phi]$ be a generally covariant action, invariant also under the Weyl transformations (1.2). Then the flat space limit of this action $I_0[\Phi]=I[g_{\mu\nu}=\delta_{\mu\nu}, \Phi]$ is invariant under the rigid conformal group of space–time transformations. The proof of this statement goes as follows (cf. [123]). Consider the product of the Weyl group and the group of general coordinate transformations, and determine the subgroup C of D × X, which leaves invariant the condition $g_{\mu\nu}=\delta_{\mu\nu}$. We have

$$e^{-2\lambda(x)}\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}\delta_{\mu\nu}=\delta_{\alpha\beta}\,. \tag{1.55}$$

To establish the algebra of C we take $\delta x^\mu=-\xi^\mu$. Equation (1.55) then reduces to (we use Latin letters to denote flat space indices)

$$\partial_a\xi_b+\partial_b\xi_a-\tfrac{1}{2}\delta_{ab}\partial_c\xi_c=0,\qquad \lambda=\tfrac{1}{4}\partial_a\xi_a\,. \tag{1.56}$$

The solution for the flat space conformal Killing vectors can be found by substituting into (1.56) the Taylor expansion

$$\xi_c=a_c+a_{cb}x_b+\tfrac{1}{2}a_{cbd}x_bx_d+\tfrac{1}{6}a_{cbde}x_bx_dx_e+\cdots\,. \tag{1.57}$$

The result

$$\begin{aligned}&\xi_c=a_c+\varepsilon_{cb}x_b+bx_c+(2\delta_{ca}k_d-\delta_{ad}k_c)x^ax^d\\&\lambda=b+2k_ax_a\,,\qquad \varepsilon^{ab}=-\varepsilon^{ba}\end{aligned} \tag{1.58}$$

corresponds to the infinitesimal transformation of the fifteen parameter conformal group C = SO(4, 2): a_c is a parameter of the translations (with generators P_a), ε_{ab} of Lorentz rotations (M_{ab}), b of dilatations (D) and k_a of proper conformal transformations or "conformal boosts" (K_a). The generators satisfy the following (conformal) algebra

$$[M_{ab}, M^{cd}] = 4M_{[a}{}^{[d}\delta^{c]}_{b]}, \qquad [M_{ab}, P_c] = 2P_{[a}\delta_{b]c},$$
$$[M_{ab}, K_c] = 2K_{[a}\delta_{b]c}, \qquad [P_a, D] = P_a, \tag{1.59}$$
$$[K_a, D] = -K_a, \qquad [P_a, K_b] = 2(\delta_{ab}D - M_{ab}),$$

while all other commutators are equal to zero. Here we are only interested in the infinitesimal conformal transformations generated by this algebra, but the statement formulated at the beginning of this section remains true for arbitrary (non-linear) conformal transformations.

As an immediate corollary of that statement and of eqs. (1.11)–(1.14) of section 1.2 we reach a well-known result about the conformal (C) invariance of the ordinary massless scalar, spinor and gauge vector actions [123]. C-invariance holds also for the flat space limits of the actions corresponding to (1.17)–(1.20), as well as for the arbitrary spin actions (1.27), (1.28).

The above discussion can be easily generalized to the supersymmetrical case. Suppose that the action $I[e^a_\mu, \psi_\lambda, \Phi]$ is invariant under X, D and also Q- and S-supersymmetry transformations (1.4), (1.5) (as well as under local Lorentz rotations). Then the flat space limit of this action $I_0[\Phi] = I[g_{\mu\nu} = \delta_{\mu\nu}, \psi_\lambda = 0, \Phi]$ is *invariant under the rigid superconformal symmetry group* (SC). The corresponding algebra contains the conformal algebra as a subalgebra. To establish its fermionic generators we have to find the subclass of the combined Q and S transformations which leave invariant the condition $\psi_\mu = 0$ (and $e^a_\mu = \delta^a_\mu$). The defining equation (cf. [263])

$$\partial_a\varepsilon - \gamma_a\eta = 0, \qquad \text{or} \qquad \partial_a\varepsilon - \tfrac{1}{4}\gamma_a\not{\partial}\varepsilon = 0, \qquad \eta = \tfrac{1}{4}\not{\partial}\varepsilon \tag{1.60}$$

and the procedure of its solution are analogous to (1.56) and (1.57). We find that the general solution is parametrized by two constant Majorana spinors ε_0 and η_0

$$\varepsilon(x) = \varepsilon_0 + \gamma_a x_a \eta_0, \qquad \eta(x) = \eta_0 . \tag{1.61}$$

ε_0 is the parameter of ordinary rigid supersymmetry (with a spinorial generator Q^α) while η_0 corresponds to special conformal supersymmetry (generated by S^α). For simplicity we shall consider here only $N = 1$ superconformal algebra. Combining Q and S with the conformal generators, we find in addition to (1.59)

$$\{Q, Q^{\mathrm{T}}\} = \tfrac{1}{2}(\gamma_a C^{-1})P_a, \qquad \{S, S^{\mathrm{T}}\} = -\tfrac{1}{2}(\gamma_a C^{-1})K_a ,$$
$$[Q, M_{ab}] = \sigma_{ab}Q, \qquad [S, M_{ab}] = \sigma_{ab}S,$$
$$[Q, K_a] = -\gamma_a S, \qquad [S, P_a] = \gamma_a Q, \tag{1.62}$$
$$[Q, D] = \tfrac{1}{2}Q, \qquad [S, D] = -\tfrac{1}{2}S .$$

To close the anticommutator $\{Q, S^{\mathrm{T}}\}$ satisfying at the same time the Jacobi identities we need to introduce an additional bosonic generator U, corresponding to the chiral U(1) transformations. It enters through

$$[Q, U] = -\tfrac{3}{4}\mathrm{i}\gamma_5 Q, \qquad [S, U] = \tfrac{3}{4}\mathrm{i}\gamma_5 S$$
$$\{Q, S^{\mathrm{T}}\} = \tfrac{1}{2}C^{-1}D - \tfrac{1}{2}\sigma^{ab}C^{-1}M_{ab} - \mathrm{i}\gamma_5 C^{-1}U . \tag{1.63}$$

In (1.62), (1.63) we used a charge conjugation matrix C (see appendix A) and suppressed spinoral indices $\alpha, \beta((Q^{\alpha})^{T} = Q_{\alpha})$. The non-vanishing (anti) commutators (1.59), (1.62), (1.63) define the superconformal algebra SU(2, 2|1) (SU(2, 2) is the covering group for SO(4, 2)) [263, 78, 65, 150].

Examples of actions invariant under the rigid superconformal group are provided by (1.51). In this way we rederive the known fact about the superconformal invariance of the vector multiplet ($s = 1$) action.

The above observations provide two important hints for the construction of conformal supergravities: (i) these theories are to be identified as "gauge theories" of the (extended) superconformal group [170, 83], (ii) the spectra of fields and the transformation rules of (extended) conformal supergravities can in principle be determined by starting from a multiplet of currents for some flat space superconformal invariant matter system (supergauge theory) and considering its coupling to the corresponding multiplet of "gauge fields" [8].

The suggestion of gauging the full (super)conformal algebra may seem strange at first sight, because local translations (local Q-transformations) already contain Lorentz rotations, dilations and conformal boosts (S-transformations). Thus, we apparently need to introduce only the gauge fields which correspond to the generators P_a, Q and U (i.e. e^a_μ, ψ_μ and A_μ, which are in fact the physical fields of $N = 1$ conformal supergravity). Being "physically" correct this argument however ignores the possibility that "redundant" generators may act non-trivially not only on the arguments (x^μ), but also on the "internal" indices of the fields. Consider for example the gauging of the Poincaré algebra (P_a, M_{ab}), realized, e.g., on spinor fields: $[\psi, M_{ab}] = \sigma_{ab}\psi$, etc. To construct the Lorentz covariant derivative for ψ we need to introduce the gauge field ω^{ab}_μ, corresponding to M_{ab} [180]. At the same time it is not necessary for ω to be independent of the "translational" gauge field e^a_μ (usually identified with vierbein). In fact, *any* function $\omega^{ab}_\mu(e)$ transforming properly, is sufficient for kinematical purposes. The absence of additional (to e^a_μ) ω^{ab}_μ-degrees of freedom provides the "irreducibility" of the theory (minimal number of propagating degrees of freedom etc.) and raises its predictive power. The advocated approach is thus to introduce gauge fields for *all* the generators of the algebra but then impose algebraic constraints on some field strengths in order to eliminate "redundant" gauge fields. In the case of the Poincaré algebra, this is the standard zero torsion constraint

$$\mathcal{R}^a_{\mu\nu}(P) = \mathcal{D}_\mu e^a_\nu - \mathcal{D}_\nu e^a_\mu = 0\,. \tag{1.64}$$

A question still remains about gauging D and K_a–transformations. All physical fields are assumed to transform only trivially under the conformal boosts. Thus the analogy with the gauging of M_{ab} formally does not take place. However *a priori* we need a dilational gauge field b_μ which *is not* inert under K_a. As was already noted in section 1.1, we do not want to have b_μ as an independent field in the action. At the same time, it turns out to be impossible to express b_μ in terms of e^a_μ by using an algebraic constraint like (1.64). The successful way out is to gauge the *full* conformal algebra (i.e. to introduce also f^a_μ, corresponding to K_a), and then to impose the constraints ((1.64) and one additional constraint on $\mathcal{R}(M)$) expressing ω^{ab}_μ and f^a_μ in terms of e^a_μ and b_μ, and finally to note that, due to the K-invariance of the theory b_μ (being the only field, which transforms under K) can always be gauged away (i.e. it *cancels* in the K-invariant action). Though this method of developing gauge theory may seem somewhat artificial it is in fact a very systematic one. For example, a large degree of freedom it is based on stream-lines the procedure used for the construction of the Weyl-invariant actions presented in section 1.2 (one uses fully conformal covariant derivatives and eliminates the gauge dependent fields only at the very end).

Analogous remarks apply to the supersymmetric case, where we have to eliminate the S-gauge field ϕ^{α}_{μ}. Having outlined the general methodology, we now turn to the more detailed presentation of the gauge approach.

2. $N = 1$ conformal supergravity

2.1. Weyl theory as a gauge theory of the conformal group

As it was noted in the previous section, the flat space limit of a generally covariant and Weyl invariant theory (e.g. Weyl gravity (1.3), interacting with conformally invariant matter (1.11), (1.13), (1.14)) is invariant under the rigid conformal group SO(4, 2). We may therefore try to inverse this relation and to derive the Weyl theory by gauging SO(4, 2). Such a gauge derivation, though not essential for this particular theory, appears to be important in the case of superconformal theories.

A first thing to be realized is that we are going to gauge a *space–time symmetry group*, containing translations. The resulting gauge theory must be generally covariant and describe gravity. Thus we are somehow to relate "internal" translations to general coordinate transformations on the base space, and to identify the translational gauge field with the vierbein. That is why the structure of a gauge theory of a space–time symmetry group is very different from that of a linear gauge theory of an internal symmetry group.

Several approaches exist for the construction of such a type of gauge theories. We shall mention only two of them.[1] The first one goes under the rubric of "non-linear realization" [25, 149, 193, 190] (see also [236, 249]). Here it is assumed that the "internal" (non-translational) subgroup H of a space–time group G is gauged by ordinary linear gauge fields,[2] while translations are gauged by a homogeneously transforming "non-linear gauge field", which is identified with the vierbein. The essence of this approach is that translations are treated as a "spontaneously broken" *internal* gauge symmetry, which leaves invariant all fields in the theory (these translations shift the tangent space vectors which the ordinary local fields are independent of). The invariance under the general coordinate transformations (X) is thus imposed as an additional assumption. The resulting local symmetry group is $H \otimes X$ (while the "broken" translations are neglected). At this stage the theory is highly "reducible" (e.g. contains torsion), so one has to impose in addition some (in this approach truly artificial or "physical") constraints in order to eliminate a number of fields (torsion, etc.) [188, 28, 170].

The second approach starts with gauging the whole group G as if it were a linear internal symmetry group. Again the role of translations is only to generate a gauge field which formally looks like a vierbein. General coordinate transformations are again introduced from outside. The final group is also identified with $H \otimes X$. However, a conceptual advantage of this approach is that here the imposition of constraints (and their form) appears to be a *consequence* of the requirement that the final theory should be also invariant under ("non-linearly realized") translations[3] (these constraints break the initial translational symmetry). Stated differently, the constraints provide the possibility to express translations, when acting on gauge fields, as a combination of X and H transformations, and thus to trade

[1] Two other and recently popular approaches are based on group manifolds [24] and soft gauge algebras [227].

[2] Sometimes the "linear" subgroup (F) is taken to be smaller than H.

[3] One may of course still consider this requirement as artificial. Then a sceptic's conclusion may be that there are no known aesthetically appealing reasons for imposing constraints (the concept of "irreducibility" seems alien to the geometrical approach) (cf. [133]).

translations for general coordinate transformations in the gauge algebra. This interpretation of constraints (independent of any particular action) appears to be especially important in the supersymmetrical case.

It is this second approach that we will use in what follows, and this is why we here summarize some general definitions on which it is based. Let $\{t_A\}$ be the basis of generators of the gauge algebra of G (for simplicity taken to be "bosonic")

$$[t_A, t_B] = f^C{}_{AB} t_C . \tag{2.1}$$

One starts with assigning a gauge field W^A_μ to each generator t_A, $W_\mu = W^A_\mu t_A$. Assuming that the gauge field transforms under G according to $\delta_G W_\mu = \partial_\mu \varepsilon - [W_\mu, \varepsilon]$ or

$$\delta_G W^A_\mu = \hat{D}_\mu \varepsilon^A \equiv \partial_\mu \varepsilon^A - f^A{}_{BC} W^B_\mu \varepsilon^C \tag{2.2}$$

(if matter fields transform as $\delta_G \Phi = \varepsilon^A t_A \Phi$, the general form of covariant derivative is $\hat{D}_\mu = \partial_\mu - W^A_\mu t_A$), we get the covariantly transforming curvature (or field strength) tensor

$$\mathcal{R}^A{}_{\mu\nu} = \partial_\mu W^A_\nu - \partial_\nu W^A_\mu - f^A{}_{BC} W^B_\mu W^C_\nu \tag{2.3}$$

$$\delta_G \mathcal{R}^A{}_{\mu\nu} = f^A{}_{BC} \mathcal{R}^B_{\mu\nu} \varepsilon^C, \qquad \hat{D}_{[\mu} \mathcal{R}_{\lambda\rho]} = 0 . \tag{2.4}$$

To prove these relations one has to make the assumption that $\partial_\mu f^A{}_{BC} = 0$ and to use the Jacobi identity $f^K_{N(A} f^N_{BC)} = 0$.[4] The transformation law under the general coordinate group X is obvious from the position of world indices

$$\delta_X W_\mu = W'_\mu(x) - W_\mu(x) = \partial_\mu \xi^\lambda W_\lambda + \xi^\lambda \partial_\lambda W_\mu . \tag{2.5}$$

This can be identically rewritten as

$$\delta_X(\xi)\, W_\mu = \delta_G(\xi^\lambda W_\lambda) W_\mu + \xi^\lambda \mathcal{R}_{\lambda\mu} , \tag{2.6}$$

i.e. as a combination of the gauge transformation with parameter $\varepsilon^A = \xi^\lambda W^A_\lambda$ and a covariant curvature term. Consider now some *four* generators among $\{t_A\}$ ("translations" P_a) and study the conditions on W_μ under which it is possible to express $\delta_P(\varepsilon^a)$ in terms of some combination of δ_X and δ_H ("H" = "G $-\{P_a\}$"). Using (2.6), these conditions can in principle be expressed as certain linear algebraic constraints on the curvature components

$$\mathcal{R}^A{}_{\mu\nu} a^{\mu\nu i}_A(W) = 0, \qquad i = 1, \dots, k . \tag{2.7}$$

The number of constraints k is in general group-dependent. The constraints (2.7) can be taken to be invariant under the final symmetry group X$\otimes$H. Next, we have to construct a geometrical (i.e. built only from curvature), dimensionless, parity conserving, X$\otimes$H-invariant action. The only possible candidate is [188]

[4] It is sufficient to make the weaker assumption that $\hat{D}_\mu f^A{}_{BC} = 0$ [227].

$$I = \int d^4x\, \lambda_{AB} \mathcal{R}^A{}_{\mu\nu} \mathcal{R}^B{}_{\lambda\rho} \varepsilon^{\mu\nu\lambda\rho}\,, \tag{2.8}$$

where λ_{AB} is some numerical tensor, which is odd under the reversal of orientation. Solving the constraints (2.7) and substituting the solution into (2.8), we finally get the action in terms of a subset of p independent gauge fields ($p = 4\dim G - k$). Though the final action is formally invariant under $X \otimes H$, the non-trivial ("physical") invariance group is actually smaller: $X \otimes F$, $F \subset H$, $\dim F = p/4$.

Specializing now to the example of conformal group $G = SO(4, 2)$, with algebra given by (1.59), we define[5]

$$W_\mu = e^a_\mu P_a + \tfrac{1}{2} w^{ab}_\mu M_{ab} + b_\mu D + f^a_\mu K_a \tag{2.9}$$

and thus find ($\varepsilon \equiv \varepsilon^a_P P_a + \frac{1}{2}\varepsilon^{ab}_M M_{ab} + \varepsilon_D D + \varepsilon^a_K K_a$)

$$\begin{aligned}
\delta_G e^a_\mu &= D_\mu \varepsilon^a_P + \varepsilon^{ab}_M e^b_\mu - \varepsilon_D e^a_\mu\,,\\
\delta_G w^{ab}_\mu &= -4\varepsilon^{[a}_P f^{b]}_\mu + D_\mu \varepsilon^{ab}_M - 4\varepsilon^{[a}_K e^{b]}_\mu\,,\\
\delta_G b_\mu &= 2\varepsilon^a_P f^a_\mu + \partial_\mu \varepsilon_D - 2\varepsilon^a_K e^a_\mu\,,\\
\delta_G f^a_\mu &= \varepsilon^{ab}_\mu f^b_\mu + \varepsilon_D f^a_\mu + D_\mu \varepsilon^a_K\,,
\end{aligned} \tag{2.10}$$

where D_μ is M- and D-covariant derivative, e.g.

$$D_\mu \varepsilon^a_K = \partial_\mu \varepsilon^a_K - w^{ab}_\mu \varepsilon^b_K - b_\mu \varepsilon^a_K\,. \tag{2.11}$$

To derive (2.10), one has simply to compute $\delta W_\mu = \partial_\mu \varepsilon - [W_\mu, \varepsilon]$ and then to equate the terms corresponding to each generator on both sides of the expression. Analogously, calculating $2\partial_{[\mu} W_{\nu]} - [W_\mu, W_\nu]$ we find the curvature components

$$\begin{aligned}
&\mathcal{R}^a_{\mu\nu}(P) = 2D_{[\mu} e^a_{\nu]}, \qquad D_\mu e^a_\nu = (\partial_\mu + b_\mu) e^a_\nu - w^{ab}_\mu e^b_\nu\,,\\
&\mathcal{R}^{ab}_{\mu\nu}(M) = 2\partial_{[\mu} w^{ab}_{\nu]} - 2w^{ac}_{[\mu} w^{cb}_{\nu]} + 8e^{[a}_{[\mu} f^{b]}_{\nu]}\,,
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
&\mathcal{R}(D) = 2\partial_{[\mu} b_{\nu]} - 2e^a_{[\mu} f^a_{\nu]}\,,\\
&\mathcal{R}(K) = 2D_{[\mu} f^a_{\nu]}\,.
\end{aligned} \tag{2.13}$$

The subgroup H is generated by M, D and K. Starting from (2.6) one can easily show that the constraint needed to express $\delta_P e^a_\mu$ as a combination of $\delta_H e^a_\mu$ and $\delta_X e^a_\mu$, is just the constraint corresponding to zero torsion

$$\mathcal{R}^a_{\mu\nu}(P) = 0\,. \tag{2.14}$$

[5] Note that in our notation the Lorentz gauge potential w^{ab}_μ and the standard Lorentz connection ω^{ab}_μ have opposite signs. Thus $\mathcal{R}(M)$ is minus the Riemann curvature plus ....

To reach the same $P \to \mathrm{X}$ conversion on all other fields it is sufficient to add one extra constraint[6] [170]

$$\mathscr{R}^{ab}_{\mu\nu}(M)\, e^{\nu}_{b} = 0 \,. \tag{2.15}$$

The solutions of the constraints in terms of the independent fields (e^a_μ, b_μ) are

$$w^{ab}_{\mu}(e,b) = -\tfrac{1}{2}(\hat{\Omega}_{\mu ab} - \hat{\Omega}_{\mu ba} - \hat{\Omega}_{ab\mu}) = -\omega^{ab}_{\mu}(e) + 2b^{[a}e^{b]}_{\mu} \,, \tag{2.16}$$

$$\hat{\Omega}_{abc} = 2e^{\mu}_{a} e^{\nu}_{b} \hat{\partial}_{[\mu} e_{\nu]c} , \qquad \hat{\partial}_\mu e^a_\nu = (\partial_\mu + b_\mu) e^a_\nu \,,$$
$$f^a_\mu(e,b) = -\tfrac{1}{4}(\hat{\mathscr{R}}^a_\mu - \tfrac{1}{6} e^a_\mu \hat{\mathscr{R}}) \,, \tag{2.17}$$
$$\hat{\mathscr{R}}^a_\mu = [\mathscr{R}^{ab}_{\mu\nu}(M)\, e^{\nu}_{b}]_{f^a_\mu = 0} , \qquad \hat{\mathscr{R}} = \hat{\mathscr{R}}^a_\mu e^\mu_a \,.$$

The expressions for $w^{ab}_{\mu}(e,b)$ and $\hat{\mathscr{R}}^{ab}_{\mu\nu}$, taken with opposite signs, coincide with the standard Lorentz connection and curvature, with all derivatives of the vierbein substituted for the Weyl (D) covariant derivatives $\hat{\partial}_\mu$. The $\mathrm{X} \otimes \mathrm{H}$-invariant geometrical action (cf. (2.8)) has a unique form [170]

$$I_{\mathrm{W}} = \frac{1}{8\alpha^2} \int \mathrm{d}^4x\, \varepsilon_{abcd} [\mathscr{R}^{ab}_{\mu\nu}(M)\, \mathscr{R}^{cd}_{\rho\sigma}(M)]_{\substack{f(e,b)\\ w(e,b)}} \varepsilon^{\mu\nu\rho\sigma} \,. \tag{2.18}$$

This action is H-invariant only on the solution of the constraint (2.14). It is interesting to note that (2.15) or (2.17) can be *derived* as a f^a_μ-field equation by varying (2.18) *before* the elimination of f^a_μ [170]. Computing $\mathscr{R}^{ab}_{\mu\nu}(M)$ with (2.16) and (2.17) taken into account we find that b_μ drops out

$$[\mathscr{R}^{ab}_{\mu\nu}(M)]_{\substack{f(e,b)\\ w(e,b)}} = -(R^{ab}_{\mu\nu} - 2e^{[a}_{[\mu} R^{b]}_{\nu]} + \tfrac{1}{3} e^a_{[\mu} e^b_{\nu]} R) = -C^{ab}_{\mu\nu},$$
$$R^{ab}_{\mu\nu} \equiv R^{ab}_{\mu\nu}(\omega(e)) = -[\hat{\mathscr{R}}^{ab}_{\mu\nu}(w)]_{b_\mu = 0,\, f^a_\mu = 0} \tag{2.19}$$

and we get exactly the (minus) Weyl tensor (see appendix B). Using the identity $C^*C^* = C^2$ we can finally rewrite (2.18) in terms of the metric $g_{\mu\nu} = e^a_\mu e^a_\nu$, and reach the Weyl action (1.3)

$$I_{\mathrm{W}} = \frac{1}{2\alpha^2} \int \mathrm{d}^4x\, \sqrt{g}\, C_{\lambda\mu\nu\rho} C^{\lambda\mu\nu\rho} = \frac{1}{\alpha^2} \int \mathrm{d}^4x\, \sqrt{g}\, (R^2_{\mu\nu} - \tfrac{1}{3}R^2) \tag{2.20}$$

(to get the second expression we used (A.22) and omitted the boundary term). The final non-trivial invariance group is $\mathrm{X} \otimes \mathrm{D}$, which consists of the general coordinate transformation and of the Weyl transformations of the metric (1.2) ($\lambda = \varepsilon_{\mathrm{D}}$). Had we started instead with the assumption of $\mathrm{X} \otimes \mathrm{D}$-invariance only the action would *not* have been unique. Namely, we could add to (2.20) also the terms $\hat{\mathscr{R}}^2_{\mu\nu}$ and $(\partial_{[\mu} b_{\nu]})^2$.[7] It is the condition of K-invariance that makes the "Weyl action" (2.20) unique.

[6] Here we assume the existence of the inverse vierbein, satisfying $e^\nu_a e^b_\nu = \delta^b_a$.

[7] The $\hat{R}^2$-invariant is not an independent term. Note that the resulting (b_μ-dependent) action was that proposed by Weyl [267] (see also [120]).

It is interesting to note that the same "geometrical" action that gives the Weyl action in the case of the conformal gauge group produces the Einstein action (plus cosmological and topological terms) in the case of the De Sitter gauge group [188]. At the same time, the principle of "geometrical" action has a rather limited range of applicability. For example, it fails to give an invariant action in the case of the $N=2$ Poincaré supergravity [247] and of the $N=1$ conformal supergravity [171, 172].

Now let us illustrate the construction of Weyl invariant matter actions in the framework of constrained gauge geometry as discussed above. For example, consider a scalar field of Weyl weight w, i.e. $\delta_H\phi = w\varepsilon_D\phi$. The derivative $\hat{D}_\mu\phi = (\partial_\mu - wb_\mu)\phi$ is no longer inert under K (because according to (2.10) $\delta_K b_\mu \neq 0$). We have

$$\delta_H(\hat{D}_a\phi) = (w+1)\varepsilon_D\hat{D}_a\phi + 2w\varepsilon^a_K\phi + \varepsilon^{ab}_M\hat{D}_b\phi\,,$$
$$\hat{D}_a \equiv e^\mu_a\hat{D}_\mu\,.$$

Employing the general rule for the construction of the full H-covariant derivatives

$$\hat{D}_\mu = \partial_\mu - \delta_M(w^{ab}_\mu) - \delta_K(f^a_\mu) - \delta_D(b_\mu) \tag{2.21}$$

(gauge fields in brackets stand for parameters) we find the following expression for the H-covariant d'Alambertian $\hat{D}^2 = \hat{D}^a\hat{D}_a$

$$\hat{D}^2\phi = e^\mu_a[(\partial_\mu - (w+1)b_\mu)\delta^{ab} - w^{ab}_\mu]\,\hat{D}_b\phi - 2wf^\mu_\mu\phi\,. \tag{2.22}$$

In order to impose the invariance of the action $\int d^4x\, e\phi\hat{D}^2\phi$, we need to take $w = -\frac{1}{2}(2-d) = 1$. Substituting in (2.22) the solutions of the constraints (2.16), (2.17) ($e^\mu_a w^{ab}_\mu = -3b^b$, $f^\mu_\mu = -\frac{1}{12}\hat{\mathcal{R}} = \frac{1}{12}R - \frac{1}{2}\partial_\mu b_\mu + \frac{1}{2}b^2_\mu$), we find that all b_μ-dependent terms cancel and we obtain just the standard Weyl-invariant scalar action (1.11) with $a = \frac{1}{6}$. We once more understand the role played by K-invariance: it provides the possibility to omit b_μ from the final invariant expressions. In fact, all the usual matter fields are K-inert and thus the only independent field which transforms under the conformal boosts is the dilational gauge field. In view of the fact that the number of components of b_μ is exactly the number of parameters of the K-transformations, we can always gauge b_μ away. Thus the requirement of K-invariance on the theory amounts to its independence of b_μ and, in this way it restricts non-trivially its structure (for example, as we have already noted K-invariance is necessary for the uniqueness of the gauge action (2.18)).

2.2. $N=1$ *conformal supergravity as the gauge theory of the superconformal group*

The aim of this section is to give a scheme of construction of $N=1$ conformal supergravity (CSG) in the context of the "constrained gauge geometry" approach, as outlined in the previous section.[1] We have simply to substitute the conformal gauge group for the $N=1$ superconformal one SU(2, 2|1) (with the algebra (1.59), (1.62), (1.63) already discussed in section 1.4) and to just follow the same steps (gauge potentials, transformation laws, curvatures, constraints, $X \otimes H$-invariant gauge action, matter couplings) as we did in the case of the conformal group.

In addition to the conformal gauge fields (2.9) we must also introduce here two types of Majorana

[1] This theory was first constructed in refs. [170–172, 248]. For pedagogical reviews see [254, 49, 50].

"gravitinos" ψ_μ and ϕ_μ (gauge fields for the ordinary Q and the special conformal S supersymmetries) and also the axial U(1) gauge vector A_μ

$$W_\mu = e^a_\mu P_a + \tfrac{1}{2} w^{ab}_\mu M_{ab} + b_\mu D + f^a_\mu K_a + \bar{\psi}_\mu Q + \bar{\phi}_\mu S + A_\mu U . \tag{2.23}$$

Computing $\delta_G W = \partial_\mu \varepsilon - [W_\mu, \varepsilon]$, we find in addition to (2.10)

$$\begin{aligned} &\delta e^a_\mu = -\tfrac{1}{2}\bar{\psi}_\mu \gamma^a \varepsilon_Q, \qquad \delta b_\mu = -\tfrac{1}{2}\bar{\psi}_\mu \varepsilon_S + \tfrac{1}{2}\bar{\phi}_\mu \varepsilon_Q , \\ &\delta w^{ab}_\mu = \bar{\psi}_\mu \sigma^{ab} \varepsilon_S + \bar{\phi}_\mu \sigma^{ab} \varepsilon_Q , \qquad \delta f^a_\mu = \tfrac{1}{2}\bar{\phi}_\mu \gamma^a \varepsilon_S , \\ &\delta \psi_\mu = D_\mu \varepsilon_Q + \varepsilon^a_P \gamma_a \phi_\mu + \tfrac{1}{2}\sigma^{ab}\varepsilon^{ab}_M \psi_\mu - \tfrac{1}{2}\varepsilon_D \psi_\mu + \tfrac{3}{4}\mathrm{i}\gamma_5 \varepsilon_U \psi_\mu - e^a_\mu \gamma_a \varepsilon_S , \\ &\delta \phi_\mu = D_\mu \varepsilon_S - \varepsilon^a_K \gamma_a \psi_\mu + \tfrac{1}{2}\sigma^{ab}\varepsilon^{ab}_M \phi_\mu + \tfrac{1}{2}\varepsilon_D \phi_\mu - \tfrac{3}{4}\mathrm{i}\gamma_5 \varepsilon_U \phi_\mu + f^a_\mu \gamma_a \varepsilon_Q , \\ &\delta A_\mu = \partial_\mu \varepsilon_U + \mathrm{i}\bar{\psi}_\mu \gamma_5 \varepsilon_S - \mathrm{i}\bar{\phi}_\mu \gamma_5 \varepsilon_Q \end{aligned} \tag{2.24}$$

where the D_μ are covariant with respect to the M, D and U transformations, e.g.

$$D_\mu \varepsilon_{Q,S} = (\partial_\mu - \tfrac{1}{2}\sigma^{ab} w_{ab\mu} \pm \tfrac{1}{2} b_\mu \mp \tfrac{3}{4}\mathrm{i}\gamma_5 A_\mu)\varepsilon_{Q,S} . \tag{2.25}$$

Note that the Weyl weight assignments

$$w(e^a_\mu) = -1, \qquad w(\psi_\mu) = -\tfrac{1}{2}, \qquad w(A_\mu) = 0 \tag{2.26}$$

are the same as in table 1.1 of section 1.2. The expressions for the curvatures are (cf. (2.12), (2.13))

$$\begin{aligned} &\mathcal{R}^a_{\mu\nu}(P) = 2D_{[\mu} e^a_{\nu]} - \tfrac{1}{2}\bar{\psi}_\mu \gamma^a \psi_\nu , \\ &\mathcal{R}^{ab}_{\mu\nu}(M) = 2\partial_{[\mu} w^{ab}_{\nu]} - 2 w^{ac}_{[\mu} w^{cb}_{\nu]} + 8 e^{[a}_{[\mu} f^{b]}_{\nu]} + \bar{\psi}_{[\mu}\sigma^{ab}\phi_{\nu]} , \\ &\mathcal{R}_{\mu\nu}(D) = 2\partial_{[\mu} b_{\nu]} - 4 e^a_{[\mu} f^a_{\nu]} - \bar{\psi}_{[\mu}\phi_{\nu]} , \\ &\mathcal{R}^a_{\mu\nu}(K) = 2D_{[\mu} f^a_{\nu]} + \tfrac{1}{2}\bar{\phi}_\mu \gamma^a \phi_\nu , \\ &\bar{\mathcal{R}}_{\mu\nu}(Q) = 2D_{[\mu}\bar{\psi}_{\nu]} - 2\bar{\phi}_{[\mu} e^a_{\nu]}\gamma_a , \\ &\bar{\mathcal{R}}_{\mu\nu}(S) = 2D_{[\mu}\bar{\phi}_{\nu]} + 2\bar{\psi}_{[\mu} f^a_{\nu]}\gamma_a , \\ &\mathcal{R}_{\mu\nu}(U) = 2\partial_{[\mu} A_{\nu]} + 2\mathrm{i}\bar{\psi}_{[\mu}\gamma_5 \phi_{\nu]} . \end{aligned} \tag{2.27}$$

Here

$$\mathcal{R}_{\mu\nu} = \mathcal{R}^a_{\mu\nu}(P)\, P_a + \cdots + \bar{\mathcal{R}}_{\mu\nu}(Q)\, Q + \cdots$$

and

$$D_\mu \bar{\psi}_\nu = \bar{\psi}_\nu (\overleftarrow{\partial}_\mu + \tfrac{1}{2}\sigma_{ab} w^{ab}_\mu + \tfrac{1}{2} b_\mu - \tfrac{3}{4}\mathrm{i}\gamma_5 A_\mu) .$$

The constraints needed to convert the "internal" P-transformations into general coordinate ones (and to achieve the maximal irreducibility of the gauge multiplet) generalize the conformal case constraints (2.14), (2.15) [172, 248]

$$\mathcal{R}^{a}_{\mu\nu}(P)=0, \qquad \bar{\mathcal{R}}_{\mu\nu}(Q)\,\gamma^{\nu}=0\,,$$
$$\mathcal{R}^{ab}_{\mu\nu}(M)\,e^{\nu}_{b}+\tfrac{1}{2}e^{\rho}_{a}(\bar{\mathcal{R}}_{\lambda\rho}(Q)\,\gamma_{\mu}\psi^{\lambda}-\mathrm{i}\mathcal{R}^{*}_{\mu\rho}(U))=0\,. \tag{2.28}$$

The constraints are taken to be covariant with respect to all transformations of H except for Q-supersymmetry[2] (by construction, they explicitly break P-symmetry which can be ignored in the following). The solutions of the constraints are given by (cf. (2.16), (2.17))

$$w^{ab}_{\mu}(e,b,\psi)=w^{ab}_{\mu}(e,b)-\tfrac{1}{4}(\bar{\psi}_{a}\gamma_{\mu}\psi_{b}+\bar{\psi}_{\mu}\gamma_{a}\psi_{b}-\bar{\psi}_{\mu}\gamma_{b}\psi_{a})$$
$$\equiv-\omega^{ab}_{\mu}(e,\psi)+2b^{[a}e^{b]}_{\mu}\,, \tag{2.29}$$

$$\phi_{\mu}(e,b,\psi,A)=\tfrac{1}{3}\gamma^{\nu}(D_{\nu}\psi_{\mu}-D_{\mu}\psi_{\nu}+\tfrac{1}{2}\gamma_{5}e\varepsilon_{\nu\mu\alpha\beta}D_{\alpha}\psi_{\beta})\,,$$
$$D_{\mu}\psi_{\nu}=(\partial_{\mu}-\tfrac{1}{2}\sigma_{ab}w^{ab}_{\mu}+\tfrac{1}{2}b_{\mu}-\tfrac{3}{4}\mathrm{i}\gamma_{5}A_{\mu})\psi_{\nu}\,, \tag{2.30}$$

$$f^{a}_{\mu}(e,b,\psi,A)=-\tfrac{1}{4}(\hat{\mathcal{R}}^{a}_{\mu}-\tfrac{1}{6}e^{a}_{\mu}\hat{\mathcal{R}})-\tfrac{1}{8}\bar{\mathcal{R}}_{\lambda a}(Q)\,\gamma_{\mu}\psi^{\lambda}+\tfrac{1}{8}\mathrm{i}\mathcal{R}^{*}_{\mu a}(U)\,,$$
$$f^{\mu}_{\mu}=-\tfrac{1}{12}\hat{\mathcal{R}} \tag{2.31}$$

where as in (2.17) $\hat{\mathcal{R}}^{\cdot\cdot}_{\cdot\cdot}=[\mathcal{R}^{\cdot\cdot}_{\cdot\cdot}(M)]_{f=0}$.

The most general "parity conserving" "geometrical" action (2.8) consists of three non-trivial terms [170, 171]

$$I=\int \mathrm{d}^{4}x\,[\alpha_{1}\,\mathcal{R}^{ab}_{\mu\nu}(M)\,\mathcal{R}^{cd}_{\rho\sigma}(M)\,\varepsilon_{abcd}+\alpha_{2}\,\bar{\mathcal{R}}_{\mu\nu}(Q)\,\gamma_{5}\,\mathcal{R}_{\rho\sigma}(S)+\alpha_{3}\,\mathcal{R}_{\mu\nu}(U)\,\mathcal{R}_{\rho\sigma}(D)]\varepsilon^{\mu\nu\rho\sigma}\,. \tag{2.32}$$

Using (2.29)–(2.31) to eliminate the dependent fields, one can check that the resulting action can be made invariant under all symmetry transformations of $\mathrm{X}\otimes\mathrm{H}$ except for Q-supersymmetry. To construct a Q-supersymmetric theory, one is thus forced to abandon the principle of "geometrical action" and to add to (2.32) some terms which explicitly involve the metric. The correspondence with the Weyl theory, and an analogy with the $N=2$ Poincaré supergravity case [247], suggest that the only extra term needed is the U(1) Maxwell action [171]

$$\Delta I=\int \mathrm{d}^{4}x\,\mathcal{R}_{\mu\nu}(U)\,\mathcal{R}_{\alpha\beta}(U)\,g^{\mu\alpha}g^{\nu\beta}\sqrt{g}\,. \tag{2.33}$$

Using the constraints we then find that the total action is fully H-invariant if

$$\alpha_{2}=\alpha_{4}=2\mathrm{i}\alpha_{3}=-8\alpha_{1}\equiv-1/\alpha^{2}\,. \tag{2.34}$$

The final step is to express the action in terms of the independent fields e^{a}_{μ}, b_{μ}, ψ_{μ}, A_{μ} (making use of

[2] The form of the constraints (2.28) is not unique. In fact, all constraints which give the possibility to express w, f and ϕ in terms of e, b, ψ, A lead to equivalent theories (related through redefinition of the fields).

(2.27)–(2.31)). As in the case of the Weyl theory, b_μ, which is the only independent field transforming under the conformal boosts, must cancel in the final expression for the K-invariant action. After various algebraic transformations one can rewrite (2.32), (2.33) in the form

$$\begin{aligned} I^{(1)}_{\rm CSG} = \frac{1}{\alpha^2}\int d^4x\,\sqrt{g}\,\{&\tfrac{1}{2}C_{\lambda\mu\nu\rho}{}^2 - \tfrac{3}{4}F_{\mu\nu}{}^2(A) - 4e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\phi_\rho\gamma_5\gamma_\sigma D_\mu\phi_\nu \\ &- 2R_{\mu\nu}[\bar\psi_\lambda\sigma_{\lambda\nu}\phi_\mu - \bar\psi_\mu\sigma_{\lambda\nu}\phi_\lambda + \bar\psi_\lambda\gamma_\nu(D_{[\mu}\psi_{\lambda]} - \gamma_{[\mu}\phi_{\lambda]})] \\ &+ \tfrac{4}{3}R\bar\psi_\lambda\sigma_{\lambda\nu}\phi_\nu - i\bar\psi_\mu(\gamma_5F_{\mu\nu} + F^*_{\mu\nu})\phi_\nu - \mathscr{D}_\mu(R_{\rho\sigma} - \tfrac{1}{6}g_{\rho\sigma}R)\bar\psi_\sigma\gamma_\rho\psi_\mu \\ &+ [R(\bar\psi\psi)^2 + F(\bar\psi\psi)^2 + (\bar\psi\mathscr{D}\psi)^2 + \bar\psi\mathscr{D}\psi(\bar\psi\psi)^2 + (\bar\psi\psi)^4 - \text{terms}]\}\,. \end{aligned} \tag{2.35}$$

The explicit form of quartic and higher order fermionic terms can be found in [172, 248]. In general, the action contains all structures that can be expected on dimensional grounds ($[\psi_\mu] = +\frac{1}{2}$, $[\alpha] = 0$). In (2.35) we assume that w^{ab}_μ and ϕ_μ are given by (2.29) and (2.30) with b_μ put equal to zero (we may also omit the $\bar\psi\psi$-torsion terms in D_μ including their contribution in higher order fermionic terms). We used the following notation

$$\begin{aligned} &F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad D_\mu\begin{pmatrix}\psi_\nu\\ \phi_\nu\end{pmatrix} = \mathscr{D}_\mu\begin{pmatrix}\psi_\nu\\ \phi_\nu\end{pmatrix} \mp \tfrac{3}{4}i\gamma_5A_\mu\begin{pmatrix}\psi_\nu\\ \phi_\nu\end{pmatrix} \\ &\mathscr{D}_\mu = \partial_\mu + \tfrac{1}{2}\omega^{ab}_\mu(e)\,\sigma^{ab}, \qquad R_{\lambda\mu\nu\rho} = R_{\lambda\mu\nu\rho}(\omega(e))\,. \end{aligned} \tag{2.36}$$

The linearized form of (2.35) coincides with the action (1.52) for the "pure" spin 2 multiplet discussed in section 1.3. We also recognize the $\bar\psi R\mathscr{D}\psi$-terms in (2.35) as already found in section 1.12 (eq. (1.20)) from the condition of Weyl invariance for the "pure" spin 3/2 action on the gravitational background.

Now let us write down the non-trivial transformation laws which form the final closed invariance algebra (X, M, D, Q, S, U) of the action (2.35), realized on the independent "physical" fields e^a_μ, ψ_μ and A_μ. To this end, we have simply to substitute the expressions for the dependent fields (2.29)–(2.31) (with $b_\mu = 0$) into (2.10), (2.24)

$$\begin{aligned} \delta e^a_\mu &= \xi^\lambda\,\partial_\lambda e^a_\mu + e^a_\lambda\,\partial_\mu\xi^\lambda + \varepsilon^{ab}_M e^b_\mu - \lambda e^a_\mu - \tfrac{1}{2}\bar\psi_\mu\gamma^a\varepsilon\,, \\ \delta\psi_\mu &= \xi^\lambda\,\partial_\lambda\psi_\mu + \psi_\lambda\,\partial_\mu\xi^\lambda + \tfrac{1}{2}\varepsilon^{ab}_M\sigma^{ab}\psi_\mu - \tfrac{1}{2}\lambda\psi_\mu + D_\mu\varepsilon - \gamma_\mu\eta + \tfrac{3}{4}i\gamma_5\alpha\psi_\mu\,, \\ \delta A_\mu &= \xi^\lambda\,\partial_\lambda A_\mu + A_\lambda\,\partial_\mu\xi^\lambda - i\bar\phi_\mu\gamma_5\varepsilon + i\bar\psi_\mu\gamma_5\eta + \partial_\mu\alpha\,. \end{aligned} \tag{2.37}$$

Here we put $\lambda = \varepsilon_D$, $\eta = \varepsilon_S$, $\varepsilon = \varepsilon_Q$ and $\alpha = \varepsilon_U$ to make contact with our previous notation (cf. (1.2), (1.4), (1.5)). D_μ and ϕ_μ are given by (2.25) and (2.30) with $b_\mu = 0$. A simple check of the closure of the algebra (2.37) is provided by the off-shell count of the degrees of freedom, already discussed in section 1.3. It was shown there that the "pure" spin multiplet (2, $\frac{3}{2}$, 1), with maximal gauge freedom, is an "off-shell multiplet" (cf. (1.51)). Repeating the argument (using (1.37) and (2.38))

$$\begin{aligned} &n(e^a_\mu) = 16 - 4(\xi^\mu) - 6(\varepsilon^{ab}_M) - 1(\lambda) = 5\,, \\ &n(A_\mu) = 4 - 1(\alpha) = 3\,, \\ &n(\psi_\mu) = -[16 - 4(\varepsilon^\alpha) - 4(\eta^\alpha)] = -8 \end{aligned} \tag{2.38}$$

we get a zero total number for the off-shell degrees of freedom $(5 + 3 - 8 = 0)$.

It is obvious from (2.37) that the Q-supersymmetry transformation laws for ψ_μ and A_μ are essentially *non-linear* (D_μ contains $\omega(e,\psi) \sim e^{-1}\,\partial e + \bar{\psi}\gamma\psi + \cdots$, and $\phi_\mu \sim e^{-1}\,\partial\psi + A\psi + \cdots$). The origin of this non-linearity (which is present also in the Poincaré supergravity) can be traced to the imposition of the constraints (2.28) which explicitly broke the initially linear P and Q symmetries. As a result, the algebra (2.37) is of a "soft" type, i.e. has structure *functions* which essentially depend on (bosonic as well as fermionic) gauge fields. This raises a non-trivial problem of maintaining the Q-supersymmetry of the quantum effective action (cf. [54]). This problem can be made more tractable and solved in the superspace approach (cf. [146, 110]).

As for the S-supersymmetry, it is realized linearly and thus can be used (in conjunction with Weyl D-symmetry) to fix various terms in the action (2.35) (avoiding a somewhat cumbersome procedure of its straightforward construction starting with (2.32), (2.33)). For example, this was the way we found the form of the $\bar{\psi}F\mathscr{D}\psi$ terms in (2.35). It is essentially simpler than that originally given in [248]. We conclude that though the construction of the complete invariant action (2.35) is rather intricate, the presence of a number of linearly realized symmetries (X, M, D, S, U) makes its final structure essentially transparent.

Here it may be useful to add the following remark. In checking the D and S invariances of (2.35), one has to take into account that, though the constraints (2.28) are D and S-covariant and thus that the D and S transformation laws of w_μ^{ab} and ϕ_μ (and f_μ^a) formally do not change, this is only true as long as we keep their dependence on b_μ. Having dropped b_μ in (2.35) we must therefore reestablish the transformation laws of w_μ^{ab} and ϕ_μ, starting directly from their expressions (2.29), (2.30) in terms of the independent fields e_μ^a, ψ_μ, A_μ (with $b_\mu = 0$).

We end this section with several additional observations concerning the CSG action (2.35). In view of the vanishing of the Weyl action (2.20) on the Einstein equations $R_{\mu\nu} = 0$, one could expect that the supersymmetric extension of the Weyl action (2.35) would also vanish on the supersymmetric extension of the Einstein equations, i.e. on the field equations of $N = 1$ Poincaré supergravity

$$
\begin{aligned}
&R_{\mu\nu}(\omega(e,\psi)) - \tfrac{1}{2}\bar{\mathscr{R}}_{\lambda\nu}(Q)\,\gamma_\mu\psi^\lambda = 0\,;\\
&A_\mu = 0; \qquad \phi_\mu = 0, \text{ i.e.,}\\
&\varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu D_\rho\psi_\sigma = 0\,.
\end{aligned}
\tag{2.39}
$$

This in fact appears to be true [172].

The action of conformal supergravity can also be derived [29] following the approach of "non-linear realization" [149, 25, 193]. Here one starts with the gauge theory of the full (super) conformal group, adds the constraints and assumes that the linearly realized subgroup F is generated only by M, D and U. If the action is required to be invariant only under F, one finds that it is not unique and depends on three free parameters in the case of gravity and on more than twenty parameters in the case of supergravity. After the additional requirement of K (and S and Q) invariance, one is uniquely led to the Weyl action (2.18) (to the CSG action (2.32)–(2.34)). This can be considered as a proof of the uniqueness of the action of conformal supergravity under the assumption that the fields w_μ^{ab}, f_μ^a and ϕ_μ are non-propagating ones. As for the more general "non-linear" actions invariant only under F, they correspond to reducible (and very probably inconsistent) theories. The uniqueness of $N = 1$ conformal supergravity is suggested also by the "pure" spin multiplet considerations of section 1.3.

The invariance of the vierbein under S-supersymmetry (see (2.37)) gives the possibility of truncating the theory (2.35) by putting $e_\mu^a = \delta_\mu^a$ without loosing the S-invariance of the action [172]. As a result, we get locally S-supersymmetrical and a U(1)-invariant theory in flat space (cf. (2.35))

$$\mathscr{L} \sim -F_{\mu\nu}{}^2 + \bar{\psi}\,\mathscr{D}^3(A)\,\psi + \bar{\psi}F\,\mathscr{D}\psi + (\bar{\psi}\psi)^4 + \cdots . \tag{2.40}$$

However, this theory suffers from a higher spin inconsistency. In fact, the free gravitino action $\bar{\psi}P_{3/2}\Box\partial\!\!\!/\psi$ is invariant under Q-supersymmetry $\delta\psi_\mu = \partial_\mu\varepsilon$. However this invariance is absent in the interacting theory (2.40). Inconsistency manifests itself also in a disbalance between the Fermi and Bose ("on" and "off" shell) degrees of freedom (in contrast to Q-supersymmetry S-supersymmetry alone does not guarantee the vanishing of the total number of degrees of freedom).

2.3. $N = 1$ superconformal actions for matter multiplets

A complete description of $N = 1$ superconformal multiplet (or tensor) calculus [86, 235] can be found in a number of reviews [254, 50, 183, 56]. Here we shall mainly concentrate on the invariant actions of "physical" multiplets (chiral, vector, tensor) in an "external field" of $N = 1$ CSG.

Let Φ transform according to a representation of the algebra of the internal gauge group G (cf. (2.1), (2.2))

$$\delta_G\Phi = \varepsilon^A t_A \Phi . \tag{2.41}$$

The corresponding covariant derivative is then (cf. (2.2))

$$\hat{D}_\mu\Phi = \partial_\mu\Phi - W_\mu^A t_A\Phi . \tag{2.42}$$

In the case of a gauge theory of a space–time symmetry group we are interested not in representations of the original (rigid) symmetry algebra but in the "*deformed*" algebra in which "internal" translations are replaced by "covariant general coordinate transformations" (i.e. by combinations of δ_X and δ_H). Yet a natural starting point is a study of the representations of the original algebra.

The simplest representation of the rigid superconformal algebra (1.59), (1.62), (1.63) is provided by a chiral multiplet (ϕ, χ, F) (A and F are complex scalar fields and χ is a Majorana spinor), that can be defined by the condition that its lowest dimensional component ϕ transforms under Q-supersymmetry into a chiral (e.g. left) projection of χ

$$\delta\phi = (w\lambda + \mathrm{i}c\alpha)\phi + \bar{\varepsilon}_L\chi_L , \qquad \chi_L = \frac{1+\gamma_5}{2}\chi \tag{2.43}$$

(we use the notation of (2.37), i.e. $\lambda = \varepsilon_D$, $\alpha = \varepsilon_U$, $\varepsilon = \varepsilon_Q$ etc.). The algebra fixes the chiral weight to be $c = w/2$ and determines the transformation laws of χ and F. If we add the condition that the invariant action should be of the standard type

$$\mathscr{L}_S = \tfrac{1}{2}(\phi^*\Box\phi - \tfrac{1}{2}\bar{\chi}\partial\!\!\!/\chi + F^*F) \tag{2.44}$$

then the Weyl weight is also fixed: $w = +1$ (cf. section 1.2). The superconformal invariant action for the multiplet with $w = 0$ (i.e. with $w(\phi) = 0$, $w(\psi) = \frac{1}{2}$, $w(F) = 1$) is based on the Lagrangian with higher derivatives [90]

$$\mathscr{L}'_{\rm S} = \phi^*\Box^2\phi - \tfrac{1}{2}\bar{\chi}\Box\not{\partial}\chi + F^*\Box F. \tag{2.45}$$

To construct locally superconformal invariant extensions of (2.44) and (2.45) we first have to check that the chiral multiplet provides also a realization of the "deformed" local superconformal algebra

$$[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] = \tfrac{1}{2}(\bar{\varepsilon}_1\gamma^a\varepsilon_2)\hat{D}_a, \tag{2.46}$$

all the other commutators are identical to those present in SU(2, 2|1). Here $\hat{D}_a = e^{\mu}_{a}\hat{D}_\mu$ is the superconformally covariant derivative (see (2.11)), which now plays the role of the generator of translations. To find the transformation rules of the chiral multiplet under (2.46) one simply has to substitute the ordinary derivatives for the covariant ones ($\hat{D}$) in the "rigid" transformation rules

$$\begin{aligned}
\delta\phi &= w\lambda\phi + \tfrac{1}{2}\mathrm{i}w\alpha\phi + \bar{\varepsilon}_{\rm L}\chi_{\rm L}, \\
\delta\chi_{\rm L} &= (w+\tfrac{1}{2})\lambda\chi_{\rm L} + \tfrac{1}{2}\mathrm{i}(w-\tfrac{3}{2})\alpha\chi_{\rm L} + (\hat{\not{D}}\phi\varepsilon_{\rm R} + F\varepsilon_{\rm L} + 2w\phi\bar{\eta}_{\rm L}), \\
\delta F &= (w+1)\lambda F + \tfrac{1}{2}\mathrm{i}(w-3)\alpha F + \bar{\varepsilon}_{\rm R}\hat{\not{D}}\chi_{\rm L} - 2(w-1)\bar{\eta}_{\rm L}\chi_{\rm L}.
\end{aligned} \tag{2.47}$$

The explicit expressions for $\hat{D}$ follow from (2.41), (2.42), (2.21) and (2.23)

$$\begin{aligned}
\hat{D}_\mu\phi &= \partial_\mu\phi - wb_\mu\phi - \tfrac{1}{2}\mathrm{i}wA_\mu\phi - \bar{\psi}_{\mu\rm L}\chi_{\rm L}, \\
\hat{D}_\mu\chi_{\rm L} &= \partial_\mu\chi_{\rm L} - \tfrac{1}{2}w^{ab}_\mu(e,\psi,b)\,\sigma^{ab}\chi_{\rm L} - (w+\tfrac{1}{2})b_\mu\chi_{\rm L} \\
&\quad - \tfrac{1}{2}\mathrm{i}(w-\tfrac{3}{2})A_\mu\chi_{\rm L} - (\hat{\not{D}}\phi)\psi_{\mu\rm R} - F\psi_{\mu\rm L} - 2w\phi\phi_{\mu\rm L}(\psi, A, b, e).
\end{aligned} \tag{2.48}$$

Specializing first to the case of $w=1$, we can construct a new chiral multiplet, with F^* as the lowest dimensional component, which is called a kinetic multiplet (F^*, $\hat{\not{D}}\chi_{\rm R}$, $\hat{D}^2\phi^*$)

$$\hat{D}^2\phi = \hat{D}_a\hat{D}^a\phi = e^{\mu}_{a}[(\partial_\mu - 2b_\mu + \tfrac{1}{2}\mathrm{i}A_\mu)\delta^{ab} - w^{ab}_\mu]\,\hat{D}_b\phi + 2f^{\mu}_{\mu}\phi - \bar{\psi}_{\mu\rm L}\,\hat{D}_\mu\chi_{\rm L} + \bar{\phi}_{\mu\rm R}\gamma_\mu\chi_{\rm L}. \tag{2.49}$$

Employing the multiplication rule for chiral multiplets and the invariant density formula, we find the Lagrangian, corresponding to the locally superconformal invariant extension of (2.44) [235, 169]

$$e^{-1}\mathscr{L}_{\rm S} = \tfrac{1}{2}(\phi^*\,\hat{D}^2\phi - \tfrac{1}{2}\bar{\chi}_{\rm L}\,\overleftrightarrow{\hat{\not{D}}}\chi_{\rm R} + F^*F) + \tfrac{1}{2}[\bar{\psi}_{\mu\rm R}\gamma_\mu(\phi\,\hat{\not{D}}\chi_{\rm R} + F^*\chi_{\rm L}) + 2\bar{\psi}_{\mu\rm R}\sigma_{\mu\nu}\psi_{\nu\rm R}\phi F^* + \text{c.c.}]. \tag{2.50}$$

Observing that all fields except b_μ are inert under K, we conclude that it cancels in (2.50). Substituting in (2.50) the expressions for ϕ_μ (2.30) and f^a_μ (2.31) we find a more explicit form of the Lagrangian (note that terms linear in F mutually cancel)

$$\begin{aligned}
e^{-1}\mathscr{L}_{\rm S} &= -\tfrac{1}{2}D_\mu\phi^*\,D_\mu\phi - \tfrac{1}{4}\bar{\chi}\,\not{D}\chi + \tfrac{1}{2}F^*F - \tfrac{1}{12}\phi^*\phi[R(\omega(e,\psi)) - \bar{\psi}_\mu R_\mu(e,\psi)] \\
&\quad + (\bar{\psi}\chi\,\partial\phi + \bar{\chi}\,D\psi\phi + \bar{\psi}\psi\phi^*\,\partial\phi + \bar{\chi}\psi\bar{\psi}\psi\phi + \bar{\chi}\chi\bar{\psi}\psi\text{-terms}), \\
R^\mu &\equiv e^{-1}\varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu\mathscr{D}_\rho(\omega(e,\psi))\,\psi_\sigma,
\end{aligned} \tag{2.51}$$

where

$$\begin{aligned}
D_\mu\phi &= (\partial_\mu - \tfrac{1}{2}\mathrm{i}A_\mu)\phi, \\
D_\mu\chi &= (\partial_\mu + \tfrac{1}{2}\sigma^{ab}\omega^{ab}_\mu(e) + \tfrac{1}{4}\mathrm{i}A_\mu\gamma_5)\,\chi.
\end{aligned}$$

This result is hardly surprizing: the "$R/6$"-term is necessary for the Weyl invariance of the scalar Lagrangian, while the standard Poincaré supergravity gravitino term is the Q-supersymmetric extension of the curvature scalar. Note that because of the closure of the superconformal algebra, the action (2.50) corresponds to an independent superconformal invariant to which we may (or may not) add the $N=1$ CSG action (2.35). If the conformal supergravity action is not added we are free to "gauge away" the chiral multiplet degrees of freedom by fixing the following gauges for "algebraic" D, U and S gauge symmetries

$$\begin{aligned} D:&\quad \operatorname{Re}\phi = a = \text{const.}\,,\\ U:&\quad \operatorname{Im}\phi = 0\,,\\ S:&\quad \chi = 0 \end{aligned} \tag{2.52}$$

(one may also assume that the trivial K-invariance of (2.50) is fixed by the gauge: $b_\mu = 0$). Then (2.51) reduces (up to field redefinitions) to a Q-invariant quantity, proportional to the Lagrangian of $N=1$ Poincaré supergravity, in the formulation with the minimal set of auxiliary fields $(A, F \sim S - \mathrm{i}P)$ [85, 234]. Instead of fixing the gauges we could make the field dependent D, S and U transformations, absorbing ϕ and χ into e^a_μ, ψ_μ and A_μ (e.g., $e^a_\mu \to (\phi^*\phi)^{1/2} e^a_\mu$ etc.). However, the *sign* of the Lagrangian ($e^{-1}\mathscr{L}_\mathrm{S} = -(a^2/12)R + \cdots$) turns out to be unphysical, i.e. is opposite to that of the Poincaré supergravity Lagrangian. The reason is that the sign of the action of the chiral multiplet was taken to be physical, while the conformal mode in the Einstein action is known to be a ghost (see e.g. (1.22), (1.23)). We conclude that it is a "ghost-like" superconformal chiral multiplet action that is gauge equivalent to the Poincaré supergravity action.

The above gauges (2.52) can in principle be used also in the case where the total action includes the conformal supergravity action (2.35) in addition to the chiral multiplet action (2.50).

Then the "pure gauge" part of the superconformal gauge multiplet (i.e. "scalar" graviton, $\gamma_\mu\psi_\mu$ and longitudinal part of A_μ) will reappear as a (ghost) chiral submultiplet of the Poincaré supergravity multiplet. The situation is clearly analogous to that in the case of the gauge invariant description of a vector gauge field mass term with the help of an additional set of scalar fields: if one gauges away the scalar fields they "reappear" after the decomposition of the vector in its "transverse" and "longitudinal" parts. We shall return to the discussion of the physical implications of this compensating mechanism later (see also [169]). For its important "technical" applications in constructing various auxiliary field versions of Poincaré supergravity and of its coupling to matter see, e.g., refs. [183, 80, 256, 50].

The locally superconformal invariant extension of the action for the $w=0$ chiral multiplet (2.45) can be found using the technique of ref. [184]. The resulting Lagrangian has the following structure

$$\begin{aligned} e^{-1}\mathscr{L}'_\mathrm{S} &= \mathscr{D}^2\phi^*\,\mathscr{D}^2\phi - 2(R_{\mu\nu} - \tfrac{1}{3}g_{\mu\nu}R)\,\mathscr{D}_\mu\phi^*\,\mathscr{D}_\nu\phi - \tfrac{1}{2}\bar\chi[\not{D}^3 + (R_{\mu\nu} - \tfrac{1}{6}g_{\mu\nu}R)\gamma_\mu D_\nu]\chi \\ &\quad - \tfrac{3}{4}\mathrm{i}\,\bar\chi\gamma_\mu\,D_\nu\chi\,F_{\mu\nu}(A) + F^*[D_\mu D_\mu - \tfrac{1}{6}(R - \bar\psi_\mu R_\mu)]F + (\text{other terms involving the gravitino})\,, \\ \mathscr{D}_\mu &= \partial_\mu + \omega_\mu(e),\ D_\mu\chi = \mathscr{D}_\mu\chi + \tfrac{3}{4}\mathrm{i}\gamma_5 A_\mu\chi,\ D_\mu F = (\partial_\mu + \tfrac{3}{2}\mathrm{i}A_\mu)F\,. \end{aligned} \tag{2.53}$$

The natural gauges for D and U symmetries here are given by: $\operatorname{Re}F = a = \text{const.}$, $\operatorname{Im}F = 0$ (K-invariance is trivial after cancellation of the b_μ-dependent terms). At the same time, there is no obvious way of fixing S-invariance (χ is invariant under S, see (2.47)). The $F = a$-gauge alone is sufficient in making contact with the Poincaré supergravity, which is however not decoupled from χ (and ϕ). To get the physical sign of the Einstein term again we have to take F to be a ghost, i.e. to change the sign of (2.53).

Our next example of a rigid superconformal multiplet is an Abelian gauge vector multiplet (B_μ, λ, $\mathscr{D}$) (generalization to the non-Abelian case is trivial). As it was noted in section 1.3, this multiplet (consisting of a gauge vector, a Majorana spinor and a real auxiliary scalar) provides the simplest example of a "pure" spin $s=1$ conformal multiplet with the flat space Lagrangian (1.52) (invariant under rigid Q-supersymmetry (1.53)). This multiplet not only forms a representation of the rigid but also of the local superconformal algebra (2.46). The corresponding D, U, S and Q transformation laws coincide with the flat space ones with the derivatives substituted for the superconformally covariant derivatives (note that all fields are inert under S)

$$\begin{aligned}
\delta B_\mu &= \tfrac{1}{2}\bar{\varepsilon}\gamma_\mu\lambda\,,\\
\delta\lambda &= \tfrac{3}{2}\varepsilon_D\lambda - \tfrac{3}{4}\mathrm{i}\gamma_5\alpha\lambda - \tfrac{1}{2}\sigma_{\mu\nu}\hat{F}_{\mu\nu}(B)\,\varepsilon - \tfrac{1}{2}\mathrm{i}\gamma_5\varepsilon\mathscr{D}\,,\\
\delta\mathscr{D} &= 2\varepsilon_D\mathscr{D} - \tfrac{1}{2}\mathrm{i}\bar{\varepsilon}\gamma_5\hat{\not{D}}\lambda\,.
\end{aligned}\tag{2.54}$$

The supercovariantizations in $\hat{F}_{\mu\nu}$ and $\hat{D}$ are found using the general rule which is a straightforward generalization of (2.21)

$$\begin{aligned}
\hat{F}_{\mu\nu} &= F_{\mu\nu} - \bar{\psi}_{[\mu}\gamma_{\nu]}\lambda, \qquad F_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu\,,\\
\hat{D}_\mu\lambda &= \mathscr{D}_\mu\lambda - \tfrac{3}{2}b_\mu\lambda + \mathrm{i}\tfrac{3}{4}A_\mu\gamma_5\lambda + \tfrac{1}{2}(\sigma_{\mu\nu}\hat{F}_{\mu\nu} + \mathrm{i}\gamma_5\mathscr{D})\psi_\mu\,.
\end{aligned}\tag{2.55}$$

Constructing the chiral multiplet with $\bar{\lambda}_{\mathrm{L}}\lambda_{\mathrm{L}}$ as a lowest dimensional component and using the F-term density formula we find the locally superconformal invariant action. The corresponding Lagrangian does not contain b_μ in view of K-invariance and has the following explicit form [87, 81]

$$\begin{aligned}
e^{-1}\mathscr{L}_{\mathrm{V}} = &-\tfrac{1}{4}F^2_{\mu\nu}(B) - \tfrac{1}{2}\bar{\lambda}\not{D}\lambda + \tfrac{1}{2}\mathscr{D}^2 - \tfrac{1}{2}\bar{\psi}_\mu(\sigma_{\lambda\rho}F_{\lambda\rho})\gamma_\mu\lambda + \tfrac{1}{4}(\bar{\psi}_\rho\sigma_{\mu\nu}\gamma_\rho\lambda)(\bar{\lambda}\gamma_\mu\psi_\nu)\\
&+ \tfrac{1}{32}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\lambda}\gamma_5\gamma_\mu\lambda\bar{\psi}_\nu\gamma_\rho\psi_\sigma, \qquad D_\mu = \mathscr{D}_\mu + \mathrm{i}\tfrac{3}{4}\gamma_5 A_\mu
\end{aligned}\tag{2.56}$$

where we have used the expressions for $\hat{F}_{\mu\nu}$ and $\hat{D}_\mu$. In order to obtain the Lagrangian for the $N=1$ super Yang–Mills theory interacting with the fields of $N=1$ conformal supergravity, we have only to make the trivial substitutions:

$$\begin{aligned}
F_{\mu\nu} &\to F_{\mu\ \nu}^{\ i} = 2\partial_{[\mu}B^i_{\nu]} - f^i_{\ jk}B^j_{\ \mu}B^k_{\ \nu}\,,\\
D_\mu\lambda &\to D_\mu\lambda^i - f^i_{\ jk}B^j_{\ \mu}\lambda^k\,, \qquad \mathscr{D} \to \mathscr{D}^i
\end{aligned}\tag{2.57}$$

and to sum over the indices $(i, j, \ldots)$ of the adjoint representation of an internal gauge group. It is possible also to construct the Lagrangian for several interacting chiral and vector multiplets (see e.g. [182, 256]). Its structure is essentially fixed by the results obtained in the limiting cases (2.50), (2.56), by D and S gauge invariances and by the absence of dimensional coupling constants. Thus the most general renormalizable action possessing rigid superconformal invariance in flat space (namely, super-Yang–Mills plus massless chiral multiplets with the standard Wess–Zumino interaction terms) has a unique $N=1$ locally superconformal extension. For example, to get the locally supersymmetric action for a massless Wess–Zumino model (with coupling constant g_1) interacting with a super-Yang–Mills system (with gauge coupling g_2) we have to combine (2.50) and (2.56) (derivatives making covariant with respect to the internal gauge group) and also to add the following terms [235, 81, 43]

$$e^{-1}\Delta\mathscr{L}_1 = g_1\bar{\chi}\hat{\phi}\chi - \tfrac{1}{2}g_1^2(\phi^*\phi)^2 - \tfrac{1}{2}g_1\bar{\psi}_\mu\hat{\phi}^{*2}\gamma_\mu\chi - \tfrac{1}{6}g_1\bar{\psi}_\mu\sigma_{\mu\nu}\hat{\phi}^{*3}\psi_\nu\,, \qquad \hat{\phi} = A + \mathrm{i}\gamma_5 B, \qquad \phi = A + \mathrm{i}B\,, \tag{2.58}$$

$$e^{-1}\Delta\mathscr{L}_2 = 2\mathrm{i}g_2\phi^*\phi\mathscr{D} - 2g_2\bar{\lambda}\hat{\phi}\chi - 2g_2\bar{\psi}_\mu\gamma_\mu\lambda\phi\phi^*\,. \tag{2.59}$$

In (2.58) we assumed that the auxiliary field F had been eliminated. Fixing the gauges (2.52), we see that (2.58) produces a cosmological term plus a De Sitter mass term for the gravitino.

One can also couple the $N=1$ conformal supergravity to the so-called $N=1$ tensor gauge multiplet [217]. It contains a real scalar field L, a Majorana spinor ζ and an antisymmetric gauge tensor $E_{\mu\nu}$. The free flat space Lagrangian for this multiplet is simply that of a chiral multiplet with a pseudoscalar rotated into $E_{\mu\nu}$ by a duality transformation[1]

$$\mathscr{L}_\mathrm{T} = -\tfrac{1}{2}(\partial_\mu L)^2 - \tfrac{1}{2}\bar{\zeta}\not{\partial}\zeta + \tfrac{1}{2}E_\mu^2\,, \qquad E^\mu = \tfrac{1}{2}\mathrm{i}e^{-1}\varepsilon^{\mu\nu\lambda\rho}\,\partial_\nu E_{\lambda\rho}\,. \tag{2.60}$$

Thus it is natural to expect that the quantum properties of this multiplet are the same as that of a chiral one, except for topological infinities and the corresponding anomalies (cf. ref. [69]). In view of the gauge nature of $E_{\mu\nu}$ its Weyl weight must be equal to zero. Then $w(\zeta)=\frac{5}{2}$, $w(L)=2$. Thus the naive action (2.60) is not scale invariant. However, there exists a scale-invariant non-linear modification of (2.60) [58]

$$\mathscr{L}'_\mathrm{T} = -\tfrac{1}{2}L^{-1}(\partial_\mu L)^2 - \tfrac{1}{2}L^{-1}\bar{\zeta}\not{\partial}\zeta + \tfrac{1}{2}L^{-1}E_\mu^2 + \tfrac{1}{4}\mathrm{i}L^{-2}\bar{\zeta}\not{E}\gamma_5\zeta - \tfrac{1}{32}L^{-3}(\bar{\zeta}\gamma_5\gamma_\mu\zeta)^2\,. \tag{2.61}$$

It is this Lagrangian that has a locally superconformal extension [58]

$$e^{-1}\mathscr{L}'_\mathrm{T} = 2\phi[\mathscr{D}^2 - \tfrac{1}{6}(R - \bar{\psi}_\mu R_\mu(\psi, A))]\phi - \tfrac{1}{2}(\bar{\zeta}\hat{\not{\mathscr{D}}}\zeta - E_\mu^2)\phi^{-2} + \tfrac{1}{2}\mathrm{i}e^{-1}\varepsilon^{\mu\nu\rho\sigma}A_\mu\,\partial_\nu E_{\rho\sigma} + (\bar{\zeta}E\zeta\phi^{-4} + (\bar{\zeta}\zeta)^2\phi^{-6} + \bar{\psi}\hat{\not{\mathscr{D}}}\zeta + \text{other terms involving gravitino}), \quad \phi \equiv L^{1/2}, \quad w(\phi) = 1\,. \tag{2.62}$$

The corresponding Q and S transformation laws are

$$\delta L = \mathrm{i}\bar{\varepsilon}\gamma_5\zeta, \qquad \delta\zeta = -\mathrm{i}\gamma_5\hat{\not{\mathscr{D}}}L\varepsilon - \not{E}\varepsilon - 2\mathrm{i}\gamma_5 L\eta\,, \qquad \delta E_{\mu\nu} = 2\mathrm{i}\bar{\varepsilon}\gamma_5\sigma_{\mu\nu}\zeta + 2L\bar{\varepsilon}\gamma_{[\mu}\psi_{\nu]}\,. \tag{2.63}$$

Fixing the D and S symmetries with the help of the gauges $L=1$ and $\zeta=0$, one can check [58] that eq. (2.62) then reduces to the action of the "new minimal" version of the $N=1$ Poincaré supergravity [229] (with auxiliary field term being $\sim A_\mu E_\mu + \frac{1}{2}E_\mu^2$). Note also that adding the $N=1$ CSG Lagrangian we

[1] We have also to omit the auxiliary field F, because $(L, \zeta, E_{\mu\nu})$ already constitutes an off-shell multiplet ($n(E_{\mu\nu}) = 6 - (4-1) = 3$).

find that the "mixing" term $A_\mu E_\mu$ in (2.62) produces a gauge invariant "mass term" ($\sim\phi^2$) for the axial vector field A_μ.

The quantum properties of the $N=1$ superconformal multiplets we discussed will be analyzed in section 6.3. Let us also remark that, in addition to the $w=0$ chiral multiplet (2.45), (2.53), there exist other higher derivative $N=1$ superconformal multiplets, e.g. (ψ_μ, $2A_\mu$, $T_{\mu\nu}$, χ) and (Λ, $T_{\mu\nu}$, χ, 2ϕ) (for the definition of the fields see (1.9), (1.10)) which will be used in sections 4.3 and 6.3.

3. $N>1$ conformal supergravities

3.1. Gauge theory of $SU(2,2|N)$

As we have already seen there exists a deep connection between theories in flat space which are invariant under the rigid superconformal group and conformal supergravity (CSG). One can thus attempt to construct a CSG by gauging the superconformal symmetry of flat space theories. Given the existence of the flat space matter multiplets with *extended* supersymmetry ($N=2$ hypermultiplet, $N\le 4$ gauge vector multiplets) it is natural to ask about the corresponding *extended conformal supergravity*. It seems rather obvious that it should contain one graviton (gauging the Poincaré invariance), N gravitino (gauging N-extended supersymmetry) and U(N) gauge vectors (gauging U(N) rigid invariance of matter multiplets) and probably some other lower spin ($s\le 1$) matter fields.

A reasonable first step towards the construction of such a theory is to develop a constrained gauge theory of extended superconformal algebra, an approach that was successful in the $N=1$ case. Starting with the $N=1$ superalgebra (1.59), (1.62), (1.63) and substituting Q for a set of N generators Q^i, we find (using $[Q,K]$) that it is necessary to introduce also N special conformal supersymmetries S^i. To close $\{Q^i, S^{jT}\}$ it appears necessary to also introduce, in addition to the U(1) generator U, the SU(N) generators G^r. The resulting superalgebra [263, 78, 150] thus contains the following subalgebras: SU(2, 2) (conformal), U(1) ($N\ne 4$)[1] and SU(N). The corresponding non-vanishing (anti)commutators include (1.59) and the straightforward generalizations of (1.62) and (1.63), i.e. [150, 83]

$$\{Q^i, Q^{jT}\} = \tfrac12\gamma^a C^{-1}P_a\delta^{ij}\,,$$

$$\{S^i, S^{jT}\} = -\tfrac12\gamma^a C^{-1}K_a\delta^{ij}\,,$$

$$\left[\begin{pmatrix}Q^i\\ S^i\end{pmatrix}, M_{ab}\right] = \sigma_{ab}\begin{pmatrix}Q^i\\ S^i\end{pmatrix},\qquad \left[\begin{pmatrix}Q^i\\ S^i\end{pmatrix}, D\right] = \pm\tfrac12\begin{pmatrix}Q^i\\ S^i\end{pmatrix},$$

$$\left[\begin{pmatrix}Q^i\\ S^i\end{pmatrix}, U\right] = \mp\frac{4-N}{4N}\,\mathrm{i}\gamma_5\begin{pmatrix}Q^i\\ S^i\end{pmatrix},$$

$$\left[\begin{pmatrix}Q^i\\ S^i\end{pmatrix}, G^r\right] = (\tau^r_- \mp \mathrm{i}\gamma_5\tau^r_+)^{ij}\begin{pmatrix}Q^j\\ S^j\end{pmatrix},$$

$$[Q^i, K_a] = -\gamma_a S^i,\qquad [S^i, P_a] = \gamma_a Q^i\,,$$

$$\{Q^i, S^{jT}\} = (\tfrac12 D - \tfrac12\sigma^{ab}M_{ab} - \mathrm{i}\gamma_5 U)C^{-1}\delta^{ij} + 2(\tau^r_- G^r_- - \mathrm{i}\gamma_5\tau^r_+ G^r_+)^{ij}C^{-1}\,. \tag{3.1}$$

[1] For $N=4$ the U(1) generator becomes a central charge which can be omitted from the algebra and thus the corresponding gauge vector is absent. The reason distinguishing this particular value of N can be traced to the number of space–time dimensions (and to the number of spinor components).

Here $r, s, \ldots = 1, \ldots, N^2-1$, τ^r_- and τ^r_+ are the antisymmetric and traceless symmetric real matrices defining the generators of the vector representation of SU(N) (any matrix V of SU(N), satisfying $V^\dagger = -V$, tr $V=0$, can be split in antisymmetric and traceless symmetric real parts, $V = V_- + \mathrm{i}V_+$, $V^{\mathrm{T}}_\pm = \pm V_\pm$, tr $V_\pm = 0$). It is useful to decompose the Majorana spinors $Q^i_{(\mathrm{M})}$ (and $S^i_{(\mathrm{M})}$) on the chiral parts (see appendix A)

$$Q_i = \Pi_+ Q^i_{(\mathrm{M})}, \qquad Q^i = \Pi_- Q^i_{(\mathrm{M})}, \qquad \Pi_\pm \equiv (1\pm\gamma_5)/2\,. \tag{3.2}$$

In view of (3.1) $\Pi_+ Q^i_{(\mathrm{M})}$ transforms according to the *conjugate* ($\bar{N}$) defining representation of SU(N). That is why we have denoted it by $Q_i\,(Q'_i = U_i{}^j Q_j)$. S^i transforms according to the N-representation and thus we put

$$S^i = \Pi_+ S^i_{(\mathrm{M})}, \qquad S_i = \Pi_- S^i_{(\mathrm{M})}, \qquad \overline{S^i} = \overline{S}_i\,. \tag{3.3}$$

Then

$$\begin{aligned}
&\left[\begin{pmatrix} Q^i \\ S^i \end{pmatrix}, G^r\right] = (\tau^r)^i_j \begin{pmatrix} Q^j \\ S^j \end{pmatrix}, \qquad \tau = \tau_- + \mathrm{i}\tau_+ \\
&\left[\begin{pmatrix} Q^i \\ S^i \end{pmatrix}, U\right] = \mathrm{i}\,\frac{4-N}{4N}\begin{pmatrix} Q^i \\ S^i \end{pmatrix}, \qquad \gamma_5 \begin{pmatrix} Q^i \\ S^i \end{pmatrix} = \mp \begin{pmatrix} Q^i \\ S^i \end{pmatrix}, \\
&\{Q^i, S^{\mathrm{T}}_j\} = \Pi_-\{(\tfrac{1}{2}D - \tfrac{1}{2}\sigma^{ab}M_{ab} + \mathrm{i}U)\delta^i_j + 2(\tau^r)^i_j G^r\}C^{-1}\,.
\end{aligned} \tag{3.4}$$

Finally, we can expand $G^r = (\tau^r)^i_j G^j{}_i$ to find[2]

$$\begin{aligned}
&[Q^i, G^n{}_k] = Q^n \delta^i_k - (1/N)\delta^n_k Q^i\,, \\
&\{Q^i, S^{\mathrm{T}}_j\} = \Pi_-\{(\tfrac{1}{2}D - \tfrac{1}{2}\sigma^{ab}M_{ab} + \mathrm{i}U)\delta^i_j - G^i{}_j\}C^{-1}\,, \\
&[G^i{}_j, G^k{}_n] = 2\delta^i_{[n} G^k_{j]}, \qquad G^n{}_n = 0, \qquad (G^i{}_j)^\dagger = -G^i{}_j\,.
\end{aligned} \tag{3.5}$$

Introducing the gauge potentials (cf. (2.23))

$$W_\mu = e^a_\mu P_a + \tfrac{1}{2}w^{ab}_\mu M_{ab} + b_\mu D + f^a_\mu K_a + (\bar{\psi}^i_\mu\, Q_i + \bar{\phi}^i_\mu S_i + \text{c.c.}) + A_\mu U + V_\mu{}^i{}_j G^j{}_i \tag{3.6}$$

where

$$\begin{aligned}
&\gamma_5 \psi^i_\mu = \psi^i_\mu, \qquad \gamma_5 \phi^i_\mu = -\phi^i_\mu, \qquad V_\mu{}^i{}_i = 0\,, \\
&(V_\mu{}^i{}_j)^* \equiv V_{\mu i}{}^j = -V_\mu{}^j{}_i\,,
\end{aligned} \tag{3.7}$$

we have to compute the gauge transformations and the curvatures. The modifications needed with respect to (2.24) are rather obvious. For example (cf. (2.37))

[2] We use the following properties of τ-matrices: $(\tau^r)^n_k(\tau^r)^i_j = -\frac{1}{2}(\delta^n_j\delta^i_k - (1/N)\delta^n_k\delta^i_j)$, $\mathrm{tr}(\tau^r\tau^s) = -\frac{1}{2}\delta^{rs}$, $i, j, \ldots = 1, \ldots, N$.

$$\begin{aligned}
&\delta\psi^i_\mu = D_\mu\varepsilon^i - \gamma_\mu\eta^i + \mathrm{i}\frac{4-N}{4N}\alpha\psi^i_\mu + \beta^i_j\psi^j_\mu\,,\\
&\delta A_\mu = \partial_\mu\alpha + \mathrm{i}(\bar\psi^i_\mu\eta_i + \phi^i_\mu\varepsilon_i - \text{c.c.})\,,\\
&\delta V_\mu{}^i{}_j = D_\mu\beta^i{}_j - [(\bar\psi^i_\mu\eta_j + \bar\phi^i_\mu\varepsilon_j - \text{trace}) - \text{h.c.}]\,,\\
&\gamma_5\varepsilon^i = \varepsilon^i, \qquad \gamma_5\eta^i = -\eta^i\,,
\end{aligned} \tag{3.8}$$

where $\beta^i{}_j$ are the parameters of SU(N) ($\varepsilon = \cdots + \beta^i{}_j G^j{}_i$) and the derivatives D_μ are covariant with respect to the Lorentz, chiral U(N) and dilational groups, e.g. (cf. (2.25))

$$D_\mu\varepsilon^i = \left[\left(\partial_\mu - \tfrac{1}{2}\sigma^{ab}w^{ab}_\mu + \tfrac{1}{2}b_\mu - \mathrm{i}\frac{4-N}{4N}A_\mu\right)\delta^i_j - V^i_j\right]\varepsilon^j\,. \tag{3.9}$$

The curvatures in (2.27) now look like

$$\begin{aligned}
&\overline{\mathcal{R}}{}^i_{\mu\nu}(Q) = 2D_{[\mu}\bar\psi^i_{\nu]} - 2\bar\phi^i_{[\mu}\gamma_{\nu]}\,,\\
&\mathcal{R}_{\mu\nu}(U) = 2\partial_{[\mu}A_{\nu]} + 2\mathrm{i}(\bar\psi^i_{[\mu}\phi_{\nu]i} - \text{c.c.})\,,\\
&\mathcal{R}_{\mu\nu}{}^i{}_j(G) = 2\,\partial_{[\mu}V_{\nu]}{}^i{}_j - 2V_{[\mu}{}^i{}_k V_{\nu]}{}^k{}_j - 2[(\bar\psi^i_{[\mu}\phi_{\nu]j} - \text{trace}) - \text{h.c.}]
\end{aligned} \tag{3.10}$$

(the structure of P, M, D, K-curvatures is unchanged, while in $\mathcal{R}(S)$ we are to use the SU(N) covariant derivative).

Following the $N=1$ pattern we may try to impose constraints which provide the conversion of the internal P-transformations into general coordinate transformations. However, this program does not work for $N>1$ [10, 8]. The constraints break down several of the SU(2, 2|N) invariances and thus deform the original algebra in such a way that it no longer closes. The reason is that the set of (independent) gauge fields alone is not sufficient for the off-shell realization of a local superalgebra of a "conventional" type (cf. (2.46)). Just as in the $N \geq 2$ Poincaré supergravity case we need some additional "matter" fields to form an off-shell multiplet. In any case, to keep the analogy with $N=1$ CSG, we do not want to have w^{ab}_μ, f^a_μ and ϕ_μ as independent fields in the theory. Their elimination can again be achieved by imposing constraints analogous to those used in the $N=1$ theory (2.28)

$$\mathcal{R}^a_{\mu\nu}(P) = 0, \qquad \mathcal{R}^{ab}_{\mu\nu}(M)e^\nu_b + \cdots = 0, \qquad \overline{\mathcal{R}}{}^i_{\mu\nu}(Q)\,\gamma_\nu + \cdots = 0 \tag{3.11}$$

(the dots stand for all modifications needed to make the constraints covariant under S-supersymmetry; note that different choices of algebraic constraints, which eliminate w^{ab}_μ, f^a_μ and ϕ_μ in favor of e^a_μ, ψ_μ, b_μ, V_μ and A_μ are related through field redefinitions). Given the independent fields e^a_μ, ψ_μ, b_μ, V_μ and A_μ it is easy to check that they fail to form an off-shell multiplet (see (1.37, (2.38))

$$\begin{aligned}
&n(e^a_\mu) = 5, \qquad n(A_\mu) = 3, \qquad n(b_\mu) = 4 - 4(\varepsilon_K) = 0\,,\\
&n(\psi^i_\mu) = -8N, \qquad n(V^i_{j\mu}) = 3(N^2-1)\,,\\
&n_{\text{tot}} = \begin{cases} 5 + 3N^2 - 8N = (N-1)(3N-5) > 0, & N \neq 4\,,\\ 2 + 48 - 32 > 0\,, & N = 4\,. \end{cases}
\end{aligned} \tag{3.12}$$

In spite of this fact, we still may try to determine the part of the N-CSG action which follow from the SU(2, 2|N) gauge approach [83]. We start again with the most general parity conserving "geometrical" action (2.32) and add to it the U(1) Maxwell (2.33) and also SU(N) Yang–Mills ($\alpha_5 \mathcal{R}^i{}_{j\mu\nu}(V)\,\mathcal{R}^j{}_{i\mu\nu}(V)$) terms. Fixing arbitrary coefficients using the conditions of M, D, K, S and U(1) invariance, we now find instead of (2.34) [83]

$$\alpha_2 = -8\alpha_1 = 2\mathrm{i}\alpha_3 = N\alpha_4 = -\alpha_5 \equiv -1/\alpha^2\,. \tag{3.13}$$

Eliminating the dependent fields we obtain the gauge part of the CSG Lagrangian [83] (cf. (2.35))

$$e^{-1}\mathcal{L}^{(N)}_{\mathrm{CSG}} = \frac{1}{\alpha^2}\{\tfrac{1}{2}C_{\lambda\mu\nu\rho}{}^2 - \frac{4-N}{4N}F_{\mu\nu}{}^2(A) + F^i_{j\mu\nu}(V)\,F^j_{i\mu\nu}(V) - 4e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\phi}_{i\mu}\gamma_\nu \overleftrightarrow{D}_\rho \phi^i_\sigma\} + \cdots \tag{3.14}$$

where the expression for $\phi_\mu(\psi)$ is given in (2.30), D_ν is Lorentz and U(N) covariant, $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ and the Yang–Mills term can also be rewritten as

$$F^i_{j\mu\nu}F^j_{i\mu\nu} = -\tfrac{1}{2}(F^r{}_{\mu\nu})^2\,, \qquad r = 1,\ldots,N^2-1\,,$$
$$V_\mu{}^i{}_j = V_\mu{}^r(\tau^r)^i{}_j\,. \tag{3.15}$$

Again b_μ drops from the action and thus the physical gauge fields are the graviton, N gravitinos and U(1) ($N \neq 4$) and SU(N) gauge vectors. Note that the overall sign in (3.14) is chosen so that the SU(N) gauge fields are *physical* (non-ghost) (recall that we work in Minkowski space with the signature ++++ and vectors with an imaginary time component). Thus the U(1) gauge field becomes a ghost for $N>4$ [83], a circumstance, sometimes considered as an argument against the existence of $N>4$ CSG's [8]. It should be clear of course that the first two terms in (3.14) contain higher (fourth and third) derivatives and thus already spoil perturbative unitarity (for all values of N). Therefore it does not seem justified to be afraid of "ordinary" ghosts in a higher derivative action like (3.14). A mechanism that may cure the unitarity problem should take care of all kinds of ghosts (including multipole and "ordinary" ones).

3.2. Field content of $N \leq 4$ conformal supergravities

In this section we are going to determine the spectra of $N \leq 4$ conformal supergravities following the method of refs. [10, 8][1] (see also [49]). This method is based on the construction of a multiplet of currents for flat space matter theories which are invariant under the rigid superconformal group SU(2, 2|N) and considering it as a source for a multiplet of fields of CSG's. $N \leq 4$ seems to be a natural limitation for this method because ordinary matter multiplets (with spins $s \leq 1$ and standard number of derivatives) do not exist for $N>4$.

Let $\mathcal{L}_0(\Phi)$ be a free matter Lagrangian in flat space. Then the coupling to an external gauge field W^A_μ (3.6) (to first order in W_μ) is described by

$$\mathcal{L} = \mathcal{L}_0 + J^A_\mu(\Phi)\, W^A_\mu\,. \tag{3.16}$$

Supposing that the action is invariant under the linearized SU(2, 2|N)-transformations of W_μ we can

[1] The field representation of $N=2$ CSG was first determined [59] starting with the auxiliary field formulation of $N=2$ Poincaré supergravity [113, 59]. An alternative method is based on the construction of superfields subject to some "reality" constraints [218, 157, 159].

deduce the corresponding properties of the currents J^A_μ computed in terms of the fields which satisfy the free field equations

$$\int \mathrm{d}^4x \left\{\delta\Phi \frac{\delta\mathscr{L}_0}{\delta\Phi} + J^A_\mu(\Phi)\,\delta W^A_\mu\right\} = 0 \Rightarrow \qquad (3.17)$$

$$\partial_\mu J^a_{\mu(P)} = 0, \qquad \partial_\mu J^{ab}_{\mu(M)} + J_{[ab](P)} = 0, \qquad \text{etc.}$$

(we assume that $e^a_\mu = \delta^a_\mu + h^a_\mu$). Next we have to substitute the "dependent" fields w^{ab}_μ, f^a_μ and ϕ_μ in (3.16) for the linearized solutions of the constraints (3.11). As a result we find the "improved" currents ($J^{\text{imp}}_{(P)} \equiv \theta$, $J^{\text{imp}}_{(Q)} \equiv S$, $J_{(G)} = j$) which couple directly to h^a_μ, ψ_μ, A_μ and V_μ and thus satisfy the conditions

$$\begin{aligned} &\text{X: } \partial_\mu\theta_{\mu\nu} = 0, \qquad \text{M: } \theta_{\mu\nu} = \theta_{[\mu\nu]}, \qquad \text{D: } \theta^\mu_\mu = 0\,, \\ &\text{Q: } \partial_\mu S^i_\mu = 0, \qquad \text{S: } \gamma_\mu S^i_\mu = 0, \qquad \text{U}(N)\text{: } \partial_\mu j_\mu = 0 \end{aligned} \qquad (3.18)$$

(as always we put $b_\mu = 0$ as a K-gauge). These currents can be represented as bilinear combinations of Φ (it is sufficient to omit self-interaction terms in $\mathscr{L}_0$). As it is clear from a counting argument analogous to the one given in (3.12) θ, S, j do not yet constitute an off-shell multiplet under rigid Q-supersymmetry. Hence in order to have a complete supermultiplet we have to add also other bilinear combinations of Φ which appear in the Q-variations of the original currents. After a finite number of steps we find an *off-shell* multiplet of currents (in spite of the fact that the matter fields are taken on shell) [49]. Finally we can determine the CSG field representation using the requirement that each current is to couple to its own gauge field. The known transformation rules for the currents help to establish (reversing the argument in (3.17)) the transformation rules for the newly introduced "matter" fields of a conformal supergravity multiplet.

In order to find the largest CSG multiplet we have to start with the largest possible matter multiplet, namely the $N = 4$ supersymmetric gauge multiplet [140, 21]. It is sufficient to consider the Abelian case and to ignore the problem of auxiliary fields (matter fields will be taken on-shell). The corresponding (free) Lagrangian is given by [140, 21]

$$\mathscr{L}_0 = -\tfrac{1}{4}F^2_{\mu\nu}(B) - \tfrac{1}{2}\bar{\lambda}_i \overleftrightarrow{\not{\partial}} \lambda^i - \tfrac{1}{2}\partial_\mu\phi^{ij}\,\partial_\mu\phi_{ij}, \qquad i, j = 1, \ldots, 4 \qquad (3.19)$$

where B_μ is an Abelian gauge vector, λ^i are four Majorana spinors (with chiral projections $\lambda^i = \gamma_5\lambda^i$ and $\lambda_i = -\gamma_5\lambda_i$ transforming as 4 and $\bar{4}$ of SU(4)) and $\phi_{ij} = -\phi_{ji}$ satisfies the restriction of "reality"

$$(\phi_{ij})^* \equiv \phi^{ij} = \tfrac{1}{2}\varepsilon^{ijkl}\phi_{kl}$$

and thus represents six spin 0 fields. The action is invariant under the following Q-supersymmetry transformations

$$\begin{aligned} &\delta B_\mu = \tfrac{1}{2}\bar{\varepsilon}^i\gamma_\mu\lambda_i + \text{c.c.}\,, \\ &\delta\lambda^i = -\tfrac{1}{2}\sigma\cdot F^-\varepsilon^i - \mathrm{i}\not{\partial}\phi^{ij}\,\varepsilon_j\,, \\ &F^\pm_{\mu\nu} \equiv \tfrac{1}{2}(F_{\mu\nu} \pm F^*_{\mu\nu}), \qquad \sigma\cdot F \equiv \sigma_{\mu\nu}F_{\mu\nu}\,, \\ &\delta\phi_{ij} = \mathrm{i}\bar{\varepsilon}_{[i}\lambda_{j]} - \tfrac{1}{2}\mathrm{i}\varepsilon_{ijkl}\bar{\varepsilon}^k\lambda^l\,. \end{aligned} \qquad (3.20)$$

It is straightforward to construct the "improved" gauge currents satisfying (on-shell) the conditions (3.18)

$$\theta_{\mu\nu} = -2F^+_{\mu\rho}F^-_{\nu\rho} - \tfrac{1}{2}\bar\lambda^i\gamma_{(\mu}\overleftrightarrow{\partial}_{\nu)}\lambda_i + \tfrac{1}{2}\delta_{\mu\nu}|\partial_\rho\phi^{ij}|^2 - \partial_\mu\phi_{ij}\,\partial_\nu\phi^{ij} - \tfrac{1}{6}(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)|\phi^{ij}|^2\,,$$

$$S_{\mu i} = -\sigma\cdot F^-\gamma_\mu\lambda_i + 2\mathrm{i}\phi_{ij}\overleftrightarrow{\partial}_\mu\lambda^j + \tfrac{4}{3}\mathrm{i}\sigma_{\mu\lambda}\partial_\lambda(\phi_{ij}\lambda^j)\,, \tag{3.21}$$

$$j_\mu{}^i{}_j = \phi^{ik}\overleftrightarrow{\partial}_\mu\phi_{kj} + \bar\lambda^i\gamma_\mu\lambda_j - \tfrac{1}{4}\delta^i_j\bar\lambda^k\gamma_\mu\lambda_k\,.$$

Employing (3.20) and the equations of motion, it is possible to determine the remaining components of the multiplet of currents [8]

$$x^{ij}{}_k = \tfrac{1}{2}\varepsilon^{ijmn}(\phi_{mn}\lambda_k + \phi_{kn}\lambda_m), \qquad e_{ij} = \bar\lambda_i\lambda_j\,,$$
$$t_{\mu\nu ij} = \tfrac{1}{2}(\bar\lambda^k\sigma_{\mu\nu}\lambda^l + 2\mathrm{i}\phi^{kl}F^-_{\mu\nu})\varepsilon_{ijkl}, \qquad c = (F^-_{\mu\nu})^2\,, \tag{3.22}$$
$$y_i = \sigma\cdot F^-\lambda_i, \qquad d^{ij}{}_{kl} = \phi^{ij}\phi_{kl} - \tfrac{1}{6}\delta^i_{[k}\delta^j_{l]}|\phi|^2\,.$$

Multiplying the currents (3.21), (3.22) by the corresponding "gauge" fields

$$J^A_\mu W^A_\mu = \tfrac{1}{2}\{\theta_{\mu\nu}h_{\mu\nu} + \bar S_{\mu i}\psi^i_\mu + j_\mu{}^i{}_j V^j{}_{i\mu} + \bar\chi_{ij}{}^k x^{ij}{}_k + T^{-ij}_{\mu\nu}t^{\mu\nu}{}_{ij} + E_{ij}e^{ij} + \bar\Lambda_i y^i + \varphi c + D^{ij}{}_{kl}d^{kl}{}_{ij} + \text{c.c.}\}\,, \tag{3.23}$$

we can establish the symmetries and quantum numbers of the fields of the $N = 4$ CSG multiplet:
fermions: ψ^i_μ, $\chi^{ij}{}_k$, Λ^i – Majorana spinors (split into chiral projections)

$$\gamma_5\psi^i_\mu = \psi^i_\mu, \qquad \chi^{[ij]}{}_k = \chi^{ij}{}_k, \qquad \chi^{ij}{}_i = 0, \qquad \chi^{ij}{}_k = \gamma_5\chi^{ij}{}_k, \qquad \gamma_5\Lambda_i = \Lambda_i\,. \tag{3.24}$$

bosons: E_{ij}, φ, $D^{ij}{}_{kl}$ – complex scalars; $T^{-ij}_{\mu\nu}$ – antisymmetric tensor (antiselfdual in μ, ν)

$$E_{ij} = E_{(ij)}, \qquad D^{ij}{}_{kl} = \tfrac{1}{2}\varepsilon^{ijmn}\varepsilon_{klpq}D^{pq}{}_{mn}\,,$$
$$D_{kl}{}^{ij} \equiv (D^{kl}{}_{ij})^* = D^{ij}{}_{kl}, \qquad D^{ij}{}_{kl} = D^{[ij]}{}_{[kl]}, \qquad D^{ij}{}_{kj} = 0\,, \tag{3.25}$$
$$T^{-ij}_{\mu\nu} = T^{-[ij]}_{\mu\nu} = -(e/2)\varepsilon_{\mu\nu\rho\sigma}T^{-ij}_{\rho\sigma}\,.$$

The canonical dimensions (k), Weyl weights (w) and SU(4) representations of fields which follow from (3.21)–(3.23) are presented in table 3.1 (cf. table 1.1 of section 1.2).

Table 3.1.
Fields of $N = 4$ conformal supergravity

	e^a_μ	ψ^i_μ	$V_\mu{}^i{}_j$	$T^{-ij}_{\mu\nu}$	$\chi^{ij}{}_k$	Λ_i	E_{ij}	φ	$D^{ij}{}_{kl}$
k	0	1/2	1	1	3/2	1/2	1	0	2
w	−1	−1/2	0	−1	3/2	1/2	1	0	2
SU(4)	1	4	15	6	20	4	10	1	$20'_{(\text{real})}$

Note that the complex conjugated tensor $T^+_{\mu\nu ij}$ is self-dual and that T^-_{ab}, with two Lorentz indices, has $w = +1$. It is sometimes useful to represent $T^-_{\mu\nu}$ as

$$T^{-\,ij}_{\ \mu\nu} = \tfrac{1}{2}(T_{\mu\nu}{}^{ij} - T_{\mu\nu}{}^{ij\,*}), \qquad T^*_{\mu\nu} = (e/2)\varepsilon_{\mu\nu\lambda\rho}T^{\lambda\rho} \tag{3.26}$$

where $T_{\mu\nu}{}^{ij}$ and $T_{\mu\nu}{}^{ij\,*}$ do not transform under particular representation of SU(4) (SU(4) "rotates" them into each other). Using the values of k and eq. (1.7) we can deduce the number of derivatives ($p = 2 - k$) in the corresponding kinetic terms, which thus coincide with that given in (1.9), (1.10) ($E \sim \phi$, $V_\mu \sim A_\mu$). We also conclude that $D^{ij}{}_{kl}$ is an auxiliary field.

Requiring the invariance of $\int d^4x\, J\cdot W$ in (3.23) under simultaneous transformations of fields and currents, we can establish the linearized Q-supersymmetry transformation rules for the fields of supergravity multiplets. A check that we thus get an *off-shell* multiplet is provided by counting off-shell degrees of freedom (1.37). In view of (2.38) and (3.12) we find

$$\begin{aligned}
&n(e^a_\mu) = 5, \quad n(\psi^i_\mu) = -8\times 4, \quad n(V_\mu{}^i{}_j) = 3\times 15,\\
&n(T^{-\,ij}_{\ \mu\nu}) = 6\times 6, \quad n(\chi^{ij}{}_k) = -4\times 20, \quad n(\Lambda_i) = -4\times 4,\\
&n(E_{ij}) = 2\times 10, \quad n(\varphi) = 2, \quad n(D^{ij}{}_{kl}) = 20, \quad n_{\rm tot} = 128 - 128 = 0\,.
\end{aligned} \tag{3.27}$$

The number of degrees of freedom (128 Bose + 128 Fermi) coincides with the number of states in the smallest (spin 2) massive multiplet of $N = 4$ supersymmetry [52] (as it should be expected for an off-shell multiplet [49], see below). The final form of the linearized Q-transformation laws is in correspondence with the SU(2, 2|N)-transformations of the gauge fields (3.8) [8]

$$\begin{aligned}
&\delta e^a_\mu = \tfrac{1}{2}\bar\varepsilon^i\gamma^a\psi_{\mu i} + \text{c.c.}, \qquad \delta\psi^i_\mu = D_\mu\varepsilon^i - \tfrac{1}{2}\sigma\cdot T^{-ij}\gamma_\mu\varepsilon_j\,,\\
&\delta V_\mu{}^i{}_j = (-\bar\phi^i_\mu\,\varepsilon_j + \tfrac{1}{2}\bar\varepsilon^k\gamma_\mu\chi_{kj}{}^i - \text{trace}) - \text{h.c.}\,,\\
&\delta\varphi = \tfrac{1}{2}\bar\varepsilon^i\Lambda_i, \qquad \delta\Lambda_i = \hat{\not{D}}\varphi\varepsilon_i + \tfrac{1}{2}E_{ij}\varepsilon^j + \tfrac{1}{2}\varepsilon_{ijkl}\sigma\cdot T^{-kl}\varepsilon^j,\\
&\delta E_{ij} = \bar\varepsilon_{(i}\hat{\not{D}}\Lambda_{j)} - \bar\varepsilon^k\chi^{mn}{}_{(i}\varepsilon_{j)kmn}, \qquad \delta D^{ij}{}_{kl} = -(4\bar\varepsilon^{[i}\hat{\not{D}}\chi_{kl}{}^{j]} - \text{trace}) + \text{h.c.}\,,\\
&\delta T^{-ij}_{\mu\nu} = \bar\varepsilon^{[i}\mathcal{R}^{j]}_{\mu\nu}(Q) + \tfrac{1}{2}\bar\varepsilon^k\sigma_{\mu\nu}\chi^{ij}{}_k + \tfrac{1}{4}\varepsilon^{ijkl}\bar\varepsilon_k\not{\partial}\sigma_{\mu\nu}\Lambda_l\,,\\
&\delta\chi^{ij}{}_k = -\tfrac{1}{2}\sigma\cdot\hat{\not{D}}T^{-ij}\varepsilon_k - \sigma\cdot\mathcal{R}^{[i}_k(V)\,\varepsilon^{j]} - \tfrac{1}{4}\varepsilon^{ijlm}\,\hat{\not{D}}E_{kl}\varepsilon_m - \text{trace} + \tfrac{1}{2}D^{ij}{}_{kl}\varepsilon^l\,.
\end{aligned} \tag{3.28}$$

Here $\hat D$ stands for a fully "H-covariant" derivative (cf. (2.21)), e.g. $\hat D_\mu\varphi = \partial_\mu\varphi - \tfrac{1}{2}\bar\psi^i_\mu\,\Lambda_i$. To find the linearized transformation rules under S-supersymmetry one first assumes that the "matter" fields (T, χ, E, Λ, φ, D) are K-inert, computes the commutator $[Q, K]$ and then checks the assumption with the help of $\{S, S^T\}$ (see (1.62)). The result is given by [8]

$$\begin{aligned}
&\delta\psi^i_\mu = -\gamma_\mu\eta^i, \qquad \delta V_\mu{}^i{}_j = -(\bar\psi^i_\mu\eta_j - \text{trace}) - \text{h.c.}\,,\\
&\delta\varphi = 0, \qquad \delta\Lambda = 0, \qquad \delta E_{ij} = 2\bar\eta_{(i}\Lambda_{j)}\,,\\
&\delta T^{-\,ij}_{\ \mu\nu} = -\tfrac{1}{2}\varepsilon^{ijkl}\bar\eta_k\sigma_{\mu\nu}\Lambda_l, \qquad \delta D^{ij}{}_{kl} = 0\,,\\
&\delta\chi^{ij}{}_k = \sigma\cdot T^{ij}\eta_k - \tfrac{1}{2}\varepsilon^{ijlm}E_{kl}\eta_m - \text{trace}\,.
\end{aligned} \tag{3.29}$$

Taking into account the dimensions of the fields and making use of (3.28) to fix the relative normalizations of the fields, it is now straightforward to establish the linearized Lagrangian of $N=4$ conformal supergravity [8] (cf. (3.14))

$$e^{-1}\mathcal{L}^{(4)}_{\rm CSG}=\frac{1}{\alpha^2}\{\tfrac{1}{2}C^2_{\lambda\mu\nu\rho}-4e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\phi}_{\mu i}\gamma_\nu \overleftrightarrow{D}_\rho\phi^i_\sigma+F^i_{j\mu\nu}(V)F^j_{i\mu\nu}(V)+8\,D_\mu T^{-ij}_{\mu\nu}\,D_\rho T^+_{\rho\nu ij}$$
$$-\bar{\chi}^{ij}{}_k\overleftrightarrow{\not{D}}\chi_{ij}{}^k-\tfrac{1}{2}D_\mu E^{ij}\,D_\mu E_{ij}-\tfrac{1}{2}\bar{\Lambda}^i\overleftrightarrow{\not{D}}{}^3\Lambda_i+2\varphi^*\mathcal{D}^4\varphi+D^{ij}{}_{kl}D^{kl}{}_{ij}\}\,. \tag{3.30}$$

Here

$$\phi^i_\mu=\tfrac{1}{3}\gamma^\nu(D_\nu\psi^i_\mu-D_\mu\psi^i_\nu+\tfrac{1}{2}e\varepsilon_{\nu\mu\alpha\beta}D_\alpha\psi^i_\beta)\,,$$
$$\gamma_5\phi^i_\mu=-\phi^i_\mu\,,$$

and we made a trivial covariantization with respect to general coordinate, Lorentz and SU(4) transformations (e.g., $D_\mu T_{\rho\nu}{}^{-ij}=\partial_\mu T_{\rho\nu}{}^{-ij}-\{{}^{\alpha}_{\mu\rho}\}T^{-ij}_{\alpha\nu}-V^i{}_{k\mu}T^{-\,kj}_{\rho\nu}+(i\leftrightarrow j,\ \mu\leftrightarrow\nu)$). Let us note once more that the SU(4) indices are raised and lowered by complex conjugation, e.g. $(E_{ij})^*\equiv E^{ij}$. As it was already stressed in section 1.2, the structure of all terms in (3.30) is uniquely dictated by the known dimensions of the fields and by the condition of scale (Weyl) invariance. Note that the "matter" part of (3.30) is not invariant under linearized S-supersymmetry (3.29). The invariance is supposed to be restored in the full non-linear theory. It is useful to observe also that the overall sign in (3.30) is taken such that the "ordinary" (non higher derivative) fields in the multiplet (V_μ,χ,E) are physical (non-ghost).

It is now easy to demonstrate that (3.30) describes a theory with a zero total number of *on-shell* degrees of freedom (as it should be in the case of an invariance under rigid supersymmetry). The number of on-shell degrees of freedom ν (1.38) was defined in section 1.3, where we found that (see (1.45), (1.49))

$$\nu(e^a_\mu)=6,\qquad \nu(\psi_\mu)=-8,\qquad \nu(V_\mu)=2,\qquad \nu(\chi)=-2 \tag{3.31}$$

for the conformal graviton, conformal gravitino, ordinary gauge vector and ordinary Majorana spinor respectively. It is also obvious that

$$\nu(E)=2,\qquad \nu(\varphi)=4,\qquad \nu(D)=0,\qquad \nu(\Lambda)=3\nu(\chi)=-6 \tag{3.32}$$

for the complex scalars and the higher derivative spinor. To find $\nu(T^-_{\mu\nu})$ we note that (see (1.16) and (3.26))

$$\mathcal{L}_{\rm T}=8\,\partial_\mu T^-_{\mu\nu}\,\partial_\rho T^+_{\rho\nu}=-\partial_\mu T_{\rho\sigma}\,\partial_\mu T_{\rho\sigma}+4\,\partial_\mu T_{\mu\nu}\,\partial_\rho T_{\rho\nu} \tag{3.33}$$

and hence [100, 101]

$$\ln\det(-\Box\delta^\mu_{[\alpha}\delta^\nu_{\beta]}+4\partial_{[\alpha}\delta^{[\nu}_{\beta]}\partial^{\mu]})\equiv\nu(T)\ln\det(-\Box),\qquad \nu(T)=6\,. \tag{3.34}$$

Multiplying the ν's by the numbers of each type of the field in the spectrum (table 3.1) we find the desired result [100]

$$\nu^{(4)}_{\rm tot} = \sum d_n \nu_n = 6-4\times 8+15\times 2+6\times 6-20\times 2+10\times 2-4\times 6+1\times 4+20\times 0 = 96-96=0\,. \tag{3.35}$$

Let us now discuss the truncation of the $N=4$ theory which gives the spectra, the linearized Lagrangians and the transformation laws for all $N<4$ conformal supergravities. As we already know from section 3.1 the $N<4$ theories contain also the U(1) gauge field A_μ. In order to provide a universal description of the "matter" fields of $N\le 4$ CSG's it is useful to substitute the $N=4$ fields of table 3.1 by the dual ones

$$E^{ijk}{}_l = \varepsilon^{ijkn} E_{nl}, \qquad \Lambda^{ijk} = \varepsilon^{ijkl}\Lambda_l, \qquad \varphi^{ijkl} = \varepsilon^{ijkl}\varphi\,, \tag{3.36}$$

which satisfy

$$E^{[ijk]}{}_l = E^{ijk}{}_l, \qquad \Lambda^{[ijk]} = \Lambda^{ijk} = \gamma_5 \Lambda^{ijk}, \qquad \varphi^{[ijkl]} = \varphi^{ijkl}$$

(and also, for $N=4$, $E^{ijk}{}_k = 0$). The fields of $N\le 4$ CSG's are then given by

$$e^a_\mu,\ \psi^i_\mu,\ A_\mu (N\neq 4),\ V_\mu{}^i{}_j,\ T_{\mu\nu}{}^{-ij},\ \chi^{ij}{}_k,\ E^{ijk}{}_l;\ \Lambda^{ijk},\ \varphi^{ijkl},\ D^{ij}{}_{kl},$$

where the SU(N) indices $i,j,k\ldots$ run from 1 to N, E and χ are traceless only for $N=4$ and D is traceless for $N=3,4$. For example, for $N=3$ we find

$$T^{-\,ij}_{\ \mu\nu} \sim \varepsilon^{ijk} T^-_{\mu\nu k}, \qquad \chi^{ij}{}_k \sim \varepsilon^{ijl}\chi_{kl}, \qquad \Lambda^{ijk} \sim \varepsilon^{ijk}\Lambda\,,$$
$$E^{ijk}{}_l \sim \varepsilon^{ijk} E_l, \qquad \varphi^{ijkl} = 0, \qquad D^{ij}{}_{mn} \sim \varepsilon^{ijk}\varepsilon_{mnp} D^p{}_k, \qquad (D^p{}_k)^* = -D^k{}_p\,,$$

while for $N=2$: $T^{-\,ij}_{\ \mu\nu} \sim \varepsilon^{ij} T^-_{\mu\nu}$, $\chi^{ij}{}_k \sim \varepsilon^{ij}\chi_k$, $E=0$, $\Lambda=0$, $\varphi=0$, $D^{ij}{}_{kl} \sim \varepsilon^{ij}\varepsilon_{kl} D$, $D^*=D$. The resulting field content of $N\le 4$ CSG's is presented in table 3.2.

Table 3.2
Field content of $N\le 4$ conformal supergravities

N	e^a_μ	ψ^i_μ	A_μ	$V_\mu{}^i{}_j$	$T^{-\,ij}_{\ \mu\nu}$	$\chi^{ij}{}_k$	Λ^{ijk}	$E^{ijk}{}_l$	φ^{ijkl}	$D^{ij}{}_{kl}$
1	1	1	1	—	—	—	—	—	—	—
2	1	2	1	3	1	2	—	—	—	—
3	1	3	1	8	3	9 = 8 + 1	1	3	—	8
4	1	4	—	15	6	20	4	10	1	20′

The corresponding transformation laws follow in an analogous way from (3.28), (3.29), while in order to get the $N\le 4$ Lagrangians, we have to add to (3.30) the U(1)-Maxwell term $(1/\alpha^2)\times [-\{(4-N)/4N\}F^2_{\mu\nu}(A)]$ found in the previous section (see (3.14)). The U(1) transformation laws of the fields are determined by the chiral weight c (cf. (3.8))

$$\delta A_\mu = \partial_\mu \alpha, \qquad \delta \Phi^{i_1\cdots i_n}_{j_1\cdots j_k} = \mathrm{i}\,\frac{4-N}{4N}\,c\alpha \Phi^{i_1\cdots i_n}_{j_1\cdots j_k}, \qquad c = n-k\,.$$

The general expression for the U(N) covariant derivative acting on bosons or left chiral fermions is

$$D_\mu \Phi^{i_1\cdots}_{j_1\cdots} = \mathscr{D}_\mu \Phi^{i_1\cdots}_{j_1\cdots} - U_\mu{}^{i_1}{}_{n_1} \Phi^{n_1\cdots}_{j_1\cdots} + \Phi^{i_1\cdots}_{m_1\cdots} U_\mu{}^{m_1}{}_{j_1} + \cdots$$
$$\equiv (\mathscr{D}_\mu - V_\mu)\Phi^{i_1\cdots}_{j_1\cdots} - \mathrm{i}\,\frac{4-N}{4N}\,cA_\mu \Phi^{i_1\cdots}_{j_1\cdots}, \qquad U_\mu{}^i{}_j \equiv V_\mu{}^i{}_j + \mathrm{i}\,\frac{4-N}{4N} A_\mu \delta^i_j\,. \tag{3.37}$$

It is helpful to summarize once more the main characteristics of the CSG fields:

Table 3.3
Weyl weights, chiral weights and numbers of off-shell and on-shell degrees of freedom

	e^a_μ	ψ^i_μ	$A_\mu, V^i_{\mu j}$	$T^{-ij}_{\mu\nu}$	$\chi^{ij}{}_k$	Λ^{ijk}	$E^{ijk}{}_l$	φ^{ijkl}	$D^{ij}{}_{kl}$
w	-1	$-\frac{1}{2}$	0	-1	$\frac{3}{2}$	$\frac{1}{2}$	1	0	2
c	0	1	0	2	1	3	2	4	0
n	5	-8	3	6	-4	-4	2	2	1
ν	6	-8	2	6	-2	-6	2	4	0

Note that, with our notations, fermions with positive γ_5 chirality have positive chiral weight. To check the consistency of the data given in tables 3.2 and 3.3 we compute the total numbers of off-shell and on-shell degrees of freedom in $N<4$ CSG's (see (3.27), (3.35) for $N=4$). The Fermi–Bose content of the resulting zeroes is given in table 3.4.

Table 3.4
Count of degrees of freedom in $N \leq 4$ conformal supergravities

	N			
	1	2	3	4
n	8–8	24–24	64–64	128–128
ν	8–8	20–20	48–48	96–96

The structure of the *off-shell* multiplets of conformal supergravities is in correspondence with the structure of the on-shell massive supermultiplets of extended supersymmetry [52, 8]. The content of the relevant massive supermultiplets is well known (see e.g. refs. [194, 91] and table 3.5).

In table 3.5 we also spelled out the content of the spin 2 massless multiplets (first columns). The number of on-shell degrees of freedom corresponding to a state of spin s is $2(-1)^{2s}$ in the massless case and $(-1)^{2s}\,(2s+1)$ in the massive case (massless scalars are assumed to be complex, while massive

Table 3.5
Some massless and massive on shell supermultiplets

	N									
s	1		2		3		4		5	
$\frac{5}{2}$										1
2	1	1	1	1	1	1	1	1	1	10
$\frac{3}{2}$	1	2	2	4	3	6	4	8	5	44
1		1	1	6	3	15	6	27	10	110
$\frac{1}{2}$				4	1	20	4	48	11	165
0				1		14	1	42	5	132
ν_{tot}	2–2	8–8	4–4	24–24	8–8	64–64	16–16	128–128	32–32	512–512

scalars are real; the numbers of massive spinors are given by the total numbers of left and right spinors). The last line gives the total number of degrees of freedom in each multiplet (Fermi degrees of freedom are counted with a negative sign). Comparing tables 3.4 and 3.5 we find agreement between the total numbers of *off-shell* degrees of freedom in CSG multiplets and of *on-shell* degrees of freedom in the massive supermultiplets. We can also establish a correspondence between the states of each particular spin. The massive supermultiplets are described by antisymmetric tensor representations of Sp($2N$) [91]. Decomposing these representations with respect to the SU(N) subgroup [8] we find that the number of spin 3/2 states is always $N + \bar{N}$, while the number os spin 1 states can be represented as

$$N = 2:\ 6 = 1 + 3 + 1 + \bar{1}, \qquad N = 3:\ 15 = 1 + 8 + 3 + \bar{3}, \qquad N = 4:\ 27 = 15 + 6 + \bar{6}.$$

These decompositions are in agreement with the spectra of CSG's in table 3.2: the off-shell spin 1 states of CSG multiplets are represented by 1 U(1) and $N^2 - 1$ SU(N) vectors and also by $\frac{1}{2}N(N-1)$ antiself-dual and $\frac{1}{2}N(N-1)$ self-dual antisymmetric tensors. Analogous decompositions are true for spin 1/2 and spin 0 states which are represented by the fields χ, Λ and E, φ, D respectively. For example, for $s = 0$ we have:

$$N = 3:\ 14 = 3 + \bar{3} + 8, \qquad N = 4:\ 42 = 10 + \overline{10} + 1 + \bar{1} + 20'.$$

There are two possible explanations for this coincidence between the structure of off-shell multiplets of conformal supergravity and that of massive supermultiplets. A "kinematical" one (see e.g. [49, 7]) is based on the remark that the classification of off-shell states with an arbitrary momentum squared p^2 should go just in the same way as that for on-shell states with $p^2 = m^2$. The idea behind a "dynamical" explanation [52] is to couple the conformal supergravity to ordinary Poincaré supergravity (PSG) and to assume that there exists an off-shell (rigid supersymmetry preserving) formulation of this coupling (i.e. that there exist an auxiliary field formulations of $N \geq 1$ PSG's). The total action is then built of two separate invariants describing two types of supergravities. The resulting linearized on-shell spectrum should consist of a combination of massless and massive supermultiplets of rigid supersymmetry. This was explicitly demonstrated in the $N=1$ [81] and $N=2$ [59] cases (see also [10, 8] for a discussion of the $N=4$ case). We have already learned from section 2.3 how to couple $N=1$ PSG to $N=1$ CSG and that it means adding the superconformally invariant action for a compensating multiplet (chiral or tensor) to the action of conformal supergravity. In a particular "Poincaré gauge" (in which "extra" superconformal symmetries are fixed by the conditions on the compensating fields) the full action looks like the "off-shell" action for PSG plus the action for CSG. As we shall see in the following sections the same pattern is true also for $N=2,3,4$. Namely, it appears that the combined PSG + CSG-theory can be constructed in the following way:

$N=2$: $\{N=2$ SO(2) Abelian vector multiplet$\}+\{N=2$ hypermultiplet (scalar, or tensor, or non-linear multiplet)$\}+\{N=2$ CSG$\}$,

$N=3$: $\{N=3$ SO(3) vector multiplet$\}+\{N=3$ CSG$\}$, (3.38)

$N=4$: $\{N=4$ SO(4) vector multiplet$\}+\{N=4$ CSG$\}$.

Speaking about *Poincaré* (ungauged) supergravity, we assume that the SO(N) group of vector multiplets is the rigid one, i.e. we are dealing with the $N(N-1)/2$ Abelian multiplets. It is now straightforward to check the validity (for all $N \leq 4$) of the following identity between the total numbers of on-shell Bose (or Fermi) degrees of freedom:

$$\nu(\text{compensating multiplets}) + \nu(\text{CSG}) = \nu(\text{massless multiplet}) + \nu(\text{massive multiplet})\,. \tag{3.39}$$

The numbers for the right hand side of (3.39) were given in table 3.5 (note that $\nu(\text{massless}) = \nu(\text{PSG})$), while $\nu(\text{CSG})$ can be found in table 3.4. Finally, we have to note that for the matter multiplets: $\nu(N=1$ chiral$) = 2-2$, $\nu(N=2$ hypermultiplet$) = 4-4$, $\nu(N=2$ vector$) = 2+2-4 = 4-4$, $\nu(N=3$ vector$) = \nu(N=4$ vector$) = 2+6-4\times 2 = 8-8$. Thus the explicit form of (3.39) is: $N=1$: $2+8=2+8$, $N=2$: $4+4+20=4+24$, $N=3$: $3\times 8+48=8+64$, $N=4$: $6\times 8+96=16+128$.

It may be interesting also to note all $N \geq 2$ compensating multiplets can be universally described as a SO(N) $N=4$ vector multiplet (a $N=2$ vector multiplet plus a $N=2$ hypermultiplet counts as one $N=4$ vector multiplet).

To clarify the mechanisms which provide the massive states let us consider first the $N=1$ case. Choosing the chiral multiplet (2.51) as a compensating one and fixing the Weyl, U(1) and S-supersymmetry gauges as in (2.52), we find the following "PSG + CSG" Lagrangian (it is sufficient to consider the linearized expressions, cf. (2.35))

$$\begin{aligned} e^{-1}\mathscr{L}^{(1)}_{\text{PSG+CSG}} &= \frac{a^2}{12}\left[R - e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu\mathscr{D}_\rho\psi_\sigma + \tfrac{3}{2}A_\mu^2 - \tfrac{1}{2}S^2 - \tfrac{1}{2}P^2\right] \\ &\quad - \frac{1}{\alpha^2}\left[\tfrac{1}{2}C^2_{\lambda\mu\nu\rho} - 4e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\phi}_\mu\gamma_5\gamma_\nu\mathscr{D}_\rho\phi_\sigma - \tfrac{3}{4}F^2_{\mu\nu}(A)\right]. \end{aligned} \tag{3.40}$$

Here $a^2/12 = 1/16\pi G \equiv 1/k^2$ and S and P are the auxiliary fields of the compensating multiplet which do not influence the on-shell count and, in principle, can be omitted. Notice that we have taken *both* the actions for the chiral multiplet and for conformal supergravity with the "ghost" signs. The ghost nature of the compensating multiplets is a universal feature (true also for $N>1$) which is necessary to get the physical sign for the gravitational constant. As we shall see shortly the "ghost" choice of the sign in front of the CSG action (under which the U(1) vector is a ghost particle) is necessary in order to have a *ghost* instead of a *tachyonic* massive multiplet.[2] Employing eqs. (1.22)–(1.26) and (1.36) we can rewrite (3.40) as

$$\mathcal{L}^{(1)}_{\mathrm{PSG+CSG}} \simeq \frac{1}{k^2}[\tfrac14 h(P_2-2P_0)\Box h - \bar\psi(P_{3/2}-2P_{1/2})\partial\!\!\!/\psi + \tfrac32 A_\mu^2 + \cdots]$$

$$-\frac{1}{\alpha^2}[\tfrac14 hP_2\Box^2 h - \bar\psi P_{3/2}\partial\!\!\!/^3\psi + \tfrac32 AP_1\Box A + \cdots]$$

$$= -\frac{1}{\alpha^2}[\tfrac14 hP_2\Box(\Box-m^2)h - \bar\psi P_{3/2}\partial\!\!\!/(\Box-m^2)\psi + \tfrac32 AP_1(\Box-m^2)A]$$

$$-\frac{1}{k^2}(\tfrac12 hP_0\Box h + \tfrac32 AP_0A - 2\bar\psi P_{1/2}\partial\!\!\!/\psi), \qquad m^2 \equiv \frac{\alpha^2}{k^2}. \tag{3.41}$$

To deduce the on-shell spectrum we can either consider the classical field equations or the corresponding partition function (cf. (1.38)). We find that it consists of the *massless physical* PSG multiplet and of the *massive ghost* multiplet corresponding, as discussed above, to the off-shell CSG multiplet. (Off-shell, in addition to the latter multiplet, one also finds the ghost chiral multiplet which is "gauge equivalent" to the original compensating multiplet.) If we had used the tensor compensating multiplet (2.62), the only difference would then have been in the description of the U(1) mass term: instead of A_μ^2 in (3.41) we would have the U(1) gauge invariant combination $\sim A_\mu E_\mu + E_\mu^2$ (cf. (2.60)–(2.62)) which is equivalent to A_μ^2 after integration over $E_{\mu\nu}$. Thus the conclusion about the multiplet content of the spectrum would not change.

The same pattern for the generation of massive *gauge* particles in the combined spectrum is also true for $N>1$: the mass terms are provided by the Einstein, Rarita–Schwinger, and A_μ^2, $(V_\mu{}^i{}_j)^2$ auxiliary parts of the PSG action. Turning to the "matter" fields sector let us consider first the case of higher derivative spinors Λ^{ijk} and scalars φ^{ijkl}. In order to find the corresponding massive states in the combined spectra, it is necessary (in complete analogy with the gravitino and graviton cases) to *identify* Λ^{ijk} *and* φ^{ijkl} *with the physical spinor and scalar fields of* $N=3$ *and* 4 *Poincaré supergravities* [10, 8] (which are known to have just the needed SO(N) assignments). Then the combined Lagrangian, generalizing (3.41), will contain the terms (cf. (3.30))

$$e^{-1}\mathcal{L}^{(N)}_{\Lambda,\varphi} \simeq \frac{1}{12k^2}(\varphi^{ijkl}\Box\varphi_{ijkl} - \bar\Lambda^{ijk}\overleftrightarrow{\partial\!\!\!/}\Lambda_{ijk}) - \frac{1}{12\alpha^2}(\varphi^{ijkl}\Box^2\varphi_{ijkl} - \bar\Lambda^{ijk}\overleftrightarrow{\partial\!\!\!/}^3\Lambda_{ijk})$$

$$= -\frac{1}{12\alpha^2}(\varphi^{ijkl}\Box(\Box-m^2)\varphi_{ijkl} - \bar\Lambda^{ijk}\overleftrightarrow{\partial\!\!\!/}(\Box-m^2)\Lambda_{ijk}), \qquad m^2 \equiv \frac{\alpha^2}{k^2}, \tag{3.42}$$

($1/k$ is again proportional to the value of one of the compensating scalars in the Poincaré gauge). Next let us write down the couplings, which provide the massive states for the scalars $E^{ijk}{}_l$, $D^{ij}{}_{kl}$ and spinors

[2] This choice of course is purely formal at this point. We shall return to the discussion of related questions in section 6.4.

$\chi^{ij}{}_k$ (see table 3.2) [10]

$$e^{-1}\mathscr{L}^{(N)}_{E,\chi,D} = \frac{1}{k^2}\left[\tfrac{1}{12}E^{ijk}{}_l E_{ijk}{}^l + (\bar{\chi}^{ij}{}_k \lambda_{ij}{}^k + \bar{\lambda}^{ij}{}_k \not{\partial} \lambda_{ij}{}^k + \tfrac{1}{3}\bar{\lambda}^{ij}{}_j \gamma_\mu R_{\mu i}(\psi) + \text{c.c.}) + \partial_\mu u^{ij}{}_{kl}\, \partial_\mu u^{kl}{}_{ij} - 2u^{ij}{}_{kl} D^{kl}{}_{ij}\right] - \frac{1}{\alpha^2}\left[\tfrac{1}{12}E^{ijk}{}_l \Box E_{ijk}{}^l - \bar{\chi}^{ij}{}_k \overleftrightarrow{\not{\partial}} \chi_{ij}{}^k + D^{ij}{}_{kl} D^{kl}{}_{ij}\right]. \tag{3.43}$$

Here λ and u are "non-conformal" auxiliary fields of Poincaré supergravity (which are propagating ghost fields of the compensating multiplets). Eliminating D, λ and the spin $\frac{1}{2}$ part of ψ_μ, we find the expected equations of motion for massive states $(\Box - m^2)(E, u, \chi) = 0$. The remaining piece of the combined Lagrangian looks like

$$e^{-1}\mathscr{L}^{(N)}_T \simeq \left[\tfrac{1}{8}(F^{+}_{\mu\nu}{}^{ij}(B))^2 - \frac{1}{k} F^{+ij}_{\mu\nu} T^{+}_{\mu\nu ij} + \frac{1}{k^2}(T^{+}_{\mu\nu ij})^2 + \text{c.c.}\right] - \frac{8}{\alpha^2}\,\partial_\mu T^{+}_{\mu\nu ij}\,\partial_\rho T^{-ij}_{\rho\nu}\,. \tag{3.44}$$

Here $F_{\mu\nu}(B) = \partial_\mu B_\nu - \partial_\nu B_\mu$, $F^{\pm} = (F \pm F^*)/2$ and B^{ij}_μ are the physical O(N) vectors of PSG. An apparent paradox is that though B_μ belong to the compensating multiplet, they have the *wrong* (ghost) sign for a kinetic term. However, it is possible to rewrite the first three terms in (3.44) as

$$\mathscr{L}_B \simeq -\frac{1}{8}(F^{-ij}_{\mu\nu})^2 + \frac{1}{k^2}\left(T^{-ij}_{\mu\nu} - \frac{k}{2}F^{-ij}_{\mu\nu}\right)^2 + \text{c.c.} \tag{3.45}$$

Thus, in the absence of the CSG part of (3.44), we nevertheless find (after the elimination or a redefinition of $T_{\mu\nu}$) the correct sign for the O(N) vector term in the PSG Lagrangian. The combined Lagrangian contains a combination of B_μ and $T_{\mu\nu}$ which provides a mass to the vector particles. It is straightforward to prove this, analyzing the equations of motion which follow from (3.44) under variations with respect to B_μ and $T_{\mu\nu}$ (we use (3.26) and (3.33)) [59]

$$\begin{aligned} &\partial_\mu F_{\mu\nu} - 4k^{-1}\,\partial_\mu T_{\mu\nu} = 0\,, \\ &-\Box T_{\mu\nu} + 2(\partial_\mu \partial_\rho T_{\rho\nu} - \partial_\nu \partial_\rho T_{\rho\mu}) + \frac{\alpha^2}{k^2} T_{\mu\nu} - \frac{\alpha^2}{2k} F_{\mu\nu} = 0\,. \end{aligned} \tag{3.46}$$

Introducing two transverse real vectors $l_\nu = \partial_\mu T_{\mu\nu}$ and $n_\nu = \mathrm{i}\partial_\rho T^{*}_{\rho\nu}$, it is easy to check that (3.46) is equivalent to

$$(\Box - m^2)\,\Box B_\mu = 0, \qquad \partial_\mu B_\mu = 0, \qquad (\Box - m^2) l_\mu = 0, \qquad (\Box - m^2) n_\mu = 0\,.$$

To clarify the structure of (3.44) it is useful to make the following change of variables: $T_{\mu\nu} \to$ two transverse real vectors $\xi^{\perp}_\mu$, $\eta^{\perp}_\mu$

$$T_{\mu\nu} = \partial_\mu \xi^{\perp}_\nu - \partial_\nu \xi^{\perp}_\mu + \mathrm{i}e\varepsilon_{\mu\nu\alpha\beta}\,\partial_\alpha \eta^{\perp}_\beta, \qquad \partial_\alpha \xi^{\perp}_\alpha = 0, \qquad \partial_\beta \eta^{\perp}_\beta = 0\,, \tag{3.47}$$

so that

$$T^{+}_{\mu\nu} = \partial_\mu \zeta^{\perp}_{\nu} - \partial_\nu \zeta^{\perp}_{\mu} + e\varepsilon_{\mu\nu\alpha\beta}\,\partial_\alpha \zeta^{\perp}_{\beta}\,,$$
$$T^{-}_{\mu\nu} = \partial_\mu \zeta^{*\perp}_{\nu} - \partial_\nu \zeta^{*\perp}_{\mu} - e\varepsilon_{\mu\nu\alpha\beta}\,\partial_\alpha \zeta^{*\perp}_{\beta}\,, \tag{3.48}$$
$$\zeta_\mu \equiv \tfrac{1}{2}(\xi_\mu + \mathrm{i}\eta_\mu), \qquad \zeta^*_\mu = \tfrac{1}{2}(\xi_\mu - \mathrm{i}\eta_\mu)\,,$$

and $\partial_\mu T^{+}_{\mu\nu} = \Box \zeta^{\perp}_{\nu}$. Then eq. (3.44) takes the form (we omit i, j-indices)

$$\mathscr{L}^{(N)}_{T} \simeq -\tfrac{1}{4} B^{\perp}_{\mu} \Box B^{\perp}_{\mu} + \left(\frac{2}{k} B^{\perp}_{\mu} \Box \zeta^{\perp}_{\mu} - \frac{4}{k^2} \zeta^{\perp}_{\mu} \Box \zeta^{\perp}_{\mu} + \text{c.c.}\right) - \frac{8}{\alpha^2} \Box \zeta^{\perp}_{\mu} \Box \zeta^{\perp}_{\mu}\,. \tag{3.49}$$

Eliminating $B^{\perp}_{\mu}(=(4/k)\,(\zeta^{\perp}_{\mu} + \zeta^{*\perp}_{\mu})$, cf. (3.46)) we obtain

$$\mathscr{L}^{(N)\prime}_{T} = -\frac{8}{\alpha^2} \zeta^{*\perp}_{\mu} \Box(\Box - m^2)\, \zeta^{\perp}_{\mu}\,. \tag{3.50}$$

To find the partition function corresponding to (3.44) we have to take into account the Jacobian corresponding to the transformation (3.47)

$$\int \mathrm{d}T_{\mu\nu} = \int \mathrm{d}\zeta^{+}_{\mu}\, \mathrm{d}\zeta^{*\perp}_{\nu} \det(\Box_1)_{\perp}, \qquad (\Box_1)_{\perp} = (\Box)_{\xi^{\perp}_{\mu}}$$

(this relation is easily proven by computing $\int \mathrm{d}T \exp T^2$). The standard gauge fixing for B_μ ($\int \mathrm{d}B_\mu \to \int \mathrm{d}B^{\perp}_{\mu}\, (\det \Box_0)^{1/2}$) and an integration over $B^{\perp}_{\mu}$, $\zeta^{\perp}_{\mu}$, $\zeta^{*\perp}_{\mu}$ yields

$$Z_{B,T} = \left[\frac{\det \Box_0}{\det(\Box_1)_{\perp}}\right]^{1/2} [\det(\Box_1 - m^2)_{\perp}]^{-1}\,. \tag{3.51}$$

This result corresponds to the contributions of one massless and two massive vector particles (hence a total of $\nu(B) + \nu(T) = 2 + (3+3)$ degrees of freedom, cf. (3.34)).

The particular form of the expressions obtained for the combined Lagrangians (3.42)–(3.44) will be justified and further discussed in the subsequent sections 3.4 and 3.5.

3.3. Non-linear structure of the $N \leq 4$ conformal supergravity actions

In this section we shall first determine some of the non-linear terms in the $N \leq 4$ CSG Lagrangians, starting with the linearized result (3.30) of the previous section, and imposing the condition of invariance under Weyl (D) transformations. Then we shall discuss the information implied by the knowledge of the complete transformation laws, as established in [8].

The covariantization of derivatives ($\partial_\mu \to D_\mu = \partial_\mu - w_\mu - U_\mu$, as done in (3.30); for the definition of the U(N) connection, see eq. (3.37)) and its analogs for $N < 4$, spoil the Weyl invariance. To make the covariant expressions D-invariant, we have to add some "non-minimal" terms which depend on the curvature $R_{\lambda\mu\nu\rho}(\omega(e))$ and

$$\mathscr{F}_{\mu\nu}{}^{i}{}_{j} \equiv (\partial_\mu U_\nu - \partial_\nu U_\mu - [U_\mu, U_\nu])^{i}_{j} = F_{\mu\nu}{}^{i}{}_{j}(V) + \mathrm{i}\,\frac{4-N}{4N}\,\delta^{i}_{j}\, F_{\mu\nu}(A)\,. \tag{3.52}$$

These terms can be uniquely determined using the method described in section 1.2 (see eqs. (1.11), (1.17)–(1.20)). In this way we find [100, 107, 109] (cf. (2.35))

$$e^{-1}\mathscr{L}^{(N)}_{\rm CSG} = \frac{1}{\alpha^2}\Big\{\tfrac{1}{2}C^2_{\lambda\mu\nu\rho} + F_{\mu\nu}{}^i{}_j(V)\,F_{\mu\nu}{}^j{}_i(V) - \frac{4-N}{4N}F^2_{\mu\nu}(A) - [4e^{-1}\varepsilon^{\mu\nu\lambda\rho}\bar\phi_{i\mu}\gamma_\nu D_\lambda\phi^i_\rho + 2R_{\mu\nu}$$
$$\times(\bar\psi^i_\lambda\sigma_{\lambda\nu}\phi_{i\mu} - \bar\psi^i_\mu\sigma_{\lambda\nu}\phi_{i\lambda} + \bar\psi^i_\lambda\gamma_\nu(D_{[\mu}\psi_{i\lambda]} - \gamma_{[\mu}\phi_{i\lambda]})) - \tfrac{4}{3}R\bar\psi^i_\lambda\sigma_{\lambda\nu}\phi_{i\nu} + 4\bar\psi^j_\mu\phi_{\nu i}\mathscr{F}^{+i}_{\mu\nu j} + \text{c.c.}]$$
$$+ 8D_\mu T^{-\,ij}_{\mu\nu}D_\rho T^+_{\rho\nu ij} - 4R_{\mu\nu}T^{-\,ij}_{\mu\rho}T^+_{\nu\rho ij} - \bar\chi_{ij}{}^k\overleftrightarrow{\not D}\chi^{ij}{}_k - \tfrac{1}{12}E^{ijk}{}_l(-D^2 + \tfrac{1}{6}R)E_{ijk}{}^l$$
$$+ [-\tfrac{1}{12}\bar\Lambda_{ijk}(\not D^3\Lambda^{ijk} + (R_{\mu\nu} - \tfrac{1}{6}g_{\mu\nu}R)\gamma_\mu D_\nu\Lambda^{ijk} + 6\mathscr{F}^{*\,i}_{\mu\nu\;m}\gamma_\mu D_\nu\Lambda^{jkm}) + \text{c.c.}]$$
$$+ \tfrac{1}{12}D^2\varphi^{ijkl}D^2\varphi_{ijkl} + \tfrac{1}{12}(R_{\mu\nu} - \tfrac{1}{3}g_{\mu\nu}R)D_\mu\varphi^{ijkl}D_\nu\varphi_{ijkl} + \cdots\Big\} \tag{3.53}$$

where the ϕ^i_μ were given in (3.31) and dots stand for all the other terms which are not required by the D-invariance of the kinetic terms (we also omit $\mathscr{D}R\bar\psi\psi$-terms and terms like $\mathscr{F}\mathscr{D}\varphi\mathscr{D}\varphi$, which vanish for $N\le 4$). The lowest order mixing terms, which are consistent with D, S and U(N) symmetries, are

$$e^{-1}\mathscr{L}^{(4)}_{\rm mix} = a_1 T^+_{\mu\nu ij}\mathscr{F}^{+\;l}_{\mu\nu\;k}E^{ijk}{}_l + a_2\bar\Lambda^{ijk}(\sigma\cdot\mathscr{F}^l{}_n)\varepsilon_{ijkm}\chi^{mn}{}_l + a_3\bar\psi_{k\mu}\gamma_\nu\chi^{jk}{}_i\mathscr{F}^{-\;i}_{\mu\nu\;j} + \text{c.c.} \tag{3.54}$$

($\bar\Lambda FD\psi$ is ruled out by D-invariance while $FT^+T^-\equiv 0$). For $N=4$ these symmetries do not exclude also the following non-minimal terms (recall that φ is D-inert)

$$e^{-1}\mathscr{L}^{(4)}_{\text{non-minimal}} = f_1(\varphi)(F_{\mu\nu}{}^i{}_j)^2 + f_2(\varphi)F_{\mu\nu}{}^i{}_jF^{*\,j}_{\mu\nu\;i} + h_1(\varphi)C^2_{\lambda\mu\nu\rho} + h_2(\varphi)C_{\lambda\mu\nu\rho}C^*_{\lambda\mu\nu\rho}\,, \tag{3.55}$$

where f_1 and h_1 (f_2 and h_2) are arbitrary real (imaginary) functions of $\varphi=\varphi_1+i\varphi_2$. There are of course plenty of other non-linear terms, the structure of which can be fixed only by Q-supersymmetry (note however, that non-polynomial terms in a non-gravitational (φ) field can be present only for $N=4$). As we shall see in section 6.2, the part (3.53) of the complete $N\le 4$ Lagrangians, which is dictated by the Weyl invariance, is already sufficient in order to computate the one-loop β-functions in conformal supergravities.

Equation (3.53) is in obvious agreement with the corresponding part of the complete $N=1$ CSG Lagrangian (2.32)–(2.35). The complete Lagrangian is known also for $N=2$ CSG [8]. It was found by constructing the non-linear $N=2$ Weyl multiplet (the corresponding non-linear transformation laws have the same form as the linearized ones (3.28), (3.29)), and applying the $N=2$ density formula [47]. In addition to the terms given in (3.53), which for $N=2$ are taken with the following normalizations

$$e^{-1}\mathscr{L}^{(2)}_{\rm CSG} = \frac{1}{\alpha^2}\{\tfrac{1}{2}C^2_{\lambda\mu\nu\rho} - \tfrac{1}{4}F^2_{\mu\nu}(A) + F_{\mu\nu}{}^i{}_j(V)F_{\mu\nu}{}^j{}_i(V) - 4e^{-1}\varepsilon^{\mu\nu\lambda\rho}\bar\phi_{i\mu}\gamma_\nu\overleftrightarrow{D}_\lambda\phi^i_\rho + 8D_\mu T^+_{\mu\nu ij}D_\rho T^-_{\rho\nu ij}$$
$$- 3\bar\chi^i\overleftrightarrow{\not D}\chi_i + 3D^2 + \cdots\}\,, \tag{3.56}$$

the $N=2$ Lagrangian contains also the following mixing terms

$$e^{-1}\mathscr{L}^{(2)}_{\rm mix} = (\tfrac{2}{3}\bar\psi_\mu{}^j\gamma_\nu\chi_i\mathscr{F}^{-\;i}_{\mu\nu\;j} + \text{c.c.}) + (T\bar\psi\psi\mathscr{F} + R\bar\psi\psi T + \bar\psi\phi\mathscr{D}T + \cdots\text{-terms}) \tag{3.57}$$

where dots stand for other terms which are quartic and of higher order in the non-gravitational fields.

One can explicitly check that $\bar{\psi}\chi R$-terms, forbiddén by D-invariance, do cancel in the expression given in [8], while all other trilinear terms are as given above. In particular, the U(1) and SU(2) field strengths occur in the coupling terms only in the U(2) combination (3.52).

The Lagrangian of $N=3$ conformal supergravity is similar to that of $N=2$ CSG. In addition to the trilinear term, given in (3.57), we here expect also $T^{-}_{\mu\nu i}\mathcal{F}_{\mu\nu}{}^{i}{}_{j}E^{j}$ (cf. (3.54)). An essential complication takes place for $N=4$, where the scalar φ (which is inert under all transformations except Q-supersymmetry) can occur in the action and in the transformation laws in a non-polynomial fashion. Let us start the analysis of the $N=4$ case with a discussion of the non-linear transformation laws established in [8] for a version of $N=4$ CSG possessing the additional local U(1) and rigid SU(1, 1) symmetries. As we already noted in the previous section, the scalar φ of $N=4$ CSG can be identified with the physical scalar of the $N=4$ Poincaré supergravity [10, 8]. The latter theory is known to possess a rigid *on-shell* SU(1, 1) symmetry (with SU(1, 1) transformations of φ being supplemented by a duality rotation of the vector field strengths and a chiral transformation of fermions). This observation suggests (in a rather indirect way) that SU(1, 1) may be realized also in $N=4$ conformal supergravity. Here it turns out to be an *off-shell* symmetry, which affects *only* the scalar fields.

The elements of SU(1, 1), i.e. the pseudounitary 2×2 matrices with unit determinant ($U^{+}\eta U=\eta$, $\eta=\mathrm{diag}(+1,-1)$, $\det U=1$) can, in general, be parametrized as

$$U(a)=\begin{pmatrix} a_1 & a_2^* \\ a_2 & a_1^* \end{pmatrix},\qquad a^{\alpha}a_{\alpha}=1,\qquad a^{\alpha}\equiv\eta^{\alpha\beta}a^*_{\beta}=(a_1^*,-a_2^*),\qquad \alpha,\beta=1,2\,. \tag{3.58}$$

In such a case $U(b)\,U(a)=U(a')$, where $a'_{\alpha}=U_{\alpha\beta}(b)\,a_{\beta}$. Consider now a doublet of complex scalar fields Φ_{α}, which satisfy the constraint

$$\Phi_{\alpha}\Phi^{\alpha}=1 \tag{3.59}$$

and transform under the rigid SU(1, 1) and local U(1) groups as

$$\Phi'=U(a)\,\Phi,\qquad \Phi'=\mathrm{e}^{\mathrm{i}\,\alpha(x)}\Phi\,. \tag{3.60}$$

If we fix the U(1) symmetry by a gauge (which also breaks the SU(1, 1))

$$\Phi_1^*=\Phi_1\,,$$

$U(\Phi)$ can then be considered as a parametrization of the coset space SU(1, 1)/U(1) (U(1) is a subgroup of SU(1, 1) which is $\mathrm{diag}(\mathrm{e}^{\mathrm{i}a},\mathrm{e}^{-\mathrm{i}a})$). A more explicit parametrization is obtained by solving (3.59)

$$\Phi_1=(1-|\varphi|^2)^{-1/2},\qquad \Phi_2=\varphi(1-|\varphi|^2)^{-1/2},\qquad |\varphi|<1\,, \tag{3.61}$$

where $\varphi=\Phi_2/\Phi_1$ is the projective coordinate of the coset space (see also [7]). Thus the main assumption will be that the $N=4$ theory can be formulated in a manifestly $\mathrm{SU}(1,1)_{\mathrm{rigid}}\times \mathrm{U}(1)_{\mathrm{local}}$ invariant way at the expense of introducing instead of one complex scalar φ, the pair of complex scalars Φ_{α}, transforming according to (3.60) and satisfying the $\mathrm{SU}(1,1)\times\mathrm{U}(1)$ invariant constraint (3.59).

This assumption provides a drastic simplification in the structure of the possible φ-dependent terms which can be present in the transformation laws (and the action). The φ-dependent terms originate from the Φ_α-dependent ones in a particular U(1) gauge (see (3.61)), while the latter terms must be polynomial in Φ_α. In fact all algebraic SU(1, 1) invariants constructed from Φ_α are equal to constants ($\varepsilon^{\alpha\beta}\Phi_\alpha\Phi_\beta = 0$, $\Phi^\alpha\Phi_\alpha = 1$, etc.). The derivative $D_\mu\Phi$ has a positive dimension and thus can occur only polynomially. As a consequence, the completion of the transformation rules requires only a finite number of steps (one starts with the linearized transformations (3.28), (3.29), adds a finite number of possible invariant terms, and, then, determines the unknown coefficients). The resulting Q-supersymmetry transformation rules look like [8]

$$\begin{aligned}
&\delta\Phi_\alpha = -\tfrac{1}{2}\bar{\varepsilon}^i\Lambda_i\varepsilon_{\alpha\beta}\Phi^\beta ,\\
&\delta\Lambda_i = \varepsilon^{\alpha\beta}\Phi_\alpha \hat{\not{D}}\Phi_\beta\varepsilon_i + \tfrac{1}{2}E_{ij}\varepsilon^j + \tfrac{1}{2}\varepsilon_{ijkl}\sigma\cdot T^{-kl}\varepsilon^j ,\\
&\delta E_{ij} = \bar{\varepsilon}_{(i}\hat{\not{D}}\Lambda_{j)} + (\bar{\varepsilon}\chi + \bar{\Lambda}\Lambda\bar{\Lambda}\varepsilon\text{-terms}) ,\\
&\delta T^{-ij}_{\mu\nu} = \bar{\varepsilon}\mathcal{R}(Q) + \bar{\varepsilon}\chi + \bar{\varepsilon}\,\hat{D}\Lambda + E\varepsilon\Lambda + \bar{\varepsilon}\Phi\,\not{D}\Phi\,\Lambda\text{-terms} ,\\
&\delta\chi = \hat{D}T\varepsilon + \mathcal{R}(V)\,\varepsilon + \cdots\text{-terms} ,\\
&\delta e^a_\mu = \tfrac{1}{2}\bar{\varepsilon}^i\gamma^a\psi_{\mu i} + \text{c.c.}, \qquad \delta V_\mu{}^i{}_j = \bar{\varepsilon}^i\phi_{j\mu} + (\bar{\varepsilon}\chi + \bar{\varepsilon}\Phi\,\not{D}\Phi\Lambda\text{-terms}) ,\\
&\delta\psi^i_\mu = D_\mu\varepsilon^i - \tfrac{1}{2}\sigma\cdot T^{-ij}\gamma_\mu\varepsilon_j - \tfrac{1}{2}\varepsilon^{ijkl}\,\bar{\varepsilon}_j\psi_{k\mu}\Lambda_l ,
\end{aligned} \tag{3.62}$$

where Φ appears explicitly only in two SU(1, 1) invariant combinations

$$\begin{aligned}
\zeta_\mu &= \varepsilon^{\alpha\beta}\Phi_\alpha\,\hat{D}_\mu\Phi_\beta = \varepsilon^{\alpha\beta}\Phi_\alpha\,\partial_\mu\Phi_\beta - \tfrac{1}{2}\bar{\psi}^i_\mu\Lambda_i\\
a_\mu &= -\tfrac{1}{2}\Phi^\alpha\,\overleftrightarrow{\partial}_\mu\Phi_\alpha - \tfrac{1}{2}\bar{\Lambda}^i\gamma_\mu\Lambda_i .
\end{aligned} \tag{3.63}$$

a_μ is a composite U(1) gauge field, which enters through covariant derivatives. The complete S-supersymmetry transformation rules [8] are independent of Φ_α and have the same form as (3.29), except for $\delta V = \sigma\cdot T\eta + E\eta + \bar{\Lambda}\Lambda\eta$. Assuming that the curvature constraints (3.11) are taken to be S-invariant one finds that the commutators of the resulting supersymmetry algebra differ from that of SU(2, 2|N) only in the case of two Q-supersymmetries (cf. (2.46))

$$[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] = (\tfrac{1}{2}\bar{\varepsilon}_1\,\gamma^\mu\varepsilon_2 + \text{c.c.})\hat{D}_\mu + \sum\delta_H(\varepsilon_H) \tag{3.64}$$

where $H = \{M, Q, S, \mathrm{SU}(4), K\}$ and ε_H are (non-linear) functions of fields, e.g. $\varepsilon_{ab} = \bar{\varepsilon}^i_1\varepsilon^j_2 T^+_{abij}$, $\varepsilon^i_Q = \bar{\varepsilon}_{1k}\varepsilon_{2j}\Lambda^{ijk}$, etc. Truncating (3.62) it is straightforward to find the complete transformation laws for $N = 2, 3$ CSG's.

Let us now turn to the implications of SU(1, 1) × U(1) invariance for the $N = 4$ Lagrangian. Again non-polynomial terms in Φ are forbidden and the action can depend on Φ only through derivative combinations (3.63), which do not depend on $\Phi = $ const. (and thus on $\varphi = $ const.). We conclude that non-minimal interaction terms like (3.55) cannot be present in the action of SU(1, 1)-invariant version of $N = 4$ CSG. There is a finite (though large) number of admissible terms, e.g.

$$\mathcal{L}_\Phi \sim (\Phi\,\partial\Phi)\,\mathcal{D}^2(\Phi\,\partial\Phi) + R(\Phi\,\partial\Phi)\,(\Phi\,\partial\Phi) + TT(\Phi\,\partial\Phi)^2 + \cdots \tag{3.65}$$

which should be in agreement with the φ-dependent terms in (3.53), after substitution of (3.61).[1]

It is interesting to compare (3.65) with the scalar part of the Lagrangian of $N=4$ Poincaré supergravity (in the O(4) formulation [41, 35])

$$e^{-1}\mathscr{L}^{(4)}_{\mathrm{PSG}} = -\frac{1}{2k^2}\frac{\partial_\mu\varphi\,\partial_\mu\varphi^*}{(1-|\varphi|^2)^2} - \frac{1}{8}\left[\left(\frac{1+\varphi^2}{1-\varphi^2}\delta_{ij}\delta_{kl} - \frac{\varphi}{1-\varphi^2}\varepsilon_{ijkl}\right)F^{+\,ik}_{\ \mu\nu}F^{+\,jl}_{\ \mu\nu} + \text{c.c.}\right] + \cdots . \tag{3.66}$$

Using (3.61) and introducing $F_{\pm\,\mu\nu}^{\ ij} \equiv \frac{1}{2}(F^{ij}_{\mu\nu} \pm \frac{1}{2}\varepsilon^{ijkl}F^{kl}_{\mu\nu})$ we can rewrite (3.66) as

$$e^{-1}\mathscr{L}^{(4)}_{\mathrm{PSG}} = -\frac{1}{2k^2}|\varepsilon^{\alpha\beta}\Phi_\alpha\,\partial_\mu\Phi_\beta|^2 - \frac{1}{8}\left\{\frac{\Phi_1-\Phi_2}{\Phi_1+\Phi_2}F^{+ij}_{+\mu\nu}F^{+ij}_{+\mu\nu} + \frac{\Phi_1+\Phi_2}{\Phi_1-\Phi_2}F^{+ij}_{-\mu\nu}F^{+ij}_{-\mu\nu} + \text{c.c.}\right\} + \cdots . \tag{3.67}$$

We see that, while the kinetic scalar term *is* invariant under the SU(1, 1)-transformations (3.60), the scalar–vector interaction terms are not. To make them invariant we have to accompany (3.60) by dual rotations of $F_{\mu\nu}$, which do not belong to a class of off-shell local transformations. This implies that the off-shell SU(1, 1) symmetry of $N=4$ CSG is to be broken in the process of deriving $N=4$ PSG within the superconformal framework. This observation is connected with the fact that, as we shall see in section 3.5, the coupling of $N=4$ super Yang–Mills to $N=4$ CSG is not SU(1, 1)-symmetric.

We are thus led to question the existence of different SU(1, 1)-non-invariant versions of $N=4$ conformal supergravity, which could have different structures in the scalar φ-sector (e.g., have non-minimal terms such as (3.55) in the Lagrangian). The importance of this question is amplified by the result [100, 107] (see section 6.2) showing that the SU(1, 1)-invariant version of $N=4$ CSG is not finite (is not anomaly free) and thus is probably inconsistent at the quantum level. The mere possibility that there may be several versions of a locally supersymmetric theory, which could have the same linearized spectrum but different non-linear Lagrangians and, thus, inequivalent quantum properties,[2] is indeed suggested by the example offered by unequivalent (gauged) $N=4$ ordinary supergravities [42, 119, 134].[3] In an attempt to answer the above question we may use the following observation: the conformal supergravity action should coincide with one of two off-shell one-loop counter-terms for the corresponding Poincaré supergravity (the supersymmetric extension of the Weyl tensor squared invariant), cf. [52, 90]. Given the $N=4$ PSG Lagrangian (3.66), we can then study the dependence of the one-loop counter-term on the background value of φ. We immediately conclude that the counter-term cannot depend on φ if $\varphi=\text{const}$. In particular, there can be infinities of order $C^2_{\lambda\mu\nu\rho}$ but no infinities of order $f(\varphi)\,C^2_{\lambda\mu\nu\rho}$. As a result, the presence of at least part of the non-minimal terms (3.55) seems to be ruled out. We cannot however exclude the possibility that the whole argument does not work for $N=4$ (for example, the coefficient of the off-shell one-loop counter-term, computed in a supersymmetric gauge, may vanish identically for $N=4$, see ref. [146], and also ref. [160]). It is also difficult to rule out terms of the type $h(\varphi)CC^*$.

In what follows we shall speak about the SU(1, 1) × U(1)-invaraint $N=4$ theory as a "minimal"

[1] It is possible to prove that non-explicitly gauge invariant couplings like $J_\mu(\Phi\,\partial_\mu\Phi)$, $\partial_\mu J_\mu = F_{\rho\sigma}F^*_{\rho\sigma}$ are also forbidden.

[2] Here we mean the unequivalence of "on-shell" quantum properties (i.e. that of S-matrices). We are already accustomed to the fact that different auxiliary field versions of supergravities (which may also be based on different field representations for on-shell physical states) may have different off-shell quantum properties (e.g., anomalies).

[3] There are also two alternative formulations of ungauged $N=4$ PSG with SO(4) [41, 35] and SU(4) [36] rigid invariances of the actions that are naively unequivalent at the quantum level. Being related through duality transformations on vectors, point transformations on scalars and chiral transformations on spinors they however can be considered as equivalent, up to the anomalous terms induced by chiral redefinitions (cf. refs. [143, 124, 111]).

version of $N=4$ CSG (in which terms like (3.55) are forbidden) while a hypothetical version of $N=4$ CSG, in which "non-minimal" terms like (3.55) can be present, will be called a "non-minimal" version.

3.4. $N=2$ superconformal matter couplings

In this section we shall continue the discussion of superconformal matter multiplets and their coupling to conformal supergravity started in section 2.3. Again we shall concentrate mainly on the actions and on their interpretations in the framework of CSG. Reviews of $N=2$ superconformal multiplet calculus, with applications to $N=2$ Poincaré (De Sitter) supergravity, can be found in refs. [62, 57, 56].

Let us first recall the explicit form of the solution for $N=2$ constraints [60, 47] which are analogous to (3.11) (cf. (2.29)–(2.31))

$$
\begin{aligned}
w_\mu^{ab} &= -\omega_\mu^{ab}(e) - \tfrac{1}{4}(\bar\psi^i_\mu \gamma_a \psi_{bi} - \bar\psi^i_\mu \gamma_b \psi_{ai} + \bar\psi^i_a \gamma_\mu \psi_{bi} + \text{c.c.}) , \\
\phi^i_\mu &= \tfrac{1}{3}\gamma^\nu (D_\nu \psi^i_\mu - D_\mu \psi^i_\nu + \tfrac{1}{3} e \varepsilon_{\nu\mu\alpha\beta} D_\alpha \psi^i_\beta) + \tfrac{1}{12}\gamma_\mu \chi^i + \tfrac{1}{4}\gamma^\lambda \sigma \cdot T^{-ij} \gamma_\mu \psi_{\lambda j} - \tfrac{1}{12}\gamma_\mu \sigma \cdot T^{-ij}\gamma_\lambda \psi_{\lambda j} , \\
f^\mu_\mu &= -\tfrac{1}{12}R - \tfrac{1}{2}D + (-\tfrac{1}{4}\bar\psi^i_\lambda \gamma_\lambda \chi_i + \tfrac{1}{12} e^{-1} \varepsilon^{\mu\nu\rho\sigma} \bar\psi^i_\mu \gamma_\nu D_\rho \psi_{\sigma i} + \text{c.c.}) + (\bar\psi\psi T\text{-term}).
\end{aligned}
\tag{3.68}
$$

Here χ^i, $T^{-\,ij}_{\mu\nu}$ and D are "matter" fields of an $N=2$ CSG multiplet (see table 3.2) normalized as in a $N=2$ CSG Lagrangian (3.56).

The $N=2$ superconformal multiplets which we shall consider include the $N=2$ vector multiplet, $N=2$ non-linear multiplet, $N=2$ scalar and $N=2$ tensor multiplets. They realize a local superconformal algebra (3.64) and can be coupled to $N=2$ CSG. The non-linear multiplet does not exist in the flat space limit and has a peculiar non-polynomial action. The scalar multiplet possesses a central charge and can be coupled to CSG only in the presence of an additional Abelian $N=2$ vector multiplet which "gauges" central charge transformations. The latter multiplet is also necessary in order to establish a correspondence with the off-shell versions of $N=2$ PSG (cf. (3.38)).

The (non-Abelian) $N=2$ vector multiplet [73, 60] consists of the gauge vector B_μ, the complex scalar X, the doublet of Majorana spinors λ_i (with chiral projections $\lambda_i = \gamma_5\lambda_i$, $\lambda^i = -\lambda_5\lambda^i$) and the auxiliary scalar fields $Y_{ij} = Y_{ji}$ (i, j are indices of SU(2)). All fields belong to the algebra of the internal gauge group G with the basis of generators $\{t_A\}$, satisfying (in the compact case) $\mathrm{tr}(t_A t_B) = -2\delta_{AB}$. Thus $B_\mu = B^A_\mu t_A, \ldots$ where B^A_μ are assumed to be real, X^A are complex and $Y^{Aij} \equiv (Y^A_{ij})^* = \varepsilon^{ik}\varepsilon^{jl} Y^A_{kl}$. This multiplet containing $(8-8)\dim G$ off-shell degrees of freedom provides a realization of the local superconformal algebra for the following assignments of the Weyl and chiral weights (see (3.37) and table 3.3)

	B_μ	λ_i	X	Y_{ij}
w	0	3/2	1	2
c	0	1	2	0

The transformations under the local Q and S supersymmetries are [62] (cf. (2.52))

$$\delta X = \tfrac{1}{2}\bar{\varepsilon}^i \lambda_i, \qquad \delta B_\mu = \tfrac{1}{2}\bar{\varepsilon}_i \gamma_\mu \lambda_j \varepsilon^{ij} + \bar{\varepsilon}_i \psi_{\mu j} X \varepsilon^{ij} + \text{c.c.}\,,$$
$$\delta\lambda_i = \hat{\not{D}} X \varepsilon_i + \tfrac{1}{2} Y_{ij}\varepsilon^j + \tfrac{1}{2}\sigma \cdot \mathscr{F}^-(B)\,\varepsilon_i - g[X, X^*]\,\varepsilon_{ij}\varepsilon^j + 2X\eta_i\,, \tag{3.69}$$
$$\delta Y_{ij} = \bar{\varepsilon}_{(i}\hat{\not{D}}\lambda_{j)} + \varepsilon_{ik}\varepsilon_{jl}\bar{\varepsilon}^{(k}\hat{\not{D}}\lambda^{l)} - 2g\varepsilon_{k(i}(\bar{\varepsilon}_{j)}[X, \lambda^k] - \bar{\varepsilon}^k[X^*, \lambda_{j)}])\,,$$

where

$$\hat{D}_\mu = D_\mu - \delta_Q(\psi_\mu) - \delta_S(\phi_\mu)\,,$$

and

$$D_\mu\lambda_i = (\partial_\mu - \tfrac{1}{2}w_\mu \cdot \sigma)\lambda_i - g[B_\mu, \lambda_i] + \lambda_j V_\mu{}^j{}_i - \tfrac{1}{4}\mathrm{i}A_\mu\lambda_i\,,$$
$$D_\mu X = \partial_\mu X - g[B_\mu, X] - \tfrac{1}{2}\mathrm{i}A_\mu X\,.$$

We also used the notation

$$\mathscr{F}_{\mu\nu} = \hat{F}_{\mu\nu}(B) - XT^+_{\mu\nu ij}\varepsilon^{ij} - X^*T^-_{\mu\nu}{}^{ij}\varepsilon_{ij}\,,$$
$$\hat{F}_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu - g[B_\mu, B_\nu] - (\bar{\psi}_{[\mu i}\gamma_{\nu]}\lambda_j\varepsilon^{ij} - \bar{\psi}_{\mu i}\psi_{\nu j}\varepsilon^{ij}X + \text{h.c.})\,. \tag{3.70}$$

The complex conjugation of fields is always understood in the component sense, e.g. $X^* \equiv (X^A)^* t_A$. The superconformal extension of the $N = 2$ super Yang–Mills action is constructed using the $N = 2$ density formula found in [47]. The result [62, 56] (see also [47, 18]) is given by (cf. (2.56))

$$\begin{aligned}
e^{-1}\mathscr{L}^{(2)}_{\mathrm{V}} = \mathrm{tr}\{&D_\mu X^* D_\mu X + X^*X(D + \tfrac{1}{6}R - \tfrac{1}{6}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_{\mu i}\gamma_\nu D_\rho\psi^i_\sigma + \tfrac{1}{2}\bar{\psi}_{\mu i}\gamma_\mu\chi^i - \tfrac{1}{3}\bar{\psi}_{\mu i}\psi_{\nu j}T^-_{\mu\nu}{}^{ij} + \text{c.c.})\\
&- \tfrac{1}{8}|Y_{ij}|^2 - g^2[X, X^*]^2 + [\tfrac{1}{8}\hat{F}^+_{\mu\nu}\hat{F}^+_{\mu\nu} - \tfrac{1}{2}\hat{F}^+_{\mu\nu}T^+_{\mu\nu ij}\varepsilon^{ij}X\\
&+ \tfrac{1}{4}X^2(T^+_{\mu\nu ij}\varepsilon^{ij})^2 + \tfrac{1}{4}\bar{\lambda}^i\not{D}\lambda_i + \tfrac{1}{2}g\varepsilon^{ij}\bar{\lambda}_i[X^*, \lambda_j]\\
&- \bar{\chi}_i\lambda^i X + \text{c.c.}] + (\bar{\lambda}\mathscr{D}\psi X + \lambda\psi XX^* + \bar{\lambda}\psi F + \bar{\lambda}\psi TX + \cdots\text{-terms})\}\,.
\end{aligned} \tag{3.71}$$

The bosonic part of the Lagrangian can be rewritten as

$$\begin{aligned}
e^{-1}\mathscr{L}^{(2)}_{\mathrm{V\,Bose}} = \mathrm{tr}\{&D_\mu X^* D_\mu X + \tfrac{1}{6}RXX^* - g^2[X, X^*]^2 - \tfrac{1}{8}|Y_{ij}|^2\\
&+ [\tfrac{1}{8}F^+_{\mu\nu}(B)\,F^+_{\mu\nu}(B) - \tfrac{1}{2}F^+_{\mu\nu}(B)\,T^+_{\mu\nu ij}\varepsilon^{ij}X + \tfrac{1}{4}(T^+_{\mu\nu ij}\varepsilon^{ij})^2X^2 + \text{c.c.}]\}\,.
\end{aligned} \tag{3.72}$$

The action, corresponding to (3.71) can be directly added to the $N = 2$ CSG action (3.56). Fixing the dilational gauge, e.g. by the condition $\mathrm{tr}(X^*X) = -c^2 = \text{const.}$, we find several terms $(R - \bar{\psi}_\mu R_\mu + \cdots)$ resembling $N = 2$ Poincaré supergravity. In order to correct the sign of these terms we have to reverse the overall sign in (3.71) (i.e. to take the vector multiplet to be a ghost one). Another (more desirable) option is to assume that the gauge group has one non-compact generator [56]. The simplest possibility is to take $\mathrm{G} = \bar{\mathrm{G}} \times \mathrm{SO}(2)$ where the SO(2) factor corresponds to a *ghost* $N = 2$ Abelian vector multiplet (the field content of this multiplet and of the $N = 2$ CSG multiplet is sometimes called the "minimal field representation" [18, 19, 62]). It is useful to introduce a special notation for the fields of this multiplet: $X^{mn} = (1/\sqrt{2})\varepsilon^{mn}a$, $\lambda^{mn}_i = (1/\sqrt{2})\varepsilon^{mn}\xi_i$, $Y^{mn}_{ij} = \sqrt{2}\,\varepsilon^{mn}S_{ij}$ (m, n are the SO(2) indices which can be identified with the SU(2) ones). Then, instead of (3.71), we get (cf. (3.45)):

$$e^{-1}\mathscr{L}^{(N=2)}_{\text{Abelian}} = \left|\left(\partial_\mu - \frac{\mathrm{i}}{2}A_\mu\right)a\right|^2 + \tfrac{1}{6}R|a|^2 + D|a|^2$$
$$- \tfrac{1}{2}|S_{ij}|^2 - \tfrac{1}{8}(F^{ij}_{\mu\nu}(B))^2 + [(\tfrac{1}{2}F^{-\ ij}_{\mu\nu}(B) - T^{-\ ij}_{\mu\nu}a^*)^2 + \text{c.c.}] + \cdots . \tag{3.73}$$

Choosing the following dilational, chiral U(1) and S-supersymmetry gauge conditions ("Poincaré gauge", cf. (2.52))

$$D, \mathrm{U}(1)\colon\ a = a^* = \text{const.}, \qquad S\colon\ \xi_i = 0 \tag{3.74}$$

and identifying B_μ with the SO(2) physical vector of $N = 2$ PSG, we indeed find from (3.73) the correct kinetic terms for the graviton, for the gravitinos, and the vector of $N = 2$ Poincaré supergravity. It is important to stress that the physical sign of the Maxwell term is essentially due to the mixing between $F_{\mu\nu}$ and $T_{\mu\nu}$, i.e. to the possibility of identifying $t^-_{\mu\nu ij}\varepsilon^{ij} + t^{+\ ij}_{\mu\nu}\varepsilon_{ij} = aT^+_{\mu\nu ij}\varepsilon^{ij} + a^*T^{-\ ij}_{\mu\nu}\varepsilon_{ij} - \frac{1}{2}\hat{F}^{ij}_{\mu\nu}\varepsilon_{ij}$ with the auxiliary field of $N = 2$ PSG [113, 59]. The fields A_μ, S_{ij} and D are then candidates for other auxiliary fields. However, the presence in (3.73) of a term linear in D leads to inconsistent equations of motion ($\det e^a_\mu = 0$). This difficulty is formally absent in the combined $N = 2$ super-Yang–Mills plus $N = 2$ conformal supergravity action (the elimination of D then leads to a cosmological constant). But then there is a non-trivial problem in the consistent truncation of the theory (in some low-energy approximation, in which we can ignore terms of the CSG Lagrangian) to a sort of $N = 2$ Poincaré supergravity. In fact, the total spectrum of the hybrid theory does not coincide with the sum of $N = 2$ massless and $N = 2$ massive supermultiplets (see (3.38), (3.39) and discussion of these relations). Though this may not be a serious drawback (the supersymmetry of the complete theory and the presence of order R term in the "low-energy" limit may be sufficient for applications in the context of CSG) it seems safer to try to add some additional matter multiplets in order to provide a possibility for having a consistent (Q-supersymmetric) truncation to $N = 2$ PSG.

The so-called non-linear (or "sigma-model") multiplet [39, 62] consists of three scalars Φ^α_i, two Majorana spinors λ^i, one complex scalar $M^{ij} = M^{[ij]}$ and a Lorentz vector V_a (total 8–8 off-shell degrees of freedom). The set of three scalar fields is represented by an SU(2) matrix Φ^α_i ($\alpha = 1, 2$)

$$\Phi^i_\alpha\,\Phi^\alpha_j = \delta^i_j, \qquad \Phi^\alpha_i\,\Phi^i_\beta = \delta^\alpha_\beta, \qquad \Phi^\alpha_i \equiv (\Phi^i_\alpha)^* = \varepsilon^{\alpha\beta}\varepsilon_{ij}\Phi^j_\beta \tag{3.75}$$

which transforms under the local SU(2) group of $N = 2$ CSG from the left (i), and under an additional rigid SU(2) group from the right (α). The fields have the following weights

	Φ^α_i	$\lambda^i = -\gamma_5\lambda^i$	M^{ij}	V_a
w	0	1/2	1	1
c	0	1	2	0

(3.76)

A peculiarity of this multiplet is that V_a transforms under the conformal boosts. The full Q, S and K transformation rules can be found in [62]; for example

$$\delta\Phi^i{}_\alpha = (\bar\varepsilon^i\lambda_j - \tfrac{1}{2}\delta^i_j\bar\varepsilon^k\lambda_k - \text{h.c.})\Phi^j{}_\alpha$$

$$\delta\lambda^i = -\tfrac{1}{4}\gamma_a V_a\varepsilon^i - \tfrac{1}{4}M^{ij}\varepsilon_j + \Phi^i{}_\alpha\hat{\not{D}}\Phi^\alpha_j\varepsilon^j + \cdots + \eta^i$$

$$\delta M^{ij} = 6\bar\varepsilon^{[i}\chi^{j]} + \cdots, \qquad \delta V^a = \tfrac{3}{4}\bar\varepsilon^i\gamma^a\chi_i + \cdots + 4\varepsilon^a_K\,.$$

The property that distinguishes the non-linear multiplet is that the superconformal algebra closes only if the fields satisfy the constraint [39, 62]

$$\hat D_a V_a - 3D - \tfrac{1}{2}V_a^2 - \tfrac{1}{4}|M_{ij}|^2 + \hat D_a\Phi^i{}_\alpha\,\hat D_a\Phi^\alpha{}_i + 2[\bar\lambda_i(\hat{\not{D}}\lambda^i + \tfrac{3}{2}\chi^i - \tfrac{1}{16}\sigma\cdot T^{-ij}\lambda_j) + \text{c.c.}] = 0\,, \tag{3.77}$$

where $\hat D_a V_a = \partial_a V_a - 4f^\mu_\mu + \cdots$. The linearized form of (3.77) is: $\partial_a V_a - D + \frac{1}{3}R + \cdots = 0$. It is important to stress that this multiplet does not exist in flat space and thus needs supergravity for consistency [39]. A natural candidate for a non-linear multiplet Lagrangian is the constraint (3.77), multiplied by the Lagrange multiplier field $H(x)$

$$e^{-1}\mathcal{L}^{(2)}_{\text{non-linear}} = H(\partial_a V_a - D + \tfrac{1}{3}R - \tfrac{1}{2}V_a^2 + D_a\Phi^i{}_\alpha\,D_a\Phi^\alpha{}_i + 2\bar\lambda_i\hat{\not{D}}\lambda^i + \cdots)\,. \tag{3.78}$$

In order to give a meaning to this odd-looking action, we have to assume that H has a non-zero vacuum value ($H = C(1+u(x))$, $C=$ const.). Then it is possible to define a perturbation theory: $\mathcal{L}^{(2)}_{\text{non-linear}} \simeq \partial_a u V_a + V_a^2 + \partial\phi\,\partial\phi + 2\bar\lambda\overleftrightarrow{\not{\partial}}\lambda + \cdots$ (we suppose also that $\Phi^i_\alpha = \delta^i_\alpha + \phi^i_\alpha$). Integrating out V_a, we find (*neglecting* the contributions of the fields of CSG) that (3.78) describes the $1+3-2\times2$ on-shell degrees of freedom (the same number that corresponds to a hypermultiplet).

Adding to (3.78) the Lagrangian of $N=2$ CSG (3.56) and integrating over D and H, we find instead of (3.78) a highly non-linear term $(\partial_a V_a - \frac{1}{2}V_a^2 + D\Phi\,D\Phi^{-1} + \cdots)^2$.[1] If instead we would add to (3.78) the (ghost) Abelian vector multiplet Lagrangian (3.73), we would then find that the elimination of D gives the constraint $H = |a|^2$ (which implies that, if a has a non-zero vacuum expectation value, the scalars and spinors in non-linear multiplet are ghosts). We then get

$$e^{-1}\mathcal{L}^{(2)}_{\substack{\text{Abelian}\\+\text{ non-linear}}} = |D_\mu a|^2 + \tfrac{1}{6}R|a|^2 + |a|^2(\partial_a V_a + \tfrac{1}{3}R - \tfrac{1}{2}V_a^2 + |D_\mu\Phi^i{}_\alpha|^2 + \cdots) - \tfrac{1}{8}(F_{\mu\nu}{}^{ij}(B))^2$$
$$+ [\tfrac{1}{2}(t_{\mu\nu}{}^{-ij}\varepsilon_{ij})^2 + \tfrac{1}{4}\bar\xi^i\not{D}\xi_i + \text{c.c.}] + \cdots \tag{3.79}$$

Imposing the Poincaré gauge (3.74) we find the off-shell formulation of $N=2$ PSG with an extra local SU(2) symmetry [62]. The role played by a "sigma-model" field $\Phi^i{}_\alpha$ is then to provide an SU(2)-invariant description of the mass term for the SU(2) gauge field of $N=2$ CSG. We can gauge this field away by choosing the SU(2) gauge: $\Phi^i{}_\alpha = \delta^i{}_\alpha$. As a result, we are led to the conventional (type I) off-shell formulation of $N=2$ PSG, with rigid SU(2) invariance [113, 59, 18]. The corresponding linearized Lagrangian is given by (we put $a^2 = 2/k^2$)

$$e^{-1}\mathcal{L}^{(2)}_{\text{PSG}} \simeq \frac{1}{k^2}\Big[R - e^{-1}\varepsilon^{\mu\nu\lambda\rho}\bar\psi^i_\mu\gamma_\nu\overleftrightarrow{D}_\lambda\psi_{\rho i} - \frac{k^2}{8}(F_{\mu\nu}{}^{ij}(B))^2 + 4(\bar\lambda^i(\not{\partial}\lambda_i + \chi_i + \tfrac{1}{3}\gamma_\mu\cdot R_{\mu i}) + \text{c.c.})$$
$$+\tfrac{1}{2}A_a^2 - 2V_a{}^i{}_j V_a{}^j{}_i - V_a^2 - \tfrac{1}{2}|M_{ij}|^2\Big] + \tfrac{1}{2}(t^{ij}_{\mu\nu}\varepsilon_{ij})^2 - \tfrac{1}{2}|S_{ij}|^2, \qquad R^\mu_i = e^{-1}\varepsilon^{\mu\lambda\rho\sigma}\gamma_\lambda D_\rho\psi_{\sigma i}\,. \tag{3.80}$$

[1] The resulting theory, considered as a new version of $N=2$ CSG was studied by I.V. Frolov (unpublished).

Comparing it with (3.43), (3.44), (3.45) we conclude that the auxiliary field terms have exactly the structure needed to generate the massive $N=2$ multiplet, after coupling (3.80) to the conformal supergravity action. To see this in more detail (and in particular, to restore the D-dependence of (3.80)) let us consider the combined action, containing the Abelian vector multiplet, the non-linear multiplet and the conformal supergravity parts (3.73), (3.78) and (3.56), $\mathcal{L}_{\rm tot} \sim 3\alpha^{-2}D^2 + |a|^2 D + H(\partial_a V_a - D - \frac{1}{2}V_a^2)$. Putting $a = \text{const.}$ and assuming that $\langle D\rangle = 0$, we get $H = a^2(1+u(x))$, $\langle u\rangle = 0$. Integrating over V_a, we find, in the linear approximation:

$$\mathcal{L}'_{\rm tot} \sim 3\alpha^{-2}D^2 + a^2[\tfrac{1}{2}(\partial_a u)^2 - uD] \tag{3.81}$$

which is in obvious agreement with (3.43).

It is also possible to consider a more general system of ($N=2$ CSG) + (non-linear multiplet) + ($N=2$ super-Yang–Mills with a gauge group with one non-compact generator). The resulting low-energy theory will be gauge-equivalent to the $N=2$ supersymmetric Einstein–Yang–Mills theory [56]. An apparent drawback of all such models, based on a non-linear multiplet, is their non-renormalizability.

Let us now turn to the $N=2$ scalar multiplet [73, 226]. When the superconformal algebra is realized on the minimal field representation, it possesses a field dependent central charge [61, 62], which acts only on the Abelian gauge field ($\delta_Z B_\mu{}^{ij} \sim \varepsilon^{ij}\partial_\mu Z$). The latter thus plays the role of the gauge field of the central charge transformations. As a result, we can try to realize the algebra on multiplets with central charge, e.g. on a scalar multiplet. The scalar multiplet consists of a set of complex scalars $A_i{}^\alpha$, $\alpha = 1, 2$, a doublet of Majorana spinors ζ^α, and auxiliary scalar fields $A^{(Z)}{}_i^\alpha$ (which can be considered as the result of the application of the central charge transformation to the $A_i{}^\alpha$, since all further iterations $A^{(ZZ)}, \ldots$ are not independent). The fields satisfy the following restrictions

$$\gamma_5\zeta^\alpha = -\zeta^\alpha , \qquad A^i{}_\alpha \equiv (A^\alpha{}_i)^* = \varepsilon^{ij}\varepsilon_{\alpha\beta}A^\beta{}_j, \text{ same for } A^{(Z)}.$$

(3.82)

	$A^\alpha{}_i$	ζ^α	$A^{(Z)\alpha}{}_i$
w	1	3/2	2
c	0	1	−1

Thus the $A^i{}_\alpha$ correspond to four real scalars and we get a total of 8–8 off-shell degrees of freedom. As in the case of the non-linear multiplet, we have a local superconformal SU(2) acting on $i, j, \ldots$ and an extra rigid SU(2) acting on $\alpha, \beta, \ldots$. The Q and S supersymmetry transformations are given by [62]

$$\begin{aligned} \delta A^\alpha{}_i &= \bar\zeta^\alpha\varepsilon_i + \varepsilon^{\alpha\beta}\varepsilon_{ij}\bar\zeta_\beta\varepsilon^j , \\ \delta\zeta^\alpha &= \tfrac{1}{2}\hat{\not{D}}A^\alpha{}_i\varepsilon^i + \tfrac{1}{2}aA^{(Z)}{}_i^\alpha\varepsilon^{ij}\varepsilon_j + A^\alpha{}_i\eta^i , \end{aligned} \tag{3.83}$$

where a is the scalar of the Abelian vector multiplet and the derivative is also covariant with respect to the central charge transformations. The superconformal Lagrangian for the scalar multiplet has the following structure [61]

$$\begin{aligned} e^{-1}\mathcal{L}^{(2)}_{\rm scalar} &= A^i{}_\alpha(\hat{D}_a^2 + \tfrac{3}{2}D)A_i^\alpha + [|a|^2 + \tfrac{1}{8}(B_a{}^{ij}\varepsilon_{ij})^2]|A^{(Z)}{}_\alpha{}^i|^2 \\ &\quad - 4(\bar\zeta^\alpha - \tfrac{1}{2}A_k{}^\alpha\bar\psi_\mu{}^k\gamma_\mu)(\hat{\not{D}}\zeta_\alpha + \tfrac{3}{2}A_\alpha{}^i\chi_i - \tfrac{1}{2}\sigma\cdot T^+_{ij}\varepsilon^{ij}\varepsilon_{\alpha\beta}\zeta^\beta) + \text{c.c.} \end{aligned} \tag{3.84}$$

where $\hat{D}_a^2 = D_a^2 - \frac{1}{6}R - D + \cdots$ and B_μ and $A^{(Z)}$ are assumed to be omitted in covariant derivatives. To construct the coupling of the scalar multiplet to conformal supergravity, we must add to (3.84) and (3.56) the Lagrangian for the Abelian central charge vector multiplet (3.73). To get the low energy truncation of this theory which coincides with off-shell (type II) $N=2$ Poincaré supergravity [62] we have to take *both* the vector multiplet and the scalar multiplet with the ghost signs (for the vector multiplet this was already done in (3.73)). We have also to impose, in addition to the Poincaré gauge (3.74), the SU(2) gauge condition (analogous to $\Phi^i{}_\alpha = \delta^i{}_\alpha$ in the case of a non-linear multiplet)

$$A^\alpha{}_i = a^2\, e^{u(x)} \delta^\alpha{}_i . \tag{3.85}$$

Here $u(x)$ is a residual real scalar field. The combined Lagrangian then takes the form

$$\begin{aligned} e^{-1}\mathscr{L}^{(2)}_{\substack{\text{Abelian}\\ +\text{scalar}}} &= \tfrac{1}{6}a^2(1+2e^{2u})R + a^2(1-e^{2u})D + 2a^2 e^{2u}(\partial_\mu u)^2 - \tfrac{1}{8}(F_{\mu\nu}{}^{ij}(B))^2 \\ &\quad - a^2 e^{2u}(1+\tfrac{1}{8}(\varepsilon_{ij}B^{ij}_\mu)^2)|G^\alpha{}_j|^2 + \cdots , \qquad G^\alpha{}_i \equiv e^u A^{(Z)\alpha}{}_i . \end{aligned} \tag{3.86}$$

The constraint implied by D is $u=0$. Expanding near this "vacuum", we find that the linearized expression for (3.86) coincides with (3.80) with $\lambda^i = e^{-2u}A^i{}_\alpha\zeta^\alpha$ and $-\frac{1}{4}|M_{ij}|^2 - \frac{1}{2}V_a{}^2 \to 2(\partial_a u)^2 - |G^\alpha{}_i|^2 - 2uD$. Again the result is in agreement with (3.43).

The importance of the scalar multiplet relies in the fact that it can be used for the construction of renormalizable "realistic" supersymmetrical models. For example, combining the $N=2$ vector and scalar multiplets we can in principle construct $N=2$ superconformal generalizations of finite $N=2$ supersymmetric theories in flat space [161, 160]. To give an example of such models let us consider the coupling of the scalar multiplet (accompanied as always by the Abelian $N=2$ vector multiplet of the minimal field representation) to the $N=2$ super-Yang–Mills. Adding together (3.71), (3.73) and (3.84) we find:

$$\begin{aligned} e^{-1}\mathscr{L}^{(2)}_{\text{YM+scalar}} &= \text{tr}(D_\mu X^* D_\mu X) + |D_\mu a|^2 - |D_\mu A^i{}_\alpha|^2 + \tfrac{1}{6}R[\text{tr}(XX^*) + |a|^2 - |A^i{}_\alpha|^2] \\ &\quad + D[\text{tr}(XX^*) + |a|^2 + \tfrac{1}{2}|A^i{}_\alpha|^2] + \cdots . \end{aligned} \tag{3.87}$$

In order to obtain a consistent truncation to an off-shell version of the $N=2$ PSG, we have to add either one extra ghost scalar multiplet or the non-linear multiplet (3.78). In the latter case the elimination of D yields: $H = \text{tr}(XX^*) + |a|^2 + \frac{1}{2}|A^i{}_\alpha|^2$. Substituting this back in the Lagrangian we obtain

$$e^{-1}\mathscr{L}^{(2)}_{\substack{\text{YM + scalar}\\ +\text{ non-linear}}} = \tfrac{1}{2}R[\text{tr}(XX^*) + |a|^2] + \text{tr}(D_\mu X^* D_\mu X) + \cdots$$

The convenient choice of dilational gauge is now $a = [2/k^2 - \text{tr}(X^*X)]^{1/2}$ [62]. It leads to the standard normalization of the Einstein term. The discussion of a general matter system of $N=2$ vector and scalar multiplets coupled to CSG can also be found in ref. [55].

The next example of $N=2$ superconformal matter multiplet is provided by the $N=2$ tensor multiplet [57] (cf. the discussion of the $N=1$ tensor multiplet in section 2.3). The $N=2$ tensor multiplet includes a triplet of scalars $L^{ij} = L^{(ij)}$, a tensor gauge field $E_{\mu\nu}$, a doublet of Majorana spinors ζ^i and a complex auxiliary scalar G, satisfying (i, j are SU(2) indices)

$$L_{ij} \equiv (L^{ij})^* = \varepsilon_{ik}\varepsilon_{jl}L^{kl}, \qquad \gamma_5\zeta^i = \zeta^i ,$$

	L^{ij}	ζ^i	$E_{\mu\nu}$	G
w	2	$\frac{5}{2}$	0	3
c	0	-1	0	-2

The flat space supersymmetric Lagrangian [59] (cf. (2.60))

$$\mathcal{L}^{(2)}_{\text{tensor}} = -\tfrac{1}{4}|\partial_\mu L^{ij}|^2 - \tfrac{1}{2}\bar{\zeta}^i \overleftrightarrow{\not\partial}\zeta_i + \tfrac{1}{2}|G|^2 + \tfrac{1}{2}E_\mu E_\mu, \qquad E_\mu = \tfrac{1}{2}\mathrm{i}e_{\mu\nu\lambda\rho}\,\partial_\nu E_{\lambda\rho} \tag{3.88}$$

describes the same number of on-shell states as the scalar multiplet (3 + 1 spin 0 and 2 spin 1/2) and it is thus a version of the hypermultiplet discussed in ref. [73]. As in the $N = 1$ case, the naive Lagrangian (3.88) is not scale invariant and cannot be used for the construction of the superconformal extension of the tensor multiplet action. The on-shell equivalent but scale invariant modification of (3.88), found in ref. [57], is analogous to (2.61), i.e. it depends on an inverse power of $L = (L_{ij}L^{ij})^{1/2}$ (with $w = 2$), which plays the role of a compensator for dilations. The corresponding superconformal extension has the form (cf. (2.62)) [57]

$$\begin{aligned} e^{-1}\mathcal{L}^{(2)}_{\text{tensor}} = &-\tfrac{1}{2}|D_\mu L^{ij}|^2 L^{-1} + |G|^2 L^{-1} + \hat{E}^2_\mu L^{-1} - \bar{\zeta}^i \overleftrightarrow{\hat{\not D}}\zeta_i L^{-1} \\ &- |\bar{\zeta}_i\zeta_j|^2 L^{-3} + 3|\bar{\zeta}_i\zeta_j L^{ij}|^2 L^{-5} + L(-\tfrac{1}{3}R + D) - V_\mu{}^k{}_i L_{kj}\varepsilon^{ij}L^{-1}\hat{E}_\mu + \cdots \end{aligned} \tag{3.89}$$

where

$$\hat{E}_\mu = \tfrac{1}{2}\mathrm{i}e\varepsilon_{\mu\nu\rho\sigma}\partial_\nu E_{\rho\sigma} - \bar{\psi}^i{}_\nu\sigma_{\mu\nu}\zeta^j\varepsilon_{ij} + \cdots$$

is the supercovariant field strength and the supersymmetry transformation rules are given by[2]

$$\begin{aligned} &\delta L_{ij} = \bar{\varepsilon}_{(i}\zeta_{j)} - \varepsilon_{k(i}\varepsilon_{j)l}\bar{\varepsilon}^k\zeta^l\,, \\ &\delta\zeta^i = \tfrac{1}{2}\hat{\not D}L^{ij}\varepsilon_j + \tfrac{1}{2}\hat{\not E}\varepsilon^{ij}\varepsilon_j - \tfrac{1}{2}G\varepsilon^i + 2L^{ij}\eta_j\,, \\ &\delta G = -\bar{\varepsilon}_i\hat{\not D}\zeta^i + \cdots + 2\bar{\eta}_i\zeta^i, \qquad \delta E_{\mu\nu} = \mathrm{i}\bar{\varepsilon}^i\sigma_{\mu\nu}\zeta^j\varepsilon_{ij} + \cdots . \end{aligned} \tag{3.90}$$

In the absence of supergravity (3.89) describes a massless non-interacting theory [57]. This suggests that (3.89) defines an (on-shell) renormalizable theory, when coupled to $N = 2$ CSG.[3] Since it corresponds to 8–8 off-shell degrees of freedom, the tensor multiplet can be used to construct a (type III) off-shell version of $N = 2$ PSG [57]. As in previous cases, to get the physical signs of the supergravity kinetic terms we have to combine the actions for the *ghost* Abelian vector multiplet (3.73), and for the *ghost* tensor multiplet. The scalars L_{ij} play the role of compensators for the SU(2) transformations of $N = 2$ CSG. Thus, in addition to (3.74), we may put (cf. (3.85))

$$L_{ij} = \frac{1}{\sqrt{2}}|a|^2\delta_{ij}\,\mathrm{e}^{2u(x)}\,. \tag{3.91}$$

[2] The fields L, ζ, G and $E^\perp_\mu$ constitute the so-called linear multiplet [18, 19].

[3] To develop a perturbation theory we are to assume that the vacuum value of the scalar matrix L_{ij} is proportional to δ_{ij}.

This SU(2) gauge leaves the residual SO(2) symmetry, and we find for the bosonic terms in the Lagrangian

$$e^{-1}\mathscr{L}^{(2)}_{\substack{\text{Abelian}\\ +\text{tensor}}} = |D_\mu a|^2 + |a|^2(\tfrac{1}{6}R + D) + 2|a|^2 e^{2u}(\partial_\mu u)^2 + (\tfrac{1}{3}R - D)|a|^2 e^{2u} + \cdots \to (a = a^* = \text{const.})$$
$$\to \tfrac{1}{6}a^2 R(1 + 2e^{2u}) + a^2(1 - e^{2u})D - \tfrac{1}{8}(F_{\mu\nu}{}^{ij}(B))^2 + a^2[\tfrac{1}{4}A_a^2 + 2e^{2u}(\partial_a u)^2$$
$$-\tfrac{1}{2}e^{-2u}|G|^2 - \tfrac{1}{2}e^{-2u}E_\mu^2 - \tfrac{1}{2}e^{2u}V_\mu{}^i{}_j(V_\mu{}^i{}_j + V_\mu{}^j{}_i) - V_\mu{}^i{}_j E_\mu \varepsilon^{ij}]\,. \tag{3.92}$$

The linearized Lagrangian looks like (3.80), with $\lambda^i \to \zeta^i$ and $V_a{}^i{}_j V_a{}^j{}_i + \frac{1}{4}|M_{ij}|^2$ substituted for $V_a{}^i{}_j V_a{}^{(j}{}_{i)} + \frac{1}{2}|G|^2 + \frac{1}{2}E_\mu^2 + V_\mu{}^i{}_j E_\mu \varepsilon_{ij}$. The $E_{\mu\nu}$-dependent terms provide a gauge invariant description of the SO(2) gauge field mass term.

It should be mentioned that, by adding some additional couplings, it is possible to describe gauged $N = 2$ supergravity [118, 112] within the superconformal framework [62]. Let us also note that other higher derivative $N = 2$ superconformal multiplets will be discussed in section 4.3. The quantum properties of the $N = 2$ multiplets will be analyzed in section 6.3.

3.5. $N = 4$ super Yang–Mills theory coupled to $N = 4$ conformal supergravity

The field content and the Lagrangian (3.19) for an Abelian $N = 4$ vector multiplet were already presented in section 3.2. In the non-Abelian case, the fields $(B_\mu, \lambda^i, \phi_{ij})$ belong to the adjoint representation of an internal gauge group G and the action includes the interaction terms [140, 141]

$$\mathscr{L}^{(4)}_{\text{YM}} = \text{tr}\{\tfrac{1}{8}F^2_{\mu\nu}(B) + \tfrac{1}{4}\bar{\lambda}_i \overleftrightarrow{\mathscr{D}}\lambda^i + \tfrac{1}{4}\mathscr{D}_\mu\phi^{ij}\mathscr{D}_\mu\phi_{ij} + \tfrac{1}{2}g(\bar{\lambda}^i[\lambda^j, \phi_{ij}] + \text{c.c.}) - \tfrac{1}{8}g^2[\phi_{ij}, \phi_{kl}][\phi^{ij}, \phi^{kl}]\}\,. \tag{3.93}$$

This theory is invariant under the rigid $N = 4$ superconformal group SU(2, 2|4) and thus is a natural "source" for the $N = 4$ CSG (cf. section 3.2). Though the corresponding supersymmetry algebra closes only on the field equations, the absence of auxiliary fields [221, 209] is not a serious obstacle on the way of extending (3.93) to a locally superconformal invariant expression (we have to admit that the local supersymmetry algebra will also close only on the mass shell).

The absence at the present time of a $N = 4$ density formula prevents us from establishing the complete expression for the $N = 4$ super Yang–Mills action on the $N = 4$ CSG background[1] (as well as for the action of $N = 4$ CSG itself). At the same time, it is however possible to construct a coupling iteratively (in powers of the fields of conformal supergravity) using the Noether procedure. The first step was already done in [8] (see section 3.2), where the coupling terms linear in the CSG fields were found. For simplicity we shall consider the case of an Abelian $N = 4$ vector multiplet (generalization to the non-Abelian case is rather straightforward). Combining (3.21)–(3.23) and (3.19), and making obvious covariantizations we find

$$e^{-1}\mathscr{L}^{(4)}_{\text{Abelian}} = -\tfrac{1}{2}D_\mu\phi^{ij} D_\mu\phi_{ij} - \tfrac{1}{12}R|\phi_{ij}|^2 + D^{kl}{}_{ij}(\phi^{ij}\phi_{kl} - \tfrac{1}{6}\delta^i_{[k}\delta^j_{l]}|\phi_{mn}|^2) + [-\tfrac{1}{4}F^{+2}_{\mu\nu}(B) - \mathrm{i}T^+_{\mu\nu ij}\phi^{ij}F^+_{\mu\nu}(B)$$
$$+ \tfrac{1}{2}\varphi F^{-2}_{\mu\nu}(B) + \text{c.c.}] + [-\tfrac{1}{2}\bar{\lambda}_i \not{D}\lambda^i + \bar{\psi}^i_\mu(-\tfrac{1}{2}\sigma\cdot F^-\gamma_\mu\lambda_i + \mathrm{i}\phi_{ij}\overleftrightarrow{\partial}_\mu\lambda^j + \tfrac{2}{3}\mathrm{i}\sigma_{\mu\lambda}\,\partial_\lambda(\phi_{ij}\lambda^j))$$
$$+ \tfrac{1}{4}\bar{\chi}^k{}_{ij}\varepsilon^{ijmn}(\phi_{mn}\lambda_k + \phi_{kn}\lambda_m) + \tfrac{1}{4}T^-_{\mu\nu}{}^{ij}\varepsilon_{ijmn}\bar{\lambda}^m\sigma_{\mu\nu}\lambda^n + \tfrac{1}{2}E^{ij}\bar{\lambda}_i\lambda_j + \tfrac{1}{2}\bar{\Lambda}^i\sigma\cdot F^-\lambda_i + \text{c.c.}] + \cdots,$$
$$D_\mu = \partial_\mu - V_\mu{}^i{}_j + \cdots\,. \tag{3.94}$$

[1] The complete action can in principle be determined by reducing to four dimensions the known superconformal action in ten dimensions [9] (see section 4.1).

To establish the correspondence between (3.94) and the $N=2$ super-Yang–Mills Lagrangian (3.71), we have to make the substitution: $\phi^{ij}\to \mathrm{i}\varepsilon^{ij}X$. Using the correspondence with the $N=2$ case as a guiding principle, it is easy to guess the structure of some higher order terms. For example, one should find the $(T^{+}_{\mu\nu ij}\phi^{ij})^2$ + c.c.-term, the Rarita–Schwinger extension of R, etc. Next, there should also be terms of higher order in the CSG scalar φ, e.g. $\varphi^2 F^{-2}_{\mu\nu}$ + c.c., etc. This is necessary for the correspondence with $N=4$ PSG (see (3.38) and below). Thus the extension of bosonic terms in (3.94) should look like

$$e^{-1}\mathcal{L}^{(4)}_{\text{Abelian, Bose}} = -\tfrac{1}{4}F^{+2}_{\mu\nu}(1-2\varphi+\varphi^2+\cdots) - \mathrm{i}T^{+}_{\mu\nu ij}\phi^{ij}F^{+}_{\mu\nu} + \tfrac{1}{2}(T^{+}_{ij}\phi^{ij})^2 + \text{c.c.} + \cdots . \tag{3.95}$$

Adding now (3.94) to the Lagrangian of $N=4$ CSG (3.53), we conclude that the resulting theory possesses the Einstein low energy limit if at least one of the $N=4$ super-Yang–Mills scalars has a non-vanishing vacuum expectation value. Stated differently, we find a term of the order of R after fixing the dilation gauge, e.g., through the condition $|\phi_{ij}|^2\sim 1/k^2$. To provide the correct sign of the Einstein term we have to take one of the $N=4$ vector multiplets to be a ghost. Note that, as in the $N=2$ case, the Maxwell term for the corresponding Abelian vector will have the physical sign after elimination of the field $t_{\mu\nu}\sim T_{\mu\nu}-F_{\mu\nu}\phi^{-1}$ (cf. (3.45), (3.73)), which is non-propagating in the low energy limit. Assuming that $N=4$ supersymmetry is broken down to $N=2$, i.e. that two "extra" gravitinos are massive, we find (by noting that one $N=4$ Abelian multiplet can be split in one $N=2$ Abelian vector multiplet and one hypermultiplet) that the theory should possess a consistent truncation to $N=2$ Poincaré supergravity.

According to the on-shell counting argument given in section 3.2 (see eqs. (3.38), (3.39)) in order to have a consistent truncation of ($N=4$ super-Yang–Mills) + ($N=4$ CSG) theory to $N=4$ PSG we must introduce *six* ghost Abelian $N=4$ vector multiplets with a rigid SO(4) group (see also [131, 159, 7]). Let $(B^{[mn]}_{\mu}, \lambda^{[mn]}_{i}, \phi^{[mn]}_{ij})$ be the fields of these multiplets ($m, n = 1,\ldots,4$ are indices of SO(4) in which the fields are antisymmetric) and let us split the $\lambda^{[mn]}_{i}$ and $\phi^{[mn]}_{ij}$ in irreducible representations of SU(4) i.e. $6\times 4 = 20+4$ and $6\times 6 = 1+15+20'$, namely:

$$\begin{aligned}
&\lambda_i^{[mn]} = \lambda_i^{\ mn} + \delta_i^{[m}\lambda^{n]}, \qquad \lambda_n^{\ mn}\equiv 0,\\
&\phi_{ij}^{[mn]} = (\delta^p_i\delta^q_j C + \tfrac{1}{2}\varepsilon^{pq}{}_{ij})(\delta^m_{[p}\delta^n_{q]}\phi + \delta^m_{[p}\phi^n_{q]} + u^{mn}{}_{pq}),\\
&\phi=\phi^*, \qquad (\phi^m{}_n)^* = -\phi^n{}_m, \qquad \phi^n{}_n = 0, \qquad u^{mn}{}_{pn}=0,\\
&(u^{kl}{}_{ij})^* = u^{ij}{}_{kl}, \qquad u^{mn}{}_{pq} = \tfrac{1}{4}\varepsilon^{mnkl}\varepsilon_{pqrs}u^{rs}{}_{kl}
\end{aligned} \tag{3.96}$$

where we have used the fact that ϕ_{ij} is "self-dual": $\phi^*_{ij} = \tfrac{1}{2}\varepsilon_{ijkl}\phi_{kl}$ (C stands for the operation of complex conjugation). The field content of $N=4$ (on-shell) PSG, $N=4$ CSG and SO(4) $N=4$ (on-shell) super Yang–Mills is summarized in table 3.6 (from now on we do not differentiate between SO(4) and SU(4) indices).

Table 3.6
Field content of $N=4$ Poincaré and conformal supergravity and SO(4) $N=4$ vector multiplets

s	$N=4$ conformal SG	SO(4) $N=4$ Yang–Mills	$N=4$ Poincaré SG
2	$e^a_\mu(1)$	—	$e^a_\mu(1)$
3/2	$\psi^i_\mu(4)$	—	$\psi^i_\mu(4)$
1	$V_\mu{}^i{}_j(15)$, $T^{ij}_{\mu\nu}(6)$	$B_\mu{}^{ij}(6)$	$B_\mu{}^{ij}(6)$
1/2	$\Lambda_i(4)$, $\chi^{ij}{}_k(20)$	$\lambda^{ij}{}_k(20)$, $\lambda^i(4)$	$\Lambda_i(4)$
0	$\varphi(1)$, $E_{ij}(10)$, $D^{ij}{}_{kl}(20')$	$\phi(1)_{\text{real}}$, $\phi^i_j(15)$, $u^{ij}{}_{kl}(20')$	$\varphi(1)$

We then understand the role played by the fields of the ghost SO(4) vector multiplets: ϕ is a compensator for local dilations, λ^i compensates S-supersymmetry and ϕ^i_j-local SU(4) transformations; $\lambda^{ij}{}_k$ and $u^{ij}{}_{kl}$ are the auxiliary fields for which $\chi^{ij}{}_k$ and $D^{ij}{}_{kl}$ are the "Lagrange multipliers" (cf. (3.43)); $B_\mu{}^{ij}$ (as well as Λ_i and φ of CSG, see table 3.1) are the physical fields of $N=4$ PSG. To obtain the Lagrangian of $N=4$ PSG from the Lagrangian (3.94) for $N=4$ vector multiplets we have to fix the D, S and SU(4) gauges: $\phi\sim 1/k$, $\lambda^i=0$, $\phi^i{}_j=\text{const.}$ (instead of fixing the D and SU(4) gauges we can make the assumption that ϕ and $\phi^i{}_j$ have suitable non-zero vacuum expectation values). The result should agree with the Poincaré supergravity part of (3.41)–(3.44) (for example, $D\phi u\to(1/k)Du$, $\bar\chi\phi\lambda\to(1/k)\bar\chi\lambda$, etc.).[2] The set of $N=4$ PSG auxiliary fields which we thus find is however incomplete: we lack the auxiliary fields of the SO(4) vector multiplets. This reveals an intimate relation between the auxiliary field problems for $N=4$ super-Yang–Mills and $N=4$ Poincaré supergravity [131, 159].

Let us add a comment about the derivation of the kinetic $F_{\mu\nu}{}^2(B)$ terms in the $N=4$ PSG Lagrangian, which is known to have the non-polynomial form (see (3.66))

$$(\mathscr{L}^{(4)}_{\rm PSG})_{B_\mu}=-\tfrac{1}{8}\Big[\Big(\frac{1-\varphi}{1+\varphi}\Big)(F^{+\,ij}_{+\,\mu\nu})^2+\Big(\frac{1+\varphi}{1-\varphi}\Big)(F^{+\,ij}_{-\,\mu\nu})^2+\text{c.c.}\Big] \tag{3.97}$$

where $F^{ij}_{\pm}=\tfrac{1}{2}(F^{ij}\pm\tfrac{1}{2}\varepsilon^{ijkl}F^{kl})$. This expression should follow from (3.95) after elimination of the auxiliary field $T_{\mu\nu}$ (note that the difference between the F_+ and F_--components is introduced by the choice of the vacuum value for $\phi_{ij}{}^{mn}$ in (3.96)).

Finally we note that all the discussion of this section can be repeated for $N=3$ (the field content of the $N=3$ super-Yang–Mills theory is the same as for $N=4$). The derivation of the $N=3$ PSG action will be much simpler than in the $N=4$ case (and analogous to the $N=2$ case) because of the absence of the non-polynomial structure associated with the scalar φ.

4. Conformal supergravity in other approaches

4.1. *Conformal supergravity in ten dimensions*

The aim of this section is to clarify the structure of $N=4$ CSG and its interaction with $N=4$ super-Yang–Mills starting with the known formulations of these theories in ten dimensions [9] and carrying out the (trivial) dimensional reduction. Examples where the study of a higher dimensional supersymmetric theory helps to understand the properties of its lower dimensional counterpart are well-known (see e.g. [140, 34, 26]).

Let us first consider the higher dimensional generalizations of the Weyl theory. Taking the conformal group SO(d, 2) as the gauge group of conformal gravity, it is straightforward to work out the "kinematics" of the corresponding gauge theory [49, 9]. The construction is parallel to that followed in the $d=4$ case (see section 2.1). For example, the constraints have the universal form (2.14), (2.15). As a result, we find that $-\mathscr{R}^{AB}_{MN}(M)$ (cf. (2.19)) coincides with the higher dimensional generalization of the Weyl tensor $C^M{}_{NPQ}$(B.6).[1] The "geometrical" action (cf. (2.18)) is unique and looks like (we shall

[2] The full non-linear derivation of the action is rather complicated; for example the SU(4) gauge degrees of freedom $\phi^i{}_j$ should be some non-polynomial functions of the original scalars ϕ_{kl}.

[1] Throughout this section we use the following notation: $M,N,\ldots=1,\ldots,d$; $\mu,\nu=1,\ldots,4$; $r,s,t=1,\ldots,d-4$; Γ_M are Dirac matrices in d dimensions; $\Gamma_{M_1\cdots M_k}=\Gamma_{[M_1}\ldots\Gamma_{M_k]}$.

consider the case of even dimensions $d = 2n$)

$$I_{\text{geom.}} \sim \int d^d x\, \varepsilon_{A_1\cdots A_{2n}} \mathcal{R}^{A_1A_2}_{M_1M_2}(M) \cdots \mathcal{R}^{A_{2n-1}A_{2n}}_{M_{2n-1}M_{2n}}(M)\, \varepsilon^{M_1\cdots M_n}$$

$$\sim \int d^d x\, e\, C^{M_1N_1}_{P_1Q_1} C^{P_1Q_1}_{M_2N_2} \cdots C^{P_nQ_n}_{M_1N_1}, \tag{4.1}$$

$$e = \det e^A{}_M .$$

There are also plenty of other "non-geometrical" generally covariant and Weyl invariant dimensionless actions which can be constructed from the Weyl tensor and appropriate conformal covariant differential operators, $\int d^d x\, e \sum_{k=2}^{n} C^k \Box^{n-k}$, e.g.

$$\int d^d x\, e\, (C_{MNPQ} \Box^{n-2} C^{MNPQ} + \cdots + C^{M_1N_1}_{P_1Q_1} \cdots C^{M_nN_n}_{P_nQ_n}) . \tag{4.2}$$

The number of off-shell states corresponding to (4.1) or (4.2) coincides with the number of states for the massive spin 2 particle ($n = \frac{1}{2}d(d+1) - d - 1 = \frac{1}{2}(d+1)(d-2)$). The only term which describes degrees of freedom for on-shell propagation is $C\Box^{n-2}C \sim h\Box^n P_2 h$ (cf. (1.36)). Their number is $\nu = \frac{1}{2}d[\frac{1}{2}(d+1)(d-2) - 2]$ and, for $d = 4$, this agrees with (1.45).

We conclude that for each particular dimension d, there is a corresponding conformal action (4.1) or (4.2) and these actions are structurally different in different dimensions. For example, the Weyl action $\int d^4x\, C^2$ is tied to four dimensions. Hence it cannot be obtained from higher dimensional actions like (4.2) by means of dimensional reduction. This is clearly different from the case of the dimensional reduction of the Einstein action. To find the direct higher dimensional analog of the Weyl action, one has to introduce an additional scalar field Φ, which should play the role of a compensator for dilatations. Then the scale invariant $d > 4$ generalization of the Weyl Lagrangian (1.3) is given by [9]

$$\mathcal{L} = -\frac{1}{\alpha_0^2} C^N{}_{MPQ} C^M{}_{NSR} \Phi^{d-4} g^{PS} g^{QR} \sqrt{g} \tag{4.3}$$

where Φ has the unit Weyl weight (D: $e^{A\prime}{}_M = e^{-\lambda} e^A{}_M$, $\Phi' = e^{\lambda}\Phi$). The theory described by (4.3) has two mutually related shortcomings: it is based on a reducible off-shell representation (massive spin 2 and massless spin 0) and it is not invariant under conformal boosts. To provide the irreducibility and K-invariance of (4.3) it is necessary to subject Φ to the conformal invariant differential constraint [9]

$$\left[-\mathcal{D}^2 + \frac{d-2}{4(d-1)} R\right] \Phi^{(d-2)/2} = 0, \qquad R = R^{MN}{}_{MN}, \tag{4.4}$$

which simply states that the Weyl-invariant vielbein $\bar{e}^A{}_M = \Phi e^A{}_M$ has a zero curvature scalar (cf. (B.5)). Supposing that $\Phi|_{x\to\infty} \to m = \text{const.}$, we can solve (4.4) to obtain a non-local expression for $\Phi(g_{MN}) \equiv m\, e^{\sigma}$,

$$\exp\left[\left(\frac{d-2}{2}\right)\sigma\right] = 1 - \left[-\frac{4(d-1)}{(d-2)} \mathcal{D}^2 + R\right]^{-1} R . \tag{4.5}$$

Here $[m]=1$, $[\sigma]=0$ (note that α_0 in (4.3) is dimensionless). Eliminating Φ from (4.3), we find a conformal invariant action which depends only on the metric but is *non-local*. Thus the analogy with the $d=4$ Weyl action is reached at the expense of introducing a non-locality, that limits the possible applications of (4.3) as a $d>4$-dimensional theory.

It is now crucial to check that the dimensional reduction of (4.3) (understood in the trivial sense $M^d=M^4\times S^1\times\cdots\times S^1$, $\partial_r g_{MN}=0$) leads to a well-defined *conformal invariant local* four-dimensional action, containing the $d=4$ Weyl action as part of it. In order to simplify the matter, let us first consider the reduction of the linearized theory

$$
\begin{aligned}
&g_{MN}=\begin{bmatrix}\delta_{\mu\nu}+h_{\mu\nu} & h_{r\nu}\\ h_{\mu s} & h_{rs}\end{bmatrix}, \quad \begin{matrix}\mu,\nu=1,\dots,4\\ r,s,t=1,\dots,d-4,\end{matrix}\\
&\text{X: } \delta h_{\mu\nu}=\partial_\mu\xi_\nu+\partial_\nu\xi_\mu, \qquad \delta h_{\mu r}=\partial_\mu\xi_r, \qquad \delta h_{rs}=0\\
&\text{D: } \delta h_{\mu\nu}=-2\lambda\delta_{\mu\nu}, \qquad \delta h_{\mu r}=0, \qquad \delta h_{rs}=-2\lambda h_{rs}\,.
\end{aligned}
\tag{4.6}
$$

Recalling that

$$
\begin{aligned}
&R_{MN}\simeq\tfrac{1}{2}(\partial_M\partial_K h_{KN}+\partial_N\partial_K h_{KM}-\partial_M\partial_N h^P_{\ P}-\Box h_{MN})+O(h^2)\\
&C_{MNPQ}C^{MNPQ}=4\left(\frac{d-3}{d-2}\right)\left(R_{MN}^{\ 2}-\frac{d}{4(d-1)}R^2\right)+(\text{total covariant derivative}),
\end{aligned}
\tag{4.7}
$$

we find

$$
\begin{aligned}
&R_{MN}\simeq\begin{bmatrix}R^{(4)}_{\mu\nu}-\tfrac{1}{2}\partial_\mu\partial_\nu h_{tt} & -\tfrac{1}{2}\partial_\sigma F^r_{\ \sigma\nu}\\ -\tfrac{1}{2}\partial_\sigma F^s_{\ \sigma\mu} & -\tfrac{1}{2}(\Box\bar{h}_{rs}+1/(d-4)\,\delta_{rs}\Box h_{tt})\end{bmatrix}\\
&F^r_{\mu\nu}\equiv\partial_\mu h^r_{\ \nu}-\partial_\nu h^r_{\ \mu}, \qquad \bar{h}_{rs}\equiv h_{rs}-\frac{1}{(d-4)}\delta_{rs}h_{tt}, \qquad R\simeq R^{(4)}-\Box h_{tt}\,.
\end{aligned}
\tag{4.8}
$$

Substitution of (4.7) and (4.8) in (4.3), and integration over the "compact dimensions", yields

$$
\mathcal{L}^{(d=4)}\simeq\frac{1}{\alpha^2}[(R^{(4)2}_{\mu\nu}-\tfrac{1}{3}R^{(4)2})+\tfrac{1}{2}(\partial_\mu F^r_{\ \mu\nu})^2+\tfrac{1}{4}(\Box\bar{h}_{rs})^2+\tfrac{1}{12}(d-1)(d-4)(\Box\phi)^2]\,,
\tag{4.9}
$$

where

$$
\Box\phi\equiv\frac{1}{(d-1)}R^{(4)}+\frac{3}{(d-1)(d-4)}\Box h_{tt}
$$

and $\alpha^{-2}=4\alpha_0^{-2}(Lm)^{d-4}(d-3)/(d-2)$ (L is the length of the "internal" circles). In order to be able to consider ϕ as an independent scalar field it is necessary to account for the constraint (4.4), which, in a linearized and "dimensionally reduced" form, reads

$$
-2(d-1)\Box\sigma+R^{(4)}-\Box h_{tt}\simeq 0\,.
\tag{4.10}
$$

Eliminating R we are led to the following identification

$$\phi = \sigma + \frac{1}{2(d-4)} h_{tt}, \tag{4.11}$$

which implies that ϕ has zero Weyl weight. The resulting $d=4$ theory contains the Weyl graviton interacting with $(d-4)$ Abelian gauge vectors $h_\mu{}^r$, $(\frac{1}{2}(d-4)(d-3)-1)$ scalars $\bar{h}_{rs}$ and the scalar ϕ. It is interesting to compare (4.9) with the standard "Kaluza–Klein" reduction of the Einstein action: both theories have the same field content (though (4.9) since being scale invariant implies a smaller number of off-shell degrees of freedom) but they differ in the number of derivatives present in the kinetic terms.

Now it is clear that the spectrum of the full non-linear extension of (4.9) should contain the metric $g_{\mu\nu}$ ($w=-2$), $d-4$ gauge vectors with $w=0$ and $\frac{1}{2}(d-4)(d-3)$ scalars with $w=0$. Using a part of the Lorentz gauge invariance $SO(d-1, 1)$, we can bring the vielbein to the "triangular" form and thus have

$$\begin{aligned} &e^{A\mu} = \begin{pmatrix} e^a{}_\mu & B_\mu{}^p \\ 0 & e^p{}_r \end{pmatrix}, \qquad B_\mu{}^r \equiv B_\mu{}^p e_p{}^r, \\ &a_{rs} = e^p{}_r e^q{}_s \delta_{pq} \Delta^{-2/(d-4)}, \qquad \det a_{rs} = 1, \\ &\Delta = \det e^p{}_r, \qquad \phi = \sigma + \frac{1}{(d-4)} \ln \Delta . \end{aligned} \tag{4.12}$$

The vectors $B_\mu{}^r$ and the scalars a_{rs} and ϕ all have the zero Weyl weight. (In the linear approximation $B_\mu{}^r \to \frac{1}{2}h_\mu{}^r$, $a_{rs} \to \delta_{rs} + h_{rs} - (1/(d-4))\delta_{rs}h_{tt}$, $\Delta \to 1 + \frac{1}{2}h_{tt}$). The reduced Lagrangian following from (4.3) is non-polynomial in a_{rs} and ϕ. For example, it contains the Weyl tensor squared term in the combination

$$\sqrt{g}\,(C_{\lambda\mu\nu\rho})^2\, e^{(d-4)\phi} \tag{4.13}$$

($e^{\sigma(d-4)}$ in (4.3) combines with Δ coming from $\det e^A{}_M$). It is important to stress that, because of the Weyl invariance of $\sqrt{g}\, C^2_{\lambda\mu\nu\rho}$, it is impossible to get rid of the "non-minimal" scalar–tensor coupling by means of redefinitions of the metric (analogous situation takes place in Einstein gravity when reduced to two dimensions). The rescaling $g_{\mu\nu} \to g_{\mu\nu}\, e^{(d-4)\phi}$ is however useful in order to put the vector term in the form $(\mathcal{D}_\mu F^r_{\mu\nu})^2 \sqrt{g}$.

To construct the $(N=1)$ conformal supergravity in ten dimensions [9], i.e. the superconformal extension of (4.3) for $d=10$ one can start with the multiplet of currents [11] for the $d=10$ Maxwell theory [170] (cf. section 3.2). The resulting multiplet of fields of the $d=10$ CSG includes [9]

$$\begin{array}{cccccc} & e^A{}_M & \psi_M & A_{M_1\cdots M_6} & \chi & \Phi \sim e^\sigma \\ w: & -1 & -\frac{1}{2} & 0 & 1/2 & 1 \end{array}\;, \tag{4.14}$$

i.e. the vielbein, a Majorana–Weyl gravitino and spinor, a gauge antisymmetric tensor ($\delta A_{M_1\cdots M_6} = \partial_{[M_1}\xi_{M_2\cdots M_6]}$) and a real scalar. Just *the same* fields constitute the spectrum of the $N=1$, $d=10$ Poincaré supergravity [140, 26, 27]. However, in contrast to the PSG case, the fields χ and Φ of conformal supergravity are not independent and are indeed subject to differential constraints analogous to (4.4) (which are necessary for the maximal irreducibility of the multiplet (4.14)). The count of the off-shell degrees of freedom then proceeds as follows ($d = 10$)

$$n(e^A{}_M) = d^2 - \tfrac{1}{2}d(d-1)(M) - d(\mathrm{X}) - 1(\mathrm{D}) = \tfrac{1}{2}(d+1)(d-2) = 44\,,$$
$$n(A_{M_1\cdots M_6}) = \sum_{k=0}^{6} \binom{d}{6-k}(-1)^k = \binom{d-1}{6} = 84\,,$$
$$n(\psi_M) = -\tfrac{1}{2}\cdot 2^{d/2}(d - 1(Q) - 1(S)) = -128, \qquad n(\chi) = 0, \qquad n(\Phi) = 0\,.$$

Thus we get the same total number of degrees of freedom (128–128) as in $N = 4$, $d = 4$ CSG (see table 3.4). The complete non-linear transformation rules of the $d = 10$ theory, established in [9], have the form

$$\begin{aligned}
&\delta e^A{}_M = \tfrac{1}{2}\bar{\varepsilon}\Gamma^A\psi_M\,,\\
&\delta\psi_M = \mathscr{D}_M\varepsilon + \tfrac{1}{4}\Phi^{-6}(\Gamma_M\Gamma_{(7)} - 2\Gamma_{(7)}\Gamma_M)\varepsilon\hat{R}(A)_{(7)} + (\bar{\varepsilon}\psi\chi + \varepsilon\bar{\chi}\chi\text{-terms}) - \Gamma_M\eta\,,\\
&\delta A_{M_1\cdots M_6} = \frac{1}{8\cdot 5!}\Phi^6\bar{\varepsilon}(\Gamma_{[M_1\cdots M_5}\psi_{M_6]} + \Gamma_{M_1\cdots M_6}\chi)\,,\\
&\delta\chi = \tfrac{1}{2}\Phi^{-1}\hat{\not{D}}\Phi\varepsilon - \tfrac{1}{12}\Phi^{-6}\Gamma_{(7)}\varepsilon\hat{R}(A)_{(7)} + \eta\,,\\
&\delta\Phi = \tfrac{1}{2}\bar{\varepsilon}\chi\Phi
\end{aligned} \tag{4.15}$$

where ε and η are the Majorana–Weyl parameters of Q and S supersymmetries, $\Gamma_{(n)}R_{(n)} \equiv \Gamma_{M_1\cdots M_n}R_{M_1\cdots M_n}$, $\hat{D}$ is the supercovariant derivative[2] and $\hat{R}(A)$ is the supercovariant field strength

$$\hat{R}(A)_{M_1\cdots M_7} = \partial_{[M_1}A_{M_2\cdots M_7]} - \frac{1}{8\cdot 5!}\Phi^6(\tfrac{1}{2}\bar{\psi}_{[M_1}\Gamma_{M_2\cdots M_6}\psi_{M_7]} + \bar{\psi}_{[M_1}\Gamma_{M_2\cdots M_7]}\chi)\,.$$

To close the gauge algebra it is necessary to impose the constraint on χ [9]

$$\Phi^{-4}\hat{\not{D}}(\Phi^4\chi) - \tfrac{1}{3}\Gamma^{(7)}\chi\,\hat{R}(A)_{(7)}\,\Phi^{-6} = 0\,. \tag{4.16}$$

The constraint on Φ follows from (4.16) by a Q transformation

$$\tfrac{1}{4}\Phi^{-4}(\hat{D}_A\hat{D}^A\Phi^4) + 2240\Phi^{-12}\hat{R}(A)_{(7)}\,\hat{R}(A)_{(7)} + \tfrac{4}{3}\Phi^{-6}\,\hat{R}(A)_{(7)}\,\bar{\chi}\Gamma_{(7)}\chi = 0\,, \tag{4.17}$$

and is direct generalization of (4.4) (note that $\frac{1}{2}(d-2) = 4$). Both (4.16) and (4.17) are inert under S and K. The linearized form of these constraints is

$$\not{\partial}\chi + \tfrac{1}{9}\Gamma^{MN}\partial_M\psi_N \simeq 0, \qquad -\Box\sigma + \tfrac{1}{18}R(e) \simeq 0\,.$$

Thus the introduction of the constrained "would-be compensators" χ and Φ opens the possibility to eliminate the "reducible", "spin $\frac{1}{2}$" and "spin 0" parts of the gravitino and the metric, maintaining at the

[2] As in $N = 1$ CSG in four dimensions, one introduces the K and S gauge fields $f^A{}_M$ and ϕ_M, which satisfy conventional constraints analogous to (2.28) (see [9] for details).

same time the locality of the transformation laws. It may happen that there exists a formulation of $d=10$ CSG, without unconventional constraints like (4.16), (4.17), which is based on a larger field representation (and which eventually may be related to a gauge theory of the $d=10$ superconformal algebra constructed in ref. [252]).

The "compensators" χ and Φ enable one to build also the superconformal invariant action analogous to (4.3). Though the complete supersymmetric action for $d=10$ CSG is unknown (as in the case of $N=4$ CSG in $d=4$) it is easy to write down the trivially covariantized form of the bilinear Lagrangian invariant under the linearized transformations (4.15) [9]

$$e^{-1}\mathcal{L}_{\mathrm{CSG}}^{(d=10)}=\frac{1}{\alpha_0^2}\{(C^{M}{}_{NPQ})^2\Phi^6+\tfrac{7}{4}(\bar\psi_{MN}\mathscr{D}\psi_{MN}-\tfrac{1}{18}\bar\psi_{MN}\Gamma^{MN}\mathscr{D}\Gamma^{PQ}\psi_{PQ})\Phi^6$$
$$-2205\times 16\times R(A)_{M_1\cdots M_7}\mathscr{D}^2R(A)_{M_1\cdots M_7}\Phi^{-6}+\cdots\},\tag{4.18}$$
$$\psi_{MN}\equiv\mathscr{D}_{[M}\psi_{N]},\qquad R(A)_{M_1\cdots M_7}=\partial_{[M_1}A_{M_2\cdots M_7]}\,.$$

The linearized expression for (4.18) reads (cf. (4.7))

$$e^{-1}\mathcal{L}_{\mathrm{CSG}}^{(d=10)}\simeq\frac{m^6}{\alpha_0^2}\Big\{\tfrac{7}{2}(R^2_{MN}-\tfrac{5}{18}R^2)+\tfrac{7}{4}(\bar\psi_{MN}\partial\!\!\!/\psi_{MN}-\tfrac{1}{18}\bar\psi_{MN}\Gamma^{MN}\partial\!\!\!/\Gamma^{PQ}\psi_{PQ})$$
$$-\frac{2205\times16}{m^{12}}R(A)_{M_1\cdots M_7}\Box R(A)_{M_1\cdots M_7}\Big\}\tag{4.19}$$

(recall that $\Phi=m\,\mathrm{e}^{\sigma}\simeq m+\cdots$ and that $[A_{M_1\cdots M_6}]=6$). The Lagrangian (4.18) is effectively non-local in $d=10$, but reduces to the local Lagrangian of $N=4$ CSG in four dimensions. Before discussing the reduction it is important to stress that the field representation of $d=10$ CSG is essentially unique. In contrast with the case of $N=1$, $d=10$ PSG [26, 27] one cannot replace $A_{M_1\cdots M_6}$ by A_{MN} using the duality transformation. In fact, as it follows from the structure of the antisymmetric tensor part of (4.18), the duality transformation leads to a non-local action for A_{MN}, which remains non-local after reduction to $d=4$ ($\mathcal{L}\sim R_{(7)}\Box R_{(7)}\to R_{(7)}\Box R_{(7)}+G_{(7)}(R_{(7)}-\partial A_{(6)})\to\tilde{\mathcal{L}}\sim R_{(2)}\Box^{-1}R_{(2)}$).

Guided by the knowledge of the spectrum of $N=4$, $d=4$ conformal supergravity (table 3.1), and by the analysis of the dimensional reduction of conformal gravity (4.3), it is easy to recognize the fields of $N=4$, $d=4$ CSG among the components of the fields of the ten-dimensional theory. The $d=10$ Lorentz group breaks down in the process of reduction to the product of the $d=4$ group SO(3, 1) and the "internal" SO(6) group, "isomorphic" to the SU(4) of $N=4$ CSG. In addition to the fields of conformal gravity (4.12) $e^a_\mu(1)$, $B^r_\mu(6)$, $a_{rs}(20')$, $\phi(1)$ (the numbers in the brackets indicate the SO(6) or SU(4) representation; $r,s=1,\ldots,6$) we get:

$$A_{M_1\cdots M_6}=\begin{cases}A_{r_1\cdots r_6}\to A(1), & (A_\mu{}^{rst}\sim\varepsilon^{rstupq}\varepsilon_{\mu\nu\lambda\rho}A_{\nu\lambda\rho upq},\ \text{etc.})\\ A_{\mu_1\cdots\mu_4 rs}\to A_{rs}(15)\\ A_{\mu\nu\lambda\, rst}\to A_\mu{}^{rst}(20)\\ A_{\mu\nu\, r_1\cdots r_4}\to A_{\mu\nu}{}^{rs}(15)\\ A_{\mu\, r_1\cdots r_5}\to A^r_\mu(6)\end{cases}\tag{4.20}$$

$$\psi_M=\begin{cases}\psi_\mu(4);\qquad \chi(4)\\ (\psi_r-\tfrac{1}{6}\Gamma_r\Gamma_s\psi_s)(20)\\ \Gamma_s\psi_s(4)\,.\end{cases}$$

In reducing the fermions, one has to note that the 32 components of the spinors are subject to the Majorana–Weyl conditions and thus every $d = 10$ spinor gives four independent Majorana spinors in four dimensions, see, e.g. [140] (this explains the numbers given in (4.20)). Taking into account the dimensions of the fields, we are led to the following identifications at the linearized level (for simplicity we neglect the relative normalizations of the fields) (cf. [9])

$$\begin{aligned}
&\psi^i_\mu \sim \psi^{(d=10)}_\mu, \qquad T_{\mu\nu}{}^{ij} \sim T_{\mu\nu}{}^r = 2\partial_{[\mu} B_{\nu]}{}^r + \mathrm{i}\varepsilon_{\mu\nu\alpha\beta}\partial_\alpha A_\beta, \\
&V_\mu{}^i{}_j \sim V_\mu{}^{rs} = \partial_\rho A_{\rho\mu}{}^{rs}, \qquad \chi^{ij}{}_k \sim \chi_r = \not\partial(\psi_r - \tfrac{1}{6}\Gamma_r\Gamma_s\psi_s), \\
&D^{ij}{}_{kl} \sim D_{rs} = \Box a_{rs}, \qquad \Lambda_i \sim \Lambda = \lambda + \Gamma_s\psi_s, \\
&\varphi \sim \phi + \mathrm{i}A, \qquad E_{ij} \sim E^{rst} = \partial_\mu A_\mu{}^{rst}
\end{aligned} \tag{4.21}$$

(the scalar A_{st} drops out from the action). The validity of these relations can be checked by comparing either the transformation laws or the linearized actions (4.19) and (3.30) for the $d = 10$ and $d = 4$ theories. We see that it is the highest dimensional spinor and scalar (Λ and φ) that absorb the "would-be compensators" (λ and σ) and thus make it possible to get a local expression for the reduced action.

It is instructive to check that (4.19) and (4.21) do produce the correct linearized Lagrangian for the antisymmetric tensor $T_{\mu\nu}$. In view of (4.9) and (4.19), we find (up to numerical coefficients)

$$\begin{aligned}
&\mathscr{L}_\mathrm{T} \sim (\partial_\rho F_{\rho\mu}{}^r)^2 - 2G_{\mu\nu}{}^r \Box G_{\mu\nu}{}^r, \\
&F_{\rho\mu}{}^r = \partial_\rho B_\mu{}^r - \partial_\mu B_\rho^r, \qquad G_{\mu\nu}{}^r = \partial_\mu A_\nu{}^r - \partial_\nu A_\mu{}^r.
\end{aligned}$$

By noting that $G_{\mu\nu}\Box G_{\mu\nu} = -\frac{1}{2}(\partial_\rho G_{\rho\mu})^2 + \mathrm{div.}$, we can rewrite this as

$$\mathscr{L}_\mathrm{T} \sim \partial_\rho(F_{\rho\mu}{}^r + \mathrm{i}G_{\rho\mu}{}^r)\partial_\sigma(F_{\sigma\mu}{}^r - \mathrm{i}G_{\sigma\mu}{}^r). \tag{4.22}$$

Our final observation is that, if we write $T_{\mu\nu} = F_{\mu\nu} + \mathrm{i}G_{\mu\nu}$, then (4.22) is proportional to the previously found expression $\partial_\rho T^+_{\rho\mu}\, \partial_\sigma T^-_{\sigma\mu}$ (see (3.30)).

In order to extend (4.21) to the non-linear level, one has to introduce fields which are invariant under internal changes of coordinates (U(1) gauge transformations of $B_\mu{}^r$) and also make the necessary rescalings giving the correct Weyl weight assignments. It is then possible to guess some of the non-linear terms in the Lagrangian for the version of $N = 4$ conformal supergravity which follows, via reduction, from $d = 10$ CSG (4.18). In analogy with (4.13) we obtain

$$e^{-1}\mathscr{L}^{(4)}_\mathrm{CSG} = \tfrac{1}{2}(C_{\lambda\mu\nu\rho})^2\, \mathrm{e}^{6\phi} + F_{\mu\nu}{}^i{}_j(V)\, F_{\mu\nu}{}^j{}_i(V)\, \mathrm{e}^{6\phi} + \tfrac{9}{2}(\Box\phi)^2 + \cdots. \tag{4.23}$$

Thus ϕ contributes to the action in a non-polynomial way. At the same time, the imaginary part of φ (the pseudoscalar A), which is a component of the antisymmetric gauge tensor, can occur in the action only polynomially. (One can imagine that the full action may contain also terms like $\mathrm{i}AC_{\lambda\mu\nu\rho}C^*_{\lambda\mu\nu\rho}$ and (or) $\mathrm{i}AF_{\mu\nu}F^*_{\mu\nu}$). The structure of the scalar couplings clearly resembles that in the SU(4) version of $N = 4$ PSG [36], which in fact can be obtained by means of reduction (and truncation) from the $N = 1$, $d = 10$ PSG [140, 26]. This version follows from the SO(4) version [41, 35] (see eq. (3.67)) after a point transformation of scalars and spinors and a duality rotation of vectors [36]. We are led to the conclusion that the parametrization of the scalar φ appearing in the version of $N = 4$ CSG connected, via reduction, with $d=10$ CSG, is *different* from the parametrization of φ in the SU(1, 1)-invariant version of $N=4$ CSG [8] discussed in section 3.3 (in which "non-minimal" terms like (4.23) were forbidden but the imaginary part of φ could contribute in a non-polynomial fashion). Taking into account that duality rotations seem to be impossible in conformal supergravity it is then natural to suggest, that these two versions of $N=4$ CSG are truly *inequivalent* (cf. section 3.3).

Next we turn to the discussion of the coupling of $d=10$ CSG to $d=10$ super Yang–Mills, which being reduced to $d=4$, corresponds to the coupling of $N=4$ CSG to $N=4$ super Yang–Mills (cf. section 3.5). The $d=10$ super Yang–Mills theory contains the fields B_μ ($w=0$) and λ ($w=\frac{3}{2}$) belonging to the adjoint representation of the internal gauge group, and has the following scale-invariant Lagrangian

$$e^{-1}\mathscr{L}_{\rm YM}^{(d=10)}=\frac{1}{g^2}\operatorname{tr}(\tfrac{1}{4}F^2_{MN}+\tfrac{1}{2}\bar\lambda\Gamma^M\mathscr{D}_M\lambda)\Phi^6\,,$$
$$F_{MN}=\partial_M B_N-\partial_N B_M-[B_M,B_N],\qquad \mathscr{D}_M=\partial_M-[B_M,\ ]+\tfrac{1}{2}\sigma^{AB}\omega_{ABM} \tag{4.24}$$

where we inserted a power of Φ in order to compensate for the non-invariance of $g^{MN}g^{PQ}\sqrt{g}$ under Weyl transformations in $d=10$. The full superconformally invariant extension of (4.24) was found in ref. [9]

$$\begin{aligned}e^{-1}\mathscr{L}_{\rm YM}^{(d=10)}=\frac{1}{g^2}\Phi^6\operatorname{tr}[&\tfrac{1}{4}F^2_{MN}+\tfrac{1}{2}\bar\lambda\Gamma^M\mathscr{D}_M\lambda\\&+\tfrac{1}{8}\bar\lambda\Gamma_M\Gamma_{NK}(F_{NK}+\hat F_{NK})(\psi_M+\Gamma_M\chi)+\tfrac{9}{64}\bar\lambda\Gamma_{(3)}\lambda\bar\chi\Gamma_{(3)}\chi\\&+\tfrac{1}{128}\bar\lambda\Gamma_{(3)}\lambda\bar\psi_M(4\Gamma_{(3)}\Gamma_M+3\Gamma_M\Gamma_{(3)})\chi]\\&+\frac{ie^{-1}}{2g^2}\varepsilon^{M_1\cdots M_{10}}\operatorname{tr}(F_{M_1M_2}F_{M_3M_4})A_{M_5\cdots M_{10}}+\frac{1}{2g^2}\operatorname{tr}(\bar\lambda\Gamma_{(7)}\lambda)\,\hat R(A)_{(7)}\,,\end{aligned} \tag{4.25}$$

where $\hat F_{MN}=F_{MN}-\bar\psi_{[M}\Gamma_{N]}$. This expression *coincides* with the Lagrangian of the super-Yang–Mills theory interacting with $N=1$, $d=10$ PSG, found in ref. [27] (the only difference is that the fields Φ and χ in (4.25) are subject to constraints absent in the PSG case). It is now in principle straightforward to reduce (4.25) to four dimensions and thus to find the *complete* Lagrangian for $N=4$ super-Yang–Mills interacting with $N=4$ CSG. In this way we get the following bosonic terms

$$e^{-1}\mathscr{L}_{\rm YM}^{(d=4)}\sim e^{6\phi}\operatorname{tr}(F^2_{\mu\nu})+i\operatorname{tr}(F_{\mu\nu}F^*_{\mu\nu})A+\cdots\,. \tag{4.26}$$

It is interesting to note that the mixing between $F_{\mu\nu}$ and $T_{\mu\nu}$, found in section 3.5, is absent here. We conclude that the version of CSG + super-Yang–Mills theory which follows via dimensional reduction from ten dimensions is very different from the "O(4)-like" theory discussed in section 3.5. This supports the conjecture that there are at least two inequivalent versions of $N=4$ conformal supergravity.

4.2. *Conformal supergravity in superspace*

The superspace description of supersymmetric theories, while useful at the classical level, is particularly important at the quantum level. Supergraph technique simplifies higher-loop calculations and opens the possibility to make some indispensable statements (e.g. non-renormalization theorems) concerning perturbation theory as a whole (for a general account of $N=1$ superspace methods see refs. [261, 129]). A superfield approach to conformal supergravity was initiated in ref. [90], where the complete treatment of the linearized $N=1$ CSG was given. The full non-linear $N=1$ conformal supergravity has been studied (in a Wess–Zumino superspace [264]) in refs. [162, 216] where the origin of super-Weyl transformations (including dilations D and S supersymmetry) as symmetries of constrained geometry was elucidated.[1] The relation between the constraints, the $N=1$ superspace superconformal group and results on components was further clarified in refs. [133, 255]. The $N=1$ CSG was also discussed within the chiral superspace approach in refs. [203, 271]. Superfield description of linearized extended $N \leq 4$ CSG was presented in refs. [10, 218, 8] (field strength superfields and actions) and in refs. [220, 159, 132] (prepotentials). The superspace constraints for the full non-linear $N \leq 4$ CSG were first formulated and solved in refs. [157, 158]. The consequences of these constraints were further explored in ref. [128]. The role of the superspace superconformal approach in the superspace description of Poincaré supergravity was studied in refs. [125, 126, 158, 127].

In spite of these developments superspace expressions for the full non-linear actions for extended conformal supergravities are still lacking at present (though the complete component action for $N=2$ CSG was established in [8] it was not yet cast in a superspace form). There seems to be no problems of principle in establishing the full off-shell actions in terms of constrained geometrical superfields. A more difficult task is to determine the actions in terms of unconstrained prepotentials. It is the latter form of actions that is necessary for a manifestly supersymmetric quantization procedure. An important intermediate step would be the formulation of extended CSG's in terms of unconstrained $N=1$ superfields. This would open the way for an application of the powerful $N=1$ supergraph methods [129] to extended conformal supergravities (cf. section 6.1).

In this section we shall summarize some of the known results about superspace formulation of extended CSG (for a comprehensive discussion of the $N=1$ case see [162, 216] and [261, 129]). We shall use the following notation: x^μ, four bosonic coordinates; $\theta^{\sigma} = (\theta^{\sigma}_i)$, $2N$ anticommuting coordinates (with $\bar\theta^{\dot\sigma i}$ as complex conjugates); $a, b = 1, \dots, 4$; $\alpha, \beta, \dots = 1, 2$; $i, j = 1, \dots N$ are the tangent space vector, two-spinor and $\mathrm{U}(N)$ internal indices with $A = (a, \boldsymbol{\alpha} = (\alpha, i))$ being a total index of a tangent space vector. The coordinate (curved space) indices are $M = (\mu, \boldsymbol{\sigma} = (\sigma, i))$ (the corresponding coordinates are $Z^M = (x^\mu, \theta^{\boldsymbol{\sigma}}, \theta^{\dot{\boldsymbol{\sigma}}})$). All other conventions are standard (see e.g. [158]); for example the flat superspace covariant derivative is $D_A = (\partial_{\alpha\dot\alpha}, D^i{}_\alpha, \bar D_{\dot\alpha i})$, $D_\alpha{}^i = \partial/\partial\theta^\alpha_i + \mathrm{i}\bar\theta^{\dot\alpha i}\partial_{\alpha\dot\alpha}$ ($\mathcal{D}_M$ will stand for the curved space derivative).

We shall first discuss the linearized theory. As it was shown in refs. [90, 10, 218, 8] the linearized $N \leq 4$ CSG can be described off-shell by a *chiral* field strength superfield $W_{(\alpha_1\cdots\alpha_{2s})[i_1\cdots i_N]} = \varepsilon_{i_1\cdots i_N} W_{\alpha_1\cdots\alpha_{2s}}$, $s = \frac{1}{2}(4-N)$, $i = 1, \dots N$, $\alpha = 1, 2$; $\bar D_{\dot\alpha} W = 0$, which is totally (anti) symmetric in α's (i's). It has dimension s, contains (off-shell) spins $s, s+\frac{1}{2}, \dots, s+N/2$ and satisfies the following additional constraints ("Bianchi identities")

$$
\begin{aligned}
N=1:&\quad \partial_{\dot\alpha}{}^{\alpha} D^{\beta} W_{\alpha\beta\gamma} = \partial_{\alpha}{}^{\dot\alpha} \bar D^{\dot\beta} \bar W_{\dot\alpha\dot\beta\dot\gamma}\,,\\
N=2:&\quad D_{\boldsymbol{\alpha}} D_{\boldsymbol{\beta}} W^{\alpha\beta} = \bar D^{\boldsymbol{\alpha}} \bar D^{\boldsymbol{\beta}} \bar W_{\dot\alpha\dot\beta}\,,\\
N=3:&\quad \varepsilon_{mkl} D^i_{(\alpha} D^j_{\beta)} D^{\alpha m} W^{\beta} = \varepsilon^{ijn} \bar D^{\dot\alpha}_k \bar D^{\dot\beta}_l \bar D_{\dot\alpha n} \bar W_{\dot\beta}\\
N=4:&\quad \varepsilon_{mnkl} D^i_{(\alpha} D^j_{\beta)} D^{\alpha m} D^{\beta n} = \varepsilon^{ijpq} \bar D_{\dot\alpha k} \bar D_{\dot\beta l} \bar D^{\dot\alpha}_p \bar D^{\dot\beta}_q \bar W\,.
\end{aligned}
\tag{4.27}
$$

[1] Superspace conformal transformations in a general context were first discussed in ref. [30] (see also [191, 192]). However, these authors did not impose the torsion constraints necessary to make contact with the irreducible set of fields of the component CSG.

The θ-expansions of $W_{\alpha_1\cdots\alpha_{2s}}$ start with fields of spin s and dimension s

$$
\begin{aligned}
&W_{\alpha\beta\gamma} = \psi_{\alpha\beta\gamma} + \cdots, \qquad W_{\alpha\beta ij} = T_{\alpha\beta ij} + \cdots, \\
&W_{\alpha ijk} = \Lambda_{\alpha\, ijk} + \cdots, \qquad W_{ijkl} = \varphi_{ijkl} + \cdots .
\end{aligned} \tag{4.28}
$$

To make contact with the field content of $N \le 4$ CSG (see table 3.2) we recall that the *gauge field strengths* and the matter fields of the $N \le 4$ conformal supergravities have the following values of (off-shell) spins and dimensions

$$
\begin{array}{lllll}
s=2 & s=\frac{3}{2} & s=1 & s=\frac{1}{2} & s=0 \\
C_{\alpha\beta\gamma\delta}(2) & \phi^{i}{}_{\alpha\beta\gamma}(\frac{5}{2}) & F_{\alpha\beta}{}^{i}{}_{j}(2) & \chi_{\alpha ij}{}^{k}(\frac{3}{2}) & D^{ij}{}_{kl}(2) \\
 & \psi_{\alpha\beta\gamma i}(\frac{3}{2}) & T_{\alpha\beta ij}(1) & \Lambda_{\alpha ijk}(\frac{1}{2}) & E_{ijk}{}^{l}(1) \\
 & & & & \varphi_{ijkl}(0) .
\end{array} \tag{4.29}
$$

We use the 2-spinor notation ($\xi_{\alpha\dot\alpha} = \sigma^{a}_{\alpha\dot\alpha}\xi_a$, $\xi_{\alpha\beta} = \sigma^{ab}_{\alpha\beta}\xi_{[ab]}$, etc.). All fields are symmetric in their spinor indices and satisfy the same restrictions as in (3.36). $C_{\alpha\beta\gamma\delta}$ is the Weyl curvature spinor, $\psi_{\alpha\beta\gamma} \sim R(Q)$ and $\phi_{\alpha\beta\gamma} \sim R(S)$ are gravitino field strengths and $F_{\alpha\beta}$ is the U(N) (SU(4)) gauge field strength. The component fields of CSG appear in W's in the covariant combinations (4.29). The meaning of the constraints (4.27) is that they ensure that C, ϕ, ψ, F are field strengths, i.e. they imply the relevant Bianchi identities

$$
\begin{aligned}
&\partial_{\alpha}{}^{\dot\beta}\bar F_{\dot\alpha\dot\beta}{}^{i}{}_{j} = \partial_{\dot\alpha}{}^{\beta}F_{\alpha\beta}{}^{i}{}_{j}, \qquad \partial_{\alpha}{}^{\dot\beta}\partial_{\beta}{}^{\dot\gamma}\bar\psi_{\dot\alpha\dot\beta\dot\gamma i} = \partial_{\dot\alpha}{}^{\gamma}\phi_{\alpha\beta\gamma i}, \\
&\partial_{\alpha}{}^{\dot\gamma}\partial_{\beta}{}^{\dot\delta}\bar C_{\dot\alpha\dot\beta\dot\gamma\dot\delta} = \partial_{\dot\alpha}{}^{\gamma}\partial_{\dot\beta}{}^{\delta}C_{\alpha\beta\gamma\delta} .
\end{aligned} \tag{4.30}
$$

Another meaning of (4.27) (for $N > 1$) is that they express the condition of "reality" for the $D^{ij}{}_{kl}$ (see (3.25)). The full expansion (4.28), for example for $N = 4$, looks like

$$
\begin{aligned}
W(x^\mu + \mathrm{i}\theta\sigma^\mu\bar\theta, \theta) = {}& \varphi + \theta\Lambda + \theta^2 E + \theta^2 T + \theta^3\chi + \theta^3\psi + \theta^4 F + \theta^4 C + \theta^4 D + \theta^5\phi \\
&+ \theta^5 \not\partial\chi + \theta^6\Box E + \theta^6\Box T + \theta^7\Box\not\partial\Lambda + \theta^8\Box^2\varphi, \qquad \varphi = \varphi(x), \ldots, D = D(x)
\end{aligned} \tag{4.31}
$$

(for simplicity we do not differentiate between θ and $\bar\theta$; for exact expressions, see [8]).

The linearized superspace actions for conformal supergravities are given by [218, 8]

$$
I^{(N)}_{\mathrm{CSG}} = \frac{1}{\alpha^2}\int \mathrm{d}^4x\, \mathrm{d}^{2N}\theta\, W_{\alpha_1\cdots\alpha_{2s}} W^{\alpha_1\cdots\alpha_{2s}} + \text{c.c.}, \qquad \left(s = \frac{4-N}{2}\right). \tag{4.32}
$$

Note that the actions (4.32) are dimensionless and scale invariant. It is easy to check that, after integration over θ, (4.32) reproduces (e.g. for $N = 4$) the linearized component Lagrangian (3.30).

To find the prepotentials for W's one has to solve the constraints (4.27). This can be done either directly [220] or by adding the constraints in the actions (4.32) with the help of the Lagrange multiplier superfields, and eliminating W in favour of the Lagrange multipliers (which are identified with prepotentials) [159]. In this way one obtains [220, 159] (for $N = 1$ see [90])

$$W_{\alpha\beta\gamma} = \bar{D}^2 \partial_{(\alpha}{}^{\dot\alpha} D_\beta V_{\gamma)\dot\alpha}, \qquad W_{\alpha\beta} = \mathrm{i}\bar{D}^4 D^2_{\alpha\beta} V,$$
$$W_\alpha = \mathrm{i}\bar{D}^6 D^3{}_{\alpha}{}^{i}{}_{j} V^j{}_i, \qquad W = \mathrm{i}\bar{D}^8 D^{4ijkl} V_{ijkl}, \tag{4.33}$$

where

$$D^2 = \varepsilon_{ij} D^i{}_\alpha D^{j\alpha}, \qquad D^3{}_{\alpha}{}^{i}{}_{j} = \varepsilon_{jkl} D^{\beta i} D_\beta{}^k D_\alpha{}^l$$
$$D^{4ijkl} = D^i_{(\alpha} D^j_{\beta)} D^{k\alpha} D^{l\beta}, \text{ etc.}$$

Substituting (4.33) in (4.32), one can express the actions in terms of the prepotentials V. The fact that (4.33) provides solutions to (4.27) does not necessarily imply that the V's are unconstrained (though it is the case for $N = 1$). For example, for $N = 2$ $V = D_\alpha \Sigma^\alpha + \text{c.c.}$, where Σ is unconstrained [132].

Another method for establishing the prepotentials (4.33) is to consider the coupling of conformal supergravity to super-Yang–Mills (SYM) theory, thus providing a superfield version of the coupling, discussed in section 3.2. The linearized $N \le 4$ SYM theories are described (on-shell) by the chiral field strength superfields Φ_α, Φ, Φ_i and Φ_{ij}, subject to the following constraints [142, 226]

$$N = 1: \quad \bar{D}_{\dot\alpha} \Phi_\beta = 0, \qquad D_{(\alpha}\Phi_{\beta)} = D_\alpha \Phi_\beta,$$
$$N = 2: \quad \bar{D}_{\dot\alpha} \Phi = 0, \qquad D^i_\alpha D^j_\beta \Phi = D^i_{(\alpha} D^j_{\beta)} \Phi,$$
$$N = 3, 4: \quad D^i_\alpha \Phi_{jk} = -\frac{2}{N-1} \delta^i_{[j} D^l_\alpha \Phi_{k]l}, \qquad \bar{D}_{\dot\alpha(i} \Phi_{j)k} = 0, \tag{4.34}$$
$$N = 3: \quad \Phi_{ij} = \varepsilon_{ijk} \bar{\Phi}^k,$$
$$N = 4: \quad \bar{\Phi}^{ij} = \tfrac{1}{2}\varepsilon^{ijkl} \Phi_{kl}.$$

These constraints imply the equations of motion and the Bianchi identity for the vector field strength. The component expansions look like (cf. (3.19)): $\Phi_{ij} = \phi_{ij} + \theta\lambda_i + \theta\theta F_{\mu\nu} + \cdots$. The corresponding (Abelian) supercurrents are given by[2] [89, 159]

$$J_{\alpha\dot\alpha} = \Phi_\alpha \bar{\Phi}_{\dot\alpha}, \qquad J = \Phi\bar{\Phi},$$
$$J^i{}_j = \Phi^i \bar{\Phi}_j - \tfrac{1}{3}\delta^i_j \Phi^k \bar{\Phi}_k \qquad (8 \text{ of } SU(3)) \tag{4.35}$$
$$J^{ij}{}_{kl} = \bar{\Phi}^{ij} \Phi_{kl} - \tfrac{1}{6}\delta^i_{[k}\delta^j_{l]} \bar{\Phi}^{mn} \Phi_{mn} \qquad (20' \text{ of } SU(4)).$$

Constructing the Noether couplings, $\int d^4x\, d^{2N}\theta\, d^{2N}\bar\theta\, V \times J$ we find the prepotentials: $V_{\alpha\dot\alpha}$, V, $V^i{}_j(8)$, $V^{ij}{}_{kl}(20')$ (the $N = 4$ prepotential in (4.33) is $V_{ijkl} \equiv \frac{1}{2}\varepsilon_{ijmn} V^{mn}{}_{kl}$). The gauge transformations of prepotentials follow from the constraints on the supercurrents, implied by (4.34). For instance, the $N = 4$ Noether coupling term is invariant under [159]: $\delta V^{ijkl} = \bar{D}_{\dot\alpha m} \zeta^{\dot\alpha mijkl} + \text{c.c.}$, where ζ belongs to the 60 representation of SU(4) (note also that $V^{ij}{}_{kl}$ satisfies the following constraints: $\bar{V}^{ij}{}_{kl} = \frac{1}{2}\varepsilon_{ijrs}\varepsilon_{klmn} V^{rs}{}_{mn}$,

[2] Let us remark in passing that under the constraints (4.34) the linearized actions for SYM theories are $\int d^4x\, d^2\theta\, \Phi^\alpha \Phi_\alpha + \text{c.c.}, \ldots,$ $\int d^4x\, d^2\theta_{ij}\, d^2\bar\theta_{kl} (\Phi^{ij}\Phi^{kl} - \frac{1}{12}\varepsilon^{ijkl}\bar\Phi^{pq}\Phi_{pq})$.

$D_\alpha V^{ij}{}_{kl} \sim 20$ of SU(4)). Instead of the scalar $N>1$ supercurrents (4.33) we may employ the vector-indexed ones [92]. For example, for $N=4$ $V_{\alpha\dot\alpha}{}^i{}_j = [D^k_\alpha, \bar D_{\dot\alpha m}] V^{im}{}_{jk}$ contains the conserved SU(4) gauge current, supersymmetry current and energy-momentum tensor as the lowest components. It should couple to a vector indexed $N=4$ prepotential $H_\mu{}^i{}_j$ containing the $N=4$ CSG gauge *fields* as its components.

Using the linearized results given above it is possible to determine the linearized geometrical superspace constraints [157, 158]. They can then be generalized to the non-linear level. Following refs. [157, 158] we shall assume that the superspace tangent space group is $SL(2, C) \times U(N)$, i.e. is the product of the Lorentz group and the gauge group of component conformal supergravity (the U(1) gauge field will be non-dynamical for $N=4$). The super-space geometry is governed by the vielbein $E_M{}^A$, and the connection $\Omega_{MA}{}^B$, which transform under the general coordinate transformations, and under the tangent space rotations, e.g.

$$\delta E_M{}^A = \xi^N \partial_N E_M{}^A + \partial_M \xi^N E_N{}^A\,, \qquad \delta E_M{}^A = E_M{}^B L_B{}^A\,, \tag{4.36}$$

where $L_B{}^A$ corresponds to $SL(2, C) \times U(N)$, i.e. $L_{ab} = -L_{ba}$, $L_{\alpha\beta} = \frac{1}{2}(\sigma^{ab})_{\alpha\beta} L_{ab}$, $(L_i^j)^* = -L^j{}_i$, etc.

In contrast with the component approach (see section 3.1) where one gauges the whole superconformal group $SU(2,2|N)$ and then imposes the curvature constraints eliminating some of the fields (gauge fields of M, K and S symmetries), here it is possible to gauge only the subgroup $SL(2, C) \times U(N)$. The local translations (x-space general coordinate transformations), the Q-supersymmetry transformations and the conformal boosts K, will then be contained in the general superspace coordinate transformations, while the Weyl D transformations and the local S-supersymmetry transformations will appear as additional symmetries of the constrained geometry. Such an approach was first suggested (for $N=1$) in refs. [162, 216].[3] Using the standard definitions of the torsion and of the curvature (see e.g. [261])

$$\mathscr{D}E^A = \tfrac{1}{2} E^C \wedge E^B T_{BC}{}^A, \qquad E^A \equiv dZ^M E_M{}^A\,,$$
$$d\Omega_A{}^B + \Omega_A{}^C \wedge \Omega_C{}^B = \tfrac{1}{2} E^D \wedge E^C R_{CDA}{}^B, \qquad \Omega_A{}^B \equiv dZ^M \Omega_{MA}{}^B\,, \tag{4.37}$$

one finds the Bianchi identities

$$\sum_{(ABC)} (R_{ABC}{}^D - \mathscr{D}_A T_{BC}{}^D - T_{AB}{}^E T_{EC}{}^D) = 0\,,$$
$$\sum_{(ABC)} (\mathscr{D}_A R_{BCD}{}^E + T_{AB}{}^F R_{FCD}{}^E) = 0 \tag{4.38}$$

where all the indices are those of the tangent space, i.e. $\mathscr{D}_A = E_A{}^M \mathscr{D}_M$, $\mathscr{D}_M = \partial_M + \Omega_M$ etc. Using the covariant objects like the torsion and the curvature it is possible to simplify the procedure used in

[3] The U(1) symmetry of $N=1$ CSG was not explicitly gauged in these works but also appeared as a symmetry of the constraints. The approach using only $SL(2, C)$ as a tangent space group with a local $U(N)$ symmetry emerging as an additional symmetry of the constrained geometry was studied in [126].

comparing with component results. Starting with the gauge superfields (the vielbein and the connection) one confronts the problem of spurious Wess–Zumino gauge degrees of freedom (the gauge parameter superfields ξ^A and $L_A{}^B$ consist, except for their first components, of the Wess–Zumino gauge parameters; the corresponding symmetries are to be fixed in order to provide an identification of the component fields).

To establish the representations of supersymmetry appropriate for (conformal) supergravities, starting with the reducible representations given by $E_M{}^A$ and $\Omega_{MA}{}^B$, one has to impose constraints on the components of the torsion[4] and then to reveal the component contents of the covariant superfields by means of spinorial derivatives [264]. Imposing the constraints and employing the Bianchi identities one finds the covariant component fields, i.e. the gauge field strengths, the "matter" fields (scalars and spinors) and the auxiliary fields. Hence one thus expects to express $T_{BC}{}^A$ and $R_{ABC}{}^D$ in terms of the (derivatives of) field strength superfields, generalizing the linearized ones appearing in (4.27) (and also of some other superfields related to additional invariances of the constrained geometry). The gauge potentials themselves are located in the lowest components of the vielbein and in the connection

$$E_\mu{}^a = e_\mu{}^a + \cdots, \qquad E_\mu{}^\alpha = \psi_\mu{}^\alpha + \cdots, \qquad \Omega_\mu{}^i{}_j = V_\mu{}^i{}_j + \cdots. \tag{4.39}$$

The torsion constraints which correspond to off-shell $N \le 4$ conformal supergravity were found in refs. [157, 158]:

$$\begin{aligned}
&\dim 0: && T_{\alpha\dot\beta}{}^a = -\mathrm{i}\delta_j^i(\sigma^a)_{\alpha\dot\beta}, \quad T_{\alpha\beta}{}^a = 0, \quad T_{\dot\alpha\dot\beta}{}^a = 0,\\
&\dim \tfrac12: && T_{\alpha\beta}{}^\gamma = 0, \quad T_{\alpha\dot\beta}{}^{\dot\gamma} = 0, \quad T_{\alpha b}{}^c = 0,\\
& && T_{\dot\alpha\dot\beta}{}^{\dot\gamma} = 0, \quad T_{\dot\alpha\beta}{}^\gamma = 0, \quad T_{\dot\alpha b}{}^c = 0,\\
&\dim 1: && T_{ab}{}^c = 0
\end{aligned} \tag{4.40}$$

(for $N = 4$ we shall need also one additional constraint [157]). As a consequence of these constraints and of the Bianchi identities (4.38) we get [158] (see also [128])

$$\begin{aligned}
&T_{\alpha\beta}{}^{\dot\gamma} = \varepsilon_{\alpha\beta}\bar\Lambda^{ijk\,\dot\gamma}, \qquad \mathscr{D}_\beta{}^{(k}\bar\Lambda^{i)jl\,\dot\alpha} = 0,\\
&T_{a\beta}{}^{\dot\gamma} \to T_{\alpha\dot\alpha}{}^{jk}_{\beta\dot\gamma} = \mathrm{i}\varepsilon_{\alpha\beta}\bar T^{jk}_{\dot\alpha\dot\gamma} + \mathrm{i}\varepsilon_{\dot\alpha\dot\gamma}B^{jk}_{\alpha\beta},\\
&T_{a\beta}{}^\gamma \to T_{\alpha\dot\alpha}{}^j{}_{\beta\gamma k} = -2\mathrm{i}\varepsilon_{\alpha\gamma}G^j_{\beta\dot\alpha k} - \frac{\mathrm{i}}{12}(N-3)\delta^j_k\,\varepsilon_{\alpha\beta}\bar\Lambda^{ijk}_{\dot\alpha}\Lambda_{ijk\gamma}.
\end{aligned} \tag{4.41}$$

The superfields Λ and T are antisymmetric in the SU(N) indices. For $N \ge 3$ $\mathscr{D}^i_{(\alpha}\Lambda_{\beta)jkl} = 3\delta^i_{[j}T_{kl]\alpha\beta}$ but for $N = 2$ Λ is zero and T is independent (both fields vanish for $N = 1$). The component expansions of Λ and T start with the dim $= \frac12$ spinor and the dim $= 1$ antisymmetric tensor of conformal supergravity (see (4.29)). The superfields $B^{jk}_{\alpha\beta} \equiv \varepsilon_{\alpha\beta}S^{(jk)} + N^{[jk]}_{(\alpha\beta)}$ and $G^j_{\beta\dot\alpha k}$ (Hermitean) play an auxiliary role (see below). The dim $= \frac32$ torsion components can be expressed in terms of the irreducible parts of the gravitino field strength superfield $\psi_{\alpha\beta\gamma i}$ and, for $N > 1$, in terms of the derivatives of the already introduced superfields. Thus the supergeometry (i.e. the components of $T_{AB}{}^C$ and thus of $R_{ABC}{}^D$) is

[4] In view of the Bianchi identities (4.38) one can express the curvature in terms of torsion (and its derivatives) [66].

described by the superfields Λ (or T for $N=2$, or ψ for $N=1$), B and G which are subject to some further constraints implied by the Bianchi identities.

Switching for a moment to a linearized theory, it is easy to understand that the additional constraint which we need in the $N=4$ case is the vanishing of the U(1) part of the curvature [157]: $R_{ABj}{}^{j}=0$ [5] (the U(1) vector is "fake" in $N=4$ CSG). One then finds that for $N=4$ [158]

$$D_{\alpha}{}^{[i}\bar{\Lambda}^{jkl]}_{\dot{\beta}}=2\mathrm{i}\partial_{\alpha\dot{\beta}}\bar{W}^{ijkl} \tag{4.42}$$

where the chiral field $W_{ijkl}=\varepsilon_{ijkl}W$ can be identified with the $N=4$ linearized field strength superfield introduced at the beginning of this section. This suggests that, in order to make contact with the component results, we have to get rid of the extraneous fields contained in B and G.

Before presenting the solution of this problem (which lies in an additional super-Weyl invariance of the full set of the constraints) let us complete the discussion of the $N=4$ theory. The additional $N=4$ constraint at the non-linear level is [157]

$$\mathscr{D}_{\alpha}{}^{[k}\Lambda^{l]\alpha}=0,\qquad \Lambda^{l}=\frac{1}{3!}\varepsilon^{ijkl}\Lambda_{ijk}\,. \tag{4.43}$$

Introducing the one-form

$$P=E^{a}P_{a}-E^{\dot{\alpha}}P_{\dot{\alpha}},\qquad P_{\alpha\dot{\alpha}}=\frac{\mathrm{i}}{4}\mathscr{D}_{\alpha}{}^{i}\bar{\Lambda}_{i\dot{\alpha}},\qquad P_{\dot{\alpha}i}=\bar{\Lambda}_{i\dot{\alpha}}$$

one establishes that the constraints imply [157]

$$\mathrm{d}\hat{\Omega}-\hat{\Omega}\wedge\hat{\Omega}=0,\qquad \hat{\Omega}=\begin{pmatrix}\frac{1}{2}\Omega^{a}{}_{a} & \bar{P}\\ P & \frac{1}{2}\bar{\Omega}_{a}{}^{a}\end{pmatrix}. \tag{4.44}$$

Therefore $\hat{\Omega}$ is a pure gauge Lie algebra valued one-form, the relevant group being the group SU(1, 1) of ref. [8] (cf. (3.58))

$$\hat{\Omega}=\mathscr{V}^{-1}\,\mathrm{d}\mathscr{V},\qquad \mathscr{V}\in\mathrm{SU}(1,1) \tag{4.45}$$

(in the linearized limit the solution of (4.44) is $P\sim\mathrm{d}W$, where W is chiral, cf. (4.42)). The $\theta=0$ component of $\mathscr{V}$ is a scalar matrix $U(\Phi)$ (3.58) with only two physical components since (4.44) implies that the U(1) gauge field does not propagate and hence that we may use the U(1) freedom to fix the third component, e.g., as in (3.61). As a result, $\mathscr{V}$ describes two real scalar superfields parametrizing the coset space SU(1, 1)/U(1) and substitutes the field strength superfield W, found at the linear level. We conclude that the $N=4$ superspace theory constructed in ref. [157] corresponds to the SU(1, 1)-invariant version of the $N=4$ CSG of ref. [8]. Still it is not clear whether the SU(1, 1)-invariant version is uniquely implied in the superspace approach (the choice of the $N=4$ constraint (4.43), connected with rigid SU(1, 1) symmetry, may not be unique).

The additional (super-Weyl) transformations which leave invariant the constrained geometry can be

[5] This constraint can of course be rewritten as a torsion constraint; note also that some of its components are not independent of (4.40).

written as follows [158]

$$\delta E_M{}^B = E_M{}^A H_A{}^B, \qquad H_a{}^b = 2\delta_a{}^b U,$$
$$H_\alpha{}^\beta = \delta_\alpha{}^\beta \delta_j^i U, \qquad H_{\dot\alpha}{}^{\dot\beta} = -\delta_\alpha{}^\beta \delta_j^i U, \tag{4.46}$$
$$H_a{}^{\dot\beta} \to H_{\alpha\dot\alpha}{}^{\dot\beta j} = 4\mathrm{i}\delta_{\dot\alpha}{}^{\dot\beta} \mathscr{D}_\alpha{}^j U = -H_{\alpha\dot\alpha}{}^\beta_j$$

(all other components of $H_A{}^B$ vanish). The corresponding connection transformations follow from the torsion constraints. Equation (4.46) can also be put in the form

$$\delta E_\alpha{}^M = -UE_\alpha{}^M . \tag{4.47}$$

Here U is a real unconstrained scalar superfield, with leading components corresponding to the ordinary Weyl transformations and to S-supersymmetry. To see this one uses (4.39) (choosing the appropriate coordinate Wess–Zumino gauge) and one is thus led to the standard D and S transformations: $\delta e^a_\mu = 2U|_{\theta=0} e^a_\mu$, $\delta\psi_\mu{}^\alpha = U|_{\theta=0}\psi_\mu{}^\alpha - 2\mathrm{i}e_\mu{}^b(\sigma_b)^{\alpha\dot\alpha}\bar{\mathscr{D}}_{\dot\alpha i}U|_{\theta=0}$. The transformations of the basic superfields under (4.47) are given by

$$\delta\mathscr{V} = 0, \qquad \delta\bar\Lambda_{\dot\alpha}{}^{ijk} = -U\bar\Lambda_{\dot\alpha}{}^{ijk}, \qquad \delta\bar T_{\dot\alpha\dot\beta}{}^{ij} = -2U\bar T_{\dot\alpha\dot\beta}{}^{ij},$$
$$\delta\bar\psi^i_{\dot\alpha\dot\beta\dot\gamma} = -3U\bar\psi^i_{\dot\alpha\dot\beta\dot\gamma}, \qquad \delta B^{ij}_{\alpha\beta} = -4\mathscr{D}_\alpha{}^i\mathscr{D}_\beta{}^j U - 2UB^{ij}_{\alpha\beta}, \tag{4.48}$$
$$\delta G^i_{\alpha\dot\beta j} = -[\mathscr{D}^i_\alpha, \bar{\mathscr{D}}_{\dot\beta j}]U - 2UG^i_{\alpha\dot\beta j} .$$

Hence the superfields $\mathscr{V}(N=4)$, $\Lambda(N=3)$, $T(N=2)$ and $\psi(N=1)$ transform homogeneously and can be regarded as superspace generalizations of the Weyl tensor. In contrast, the "auxiliary" superfields B and G transform inhomogeneously and hence their non-covariant components can be gauged away, going to a Wess–Zumino gauge for the super-Weyl transformations. Taking into account all the constraints one can check [158] that the independent superfields ($N=4$: $\mathscr{V}$, B, G; $N=3$: ...) describe the known (off-shell) component degrees of freedom (4.29) together with components of a real scalar superfield. Thus we understand why the component CSG theories were given in the Wess–Zumino gauge defined with respect to supercoordinate and super-Weyl transformations.

The conclusion is that off-shell $N \le 4$ conformal supergravity may be described in a superspace with $\mathrm{SL}(2,C)\times \mathrm{U}(N)$ as the tangent space group in terms of a conformally covariant superfield which is $\mathscr{V}$ for $N=4$, $\Lambda_{\alpha ijk}$ for $N=3$, $T_{\alpha\beta ij}$ for $N=2$ and $\psi_{\alpha\beta\gamma}$ for $N=1$ (which are generalizations of the linear field strength superfields W, W_α, $W_{\alpha\beta}$, $W_{\alpha\beta\gamma}$) together with the additional non-Weyl-covariant superfields B and G.

The covariant superfields satisfy a chirality condition[6] which should be accounted for in any attempt to construct non-linear superspace actions which generalize (4.32). These actions should be invariant under superspace coordinate transformations and also under super-Weyl transformations (4.47). Hence they are to be built out of geometrical quantities with definite transformation laws under (4.47). Correspondence with the linearized expressions (4.32) suggests that the non-linear actions for $N=$

[6] Namely, $N=1$: $\bar{\mathscr{D}}_{\dot\gamma}\psi_{\alpha\beta\gamma}=0$; $N=2$: $\bar{\mathscr{D}}_{\dot\gamma}T_{\alpha\beta}=0$; $N=3$: $\bar{\mathscr{D}}_{\dot\gamma}\Lambda_\alpha=0$; $N=4$: the analog of the chirality condition for $\mathscr{V}$ follows from (4.44), (4.45). Note also that these fields are not subject to further constraints like (4.27) because their components are by definition field strengths. The constraints (4.27) are automatically satisfied in the linear approximation.

1, 2, 3 should look like

$$I^{(1)}_{\text{CSG}} \sim \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \varepsilon_1 \psi_{\alpha\beta\gamma}\psi^{\alpha\beta\gamma} + \text{c.c.}\,,$$
$$I^{(2)}_{\text{CSG}} \sim \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \varepsilon_2 T_{\alpha\beta}T^{\alpha\beta} + \text{c.c.}, \qquad I^{(3)}_{\text{CSG}} \sim \int \mathrm{d}^4x\, \mathrm{d}^6\theta\, \varepsilon_3 \Lambda_\alpha \Lambda^\alpha + \text{c.c.} \tag{4.49}$$

where $T_{\alpha\beta} = \frac{1}{2}T_{\alpha\beta ij}\varepsilon^{ij}$, $\Lambda_\alpha = (1/3!)\Lambda_{\alpha ijk}\varepsilon^{ijk}$ and ε_N are covariant chiral densities of zero dimension (the form of the $N=4$ action which should be non-polynomial in $\mathscr{V}$ is harder to guess). The expression for ε_N (and hence for the complete action) is known only for $N=1$: $\varepsilon_1 \sim e + \theta\psi + \cdots$ is the chiral density superfield (see e.g. [261]) or $\varepsilon_1 = \phi^3$, where ϕ is the chiral compensator [129]. The $N=1$ CSG action was first written down as a manifestly covariant integral over the whole superspace [90, 162, 215, 216]

$$I^{(1)}_{\text{CSG}} \sim \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, ER^{-1}\psi_{\alpha\beta\gamma}\psi^{\alpha\beta\gamma} + \text{c.c.} \tag{4.50}$$

where $E = \operatorname{sdet} E_M{}^A$ and $R = R_{\dot\alpha\dot\beta}{}^{\dot\alpha\dot\beta}$ is the chiral $N=1$ superfield of refs. [264, 142] (it is the $N=1$ analog of the auxiliary superfield B). Under the super-Weyl transformation (4.47) $\delta E = 4UE$, $\delta R = -2UR + 24\bar{\mathscr{D}}^2U$ and thus (4.50) is invariant. The equivalent form of (4.50) can be obtained using the super Gauss–Bonnet theorem [90]

$$I^{(1)}_{\text{CSG}} \sim \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, E(G^{\alpha\dot\alpha}G_{\alpha\dot\alpha} - 4R\bar{R})\,. \tag{4.51}$$

Here $G_{\alpha\dot\alpha}$ is the second "auxiliary" superfield of ref. [264] (the $N=1$ analog of the auxiliary superfield G). The possibility to rewrite the chiral action as an integral of a local density over the full superspace seems to be a peculiarity of the $N=1$ case. This follows from dimensional analysis:

$$[x] = -1, \quad [\theta] = -\tfrac{1}{2}, \quad [\mathrm{d}\theta] = +\tfrac{1}{2}, \quad [E] = 0,$$
$$[R] = +1, \quad [\psi] = \tfrac{3}{2}, \quad [T] = 1, \quad [\Lambda] = \tfrac{1}{2}\,. \tag{4.52}$$

This shows that the full superspace measure (as well as the dimensions of the geometrical fields) is non-negative for $N \geq 2$.[7] At the same time, the action expressed in terms of prepotentials should be representable as an integral over the full superspace. Needless to say, the structure of superspace actions for $N > 1$ conformal supergravities deserves further study.

The final question we would like to address in this section is concerned with the possibility of higher N generalization of the above superspace construction of conformal supergravity (see also the next section). One can make the assumption that the constraints (4.40) are valid also for $N > 4$ and then try to study the consequences of these constraints for arbitrary N. A natural question is then whether or not the constraints describe an off-shell theory, i.e. are purely kinematic (do not imply equations of motion

[7] Note that the chiral measure $\mathrm{d}^4x\, \mathrm{d}^{2N}\theta$ is non-positive only for $N \leq 4$; thus the actions of hypothetical $N > 4$ conformal supergravities should be integrals over some other subspaces of superspace, or they should include fields of negative dimensions.

which lead to restrictions on x-space geometry). This latter problem was investigated in ref. [128], where it was shown that eqs. (4.40) are kinematic in nature only for $N \leq 4$, while, for $N > 4$, some of their consequences are equivalent to the conformal supergravity equations of motion. More precisely, treating the theory in the linearized limit, it was shown that eqs. (4.40) imply that the auxiliary field $\bar{T}^{jk}_{\dot\alpha\dot\beta}$ (see (4.41)) satisfies the following equation

$$(N-3)(N-4)\partial_\alpha{}^{\dot\alpha}\, \partial_\beta{}^{\dot\beta}\, \bar{T}^{ij}_{\dot\alpha\dot\beta} = 0, \qquad N \geq 3 . \tag{4.53}$$

Making use of the fact that the gravitino field strength and the Weyl tensor can be obtained from $\bar{T}^{ij}_{\dot\alpha\dot\beta}$ as:

$$\begin{aligned} \bar{\psi}^i{}_{\dot\alpha\dot\beta\dot\gamma} &= \frac{1}{6(N-1)} \sum_{(\dot\alpha\dot\beta\dot\gamma)} D_{\dot\gamma j}\, \bar{T}^{ji}_{\dot\alpha\dot\beta}\big|_{\theta=0} , \\ \bar{C}_{\dot\alpha\dot\beta\dot\gamma\dot\delta} &= \frac{1}{144N(N-1)} \sum_{(\dot\alpha\dot\beta\dot\gamma\dot\delta)} D_{\dot\delta i} D_{\dot\gamma j} \bar{T}^{ji}_{\dot\alpha\dot\beta}\big|_{\theta=0} , \end{aligned} \tag{4.54}$$

one finds [128] that the gravitino and graviton fields must satisfy the linearized conformal equations of motion [90], multiplied by the factor $(N-3)(N-4)$

$$\begin{aligned} &(N-3)(N-4)\, \partial_\gamma{}^{\dot\gamma}\, \partial_\beta{}^{\dot\beta}\, \bar{\psi}^i{}_{\dot\alpha\dot\beta\dot\gamma} = 0 , \\ &(N-3)(N-4)\, \partial_\delta{}^{\dot\delta}\, \partial_\gamma{}^{\dot\gamma}\, \bar{C}_{\dot\alpha\dot\beta\dot\gamma\dot\delta} = 0 . \end{aligned}$$

Thus the constraints (4.40) describe the *off-shell* $N = 3, 4$ theories, this being due to the presence in (4.53) of the curious numerical factors. For $N > 4$, we presumably get a conformal supergravity multiplet which is *on-shell* with respect to the conformal equations of motion. To find the off-shell theory one has probably to relax the second $\dim = 0$ constraint in (4.40), substituting it for $T_{\alpha\beta}{}^a = \varepsilon_{\alpha\beta} y^{[ij]a}$ [128], where the superfield y corresponds to a "non-linear multiplet" (see also ref. [127]). The $N > 4 = 2s_{\max}$ linearized conformal supergravities may be supposed to be described by at least one real scalar field strength superfield $W_{[i_1\cdots i_4]}$ [218], which should be covariant under super-Weyl transformations. Then y will be the *second* conformally covariant field strength superfield in the theory (which was absent for $N \leq 4$). Thus a superspace approach, while clearly distinguishing the $N \leq 4$ CSG's, may not completely rule out the possibility of the existence of some non-trivial $N > 4$ conformal supergravities.

4.3. $N > 4$?

In this section we shall first summarize the arguments that are usually considered as giving indications that "standard" conformal supergravities "do not exist" for $N > 4$ and then discuss some hypothetic possibilities concerning how $N \geq 5$ CSG could look like.

An earliest indication that some drastic changes should take place above $N = 4$ is provided by the superconformal algebra $SU(2, 2|N)$ itself (see section 3.1): some of the structure constants are proportional to $(N-4)$ and thus change sign for $N > 4$. As a consequence, the U(1)-term in the naive CSG Lagrangian (3.14) contributes for $N > 4$ with a "wrong" (ghost) sign [83]. We have already mentioned this fact in section 3.1 where it was noted, however, that the appearance of an additional ghost vector particle in the higher derivative (perturbatively non-unitary) theory under consideration, should not probably be considered as a disaster.

The (chronologically) second argument for the upper limit of $N=4$ for extended conformal supergravities had been given in ref. [52].[1] It was based on a generalization of the procedure for the counting of the states for the system of Poincaré and conformal supergravities described in section 3.2. The main assumption that underlies the reasoning in ref. [52] is that there exists *an off-shell formulation* of this O(N) PSG + U(N) CSG coupling, i.e. that there exist the corresponding PSG auxiliary fields.[2] This assumption is of course a highly speculative one in view of the absence, for $N>4$, of natural candidates for compensating matter multiplets that could play the role of, e.g., the $N=4$ super-Yang–Mills in the $N=4$ case, cf. sections 3.2, 3.5. It is also worth recalling that the existence of (a *finite* number of) $N>2$ Poincaré supergravity auxiliary fields is doubtful at present in view of "no go" theorems [221, 209]. Under this assumption the spectrum of the "hybrid" theory should consist of a combination of massless and massive *on-shell* multiplets of N-extended (rigid) supersymmetry [194, 91] (see table 3.5). The results of the analysis of the spectrum (see section 3.2 for $N\leq 4$) change completely for $N>4$ because there are no massive supermultiplets with $N>4$ and the highest spin $s\leq 2$. In this situation one can envisage at least three alternatives (cf. [52]):

(1) there exists no off-shell formulation for the PSG + CSG-coupling;

(2) $N>4$ CSG's exist but contain propagating higher-spin fields;

(3) $N>4$ CSG's do not exist (there exist no SO(N)-supersymmetric extensions of the Weyl action).

It is the third possibility that was advocated in ref. [52] (the reasons were: difficulties with higher spin PSG auxiliary fields; difficulties with propagating higher spin conformal fields in CSG; naturality of the bound $s\leq 2$ in the CSG, interpreted as a gauge theory of the superconformal group). Still the authors themselves considered their arguments as far from being completely conclusive (see also [8]).[3]

The presence of the $N=4$ "boundary" was also noted in the context of linearized superfield description of CSG's [218]. Let $s_{\max}$ be the maximal spin in a supermultiplet. Then the field strength superfields that describe the *on-shell* PSG's are chiral $W_{(\alpha_1\cdots\alpha_{4-N})[i_1\cdots i_N]}$ for $N\leq 2s_{\max}=4$ (see (4.27)) and general scalar W_{ijkl} for $N>2s_{\max}=4$, where $i,j,\ldots$ are SU(N) indices. In the latter case one has instead of (4.27) [218]

$$D_\alpha{}^i W_{jklm} = -\frac{4}{N-3}\,\delta^i_{[j} D_\alpha{}^n W_{klm]n}\,. \tag{4.55}$$

Given that the chiral superfields $W_{\cdots\cdots}$ describe also *off-shell* $N\leq 4$ CSG's (see section 4.2), one can

[1] This upper limit was also conjectured by M. Gell-Mann, unpublished (as quoted in [52]).

[2] The argument that follows is insensitive to a subset of auxiliary fields which are auxiliary fields of CSG and of compensating matter multiplet used to generate PSG in the Poincaré gauge.

[3] One apparent "contradiction" noted in ref. [52] is that the contribution of spins ≤ 2 to the one-loop Weyl tensor squared counter-term is "always positive" in a standard class of gauges. This seems to imply that quantum corrections will give rise to this invariant (and thus to conformal supergravity action) irrespective of the value of N. Such reasoning, however, is not correct in two points. First, the contribution of the ordinary gravitino (quantized in the standard *background* gravitational gauge) in the Weyl infinity has *negative* sign [105] and thus the total coefficient of the C^2-term may well turn out to be zero even in a standard class of gauges (as it happens indeed in the $N=8$ PSG [105]). Second, the "standard gauges" may not be consistent with supersymmetry (they do not form a "supergauge"). As for coefficients of infinities in arbitrary gauges they can take any possible value and sign (see e.g. [175]). Whether the one-loop off-shell infinities actually cancel in $N>4$ PSG's once quantized in a manifestly "$N/2$"-supersymmetric fashion, is presently unclear [160] (though they have to cancel for $N\geq 3$ if these theories can be formulated off-shell in terms of N-extended superfields [146]).

conjecture that the scalar superfields W_{ijkl} describe when taken off-shell $N>4$ conformal supergravity (the corresponding action then should differ from (4.32) because W_{ijkl} is non-chiral). However, the analysis of ref. [128] (see the end of the previous section 4.2) suggests that this scalar superfield alone is not sufficient for the off-shell representation of $N>4$ CSG: one needs at least one additional conformally covariant superfield. It may turn out eventually that the off-shell field representation of $N>4$ CSG may exist (and may even not contain higher spin fields). Then we will still be left with the problem of how to construct the corresponding conformally invariant action.

Presently a safe conclusion is that there are no methods for the construction of conformal supergravities with $N>4$. The method of "coupling to supercurrents" [8, 159], that was successful for $N \leq 4$ [8], cannot be straightforwardly applied for $N>4$ (if supercurrents are constructed from on-shell $N>4$ PSG superfields, $J \sim W\bar{W}$, we are led to higher spin (e.g. $s=4$) gauge multiplets. However, there are ambiguities in the interpretation and definition of the supergravity supercurrent at the non-linear level, cf. ref. [49]. An interesting possibility that falls outside the scope of the methods of ref. [159] is that $N>4$ CSG's exist and contain only spins ≤ 2 but that the supersymmetry algebra closes on the corresponding field representation only *on the mass shell* (i.e. that the auxiliary fields are absent). The point is that the following theories: $N=2$ hypermultiplet, $N=4$ super-Yang–Mills and $N>4$ conformal supergravities without higher spins have much in common in a sense that they all belong to a class of theories with $N>2s_{\max}$ and thus cannot be formulated off-shell within the method of ref. [159]. This hypothetical possibility (supported also by the observation that the usual superspace constraints for $N>4$ define the on-shell theory [128]) is a version of the first alternative mentioned above:

(1′) there exist $N>4$ conformal supergravities with $s_{\max} \leq 2$ but without (a finite number of) auxiliary fields, so that they cannot be coupled to Poincaré supergravity in a manifestly supersymmetric manner. The natural spectrum of fields of such a theory should probably consist of the same fields as those which enter $N \leq 4$ CSG's, see (3.36) (the corresponding field strength superfield is $W_{ijkl} \sim \varphi_{ijkl} + \theta_i \Lambda_{jkl} + \cdots$). For example, for $N=5$ we suggest

$$\begin{aligned} &e^a_\mu; \quad \psi^i_\mu(5); \quad A_\mu; \quad V_\mu{}^i{}_j(24); \quad T_{\mu\nu}{}^{-ij}(10); \\ &\Lambda^{ijk}(1+10); \quad \varphi^{ijkl}(5); \quad \chi^{\cdots}(N_{1/2}); \quad E^{\cdots}(N_0). \end{aligned} \tag{4.56}$$

The numbers in brackets indicate the SU(5) representations and have natural interpretation: V_μ is the gauge field of SU(5), while the numbers of ψ_μ, $T_{\mu\nu}$, Λ and φ are equal to the numbers of gravitinos, SO(5) vectors, spinors and scalars of the SO(5) PSG. The numbers of "ordinary" spinors (χ) and scalars (E) are less certain. The condition of zero total number of *on-shell* degrees of freedom in the multiplet (4.56) gives (see table 3.3)

$$\nu_{\text{tot}} = 30 - 2N_{1/2} + 2N_0 = 0\,. \tag{4.57}$$

Instead of assigning component fields to some "natural" SU(N) representations one can try to construct the multiplet of fields of higher N CSG using $N \leq 4$ supermultiplets as building blocks. For example, the relevant $N=1$ supermultiplets have the following field content:

$$\begin{aligned} &\{2\}_1 = (e^a_\mu, \psi_\mu, V_\mu), \qquad \{\tfrac{3}{2}\}_1 = (\psi_\mu, 2V_\mu, T_{\mu\nu}, \chi)\,, \\ &\{1\}_1 = (V_\mu, \chi, D), \qquad \{\tfrac{1}{2}\}_1 = (\Lambda, T_{\mu\nu}, E, \chi)\,, \\ &\{0\}_1 = (E, \chi, 2D), \qquad \{0'\}_1 = (\varphi, \Lambda, E)\,. \end{aligned} \tag{4.58}$$

Here all the fields are the same as in table 3.3, but now they are taken without SU(N) indices (e.g., V_μ is a vector, χ is an ordinary spinor, E is a complex scalar, Λ is a "$\partial\!\!\!/^3$"-spinor, φ is a complex "$\Box^2$"-scalar, D is a real auxiliary field, etc.). The multiplets $\{1\}_1$ and $\{0\}_1$ are the standard vector and scalar $N=1$ multiplets, while $\{\frac{1}{2}\}_1$ and $\{0'\}_1$ are the higher derivative multiplets (the latter was already discussed in section 2.3). All multiplets are *off-shell* multiplets, i.e. have zero total number of off-shell (as well as on-shell) degrees of freedom (this can be readily checked using table 3.3). The superfields that correspond to (4.58) are the three gauge superfields with maximum gauge invariance $\hat{h}_{\alpha\dot{\alpha}}$, $\hat{\psi}_\alpha$, $\hat{V}$ (taken in the Wess–Zumino gauge) and the non-gauge chiral superfields $\hat{\Lambda}$, $\hat{E}$, $\hat{\varphi}$ (cf. [233]). The $N=1$ decomposition of the spectra of $N\le 4$ CSG's looks like (see table 3.2)

$$\begin{aligned}
&N=1\ \text{CSG}=\{2\}_1\,,\\
&N=2\ \text{CSG}=\{2\}_1+\{\tfrac{3}{2}\}_1+\{1\}_1\,,\\
&N=3\ \text{CSG}=\{2\}_1+2\{\tfrac{3}{2}\}_1+(3+1)\{1\}_1+\{\tfrac{1}{2}\}_1+2\{0\}_1\\
&N=4\ \text{CSG}=\{2\}_1+3\{\tfrac{3}{2}\}_1+8\{1\}_1+3\{\tfrac{1}{2}\}_1+6\{0\}_1+\{0'\}_1\,.
\end{aligned}\tag{4.59}$$

Generalizing this pattern (observing that the numbers of $N=1$ multiplets conicide with dimensions of SU($N-1$) representations) one can suggest that

$$N=5\ \text{CSG}=\{2\}_1+4\{\tfrac{3}{2}\}_1+(15+1)\{1\}_1+6\{\tfrac{1}{2}\}_1+20\{0\}_1+4\{0'\}_1 \tag{4.60}$$

(the number of $\{0\}_1$ is again ambiguous). It is possible to play the same game with $N=2$ and $N=3$ multiplets. For example, the $N=2$ multiplets have the following field content (cf. [106, 109])

$$\begin{aligned}
&\{2\}_2=(N=2\ \text{CSG})=(e^a_\mu, 2\psi_\mu, 4V_\mu, T_{\mu\nu}, 2\chi, D),\\
&\{\tfrac{3}{2}\}_2=(\psi_\mu, 2T_{\mu\nu}, 3V_\mu, 3\chi, \Lambda, E, D)\,,\\
&\{1\}_2=(V_\mu, 2\chi, E, 3D),\qquad \{\tfrac{1}{2}\}_2=(2\Lambda, T_{\mu\nu}, 2\chi, 3E, \varphi)\,,\\
&\{0\}_2=(2E, 2\chi, 4D)\,.
\end{aligned}\tag{4.61}$$

Even having "guessed" some natural higher N CSG field content we would still be left with the problem of how to check that the corresponding supersymmetric theory can actually be constructed.

Next let us discuss the second alternative mentioned above, namely that $N>4$ conformal supergravities contain higher propagating fields. If we assume that such a theory can be coupled to Poincaré supergravity, then the $s>2$ CSG fields should have counterparts among the auxiliary fields of PSG. In order to have these higher spin fields as components of the massive supermultiplets, we are apparently led to the conclusion that they should occur in the CSG action with the "*standard*" number of derivatives: $\phi\ldots\Box\phi\ldots+\bar{\zeta}\ldots\partial\!\!\!/\zeta\ldots$. This, however, contradicts the basic postulate of the Weyl invariance of the CSG action. In fact, the conformally invariant arbitrary spin actions are the "pure spin" actions of section 1.3 (see (1.27)), $\mathscr{L}_s\sim\Phi\Box^s\Phi$. Only such higher derivative higher spin fields (with dimensions $2-s$) can be combined into one supermultiplet with the conformal graviton and gravitino. Thus the conformally invariant CSG action (without dimensional constants) should contain terms like

$$\mathscr{L}\sim\cdots+\Phi_3\Box^3\Phi_3+\bar{\zeta}_{5/2}\partial\!\!\!/\Box\zeta_{5/2}+f_2\Box^2 f_2+h\Box^2 h+\bar{\psi}\partial\!\!\!/^3\psi+\cdots\,. \tag{4.62}$$

Given that the $s > 2$ fields are propagating higher derivative fields of CSG and can correspond only to the auxiliary fields of $N \leq 8$ PSG, it seems difficult to imagine how they can form a part of the massive higher N multiplets supposed to appear in the spectra of the combined CSG + PSG system. The mechanisms that generate massive states for $N \leq 4$ (see eqs. (3.41)–(3.44)) fail to do this in the higher derivative case.

This suggests that, even if $N > 4$ CSG's with higher spin fields can exist, they cannot be consistently coupled to the corresponding PSG, i.e. that the reasoning of ref. [52] which originally led to the second alternative (CSG with $s > 2$ fields) cannot be applied. This in turn may lend support to the conjecture that $N > 4$ CSG should exist without higher spin fields. As for a conformally invariant theory with higher spins (4.62), it is difficult to interpret it as a kind of conformal supergravity in the absence of a reasonable mechanism that can separate the physical graviton from other spin 2 fields, which together should form a representation of SU(N). Also the problem of consistency of higher spin couplings, though probably milder in a higher derivative case, still deserves further study.

The growth of the number of derivatives in (4.62) with the increase of the spin value suggests the following speculation: the "true" theory should contain *all* spins $s < \infty$ (an infinite "tower" of fields) and thus derivatives of all powers in the action. It is the *low-energy* approximation that separates a small set of lower spins from all the others (higher derivative terms are irrelevant in the low energy domain). The full action including all values of spin and all powers of derivatives may resemble some "non-local" (or "effective" "string") action.

The conclusion that can be drawn from this somewhat confusing section is that the subject of $N > 4$ conformal supergravities (with or without higher spin fields), though speculative, is not completely fantastic because there seems to be no safe arguments that can rule out the existence of such theories.

5. Quantum Weyl gravity

5.1. *Perturbation theory: asymptotic freedom and anomaly*

We shall start the discussion of quantum conformal supergravities with the $N = 0$ case, i.e. with the Weyl theory (1.3). We shall concentrate mainly on a number of points essential for the subsequent treatment of $N \geq 1$ theories.

For the sake of full generality we shall include in the basic action the scalar curvature squared term

$$I = \int \mathrm{d}^4x \sqrt{g}\,(aC^2 + bR^2) \tag{5.1}$$

(the Weyl theory corresponds to $b = 0$). For topologically non-trivial situations it is necessary to add to (5.1) the topological (Euler number) term

$$I_{\text{top}} = \int \mathrm{d}^4x \sqrt{g}\,\sigma R^*R^* = 32\pi^2\sigma\chi\,. \tag{5.1a}$$

The theory (5.1) in this section is supposed to be the fundamental theory which describes gravity at small distances and effectively reduces to the Einstein theory at large distances.

To quantize (5.1) we shall employ the Euclidean path integral approach. One can a priori question the validity of this approach in the case of higher derivative theories like (5.1). To "justify" the configuration space path integral quantization, one has first to work out the canonical formalism for the action (5.1) [177, 178, 166, 13], making use of the generalization of the Ostrogradski method in classical mechanics with higher derivatives (see e.g. [156] and references mentioned there). An equivalent approach is to consider the connection as an independent field, adding at the same time the Lagrange multiplier term $\Lambda^{\mu\nu}_{\lambda}(\Gamma^{\lambda}_{\mu\nu} - \{^{\lambda}_{\mu\nu}\})$. Then the Hamiltonian is defined in the standard way and is bilinear in the momenta. Quantizing the theory by means of the canonical path integral procedure (which is equivalent to the operator quantization), and carrying out integrations over the momenta (with proper account of constraints due to gauge invariance), we do finish with a configuration space path integral (cf. ref. [93]). Such a canonical derivation was already given in ref. [13], where it was stressed that it is "formal" in that respect that, for some of the canonical variables, the integration is to be carried over imaginary values. In the operator quantization language this means that one has to use an indefinite metric Hilbert space (see e.g. refs. [139, 15]).

It is also worth mentioning the point of view sometimes expressed that the heuristic Euclidean path integral quantization is in some sense more general than the canonical one in Minkowski space because it is more suitable for taking account of non-perturbative phenomena (for example, it can describe the processes with changes of topology which are excluded in the canonical approach [238]). This is supposed to be true only if the Euclidean theory is well-defined. Though this is not the case for the Einstein action (known to be non-positive definite) the positivity of the Euclidean action does take place in the Weyl theory (5.1) (we assume that in the *fundamental* action $a>0$ and $b\geq 0$).[1] Having found the well-behaved Euclidean correlation functions, we can then *define* the Minkowski Green's functions by analytic continuation [206]. The central problem is to check that the latter correspond to a consistent field theory in Minkowski space (for example, that the resulting S-matrix is unitary). We defer the discussion of this problem to section 5.2.

The basic object that contains all necessary information about the (Euclidean) quantum field theory is the effective action which can be symbolically defined as follows (see e.g. refs. [64, 258]),

$$\exp\left(-\frac{1}{\hbar}\Gamma[g]\right) = \int [\mathrm{d}h] \exp\left\{-\frac{1}{\hbar}\left(I[g+h] - \frac{\delta\Gamma[g]}{\delta g}h\right)\right\} \tag{5.2}$$

where the measure $[\mathrm{d}h]$ includes all the necessary gauge fixing and ghost factors. Γ is assumed to be expandable in powers of $\hbar$, $\Gamma[g] = I[g] + \hbar\,\Sigma_i\, c_i \ln\det\Delta^{(i)} + \cdots$. It is now possible to discuss the formal "kinematical" properties of the Euclidean effective action corresponding to (5.1). First, it is renormalizable (in four dimensions) if $a\neq 0$ *and* $b\neq 0$ [231].[2] In general renormalizability is a trivial consequence of the dimensionless nature of the couplings a and b and of the metric. Expanding near flat space we find $\mathscr{L} \sim h\Box^2 h + h\,\partial^2 h\,\partial^2 h + \cdots$. Thus the index of divergence, e.g. of the vacuum diagrams is $D = 4L + 4V - 4I = 4$ (L is the number of loops, V the number of vertices and I the number of internal lines). The necessary condition for renormalizability is the $1/p^4$ behavior of the propagator in *both* the gauge-invariant off-shell spin 2 and spin 0 sectors. This is the reason why the theories with $a=0$ or $b=0$ are a priori non-renormalizable.

To clarify this point let us introduce the following splitting of the quantum field

[1] The sign of b should turn out to be opposite in the effective low-energy action.

[2] To achieve renormalizability in the case of topologically non-trivial backgrounds, one has also to include the topological term (5.2) [98, 99].

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + \tfrac{1}{4}g_{\mu\nu}\phi, \qquad \bar{h}^{\mu}_{\mu} \equiv 0, \qquad \phi \equiv h^{\mu}_{\mu},$$
$$\bar{h}_{\mu\nu} = \bar{h}^{\perp}_{\mu\nu} + \mathcal{D}_{\mu}\eta^{\perp}_{\nu} + \mathcal{D}_{\nu}\eta^{\perp}_{\mu} + \mathcal{D}_{\mu}\mathcal{D}_{\nu}\sigma - \tfrac{1}{4}g_{\mu\nu}\mathcal{D}^2\sigma, \qquad \mathcal{D}_{\mu}\bar{h}^{\perp}_{\mu\nu} = 0, \qquad \mathcal{D}_{\mu}\eta^{\perp}_{\mu} = 0, \tag{5.3}$$

where $\mathcal{D}_{\mu}$ is covariant with respect to an arbitrary background metric $g_{\mu\nu}$. For example, for $g_{\mu\nu} = \delta_{\mu\nu}$ one finds (cf. (1.19), (1.22), (1.36))

$$-R \simeq \tfrac{3}{4}\Box\bar{\phi} + \tfrac{1}{4}[\bar{h}(-\Box)\bar{h} - \tfrac{3}{8}\bar{\phi}(-\Box)\bar{\phi}] + \cdots,$$
$$\bar{\phi} \equiv \phi - \Box\sigma, \tag{5.4}$$

$$R^2 \simeq \tfrac{9}{16}(\Box\bar{\phi})^2 = 3hP_0\Box^2 h, \qquad C^2_{\lambda\mu\nu\rho} \simeq \tfrac{1}{2}(\Box\bar{h}^{\perp})^2 = \tfrac{1}{2}hP_2\Box^2 h. \tag{5.5}$$

The cancellation of $\eta^{\perp}$, σ and ϕ in the expression for $C^2_{\lambda\mu\nu\rho}$ is the consequence of the general coordinate and Weyl invariances (note that the off-shell spin content of $h_{\mu\nu}$ is: $2(\bar{h}^{\perp})$, $1(\eta^{\perp})$, $0(\bar{\phi})$, $0(\sigma)$, total $5+3+1+1=10$ degrees of freedom). The inconsistency of the R^2-theory ($a=0$ in (5.1)) follows from the observation that $\bar{h}^{\perp}$ is absent in the propagator but contributes in the vertices. The situation in the Weyl theory ($b=0$) is more subtle. Here the additional local conformal symmetry dictates the classical decoupling of the scalar degree of freedom ϕ. However, it reappears through the finite terms which *turns* out to be necessary to add to Γ in order to maintain the Ward identity for general convariance, and which, at the same time, break the conformal symmetry. This is usually expressed by saying that the conformal (Weyl) symmetry is anomalous at the quantum level, i.e. that classically Weyl invariant theories are, as a rule, quantally inconsistent in perturbation theory (non-renormalizable in a conformal invariant fashion) [23, 106].

Leaving aside for a moment the topic of conformal anomaly in Weyl theory (which makes the theory inconsistent by producing R^2-infinities starting at the two-loop level [106]), let us now discuss the one-loop approximation and check some of the conclusions presented above. The computation of the one-loop infinities in the theory (5.1) (possibly augmented with the Einstein and cosmological terms) was initiated in ref. [165] and completed in refs. [98, 99] (for a detailed exposition of some points see also refs. [31, 3]). To carry out such a program one has to solve a number of problems of principle (covariant gauge fixing in higher derivative theories, algorithm for infinities in fourth-order differential operators etc.) as well as to do a great deal of algebraic computations. We shall below review a number of relevant questions and describe a simple (short-cut) way [250] to the results of ref. [99]. Let us first consider the one-loop effective action on a flat background. In terms of irreducible variables (5.3) we have

$$\mathcal{L} \simeq \tfrac{1}{2}a(\Box\bar{h}^{\perp})^2 + \tfrac{9}{16}b(\Box\bar{\phi})^2,$$
$$[\mathrm{d}h_{\mu\nu}] \to \mathrm{d}h^{\perp}_{\mu\nu}\,\mathrm{d}\eta^{\perp}_{\mu}\,\mathrm{d}\sigma\,\mathrm{d}\phi\,[\det\Delta_{1\perp}(\det\Delta_0)^2]^{1/2}, \tag{5.6}$$
$$\Delta_0 = -\Box, \qquad \Delta_{1\perp\mu\nu} = (-\delta_{\mu\nu}\Box)_{\eta^{\perp}_{\nu}}.$$

Fixing the coordinate gauge by inserting $\delta(\partial_{\mu}h_{\mu\nu})\det\Delta_{\mathrm{gh}}$, $(\Delta_{\mathrm{gh}})_{\mu\nu} = -\delta_{\mu\nu}\Box - \partial_{\mu}\partial_{\nu}$, $\det\Delta_{\mathrm{gh}} = \det\Delta_{1\perp}\det\Delta_0$ and integrating over $\bar{h}^{\perp}$, $\eta^{\perp}$, σ and ϕ, we find for the partition function ($\Gamma_1 = -\ln Z$)

$$Z = \left[\frac{\det\Delta_{1\perp}}{(\det\Delta_{2\perp})^2\det\Delta_0}\right]^{1/2}, \qquad \Delta_{2\perp} = (-\Box)_{\bar{h}^{\perp}_{\mu\nu}}. \tag{5.7}$$

Thus the number of dynamical degrees of freedom described by the theory (5.1) with $a \neq 0$ and $b \neq 0$ is equal to $\nu = 2\times 5 + 1 - 3 = 8$ [99]. In the case of Weyl theory ($b = 0$) the $(\det \Delta_0)^{-1}$ contribution of $\bar{\phi}$ is absent (the integral over ϕ is regularized by the ghost-free conformal gauge $\delta(\phi)$) [99, 100]

$$Z_W = [\det \Delta_{1\perp} \det \Delta_0 / (\det \Delta_{2\perp})^2]^{1/2} \tag{5.8}$$

and hence $\nu = 6$ [99] in agreement with (1.41)–(1.45).[3] The expressions (5.7), (5.8) (and thus the results for ν) are clearly gauge-independent because flat space is the classical solution of the theory (5.1) (the effective action is gauge-independent on-shell).

The derivation of (5.7) or (5.8) can be repeated also in a more customary way which generalizes to the case of an arbitrary metrical background. The starting point is the linearized Lagrangian (5.1) (e.g. for the pure Weyl theory $b = 0$) written in terms of the unconstrained variable $h_{\mu\nu}$

$$\begin{aligned} &\mathcal{L}_W = \tfrac{1}{2} a \{h_{\mu\nu}\Box^2 h_{\mu\nu} - (\partial_\mu h_{\mu\nu}) H_{\nu\rho}(\partial_\sigma h_{\sigma\rho}) - \tfrac{1}{6} h \Box^2 h\} \\ &H_{\mu\nu} \equiv -\delta_{\mu\nu}\Box + \tfrac{1}{2}\partial_\mu\partial_\nu, \qquad h = h^\mu_\mu - \partial_\mu \Box^{-1}\partial_\nu h_{\mu\nu}. \end{aligned} \tag{5.9}$$

To obtain the diagonal kinetic operator we have to cancel the second and the third terms by fixing the coordinate $(\partial_\mu h_{\mu\nu} = \xi_\nu(x))$ and conformal $(h = \varphi(x))$ gauges. Averaging over the gauges with appropriate weight operators (using the 't Hooft averaging trick: $\delta(\chi - \xi) \to \int d\xi\, e^{-\xi H \xi} (\det H)^{1/2} \delta(\chi - \xi) \to e^{-\chi H \chi} (\det H)^{1/2}$), and not forgetting to include the contributions of these operators in the path integral, we find

$$Z_W = \det \Delta_{gh} (\det H_{\mu\nu} \det \Box^2)^{1/2} [\det(\Box^2)_{h_{\mu\nu}}]^{-1/2} \tag{5.10}$$

where $(\Delta_{gh})_{\mu\nu} = -\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu$ is the ghost operator for the coordinate gauge. Diagonalizing the operators we are again led to $Z_W = \{[\det(-\Box)]^{1/2}\}^6$, $\nu = 6$, in agreement with (5.8) [99].[4] The expression for the partition function on a general background is analogous to (5.10) with all operators substituted for their covariant analogs, with coefficients depending on the curvature tensor (cf. (1.19)). Thus the one-loop correction in the effective action is given by

$$\Gamma_1[g] = \sum_i c_i \ln \det(\Delta^i[g]/\mu^{p_i}), \tag{5.11}$$

where $\Delta^{(i)}$ are second and fourth-order differential operators ($p_i = 2$ or 4). Let us first recall a number of standard facts concerning the infinities of $\ln \det \Delta$ for the diagonal second order operator. Employing the definition

$$\ln \det(\Delta/\mu^2) = -\int_\varepsilon^\infty \frac{ds}{s} \operatorname{tr} e^{-s\Delta/\mu^2}, \qquad \varepsilon \to +0 \tag{5.12}$$

[3] For pure R^2 theory ($a = 0$) we get $Z = (\det \Delta_{1\perp}/\det \Delta_0)^{1/2}$, i.e. $\nu = -2$.

[4] For discussions of degrees of freedom count in higher derivative (super) gravity theories within the canonical formalism see [185, 14].

where $\varepsilon = \mu/L$, $L \to \infty$ is the "proper time cut-off", and the asymptotic expansion (see e.g. [63, 137])

$$\operatorname{tr} e^{-s\Delta/\mu^2}|_{s\to 0} \simeq \sum_{p=0}^{\infty} \left(\frac{s}{\mu^2}\right)^{(p-4)/2} B_p\,,$$
$$B_p = \frac{1}{(4\pi)^2}\int b_p \sqrt{g}\, d^4x, \qquad B_{2k+1} = 0\,. \tag{5.13}$$

(Δ is assumed to be an elliptic self-adjoint operator, defined on a vector bundle over a four-dimensional compact Riemannian manifold), we can establish the ultra-violet divergent part of the effective action which reads:

$$\tfrac{1}{2}(\ln \det(\Delta/\mu^2))_\infty = -\tfrac{1}{2}(\tfrac{1}{2}L^4 B_0 + L^2 B_2 + \ln(L^2/\mu^2) B_4) \tag{5.14}$$

(for comparison we note that quartic and quadratic divergences are absent in dimensional regularization where $\frac{1}{2}(\ln \det(\Delta/\mu^2))_\infty = -B_4/(4-n)$, $4-n \to +0$). If

$$\Delta = -\mathscr{D}^2 + X, \qquad \mathscr{D}_\mu = \partial_\mu + A_\mu + \left\{ {\lambda \atop \mu\rho} \right\}, \tag{5.15}$$

where A_μ is a connection on the vector bundle (with curvature $F_{\mu\nu}$), the coefficients b_p in the infinite terms are given by [63, 241, 137]

$$b_0 = \operatorname{tr} \mathbb{1}, \qquad b_2 = \operatorname{tr}(\tfrac{1}{6}R\cdot \mathbb{1} - X)\,,$$
$$b_4 = \operatorname{tr}(\tfrac{1}{12}F_{\mu\nu}{}^2 + E\cdot\mathbb{1} + \tfrac{1}{2}X^2 - \tfrac{1}{6}RX - \tfrac{1}{6}\mathscr{D}^2 X) \tag{5.16}$$

$$E \equiv \tfrac{1}{180}R^*R^* + \tfrac{1}{60}W + \tfrac{1}{72}R^2 + \tfrac{1}{30}\mathscr{D}^2 R\,, \tag{5.17}$$

where we have introduced the notation

$$W \equiv R_{\mu\nu}{}^2 - \tfrac{1}{3}R^2 = \tfrac{1}{2}(C^2_{\lambda\mu\nu\rho} - R^*R^*)\,. \tag{5.18}$$

The formulas (5.14), (5.16), (5.17) make it possible to establish the gravitational infinities for various quantum field models defined on a gravitational background. In general

$$b_0 = \nu, \qquad b_2 = \rho R\,,$$
$$b_4 = \beta_1 R^*R^* + \beta_2 W + \beta_3 R^2 + \beta_4 \mathscr{D}^2 R \equiv (\beta_1 - \tfrac{1}{2}\beta_2)R^*R^* + \tfrac{1}{2}\beta_2 C^2 + \beta_3 R^2 + \beta_4\mathscr{D}^2 R\,. \tag{5.19}$$

The results for the coefficients ν, ρ, β_i in the case of a real conformal scalar, Majorana spinor, gauge vector, Einstein graviton and ordinary gravitino (the latter three quantized in the standard background gauges) are presented in table 5.1. The results for $s = 0, \frac{1}{2}, 1$ are well-known (see e.g. [12]); the off-shell ($R_{\mu\nu} \neq \Lambda g_{\mu\nu}$) infinities for the ordinary gravitino were found in ref. [105], the infinities in the Einstein theory were computed in ref. [241] (β_2, β_3), ref. [37] (β_1) and in refs. [117, 99] (ρ, ν).[5]

[5] Though the results for β_2 (as well as for ρ, β_3 and β_4) are in general gauge dependent (in Einstein theory) it is gratifying to find that the total β_2-coefficient vanishes in $N = 8$ supergravity when computed in the standard class of gauges [105].

Table 5.1
Gravitational infinity coefficients for the ordinary matter and gauge fields

s	ν	ρ	β_1	β_2	β_3	β_4
0	1	0	1/180	1/60	0	1/180
$\frac{1}{2}$	−2	0	7/720	1/20	0	1/60
1	2	−2/3	−13/180	1/5	0	−1/10
$\frac{3}{2}$	−2	1/6	−233/180	−77/60	−1/9	1/60
2	2	−23/3	53/45	7/10	1/4	−19/15

The same formula (5.14) is also true in the case of the fourth-order differential operator

$$\Delta = \mathcal{D}^4 + V^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu + U, \tag{5.20}$$

were instead of (5.16), (5.17) one has [98, 99][6]

$$b_0 = 2\,\mathrm{tr}\,\mathbb{1}, \qquad b_2 = \mathrm{tr}(\tfrac{1}{3}R\cdot\mathbb{1} + \tfrac{1}{4}V), \qquad V \equiv V^\mu_\mu, \tag{5.21}$$

$$b_4 = \mathrm{tr}(\tfrac{1}{6}F_{\mu\nu}{}^2 + 2E\cdot\mathbb{1} + \tfrac{1}{24}V_{\mu\nu}{}^2 + \tfrac{1}{48}V^2 - \tfrac{1}{6}V^{\mu\nu}R_{\mu\nu} + \tfrac{1}{12}RV - U + \tfrac{1}{2}\mathcal{D}_\mu\mathcal{D}_\nu V^{\mu\nu} + \tfrac{1}{12}\mathcal{D}^2 V), \tag{5.22}$$

where E is the same as in (5.17). Given the covariant analog of eq. (5.10), the algorithm (5.21), (5.22), and reducing the non-diagonal second order operators (like Δ_{gh} and $H_{\mu\nu}$ in (5.10)) to the diagonal second-order ones, as discussed in [98, 99] it is in principle straightforward (though tedious) to compute the infinity coefficients (ν, ρ, β_i in (5.19)) for higher derivative gravity (5.1) with $a \neq 0$ and $b \neq 0$ [98, 99]

$$\begin{aligned} &\nu = 8, \quad \rho_1 = -\tfrac{10}{3}\omega - 5, \quad \beta_1 = \tfrac{413}{180}, \\ &\beta_2 = \tfrac{133}{10}, \quad \beta_3 = \tfrac{10}{9}\omega^2 + \tfrac{5}{3}\omega - \tfrac{1}{36}, \quad \omega = -3b/a \end{aligned} \tag{5.23}$$

(the coefficient β_4 of the total derivative term was not yet computed).

To find the infinities in Weyl theory it is not sufficient to put $b = 0$ in (5.23). In fact, it is necessary to impose the quantum Weyl gauge (e.g. $h^\mu_\mu = 0$) and thus to subtract from (5.23) the contribution of *two* scalar degrees of freedom (cf. (5.7), (5.8)): $\Delta\nu = 2$, $\Delta\beta_1 = \frac{1}{90}$, $\Delta\beta_2 = \frac{1}{30}$. The coefficients ρ, β_3 (and β_4) are gauge dependent in Weyl theory. The corresponding conformal symmetry breaking infinities can be avoided in the one-loop approximation by a proper choice of gauge. In general, there are two sources of conformal non-invariance in the effective action for a classically conformal invariant theory: non-invariance introduced by dependence upon the choice of gauges and anomalous non-invariance due to regularization (due to infinities). The latter influences only the *finite* part of the *one-loop* effective action, while the former contributes also to its infinite part. To preserve the background Weyl invariance of the one-loop infinities, we have to employ the background conformal covariant gauges. To construct such gauges we shall adopt the procedure of refs. [116, 98, 99]: given an arbitrary functional of the background metric $F[g]$ one can define its conformal invariant analog $\tilde{F}[g]$

[6] The same results (as well as a rigorous justification of formulae like (5.12), (5.13) for higher order operators) are implicit in ref. [138] which was unknown to the authors of refs. [98–100]. The derivation of (5.21), (5.22), given in refs. [98, 99] is simpler and more transparent than that in ref. [138].

$$\tilde{F}[g] = F[\tilde{g}], \qquad \tilde{g}_{\mu\nu} = \Phi^2(g)\, g_{\mu\nu},$$
$$\Phi(g) = 1 - (-\mathcal{D}^2 + \tfrac{1}{6}R)^{-1}\tfrac{1}{6}R, \qquad (-\mathcal{D}^2 + \tfrac{1}{6}R)\Phi = 0, \qquad R(\tilde{g}) = 0\,. \tag{5.24}$$

Here we assume that $g_{\mu\nu}$ is asymptotically flat and that Φ (as well as the parameter of the Weyl transformation) satisfies the boundary condition $\Phi|_\infty \to 1 + O(1/x)$ (in the compact case one can use the condition $R(\tilde{g}) = 4\Lambda$). Then $\tilde{g}_{\mu\nu}$ is invariant under non-singular Weyl transformations and represents the metric with zero curvature scalar in each class of conformally equivalent metrics ($R = 0$ is a "classical" conformal gauge). The quantum coordinate gauge should then be taken of the form: $\chi_a[\tilde{g}_{\mu\nu}, \bar{h}_{\lambda\rho}] = \xi_a(x)$ (one has to substitute $g_{\mu\nu}$ for $\tilde{g}_{\mu\nu}$ in the ghost and gauge breaking $\chi H\chi$-terms). Choosing $\phi = 0$ as a quantum conformal gauge, and changing the variable $\bar{h}_{\mu\nu}$ one formally finds (neglecting the problem of the breakdown of Weyl invariance through regularization) that $\Gamma[g] = \Gamma^{(0)}[g]$ where $\Gamma^{(0)}$ is the effective action, computed in an arbitrary background gauge $\chi_a[g_{\mu\nu}, \bar{h}_{\lambda\rho}] = \xi_a(x)$. The whole procedure amounts to putting $R = 0$ in the (infinite part of) resulting effective action. Here it is worth recalling that the classical solutions of Weyl theory can be split into conformal classes, each containing a metric with $R = 0$ (all Einstein metrics $R_{\mu\nu} = \Lambda g_{\mu\nu}$ satisfy the Weyl theory field equations). This is why specializing to $R = 0$ resembles going on-shell, where the effective action is gauge independent. The final result for the one-loop infinities in Weyl theory, computed in the background conformal covariant gauges reads [98, 99]

$$\nu = 6, \qquad \beta_1 = \tfrac{137}{60}, \qquad \beta_2 = \tfrac{199}{15}, \qquad \rho = \beta_3 = \beta_4 = 0 \tag{5.25}$$

or, in dimensional regularization:

$$\Gamma_{1\infty} = \frac{1}{(n-4)(4\pi)^2}\int b_4 \sqrt{g}\, \mathrm{d}^4x, \qquad b_4 = -\tfrac{87}{20}R^*R^* + \tfrac{199}{30}C^2_{\lambda\mu\nu\rho}\,. \tag{5.26}$$

Thus Weyl theory is renormalizable in the one-loop approximation (β_2 gives the one-loop β-function). Taking $a = 1/2\alpha^2 > 0$ (which leads to positive Euclidean action) we find that the positivity of β_2 in (5.25) implies the *asymptotic freedom* of Weyl theory [99] (for earlier suggestions about asymptotic freedom in Einstein gravity and in Weyl theory see refs. [117, 245, 152])

$$\alpha^2(t) = \frac{\alpha^2(0)}{1 + \beta_2\,\alpha^2(0)\,t}, \qquad t \equiv \frac{1}{32\pi^2}\ln(L^2/\mu^2)\,. \tag{5.27}$$

This asymptotically free behavior is supported by the contributions of the standard $s \leq 1$ matter fields, which have $\beta_2 > 0$ (table 5.1). The *negative* sign of the ordinary gravitino contribution is a first indication that β_2 can be reduced, or even made zero, in a supergravity theory (see [105]). We also note that, for the general theory (5.1), asymptotic freedom for α (see (5.23)) implies asymptotic freedom for the second coupling $b^{-1/2}(b^{-1/2} \sim \omega^{-1/2}\alpha$, where ω has an ultraviolet fixed point [99]).

Let us now describe how one can easily derive the results for infinities (5.25) (or (5.23)) using *only* the algorithm (5.16), (5.17) for the *second*-order diagonal operators (5.15), thus avoiding the complications due to the fourth-order and non-diagonal second-order operators [250, 107]. The main idea is to use the special (on-shell) backgrounds on which the effective action can be written in terms of diagonal second-order operators. Let us first determine the coefficient of the topological infinity. To this end we shall compute the Weyl theory one-loop effective action for the Einstein background $R_{\mu\nu} = 0$.

Direct inspection of eq. (1.19) reveals that the bilinear term of the Weyl action then takes the form

$$C^2 \simeq (-\mathscr{D}^2\bar{h}_{\mu\nu} + 2C_{\alpha\mu\nu\beta}\bar{h}_{\alpha\beta})^2 + \text{gauge terms}\,. \tag{5.28}$$

The operator $\Delta_2 = -\mathscr{D}^2 + 2C\ldots$, defined on $\bar{h}_{\mu\nu}$, is exactly the kinetic operator that governs the one-loop on-shell effective action for Einstein theory [241, 136]

$$Z_{\rm E} = \frac{\det \Delta_1}{[\det \Delta_2 \det \Delta_0]^{1/2}}\,, \qquad \Delta_{1\mu\nu} = -\mathscr{D}^2_{\mu\nu}\,, \qquad \Delta_0 = -\mathscr{D}^2\,. \tag{5.29}$$

Gauge fixing and averaging over gauges (cf. (5.8), (5.10)), brings us to the one-loop (on-shell) effective action for Weyl theory [250, 107]

$$Z_{\rm W} = \left[\frac{(\det \Delta_1)^3}{(\det \Delta_2)^2}\right]^{1/2} = \left\{\frac{\det \Delta_1}{[\det \Delta_2 \det \Delta_0]^{1/2}}\right\}^2 \frac{\det \Delta_0}{[\det \Delta_1]^{1/2}}\,, \tag{5.30}$$

which is thus *equal* to the sum of the effective actions for *two* Einstein gravitons (5.29) and *one* gauge vector (total $2+2+2=6$ degrees of freedom). The infinities of (5.30) which for $R_{\mu\nu}=0$ are determined by β_1 (see (5.19)) can thus be computed using (5.17) or directly the data of table 5.1

$$\beta_1 = 2\times\tfrac{53}{45} + (-\tfrac{13}{180}) = \tfrac{137}{60}\,. \tag{5.31}$$

This number agrees with the result given in (5.25).

To find β_2, we shall specialize to the De Sitter background

$$R_{\lambda\mu\nu\rho} = \frac{2\Lambda}{3} g_{\lambda[\nu}g_{\rho]\mu}\,, \qquad C_{\lambda\mu\nu\rho} = 0\,, \qquad R_{\mu\nu} = \Lambda g_{\mu\nu}\,, \qquad \int d^4x\,\sqrt{g} = 24\pi^2/\Lambda^2\,. \tag{5.32}$$

The expression for the effective action of the Weyl theory on the De Sitter background is most easily derived using the "irreducible" variables defined in (5.3), and introducing the "constrained" operators

$$\Delta_0(X) = -\mathscr{D}^2 + X\,, \qquad \Delta_{1\perp}(X)_{\mu\nu} = (-\mathscr{D}^2_{\mu\nu} + Xg_{\mu\nu})_{\eta^\perp_\nu}$$
$$\Delta_{2\perp}(X)^{\mu\nu}_{\alpha\beta} = (-\mathscr{D}^{2\mu\nu}_{\alpha\beta} + X\delta^\mu_\alpha\delta^\nu_\beta)_{\bar{h}^\perp_{\mu\nu}}\,. \tag{5.33}$$

Then the bilinear part of the action (see (1.19) and also [45]) and the Jacobian in (5.6) [108] are [250]

$$C^2 \simeq \bar{h}^\perp \Delta_{2\perp}(\tfrac{2}{3}\Lambda)\Delta_2(\tfrac{4}{3}\Lambda)\bar{h}^\perp\,,$$
$$J = [\det \Delta_{1\perp}(-\Lambda)\det \Delta_0(-\tfrac{4}{3}\Lambda)\det \Delta_0(0)]^{1/2}\,. \tag{5.34}$$

The background gauge factor is

$$\delta(\mathcal{D}_\mu \bar{h}_{\mu\nu})\,\delta(\phi)\det\Delta_{gh} = \delta[\Delta_{1\perp}(-\Lambda)_{\mu\nu}\eta^{\perp}_{\nu} - \tfrac{3}{4}\mathcal{D}_\mu\Delta_0(-\tfrac{4}{3}\Lambda)\sigma]\,\delta(\phi)\det\Delta_{1\perp}(-\Lambda)\det\Delta_0(-\tfrac{4}{3}\Lambda)\,,$$
$$(\Delta_{gh})_{\mu\nu}\xi_\nu = -\mathcal{D}_\mu(\mathcal{D}_\mu\xi_\nu + \mathcal{D}_\nu\xi_\mu - \tfrac{1}{2}g_{\mu\nu}\mathcal{D}_\rho\xi_\rho)\,. \tag{5.35}$$

Integrating over $\bar{h}^{\perp}$, $\eta^{\perp}$, σ and ϕ we find the generalization of the flat background result (5.8) [250, 107]

$$Z_W = \left[\frac{\det\Delta_{1\perp}(-\Lambda)\det\Delta_0(-\frac{4}{3}\Lambda)}{\det\Delta_{2\perp}(\frac{2}{3}\Lambda)\det\Delta_{2\perp}(\frac{4}{3}\Lambda)}\right]^{1/2}. \tag{5.36}$$

The same expression gives the effective action for a general $R_{\mu\nu} = \Lambda g_{\mu\nu}$ background if $\Delta_{2\perp}$ includes also the Weyl tensor piece (cf. (5.28)). Equation (5.36) is a correct one only when supplemented by the contribution of the zero modes which could be omitted in the process of changing variables (cf. the discussion in ref. [108]). These zero modes are automatically included in the partition function expressed in terms of unconstrained operators $\Delta_s(X)$ defined on scalars, vectors and traceless tensors. Using the relations [108]

$$\det\Delta_{1\perp}(X) \sim \det\Delta_1(X)/\det\Delta_0(X-\Lambda)\,,$$
$$\det\Delta_{2\perp}(X) \sim \det\Delta_2(X)/\det\Delta_1(X-\tfrac{5}{3}\Lambda) \tag{5.37}$$

we find

$$Z_W = \left[\frac{[\det\Delta_1(-\Lambda)]^2\det\Delta_1(-\frac{1}{3}\Lambda)\det\Delta_0(-\frac{4}{3}\Lambda)}{\det\Delta_2(\frac{2}{3}\Lambda)\det\Delta_2(\frac{4}{3}\Lambda)\det\Delta_0(-2\Lambda)}\right]^{1/2}. \tag{5.38}$$

For comparison, we also recall the structure of the partition function (on the De Sitter background) for Einstein theory with a cosmological term ($\mathcal{L}_E = -(1/k^2)(R-2\Lambda)\sqrt{g}$)

$$Z_E = \frac{\det\Delta_1(-\Lambda)}{[\det\Delta_2(\frac{2}{3}\Lambda)\det\Delta_0(-2\Lambda)]^{1/2}}. \tag{5.39}$$

Thus, in contrast to the case of the $R_{\mu\nu} = 0$ background (5.30), the Weyl theory partition function on the De Sitter background cannot be simply represented as the product of partition functions for two equivalent Einstein "gravitons" and one vector (the second "graviton" and the vector acquire the "masses": $m_2^2 = \frac{2}{3}\Lambda$, $m_1^2 = -\frac{4}{3}\Lambda$ respectively). It should be stressed that the effective action corresponding to (5.38) is gauge independent (the De Sitter space is a trivial *on-shell* background). It is instructive to check this adopting a different choice of gauge[7]

$$\mathcal{D}_\mu h_{\mu\nu} - \tfrac{1}{2}\mathcal{D}_\nu h^{\mu}_{\mu} = 0, \qquad R(g+h) = 4\Lambda\,. \tag{5.40}$$

The conformal gauge $R = 4\Lambda$ which, in the one-loop approximation, can be rewritten as $\mathcal{D}_\mu\mathcal{D}_\nu h_{\mu\nu} - \mathcal{D}^2 h_{\mu\mu} - R_{\mu\nu}h_{\mu\nu} = 0$ has the advantage of being invariant under the coordinate transformations but is no longer ghost free (the ghost determinants are correspondingly

[7] The use of these gauges was first suggested in ref. [238]; however the expression for the partition function on the $R_{\mu\nu} = \Lambda g_{\mu\nu}$ background, as found in this reference, appears to be incorrect.

$$\det(-\mathscr{D}^2_{\mu\nu} - R_{\mu\nu}) = \det \Delta_{1\perp}(-\Lambda) \det \Delta_0(-2\Lambda) \qquad \text{and} \qquad \det \Delta_0(-\tfrac{4}{3}\Lambda)\,.$$

Substituting (5.3) in the gauge δ-functions, we establish that the integral over ϕ and σ can be put in the form

$$\int \mathrm{d}\sigma\, \mathrm{d}\phi\, \delta(\mathscr{D}_\mu \Sigma_\mu - \tfrac{3}{4}\mathscr{D}^2\phi - \Lambda\phi)\, \delta(\Sigma_\mu - \tfrac{1}{4}\mathscr{D}_\mu\phi)$$

$$\Sigma_\mu = \mathscr{D}_\rho(\mathscr{D}_\rho\mathscr{D}_\mu\sigma - \tfrac{1}{4}g_{\rho\mu}\mathscr{D}^2\sigma) = -\tfrac{3}{4}\mathscr{D}_\mu\Delta_0(-\tfrac{4}{3}\Lambda)\sigma\,.$$

The result of the integration is

$$\{\det \Delta_0(-\tfrac{4}{3}\Lambda)\, [\det \Delta_0(0)]^{1/2} \det \Delta_0(-2\Lambda)\}^{-1}\,.$$

Checking that the two $\det \Delta_0(-2\Lambda)$-factors cancel in the final expression, and using (5.34), we again finish with (5.36) and thus with (5.38).

All that is left is to calculate the infinities of ln det's in (5.38), making use of the algorithm for the second-order diagonal operators (5.16), (5.17). In this way one finds that the total coefficient of the logarithmic infinity is [250]: $b_4 = -\frac{58}{5}\Lambda^2$. This is to be compared with the general expression $b_4 = \beta_1 R^*R^* + \beta_2 W$ (cf. (5.19)) computed on the De Sitter background: $b_4 = \frac{8}{3}(\beta_1 - \frac{1}{2}\beta_2)\Lambda^2$. The important point is that the conformal non-invariant R^2-term in b_4 *must* be absent for an *on-shell* background. Using the already obtained value for β_1 (5.31) we find that $\beta_2 = \frac{199}{15}$, in agreement with the result of the straightforward computation for an arbitrary background (5.25). Yet another check of the value of Weyl's theory β-function β_2 is provided by embedding Weyl theory in a ($N = 1$) conformal supergravity and computing the β-function in the latter theory in two independent ways (see ref. [100] and section 6.2).

One can also compute the finite part of the effective action, corresponding to (5.36) (or 5.38) [250]

$$\Gamma_{1\,\text{finite}} = \tfrac{1}{2}B_4 \ln \frac{\Lambda}{3\mu^2} + C, \qquad C = -\sum_i \zeta_i'(0) = \text{const.}$$

where $B_4 = 4\beta_1 - 2\beta_2$ and the number C can be calculated in terms of ζ-functions of the operators in (5.36). Observing that the De Sitter background is conformally flat and that classical Weyl theory is conformally invariant, we understand that the dependence of (5.40) on Λ is completely due to the *conformal anomaly*. The general expression for the one-loop conformal anomaly in a classically conformal invariant theory is (see e.g. ref. [67] and in particular for the Weyl theory [106] and also [270])

$$\theta^\mu_\mu \equiv \frac{2}{\sqrt{2}} g_{\mu\nu} \frac{\delta\Gamma}{\delta g_{\mu\nu}} = \frac{1}{(4\pi)^2} b_4\,, \tag{5.41}$$

where b_4 determines the infinite part of Γ according to a relation like (5.26). It is only the *finite* part of Γ that contributes to the anomaly (the infinite part is an invariant for proper, conformally invariant, background gauges). The general expression for b_4 was given in (5.19). As was already said, β_3 vanishes in conformal background gauges (cf. in this connection ref. [208]). As for β_4, its value is known to be ambiguous (and in general depends on the choice of the gauge and of boundary conditions). If one adopts the prescription of ref. [67] (i.e. employs dimensional regularization and defines the Weyl tensor in n dimensions) then $\beta_4 = \frac{1}{3}\beta_2$. Though R^2-infinities are absent at the lower loop order for classically conformal invariant matter theories, they usually appear at higher loop orders because of matter self-interactions (see e.g. [22, 154]). This suggests that (higher loop) R^2-infinity will also be induced in Weyl theory. But here the appearance of R^2-infinity is even hastened – the anomaly of the conformal symmetry, which is the *gauge* symmetry of Weyl theory, implies a perturbative inconsistency (first anticipated in ref. [23]) already at the two-loop level [106]. The non-invariance of the finite part of the one-loop effective action implies the presence of a kinetic ($\phi\Box^2\phi$) term, for the anomalous conformal degree of freedom, which produces an additional non-renormalizable R^2-infinity in the two-loop approximation [106].

Given the perturbation theory inconsistency of Weyl theory (either pure or with the standard $s \le 1$ matter fields), one is left with several options:

1. One may give up local conformal symmetry and add the R^2-term in the bare action ($b \neq 0$ in (5.1)). This seems to be not a very attractive possibility because it raises the degree of arbitrariness of the theory (one additional coupling constant).

2. One may try to *define* quantum Weyl theory (its effective action) so that it would be automatically conformal invariant and hence be renormalizable in terms of only one (Weyl) coupling constant. Such a definition was first suggested in refs. [116, 98, 99]: one has simply to substitute the argument of $\Gamma[g]$ for the Weyl invariant background metric $\tilde{g}$ in (5.24) (one computes the one-loop Γ_1, substitutes g for $\tilde{g}$, then computes the two-loop correction, etc.). A possible drawback of this recipe is that it probably cannot be realized directly in a standard regularized diagrammatic expansion (while its merit is that it makes it possible to avoid the gravitational conformal anomaly in all matter theories and is therefore particularly attractive in Weyl theory).

3. One may hope that the appearance of the R^2 term, with an infinite coefficient, is a drawback of the loop expansion, while in the complete theory, the R^2-term appears with a finite coefficient [270]. Such a phenomenon was found to take place in an asymptotically free gauge theory in an external gravitational field [269]. Though the situation in Weyl theory is much more complicated, such a conjecture is very attractive in view of asymptotic freedom (5.27) and because it may help to define the quantum Weyl theory in a non-perturbative way. Still it should be stressed that this suggestion goes beyond the present criterion of perturbative renormalizability.

4. The most conservative (and yet the most interesting) possibility is to try to *cancel* the anomaly (and thus *all* the infinities) by combining Weyl theory with other fields and thus completing it to an ultraviolet *finite* theory [106]. It is striking that the condition of consistency for a locally conformal invariant theory is at the same time the condition for its ultraviolet finiteness. The prime candidates for such a finite theory are of course the locally superconformal theories discussed in the previous sections. By finiteness we a priori mean $B_4 = 0$, i.e. the cancellation of all, including topological, infinities. This in principle may turn out to be a sufficient but not a necessary condition for the consistency of a locally conformal invariant theory. If $b_4 = (\beta_1 - \frac{1}{2}\beta_2)R^*R^* + \frac{1}{2}\beta_2(C^2 + \frac{2}{3}\mathcal{D}^2 R)$ (cf. (5.19)) then the obvious necessary condition is $\beta_2 = 0$, i.e. the zero β-function for the Weyl coupling.[8] If $\beta_2 = 0$ but $\beta_1 \neq 0$ and the space–time is asymptotically flat the theory is free of (one-loop) infinities and thus is likely to be consistent (no R^2-infinities) also at the two-loop level.

The ideology of infinity/anomaly cancellation (which is known to be successful in the construction of

[8] The simplest way to understand why $\beta_2 \neq 0$ leads to R^2-infinities at the two-loop level is to note that the $\mathcal{D}^2R$-term in the anomaly (5.41) implies the presence of the finite R^2-term in the effective action which in turn produces the R^2-infinities (cf. (5.1), (5.23)).

unified theories of non-gravitational interactions) is attractive also in that respect that it points out a direction of a search for a truly unified theory (the condition of mathematical consistency imposes stringent restrictions on the field content). This motivates the study of infinities in conformal supergravities. As we shall see in section 6.2 the conformal gravitino does give a *negative* contribution in the Weyl coupling β-function, thus opening in principle a possibility to make a theory finite.

5.2. Non-perturbative approaches and unitarity problem

Leaving aside for a moment the "kinematical" problem of anomalies (supposing that it has one of the solutions described at the end of the previous section) Weyl theory has also two dynamical (and probably inter-related) problems: (i) how to establish the correspondence with Einstein theory, i.e. to prove that the low-energy effective action contains a term linear in the scalar curvature[1]; (ii) how to prove that the resulting theory can be identified with a consistent quantum field theory in Minkowski space, i.e. that there is a stable vacuum and that all physical ("composite") asymptotic states have positive norms so that the physical S-matrix is unitary. Both problems concern the infrared dynamics of an asymptotically free field theory and thus are highly non-trivial. The first problem seems simpler in that, in principle, it can be formulated and resolved directly in the framework of the Euclidean path integral approach. The possibility of inducing the Einstein term starting with a scale invariant matter field theory (with or without the Weyl term) or with a pure Weyl theory, has been extensively discussed in the literature (see e.g. refs. [1, 269, 151, 223] and references therein). The main idea is that quantum corrections (in a *non-finite* theory) naturally break scale invariance (e.g. dynamically, through condensates $\bar{\psi}\psi$, $F_{\mu\nu}{}^2$, etc.) and thus induce all possible generally covariant terms, which are forbidden by scale invariance at the classical level, with finite calculable coefficients. The physical picture of this phenomenon is rather transparent in the case of a gauge theory in an external metric (condensates, and thus the induced gravitational constant, are proportional to the dynamical scale, which is provided by asymptotic freedom) but should be more involved in the case of the pure Weyl theory. In the latter case, the roles of the "external metric" and of the asymptotically free quantum field are played by one and the same "Weyl graviton". These "ultraviolet" gravitons probably form some bound states, to be identified with the Einstein "infrared" gravitons (cf. [168, 223, 197]).

The asymptotic freedom of Weyl theory (5.27) (or of general theory (5.1)) implies that it is characterized by a scale. It is natural to assume that, in our world, the value of this scale is of the order of the Planck scale.[2] Then the main problem is to understand the transition from the power counting renormalizable, asymptotically free Weyl gravity, describing sub-Planck distances, to the effective theory of low-energy non-renormalizable Einstein gravity. The fluctuations that may play an important role in this transition are described by instantons (cf. [151]). In this connection we note that instantons in Weyl theory were recently discussed in refs. [46, 224, 238, 237].

An apparent lack of (perturbative) unitarity is a common problem of theories with higher derivatives, like Weyl theory (and conformal supergravities). Starting with a linearized $\Box^2$-theory in Minkowski space (cf. (5.5)), and carrying out the straightforward canonical quantization (introducing the auxiliary variables), one can have either a non-positive Hamiltonian or a positive Hamiltonian but an indefinite metric in a Hilbert space. In the latter case, the one-particle dipole ghost states have positive energy but

[1] A related problem (and sometimes considered as the most important) is how to avoid inducing a cosmological term of Planck order. This problem may find a solution in conformal supergravities or through some "space–time foam" averaging mechanism.

[2] This then implies that the most important contribution in the induced gravitational constant is provided not by the gauge matter fields but by the Weyl theory itself.

zero norm (see e.g. [181, 135, 90, 195]). The pathology of the linearized quantum theory can be related to the indefiniteness of the linearized expression for the classical energy [13] and thus to the presence of "runaway" solutions of the linearized classical equations [163]. However, it was found [16] that the energy of the asymptotically flat solutions of the full *non-linear* classical equations of the theory (5.1) (with $ab \geq 0$) is exactly equal to zero. The physical explanation of this result is provided by the observation that the classical potential (corresponding to $\Box^{-2}$) between two sources grows linearly with the distance. Thus in analogy with "quantum Yang–Mills theory" ($\mathscr{L} \sim F_{\mu\nu}\mathscr{D}^2 F_{\mu\nu} + \cdots$) describing the confinement of colour, we have a sort of confinement of energy (the energy is either infinite or zero). This suggests that the conclusions drawn from the linearized theory are completely misleading.

A further support to this point of view comes from the asymptotic freedom property of Weyl theory, which implies the presence of an infrared instability in perturbation theory (infrared divergences require a reinterpretation of the naive Hilbert space). Thus it is senseless to speak about unitarity in terms of the in–out states of the linearized theory.[3] In fact, unitarity is a statement about the true asymptotic states and, hence, to decide whether an asymptotically free theory is unitary or not, one has first to understand its infrared dynamics (see also the discussion in [206]). The first attempts in this direction were made in refs. [167, 168, 197]. In analogy with QCD, one conjectures that for energies of the order of (or smaller than) the Planck energy, where the Weyl coupling becomes strong, ("string-like") bound states are formed. Their typical mass is of the order of the Planck mass, but there should also be a massless spin 2 bound state, which can be identified with the Einstein graviton. The effective low-energy action (the analog of the chiral model action for QCD) for this spin 2 state should contain all powers of the curvature, including a linear term, and should thus describe a unitary theory.

In order to reexamine the dynamical question of unitarity at the non-perturbative level, one needs a non-perturbative definition of the theory and some reasonable approximation scheme. The former is provided by a lattice formulation [168, 246]. Asymptotic freedom guarantees that Weyl theory may actually exist as a truly interacting continuum theory which is cut-off independent in four dimensions. The Euclidean lattice version of Weyl theory, developed in ref. [246], has the property of boundedness and the property of reflection positivity and thus is claimed to define a unitary theory in Minkowski space with positivity condition on its spectrum. The formal argument in ref. [246], though completely non-perturbative, does not however, reveal the actual structure of the physical states and the mechanism by which the negative energy excitations of the linearized theory disappear in the physical asymptotic states. As for the approximation schemes, we note the Hamiltonian lattice strong coupling approach developed in refs. [167, 168]. It was shown there that the eigenstates of the linearized Hamiltonian (dipole ghosts) do not solve the non-linear quantum gauge symmetry constraints to leading order in the strong coupling expansion. These Dirac operator constraints are solved only by the "closed string-like" composite objects ("graviballs") so that the troublesome dipole ghost states, found in the weak coupling limit, are confined in the full theory. The question which remains open in this approach concerns the positivity of energy for the true eigenstates.[4] Another problem is connected with the high-energy (high-temperature) behavior of the correlation functions of the resulting theory, which should (in analogy with QCD) indicate the presence of ghosts confined within a "bag" of Planck radius.

[3] It is worth stressing that it is even senseless to say that a theory is non-unitary at the perturbation level, because there is no stable ground state. These remarks do not apply, however, to a theory containing also the Einstein R-term. This latter theory is truly non-unitary in perturbation theory.

[4] The main point emphasized in refs. [167, 168] that gauge symmetry constraints rule out the naive dipole ghost states was illustrated with the example provided by the higher derivative gauge theory $\mathscr{L} \sim F_{\mu\nu}\mathscr{D}^2 F_{\mu\nu}$. However, the strong coupling Hamiltonian for this theory turned out to be non-positive [168].

The main conjecture (yet to be proven) is that, though the high-energy (weak coupling) theory looks as pathological, this does not imply any inconsistency in the high-energy behavior of physically measurable quantities. Here it is useful to stress that the singularity of the propagator of the linearized theory $(p^2)^{-2}$ appears at small momenta (i.e. in the strong coupling region) and therefore not in the region of applicability of the linearized theory. As for the high-energy domain, the $(p^2)^{-2}$ propagator may here be harmless. It may happen that some inconsistencies are still present but that they show up only at sub-Planck energies, where they cannot be directly detected. Stated differently, unitarity may appear to be only a property of the effective low-energy theory.

One can also try to construct a ghost free renormalizable quantum gravity theory, adopting less sophisticated "perturbative" resummation mechanisms. For example, one can add the term of order R to the Weyl action from the very beginning (e.g. assuming that this term is induced by the matter fields) and quantize the resulting theory. The linearized spectrum then consists of the physical massless spin 2 graviton and of the massive spin 2 ghost [232] (the relative signs in the Euclidean Lagrangian are $\mathscr{L} = (-(1/k^2)R + (1/2\alpha^2)C^2)\sqrt{g}$ [5] and thus corresponds to a non-unitarity theory). Several mechanisms were proposed to cure the massive ghost problem: infrared fixed points for the dimensionless couplings [165, 212] (see also refs. [259, 210]), the Kugo–Ojima formalism [176, 177], a shift of the ghost pole away from the real axis as a result of radiative corrections, and use of a Lee–Wick-type prescription [186, 187, 33, 139, 15] to define a resummed unitary theory [152, 99]. The third and at first sight most attractive suggestion is however difficult to realize in a consistent manner, in view of our ignorance of the *infrared* behavior of the corrected propagator [99, 206]. At the same time, an analogous program *can* be carried out in a self-consistent way within the $1/N$ expansion for the $(R + C^2 + \text{matter})$-theory [243–245, 239] ($N$ is the number of scale invariant, e.g. massless spinor, matter fields). In the leading $N = \infty$ approximation, one ignores gravitational loops and computes the matter loop correction to the tree gravitational action. This correction has a closed form (controllable in the infrared) and indicates that the new propagator possesses no unphysical poles on the real $(\text{momentum})^2$ axis. The ghost pole is split into a pair of complex conjugate poles (the tree ghost *decays* due to interaction with matter). As a result, imposing natural boundary conditions (eliminating runaway solutions), one gets a viable semiclassical theory [179] (cf. [153]), i.e. the theory is unitary to leading order in the $1/N$ approximation. To achieve unitarity at higher orders in the $1/N$ expansion (which correspond to higher loop orders for the theory with $\mathscr{L} \sim N(R + C^2 + \ln\det \not{D})$) one has to use a Lee–Wick-type prescription in order to define the theory (S-matrix) in terms of Feynman diagrams with a modified graviton propagator. This prescription [186, 187, 243, 15] involves several points: (i) because of the complex poles, the effective Hamiltonian may not be Hermitean and as a result one has to use an indefinite metric in Hilbert space; the real energy eigenstates have then positive energies and positive norms but there are also runaway modes with complex energies; (ii) these eigenstates which are exponentially growing with time are eliminated from the space of asymptotic states by a boundary condition set in the future; (iii) to preserve unitarity, one has also to exclude the corresponding growing virtual modes, using the special rules for a diagrammatic definition of the S-matrix, order by order in perturbation theory (with modification of the absorptive parts of some diagrams). The resulting theory is unitary but has unusual causality properties (however, non-causal effects are negligible and are present only at Planck energies [33, 244]).

The role played by the $1/N$ expansion (see also [225]) is to provide a gauge invariant resummation of the theory, which makes it possible to solve the ghost problem. As for the necessity to recourse to an ad

[5] To have a renormalizable theory, one has also to include the cosmological term and the R^2-term [99].

hoc Lee–Wick prescription (which in contrast to indefinite metric quantization, cannot be implemented directly in the path integral [15]) this seems to be an unsatisfactory feature of the $1/N$-approach. It is desirable to have a dynamical mechanism for "ghost confinement" *automatically* preserving unitarity (probably at the expense of weaker causality properties). The existence of such a mechanism seems very probable but only if one starts with the pure higher derivative Weyl action (5.1) (see the previous discussion in this section). Repeating oneself one should say that, for this theory, there is no meaning for unitarity in the perturbative expansion: the graviton propagator behaves as $(p^2)^{-2}$ to any finite order in the coupling constant (and thus a Lee–Wick-type prescription is not applicable). To settle the question of unitarity, one has to study the infrared dynamics of this asymptotically free theory using e.g., the non-perturbative methods already developed for Yang–Mills theory. The very first steps only have been made so far in this direction [168, 197, 246]. The point we would like to emphasize is that one cannot a priori rule out Weyl theory (5.1) as "inconsistent" on the basis that the unitarity property of this theory is not obvious and that it cannot be established in the framework of a trivial perturbation theory.

New possibilities for the solution of the unitarity problem may appear in supersymmetric generalizations of Weyl theory (conformal supergravity theories, cf. [90, 100, 167]). It may well happen that the *finiteness* of such a theory is a necessary requirement, not only for its formal consistency (superconformal invariance at the quantum level, cf. section 5.1) but also for its physical consistency, i.e. unitarity. New possibilities also appear in the problem of achieving the low-energy correspondence with Einstein theory: the natural presence of conformally coupled scalars, and of local (super) conformal symmetry, suggest that the Einstein term can be generated "spontaneously" by gauging one of the scalars to its vacuum value (see refs. [169, 106] and the discussion in section 6.4). The corresponding Weyl gauge transformation can be carried out consistently (at the quantum level) only in a conformal anomaly free, i.e. *finite*, superconformal theory.

6. Quantum conformal supergravities

6.1. Superspace argument for higher loop finiteness

Conformal supergravities being supersymmetric theories should be most naturally quantized directly in superspace.[1] It is straightforward to carry out the background field method for the superspace quantization of $N=1$ CSG, starting with the known classical superspace action (4.50) expressed in terms of the known unconstrained (axial superfield) $N=1$ prepotential [203–205, 219] and extending the methods of refs. [145, 146, 129]. It should be also possible to quantize $N>1$ conformal supergravities in terms of the $N=1$ superfields corresponding to the $N=1$ multiplets introduced in (4.58) (see also [233]). Though a complete unconstrained extended superfield formulation of extended CSG's is not available at present (but should certainly exist since the component auxiliary fields, and the off-shell superspace constraints, are known for $N\leq 4$, see section 4.2) one can draw some important conclusions about the corresponding quantum theories, merely assuming the existence of such a formulation.

[1] Here by quantization we understand the configuration space path integral quantization. As in the case of the Weyl theory (section 5.1) we anticipate no problems in the formal derivation of the configuration space path integral from a phase space one (for a general method of quantization of dynamical systems with arbitrary Fermi–Bose constraints, see [94, 114, 115, 4, 95]). For discussions of the canonical approach to $N=1$ CSG see [185, 167, 14].

We shall start with a review of some basic facts concerning the superspace effective action within the background field method for the superspace quantization of geometrical supersymmetric theories (super Yang–Mills or any sort of supergravity) [145, 146, 129, 160]. A generic theory is described in superspace in terms of the vielbein $E_M{}^A$ and the connection $\Omega_{MA}{}^B$ [2] (see section 4.2), which satisfy certain off-shell constraints (that are usually algebraic in terms of the torsion or of the curvature). Suppose that these constraints can be solved expressing E and Ω in terms of a set of unconstrained N-extended superfields (prepotentials) which we shall denote by H. Then, the classical action can be expressed as a functional of H and so H is an integration variable in the path integral that defines the effective action $\Gamma[\bar{H}]$ ($\bar{H}$ is the background value of H). The so-called "non-renormalization theorem" for Γ [164, 82, 146, 160] rests on several assumptions (which have been explicitly checked for a number of theories [160]):

(i) the part of the classical action from which the Feynman rules for the background field method are derived $I[H, \bar{H}]$ can be represented as a power series in terms of unconstrained quantum prepotentials H;

(ii) all terms in this action are integrals over the full N-extended superspace[3];

(iii) $I[H, \bar{H}]$ depends on the background superfields $\bar{H}$ only through the "geometrical" constrained superfields (the vielbein and the connection) and not explicitly on them.

To define the path integral one has to fix the gauges for the quantum prepotentials H and thus to introduce the corresponding (first generation) ghosts. It was found (on the examples of super-Yang–Mills and supergravity) that these ghosts have their own gauge symmetries and thus need additional (second generation) ghosts. As a result, one gets an infinite sequence of ghosts for the ghosts [145, 146]. For $N>1$ all ghosts couple to the background fields, but only a finite number of ghosts contribute at the two loop and higher loop levels. Thus, for extended supersymmetric theories, there is a complication at the one-loop level. It is possible to truncate the infinite series of ghosts (i.e. to decouple all but the first two generations from the background fields) by fixing the gauges so that they explicitly depend on the background prepotentials, i.e. at the expense of introducing explicit interactions between the second generation ghosts and the background prepotentials [146, 160]. At the same time, all higher loop contributions to the effective action can be computed in a manifestly background covariant form (with no explicit dependence on the background prepotentials) and obey the natural power counting rules (assuming as we shall do that there exists a regularization that preserves the background general covariance in superspace).

The non-renormalization theorem (which is a generalization to extended supersymmetry of the well-known $N=1$ non-renormalization statements [262, 164, 82]) can now be formulated as follows [146, 160]: any L-loop, $L>1$, contribution to the effective action, computed in the N-extended superspace background formalism:

(i) must be an integral over the whole N-superspace $\{x^\mu, \theta^i, \bar{\theta}^i\}$ (a non-local part of Γ must contain also additional integrals over the space–time coordinates);

(ii) must be a covariant functional of the background values of the vielbein and of the connection but not of the "naked" prepotentials;

(iii) must be such that the infinite part of this contribution is a full N-superspace integral of a local function of constrained geometrical objects which is a polynomial in the connection superfield

[2] If an N-supersymmetric theory is described in the context of an $M(<N)$-extended superspace, then, in addition to the vielbein and the connection, one has also to introduce a number of "matter" superfields Φ and substitute N for M in the discussion that follows.

[3] Though it may be impossible (for $N\geq 2$) to rewrite the action expressed in terms of constrained superfields (E, Ω, torsion) as an integral of a local integrand over the full superspace, this should be possible when the action is expressed in terms of the prepotentials.

$$\Gamma_\infty^{(L)} = \hbar^{L-1} \int d^4x\, d^{2N}\theta\, d^{2N}\bar{\theta}\, \mathscr{L}_\infty^{(L)}(E, \Omega, \varepsilon), \qquad L > 1 \tag{6.1}$$

(ε is a regularization parameter).

The one-loop exception (absent in the $N = 1$ case) is connected with the complication in the gauge fixing procedure which was mentioned above. The structure of infinities in the theory now follows from dimensional analysis. Suppose for example that the classical action contains no dimensional coupling constants. Then, observing that E and Ω have non-negative dimensions and that $[x] = -1$, $[d\theta] = +\frac{1}{2}$, so that the dimension of the integration measure in (6.1) ($[d^4x\, d^{4N}\theta] = -4 + 2N$) is positive for $N \geq 3$, we conclude that such a theory is *finite* at all orders higher than the one loop one. This conclusion is also true for $N = 2$ since the only available $N = 2$ counter-term $\sim \int d^4x\, d^4\theta\, d^4\bar{\theta}$ sdet E vanishes identically [230]. In particular, this argument proves the $L > 1$ loop finiteness of the general "$N = 2$ super-Yang–Mills plus $N = 2$ matter" theories [146, 160], including the $N = 4$ super-Yang–Mills model, represented in terms of $N = 2$ superfields.[4] To establish a class of $N = 2$ theories (including $N = 4$ super-Yang–Mills) which are finite also at the one-loop level [161], one has to carry out a separate one-loop calculation (using either the component formulation or the $N = 1$ superfield formulation). When applied to ordinary Poincaré supergravity with dimensional coupling $[k] = -1$, the non-renormalization theorem ($\Gamma_\infty^{(L)} = (k^2\hbar)^{L-1} \int d^4x\, d^{2N}\theta\, d^{2N}\bar{\theta}\, \mathscr{L}_\infty^{(L)}$) indicates the presence of candidates for infinities, starting with a high enough loop order [146] (e.g. three loops for $N = 8$ supergravity, written in terms of $M = 4$ superfields [160]).

Supposing that the assumptions of the non-renormalization theorem are also valid for power counting renormalizable supergravity theories, which are N-supersymmetric extensions of higher derivative gravity theories (5.1) ($\mathscr{L} = (aC^2 + bR^2)\sqrt{g}$), and thus are parametrized by (two) *dimensionless* coupling constants, we immediately conclude that all such theories with $N \geq 2$ (in particular, conformal supergravities with $b = 0$) are finite at the two and higher loop levels. As for the $N = 1$ theory, it can have infinities at all loop orders (the dimension of the integration measure in (6.1) is -2 and thus an admissible counter-term is the action itself, since it can be rewritten as an integral over the full superspace, cf. eq. (4.51)). The remark that the non-renormalization theorem of extended supersymmetry implies a higher loop finiteness of $N \geq 2$ conformal supergravities, was made in ref. [233] and also in refs. [265, 160].

To settle the question of manifest finiteness, one has again to do an explicit computation of one-loop divergences. This can be accomplished either through a formulation in terms of component fields (as it has been the case in refs. [100, 101, 107], and this will be described in section 6.2), or using the $N = 1$ superfield approach.[5] In the latter case, one could hope to recognize a finite theory by the presence in the action of a term written as an integral of a local combination of fields over the chiral $N = 1$ sub-superspace. The coefficient of such a term cannot be renormalized according to the ($N = 1$) non-renormalization theorem, while extended supersymmetry relates this coefficient to coefficients of other terms in the action. Such an explanation of the one-loop finiteness works for the $N = 4$ super-Yang–Mills theory [144, 233] (though it should be understood that it does not work for a general class of finite $N = 2$ super gauge theories found in ref. [161]). It was noted in ref. [233] that a chiral

[4] It would show also the finiteness of $N = 4$ super-Yang–Mills in a direct way if an $N = 4$ unconstrained superfield formulation of this theory were possible [146].

[5] Though the expressions for the $N > 1$ CSG actions, written in terms of $N = 1$ superfields (cf. (4.58)), are presently unknown it is certainly possible to establish the structure of the terms bilinear in the quantum fields in the actions, using the background gauge invariances of the theory (in analogy with the procedure used in refs. [100, 101] in the framework of the component approach).

"potential term" cannot exist in the $SU(1,1)$-*invariant version* of $N = 4$ CSG (see [8] and section 3.3). The argument was based on chiral U(1) weight assignments for $N = 1$ superfields, where $U(1) \subset SU(1,1)$ is a symmetry of the full supersymmetry transformation laws (3.62). This negative conclusion appears to be in agreement with the result of an explicit computation (see refs. [100, 101, 107] and section 6.2) that reveals that the SU(1, 1)-invariant ("minimal") version of $N = 4$ CSG is *not* finite at the one-loop order. At the same time, the reasoning based on the existence of a chiral term in the action may be helpful in order to explain the one-loop finiteness of a "non-minimal" version of the $N = 4$ theory (cf. sections 3.3 and 4.1). The action for the latter version should contain "non-minimal" terms (3.55) which, when rewritten in terms of $N = 1$ superfields, look like $\int d^4x\, d^2\theta\, f(\hat{\varphi})\, \psi^{\alpha\beta\gamma}\, \psi_{\alpha\beta\gamma} + \text{c.c.} + \cdots$ (here $\hat{\varphi}$ is the chiral superfield of dimension zero, corresponding to the $\{0'\}_1$ $N = 1$ chiral multiplet in (4.58), and $\psi_{\alpha\beta\gamma}$ is the $N = 1$ CSG chiral field strength superfield, see (4.50)). In contrast to the action (4.50) of pure $N = 1$ CSG, such terms cannot be rewritten as integrals over the full $N = 1$ superspace (here one cannot make use of the super-Gauss–Bonnet theorem [90], which was essential in the passage from (4.50) to (4.51)).

It is important to stress that the above statement about the $L \geq 2$-loop finiteness of the renormalizable $N \geq 2$ supergravity models was based only on the general coordinate invariance and on ordinary Q-supersymmetry, i.e. on general covariance in superspace and not on any kind of additional symmetry (like, e.g., scale transformations of the vielbein (4.47)). Thus the finiteness property holds for the general $C^2 + R^2$-type models, as well as for the C^2-type conformal supergravities, invariant also under proper superconformal (local Weyl, U(N) and S-supersymmetry) transformations. Irrespectively of whether or not these additional symmetries are broken at the quantum level, the assumption that regularization preserves general covariance and Q-supersymmetry is formally sufficient for the finiteness of the theory at the $L \geq 2$ loop level. This suggests that, even if a theory is anomalous in the one-loop approximation (and we shall find in section 6.2 that the $N = 2, 3$ and $N = 4$ "minimal" CSG's do have one-loop infinities and hence anomalies) it may still be consistent (finite) at the higher loop level.[6] There may be of course a number of subtleties in the application of the non-renormalization theorem to one-loop anomalous conformal supergravities. As one can learn on the example of Weyl theory (see section 5.1 and ref. [106]), the conformal (chiral U(N), S-supersymmetry) anomaly in the finite part of the one-loop effective action implies that the anomalous degree of freedom h^{μ}_{μ} ($\Box^{-1}\partial_\mu A_\mu$ for U(1), $\gamma_\mu\psi_\mu$ for S) becomes dynamical at the one-loop level[7] and produces non(super)-conformal invariant (R^2-type) infinities at the two-loop level. We are thus led to the following dilemma: either all non-invariant infinities mutually cancel at the two and higher loop levels in $N \geq 2$ CSG's, in agreement with the prediction of the non-renormalization theorem, or the non-renormalization theorem does not in fact apply in the anomalous situation (this may be, for example, due to problems in gauge fixing for additional symmetries or due to the fact that two-loop infinities produced by anomalous degrees of freedom are one-loop in nature, while the non-renormalization theorem formally applies only at the $L \geq 2$ loop level). If the first possibility is realized, the one-loop anomaly is probably harmless for $N \geq 2$, and we get examples of theories which have dynamically broken proper superconformal gauge symmetries but are yet consistent at the quantum level.[8] In the second case, we have either to look for a

[6] It should be kept in mind however, that the absence of dangerous infinities may not be sufficient for a formal consistency because of a breakdown of proper superconformal gauge symmetries by finite quantum corrections. This interesting situation deserves further study.

[7] Note the analogy with the situation in the quantum $N = 1$ [207] and especially $N = 2$ [103] "conformal supergravity" in two dimensions.

[8] We recall (see section 5.1) that one-loop infinities *are* invariant in a classically invariant theory if the quantum gauges respect *all* the background classical symmetries.

truly (one-loop) finite, and thus anomaly free, superconformal theory, or we may try to *define* quantum $N \geq 2$ CSG's adding in the classical action a super-extension of the R^2-term, with an infinitesimal coefficient $b \to 0$. This R^2-term explicitly breaks proper superconformal symmetries but makes the resulting theory unambiguously finite at the $L \geq 2$ loop level. The limit $b \to 0$ is surely not a regular one in the full quantum theory (cf. eqs. (5.23) and (5.25)).

Here it is worth drawing attention to the following conjecture: the (Poincaré) supersymmetric extension of the troublesome R^2-term *may not exist for* $N \geq 3$. The argument that leads to this statement is the same as the one that suggests (see [52] and section 4.3) that conformal supergravities (superextensions of the C^2-term) may not exist for $N > 4$. Namely, one considers the coupling of the N-extended R^2-theory to the O(N) Poincaré supergravity and assumes that this coupling preserves rigid N-supersymmetry. Then the linearized spectrum of the total theory should consist of a combination of a massless spin 2 PSG multiplet and of a massive N-supermultiplet, containing the massive spin 0 state present in $R + R^2$-gravity [232], cf. (5.5). For example, for $N = 1$ one has at the linearized level [81] (cf. (3.40), (3.41))

$$\mathcal{L}^{(1)}_{R+R^2} = \frac{1}{k^2}(R - \bar{\psi}_\mu R_\mu + \tfrac{3}{2}A_\mu^2 - \tfrac{1}{2}S^2 - \tfrac{1}{2}P^2) + b[R^2 + 2\bar{\psi}_\mu(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu)\gamma_\nu\gamma_\rho R_\rho + 9(\partial_\mu A_\mu)^2 - 3(\partial_\mu S)^2 - 3(\partial_\mu P)^2], \qquad R^\mu = e^{-1}\varepsilon^{\mu\nu\lambda\rho}\gamma_5\gamma_\nu\mathscr{D}_\lambda\psi_\rho . \tag{6.2}$$

Using (5.4) and splitting ψ_μ and A_μ as in (5.3)

$$\begin{aligned} &\psi_\mu = \varphi_\mu + \tfrac{1}{4}\gamma_\mu\psi, \qquad \gamma_\mu\varphi_\mu = 0, \\ &\varphi_\mu = \varphi_\mu^\perp + \mathscr{D}_\mu\zeta - \tfrac{1}{4}\gamma_\mu\not{\mathscr{D}}\zeta, \qquad \mathscr{D}_\mu\varphi_\mu^\perp = 0, \\ &A_\mu = A_\mu^\perp + \mathscr{D}_\mu\rho, \qquad \mathscr{D}_\mu A_\mu^\perp = 0, \end{aligned} \tag{6.3}$$

one can rewrite (6.2) in the form (cf. (3.41))

$$\mathcal{L}^{(1)}_{R+R^2} = \frac{1}{k^2}(-\tfrac{3}{32}\bar{\phi}\Box\bar{\phi} + \tfrac{3}{8}\bar{\alpha}\not{\partial}\alpha - \tfrac{3}{2}\rho\Box\rho - \tfrac{1}{2}S^2 - \tfrac{1}{2}P^2 + \tfrac{1}{4}\bar{h}^\perp_{\mu\nu}\Box\bar{h}^\perp_{\mu\nu} - \bar{\varphi}^\perp_\mu\not{\partial}\bar{\varphi}^\perp_\mu + \tfrac{3}{2}A_\mu^{\perp 2}) + b[\tfrac{9}{16}(\Box\bar{\phi})^2 + 9(\Box\rho)^2 - \tfrac{9}{4}\bar{\alpha}\not{\partial}^3\alpha + 3S\Box S + 3P\Box P], \qquad \alpha \equiv \psi - \not{\partial}\zeta . \tag{6.4}$$

Thus the massive $N = 1$ multiplet which we get is a scalar multiplet ($m^2 = (6bk^2)^{-1}$) consisting of two spin 0 fields ($\bar{\phi}$ and ρ), of one Majorana spinor (α) and of two auxiliary fields (S and P). It is interesting to note that the linearized $N = 1$ R^2-action coincides with the action (2.45) for the higher derivative $N = 1$ scalar multiplet. Analogous results are found for $N = 2$ [59] (cf. (3.80))

$$\mathcal{L}^{(2)}_{R+R^2} \simeq \frac{1}{k^2}\Big\{R - (\bar{\psi}^i_\mu R_{\mu i} + \text{c.c.}) - \frac{k^2}{8}(F_{\mu\nu}{}^{ij}(B))^2 + \Big[4\bar{\lambda}^i(\not{\partial}\lambda_i + \chi_i + \tfrac{1}{3}\gamma_\mu R^i_\mu) + \frac{k^2}{2}(t_{\mu\nu}{}^{-ij}\varepsilon_{ij})^2 + \text{c.c.}\Big] + \tfrac{1}{2}A_\mu^2 - V_\mu^2 - \frac{k^2}{2}|S_{ij}|^2 + \cdots\Big\} + \tfrac{3}{2}b\Big\{\frac{k^2}{4}[(\partial_\mu(t_{\mu\nu}{}^{-ij} - \tfrac{1}{2}F_{\mu\nu}{}^{-ij}(B)))^2 + \text{c.c.}] + \tfrac{1}{2}(\partial_\mu V_\mu + \tfrac{1}{2}R)^2 + \tfrac{1}{8}(\partial_\mu A_\mu)^2 - \tfrac{1}{2}\bar{\chi}^i\overleftrightarrow{\not{\partial}}\chi_i - \frac{k^2}{8}|\partial_\mu S_{ij}|^2\Big\} . \tag{6.5}$$

Here the $N=2$ "R^2-term"[9] is represented by the Lagrangian of the $N=2$ "tensor gauge multiplet" of ref. [59] (it is a combination of the $N=1$ tensor gauge multiplet and of the $N=1$ R^2-term in (6.2)). In order to find the massive part of the spectrum, one has to note that it is determined by the "longitudinal" parts of the fields: $R=-\frac{3}{4}\Box\bar{\phi}+\cdots$, $V_\mu=\partial_\mu\kappa+\cdots$, $A_\mu=\partial_\mu\rho+\cdots$, $t_{\mu\nu}=\partial_\mu\xi^\perp_\nu-\partial_\nu\xi^\perp_\mu+\cdots$, etc. The mixing terms can be diagonalized introducing the variables $\hat{\kappa}=\kappa-\frac{3}{8}\bar{\phi}$, $\hat{\xi}_\mu=\xi_\mu-\frac{1}{2}B_\mu$, $\sigma^i=\chi^i-\frac{3}{4}\not{\partial}\alpha^i$. The corresponding massive multiplet is the $N=2$ vector multiplet consisting of one vector ($\hat{\xi}_\mu$), of two spinors (σ^i), of two spin 0 fields (ρ and $\hat{\kappa}$) and of three auxiliary fields (S^{ij}). Proceeding in the same way, for $N\geq 3$, we expect to find a massive multiplet of N-extended supersymmetry. The crucial observation [52] is that all such multiplets with $N\geq 3$ include states of *spins greater than one* (for example, for $N=3$, the highest value of the spin is $s_{\max}=\frac{3}{2}$ while, for $N=4$, it is $s_{\max}=2$). The corresponding $s>1$ fields do not coincide with the gravitino and graviton fields and should be auxiliary in the Poincaré supergravity action.[10] At the same time, it appears difficult to imagine how these *additional* $s=\frac{3}{2}$, 2 fields can be *propagating* in the R^2-type action. Though such a reasoning may be too naive, the possibility that the R^2-type supersymmetric invariants may not exist for $N=3,4$ seems very interesting. If this conjecture is true that $N=3,4$ CSG's may appear to be consistent at the quantum level and finite at the $L\geq 2$ loop level irrespectively of the values of their one-loop β-functions. (The absence of $N\geq 3$ supersymmetric extensions of the R^2-term will manifest itself through cancellations among non-conformal-invariant infinities).

The most satisfactory solution to the problem of anomalies would of course be to find an $N\geq 2$ theory which is finite also at the one-loop level. This circumstance (and more generally, the one-loop omission in the non-renormalization theorem) motivates the detailed study of one-loop divergences in conformal supergravities, which is the topic of the next section.

6.2. *One-loop approximation (β-functions, effective actions on instanton backgrounds, etc.)*

We shall start with some general remarks concerning the quantization of the $N\leq 4$ conformal supergravities in ordinary space–time. The basic object is the path integral in configuration space, with the classical action in Minkowski space (see (3.53))

$$I^{(N)}_{\rm CSG}=\frac{1}{\alpha^2}\int \mathrm{d}^4x\, e\left\{\tfrac{1}{2}C^2_{\lambda\mu\nu\rho}-\frac{4-N}{4N}F^2_{\mu\nu}(A)+F^i{}_{j\mu\nu}(V)F^j{}_{i\mu\nu}(V)+\cdots\right\}. \tag{6.6}$$

Here, the overall sign is chosen so that all ordinary non-higher derivative kinetic terms in the action (corresponding to U(N) gauge vectors, spinors χ and scalars E) have physical (non-ghost-like) signs. To carry out the continuation to Euclidean space, one has to "rotate" the time components of the fields and coordinates to make them real (we use the ++++ metric in Minkowski space), and also to reverse the sign of the Minkowski Lagrangian ($x_4\to -\mathrm{i}x_4$, $\exp(\mathrm{i}I_{\rm M})=\exp(-I_{\rm E})$. The resulting Euclidean action coincides with (6.6), taken with the *opposite* sign. We conclude that the Euclidean action for $N\leq 4$

[9] It is interesting to point out here that the fields present in the superextension of the R^2-term correspond to the fields of a multiplet of superconformal anomalies ("gauged" in the CSG case and "global" in the case of super-Yang–Mills). Note that the $N=2$ multiplet does not contain the field corresponding to $\partial_\mu V_\mu{}^i{}_j$, and that may be connected with the fact that SU(2) is a "safe" group (see also the end of section 6.2 and ref. [228] for some useful remarks concerning the $N=4$ case).

[10] These fields should then be auxiliary fields of $N=4$ super-Yang–Mills if it were possible to find an off-shell formulation of $N\geq 3$ PSG by coupling $N\geq 3$ super-Yang–Mills to conformal supergravity, as discussed in section 3.5.

CSG's is obviously *non-positive* in the bosonic sector[1]: the Weyl term and the Yang–Mills terms contribute with opposite signs. This indefiniteness of the action seems to be a characteristic property of superconformal theories (for example, it is true also for the effective $N = 2$ conformal supergravity action in two dimensions [103, 104]). In the case of four dimensional CSG's, one can provide the following "physical" explanation: the contributions of ordinary matter fields (e.g. constituting an $N = 1$ superconformal chiral multiplet (2.51)) are known to be positive (asymptotically free) in the Weyl infinities (see table 5.1 and section 6.3) and negative (non-asymptotically free) in the U(1) or SU(N) gauge infinities. The indefiniteness of the (bosonic) action is related to the non-positivity of the energy in the canonical approach: the formal positive energy argument in a supersymmetric theory breaks down if the Hilbert space metric is indefinite (the non-positivity of the Hilbert space metric in the fermionic sector of $N = 1$ CSG was demonstrated in [14]).

To define the Euclidean path integral, one has to use the prescription that the integrations over the "indefinite metric" quantum fields (contributing with negative signs in the action) are carried out over imaginary values. In the one-loop approximation, this is equivalent to first computing the formal path integral in Minkowski space and then to define the resulting determinants of the differential operators in Euclidean space. To keep the correspondence with the asymptotically free Weyl theory (see section 5.1), it is desirable to reverse the sign in the Minkowski action (6.6) so that the *Euclidean* Lagrangian will look like (cf. (3.14), (3.15))

$$e^{-1}\mathscr{L}^{(N)}_{\mathrm{CSG\,(E)}} = \frac{1}{\alpha^2}\left\{\tfrac{1}{2}C^2_{\lambda\mu\nu\rho} - \frac{4-N}{4N}F^2_{\mu\nu}(A) - \tfrac{1}{2}(F^r_{\mu\nu}(V))^2 + \cdots\right\}, \qquad (r = 1, \ldots, N^2-1) \tag{6.7}$$

(then (6.6) coincides with the *Euclidean* action). Observing that conformal supergravities should be renormalizable in the one-loop approximation (under a proper choice of gauge), we can write the infinite part of the one-loop effective action in the form (cf. (5.26))[2]

$$\begin{aligned} \Gamma_{1\infty} &= \frac{1}{(4\pi)^2(n-4)}\int \mathrm{d}^4x\,\sqrt{g}\,b_4\,, \\ b_4 &= \beta_1 R^*R^* + \beta\left(W - \frac{4-N}{4N}F^2_{\mu\nu} - \tfrac{1}{2}F^{r2}_{\mu\nu} + \cdots\right), \qquad \beta \equiv \beta_2\,. \end{aligned} \tag{6.8}$$

The only total derivative term which we included in b_4 is a topological term which has no non-trivial superextension. In background non-covariant gauges, b_4 may also contain the superextension of the R^2-term. The coefficient $\beta = \beta_2$ determines the β-function for the coupling α, while β_1 governs the renormalization of the topological coupling term (5.2) which should be added in the bare action to provide renormalizability on topologically non-trivial backgrounds.

Supersymmetry makes it possible to establish β by computing the infinities of the effective action in any suitable background sector. The most convenient ones are the following bosonic backgrounds: gravitational (G), U(1) and SU(N) (we indicate only fields that have non-vanishing background values)

[1] There is also an additional subtlety in formulating and quantizing supersymmetric theories in Euclidean space [198, 155]: one has to impose a weaker hermiticity property in the fermionic sector [198].

[2] We note that, if a proper-time cut-off were used (cf. (5.12)), then there would be no quartic and quadratic infinities: $b_0 = 0$ (because of the zero total number of on-shell degrees of freedom, table 3.4) and $b_2 = 0$ (since, in superconformal symmetry preserving gauges, there can be no terms proportional to R).

$$\begin{aligned} &\text{G:} \quad g_{\mu\nu} \neq \delta_{\mu\nu}\,, \\ &\text{U(1):} \quad A_\mu \neq 0, \quad g_{\mu\nu} = \delta_{\mu\nu}\,, \\ &\text{SU}(N)\text{:} \quad V_\mu \neq 0, \quad g_{\mu\nu} = \delta_{\mu\nu}\,. \end{aligned} \tag{6.9}$$

The calculation in the G-sector makes it possible to find also β_1. We shall determine β independently in these three sectors, thus checking the consistency (supersymmetry) of the computational scheme. The part of the $N \leq 4$ CSG action bilinear in the quantum fields, which is sufficient for establishing the effective action Γ_1 in the bosonic background sectors (6.9), is provided by eqs. (3.53), (3.54) (see also (1.17)–(1.21)). It can be schematically written as follows

$$e^{-1}\mathscr{L}_q = \sum_n \Phi_n \Delta^{(n)} \Phi_n + \sum_{m\neq n} \Phi_m X_{mn} \Phi_n \tag{6.10}$$

where Φ_n stand for the quantum fields ($h_{\mu\nu}$, ψ_μ, V_μ, A_μ, $T_{\mu\nu}^{-ij}$, $\chi^{ij}{}_k$, $E^{ijk}{}_l$, Λ^{ijk} and φ^{ijkl}), $\Delta^{(n)}$ are "kinetic" operators and X_{mn} correspond to the mixing terms (3.54) (Δ and X depend only on the background fields). Thus in general Γ_1 is determined by an operator which is non-diagonal in the original basis of quantum fields. However, it can be proven [100, 101, 107] that the contribution of mixing $\Phi X \Phi$-terms in the *infinite* part of the effective action vanishes for the backgrounds (6.9) (for simplicity here we ignore the $h_{\mu\nu}$–A_μ, $h_{\mu\nu}$–V_μ and φ–V_μ mixings which will be discussed below). For example, the gravitino–spinor mixing term $\bar\psi_\mu X_\mu \chi$ + c.c. produces a $\mathrm{tr}(X_\mu \gamma_\alpha X_\mu \gamma_\alpha)$-term in the infinities, which vanishes for $X_\mu \sim (\sigma\cdot\mathscr{F})\gamma_\mu$, corresponding to (3.54). As a result, $\Gamma_{1\infty} \sim \Sigma_k\, c_k (\ln\det \hat\Delta^{(k)})_\infty$ where $\hat\Delta^{(k)}$ include "gauge fixed" analogs of $\Delta^{(n)}$ in (6.10), as well as the necessary ghost and averaging over gauge operators. The gauges we are to use should respect all the background bosonic symmetries: general covariance, D and U(N) invariances. To provide the invariance of $\Gamma_{1\infty}$ under Weyl transformations, we can adopt the same prescription that was used in the case of Weyl theory (section 5.1), i.e. to assume that the gauges depend on the background metric with a zero curvature scalar (cf. (5.24)).

The infinite part of Γ_1 is then given by (6.8), where the expressions for the β-functions are

$$\begin{aligned} &\text{G:} \quad \beta = \beta_2 = \sum_n \beta_{2n} d_n, \quad \beta_1 = \sum_n \beta_{1n} d_n\,, \\ &\text{U(1):} \quad \beta = \frac{4-N}{4N} \sum_n \beta_{0n} c_n^2 d_n\,, \\ &\text{SU}(N)\text{:} \quad \beta = 2\sum_n \beta_{0n} C_{2n}\,. \end{aligned} \tag{6.11}$$

Here β_{0n}, β_{1n} and β_{2n} are the partial contributions of each type of fields, d_n is a number of fields of a given type in a particular CSG theory (or the dimension of the corresponding SU(N) representation, see table 3.2), c_n are the chiral U(1) weights (collected in table 3.3), C_{2n} are the second Casimir invariants of the SU(N) representations (see e.g. [222]). The dimensions, the Casimir invariants and the results for β_{1n} [101, 107], β_{2n} [100, 101], β_{0n} [100, 107] are presented in table 6.1. The values of β's for the scalar φ are given both for the "minimal" and "non-minimal" versions of $N = 4$ CSG (see below). The derivation of (6.11) is based on the fact that all U(1) and SU(N) couplings have the common U(N) structure (see eqs.

Table 6.1
Dimensions (Casimir invariants) of the SU(N) representations and contributions in the one-loop β-functions of fields in $N \le 4$ conformal supergravities

N	$h_{\mu\nu}$	$\psi_\mu{}^i$	A_μ	$V_\mu{}^i{}_j$	$T^{-ij}_{\mu\nu}$	$\chi^{ij}{}_k$	Λ^{ijk}	$E^{ijk}{}_l$	φ^{ijkl}	
1	1(0)	1(0)	1(0)	—	—	—	—	—	—	
2	1(0)	2(1/2)	1(0)	3(2)	1(0)	2(1/2)	—	—	—	
3	1(0)	3(1/2)	1(0)	8(3)	3(1/2)	8(3) + 1(0)	1(0)	3(1/2)	—	
4	1(0)	4(1/2)	—	15(4)	6(1)	20(13/2)	4(1/2)	10(3)	1(0)	
β_1	137/60	−173/180	−13/180	−13/180	11/30	7/720	7/240	1/90	1/45	46/45
β_2	199/15	−149/30	1/5	1/5	1/10	1/20	−1/60	1/30	−4/15	26/15
β_0	0	34/3	0	−11/6	−3/2	1/3	−1	1/6	0	2

(3.37), (3.52), (3.53)). Thus the coefficient $(b_4)_n$ in $(\Gamma_{1\infty})_n$ computed in the U(1) × SU(N)-sector, has the form

$$(b_4)_n = \beta_{0n}\, \mathrm{tr}(\mathscr{F}^2_{\mu\nu}) = -\beta_{0n} \left[C_{2n}(F^r_{\mu\nu}(V))^2 + d_n c_n \left(\frac{4-N}{4N}\right)^2 (F_{\mu\nu}(A))^2 \right]. \tag{6.12}$$

Comparing this with (6.8) we are led to (6.11). Equation (6.12) shows also that the actual computation of β_{0n} can always be done in the U(1) sector.

All the information necessary for establishing the values of β_n's is provided by the algorithms for infinities of ln det Δ for the second (5.17) and fourth (5.22) order differential operators Δ_2 and Δ_4, or of their trivial modifications. Using the Δ_2-algorithm, it is easy to check the standard results (see, e.g. [240, 12]) for β's in the cases of the gauge vector (A_μ, V_μ), Majorana spinor (χ) and complex conformal scalar (E) (see also table 5.1). The values of β_1 and β_2 for the Weyl graviton [99] were already discussed in section 5.1 (eq. (5.25)). The important fact that the interaction of the Weyl graviton with a vector gauge field *does not modify the one-loop β-function of the latter* (i.e. $\beta_{0h_{\mu\nu}} = 0$) was first established in refs. [98, 99] (see also ref. [100]). The relevant piece in the interaction Lagrangian looks like

$$\mathscr{L}_{hA} \sim \bar{h}_{\mu\nu}\Box^2\bar{h}_{\mu\nu} + A_\mu\Box A_\mu + \bar{h}U\bar{h} + \bar{h}Y\,\partial A \tag{6.13}$$

where $U \sim F^2(A)$, $Y \sim F(A)$ and $F_{\mu\nu}(A)$ is the background gauge field strength. The contribution of (6.13) in the infinities is easily found, e.g., by the diagram method, $b_4 \sim -U + Y^2$, and appears to be identically vanishing due to the cancellation of the $F^2_{\mu\nu}$-terms coming from the U and Y vertices.

In order to clarify the procedure of quantization of the conformal gravitino, let us first consider the corresponding partition function on a trivial flat background. The gravitino kinetic term in (1.20) can be rewritten in the following way

$$\begin{aligned} &\mathscr{L}_\psi \simeq -\bar{\psi}_\mu \not{\partial}^3 \psi_\mu - \tfrac{2}{3}\bar{\eta}\not{\partial}\eta + \tfrac{1}{2}\bar{\psi}\not{\partial}^3\psi\,, \\ &\eta \equiv \partial_\mu\psi_\mu + \tfrac{1}{2}\not{\partial}\psi, \qquad \psi = \gamma_\mu\psi_\mu\,. \end{aligned} \tag{6.14}$$

The last two terms in (6.14) can be cancelled by fixing the gauges for Q- and S-supersymmetry: $\partial_\mu\psi_\mu + \frac{1}{2}\not{\partial}\psi = a(x)$, $\psi = b(x)$. Averaging over these gauges with the help of the operators $\not{\partial}$ and $\not{\partial}^3$ (for an analogous procedure in the case of the ordinary gravitino see refs. [200, 173]) we find [100] (cf. (1.46), (5.10)),

$$\begin{aligned} &Z_\psi = [\det \Delta_3]^{1/2} [\det H_1 \det H_2]^{-1/2} [\det \Delta_{\rm gh}]^{-1} , \\ &\Delta_3 = (-\not\partial^3)_{\psi_\mu}, \qquad H_1 = \not\partial, \qquad H_2 = -\not\partial^3 \end{aligned} \tag{6.15}$$

where $\Delta_{\rm gh} = -\not\partial^2$ is the ghost operator (the Q- and S-gauge transformations are parametrized by two spinors: $\psi'_\mu - \psi_\mu = \delta\psi_\mu = \partial_\mu \varepsilon - \gamma_\mu \eta$, so that $\int d\varepsilon\, d\eta\, \delta[\partial_\mu \psi'_\mu + \frac{1}{2}\not\partial \psi' + \cdots]\, \delta[\psi' + \cdots] \sim \det(-\not\partial^2)$). Comparing (6.15) with (1.38), we once more conclude that the conformal gravitino corresponds to -8 dynamical degrees of freedom. The partition function on a non-trivial background is obtained in a similar way. For example, in the U(1)-sector one obtains, starting with the relevant terms in the $N=1$ CSG action (2.35), the same expression (6.15), where, in this case (see ref. [100] for details):

$$\begin{aligned} &\Delta_{3\mu\nu} = -\delta_{\mu\nu}\not{\mathscr{D}}^3 + V^\alpha_{\mu\nu}\mathscr{D}_\alpha, \qquad V^\alpha_{\mu\nu} = -\tfrac{10}{3} B_{\mu\nu}\gamma^\alpha + \text{analogous terms}, \\ &\Delta_{\rm gh} = -\mathscr{D}^2 - \tfrac{1}{3}\sigma\cdot B\gamma_5, \qquad H_1 = \not{\mathscr{D}}, \qquad H_2 = -\not{\mathscr{D}}^3 , \\ &\mathscr{D}_\mu = \partial_\mu - \gamma_5 B_\mu, \qquad B_\mu = \tfrac{3}{4}\mathrm{i} A_\mu, \qquad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu . \end{aligned} \tag{6.16}$$

To establish $\beta_{0\psi}$ as the coefficient of the $F^2_{\mu\nu}$-term in the infinite part of $\ln Z_\psi$, we have to find the contributions in the infinities coming from the third order operator Δ_3 (the contributions of $\Delta_{\rm gh}$, H_1 and H_2 are easily found using the "Δ_2-algorithm" (5.17)). This can be done by multiplying Δ_3 by the operator $\not{\mathscr{D}}$, thus reducing it to a fourth-order Δ_4-operator of the type (5.20) for which we have the "Δ_4-algorithm" (5.22), $b_4(\Delta_3) = b_4(\Delta_4) - \frac{1}{2} b_4(\Delta_2)$, $\Delta_2 = -\not{\mathscr{D}}^2 = -\mathscr{D}^2 + \sigma\cdot B\gamma_5$. The final result for $(b_4)_4 = -[b_4(\Delta_3) - 2b_4(\Delta_{\rm gh}) - 2b_4(\Delta_2)]$ is $\frac{34}{3}B^2_{\mu\nu} = -\frac{34}{3}(\frac{3}{4})^2 F^2_{\mu\nu}(A)$, or, according to (6.12), $\beta_{0\psi} = \frac{34}{3}$. The computation of $\beta_{1\psi}$ and $\beta_{2\psi}$ in the gravitational sector is carried out in an analogous manner. Instead of (6.16) one finds here [100]

$$\begin{aligned} &\Delta_{3\,\mu\nu} = -\not{\mathscr{D}}^3_{\mu\nu} + V^\alpha_{\mu\nu}\mathscr{D}_\alpha, \qquad V^\alpha_{\mu\nu} = \tfrac{4}{3} R_{\mu\nu}\gamma^\alpha + \text{analogous terms}, \\ &\Delta_{\rm gh} = -\mathscr{D}^2 - \tfrac{1}{12}R, \qquad H_1 = \not{\mathscr{D}}, \qquad H_2 = -\not{\mathscr{D}}^3 , \\ &\Delta_2 = -\not{\mathscr{D}}^2 = -\mathscr{D}^2 + \tfrac{1}{4}R, \qquad \mathscr{D}_\mu = \partial_\mu + \left\{ {\lambda \atop \mu\rho} \right\} + \tfrac{1}{2}\sigma_{ab}\omega^{\;ab}_{\mu} . \end{aligned} \tag{6.17}$$

Representing Γ_1 in terms of the second- and fourth-order operators, and using the Δ_2- and Δ_4-algorithms for infinities, we are led to the values $\beta_{1\psi} = -173/180$, $\beta_{2\psi} = -149/30$, given in table 6.1. It is important to note that the conformal gravitino contribution in the Weyl infinities appears to have the *opposite* sign as compared to those of the contributions of the conformal graviton and of the ordinary matter fields. This fact has important implications for the finiteness of extended superconformal theories.

Next let us consider the infinities produced by the terms in (3.53) which contain the antisymmetric tensor $T_{\mu\nu}$. Splitting $T_{\mu\nu}$ in harmonic, coexact and exact parts (cf. (3.47), (3.48))

$$T^\pm_{\mu\nu} = T^{(\mathrm{H})\pm}_{\;\mu\nu} + \mathscr{D}_\mu \zeta^\pm_\nu - \mathscr{D}_\nu \zeta^\pm_\mu \pm e\,\varepsilon_{\mu\nu\alpha\beta}\mathscr{D}_\alpha \zeta^\pm_\beta, \qquad \zeta^\pm_\mu = \tfrac{1}{2}(\xi_\mu \pm \mathrm{i}\eta_\mu), \tag{6.18}$$

and changing variables $T_{\mu\nu} \to (T^{(\mathrm{H})}_{\mu\nu}, \zeta^\pm_\mu)$ (not forgetting to account for the additional gauge freedom introduced by (6.18)), one gets in the gravitational sector [101]

$$Z_T = C \det \Delta_2 \det \Delta_0 (\det \Delta_4)^{-1} \tag{6.19}$$

where $\Delta_0 = -\mathscr{D}^2$, $\Delta_{2\mu\nu} = -\mathscr{D}^2_{\mu\nu} + R_{\mu\nu}$ and $\Delta_{4\mu\nu}$ has the general structure (5.20) with $V^{\alpha\beta}_{\mu\nu} = g^{\alpha\beta}R_{\mu\nu}$ + analogous terms, $U_{\mu\nu} = \frac{1}{4}R(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R)$. The factor C stands for a contribution of the harmonic zero modes, $C = \int \mathrm{d}T^{(\mathrm{H})}_{\mu\nu}$, important for establishing the value of β_1 (see in this connection ref. [69]). Applying the algorithms (5.17), (5.22) one finds [100, 101]: $\beta_{1T} = -\frac{2}{15} + \frac{1}{2} = \frac{11}{30}$ ($\frac{1}{2}$ is due to the harmonic zero modes), $\beta_{\alpha T} = \frac{1}{10}$. The analog of (6.19) in the U(1)-sector is [107]

$$\begin{aligned} &Z_T = (\det \Delta_{+\perp} \det \Delta_{-\perp})^{-1/2}, \qquad \Delta_{\pm\perp\mu\nu} = -\delta_{\mu\nu}\mathscr{D}^2 \mp B^{\mp}_{\mu\nu}, \\ &B_\mu = \mathrm{i}c^2 \frac{4-N}{4N} A_\mu, \qquad \mathscr{D}_\mu = \partial_\mu - \gamma_5 B_\mu \end{aligned} \tag{6.20}$$

where $\Delta_{\pm\perp}$ are defined on covariantly transverse vectors. As a result [107], $\beta_{0T} = -\frac{3}{2}$.

An analogous treatment is possible in the case of a higher derivative Majorana spinor Λ and of a complex scalar φ. The expression for the third-order operator acting on Λ follows directly from (3.53). Using the same tricks as in the gravitino case, we are led to the coefficients β_Λ given in table 6.1 (see refs. [101] and [107] for a discussion of the calculations in the G and U(1) sectors respectively). The contribution in infinities due the scalar φ (present only in $N = 4$ CSG) is sensitive to the structure of φ-dependent terms in the action (see (3.53)–(3.55)). Let us study first the case of the "minimal" SU(1, 1)-invariant version of $N = 4$ CSG in which the non-minimal terms (3.55) are excluded (see section 3.3). Then φ being an SU(4)-singlet does not produce infinities in the SU(N)-sector, i.e. $\beta_{0\varphi} = 0$. The gravitational infinities are easily found using the "Δ_4-algorithm" (5.22) and correspond to $\beta_{1\varphi} = \frac{1}{45}$, $\beta_{2\varphi} = -\frac{4}{45}$.[3] Consider now the case of a "non-minimal" version of $N = 4$ CSG, in which non-minimal terms (3.55) are to be present (see sections 3.3 and 4.1). The parts of (3.53) and (3.55), which are relevant for establishing the additional contribution in infinities due to the non-minimal terms, can be parametrized as follows

$$\begin{aligned} \tfrac{1}{2}\mathscr{L} \simeq{} & \phi\Box^2\phi + d_1(F^r_{\mu\nu})^2 + d_2 C^2_{\lambda\mu\nu\rho} + \phi[a_1(F^r_{\mu\nu})^2 + b_1 F^r_{\mu\nu}F^{r}_{\mu\nu}{}^{*} + a_2 C^2_{\lambda\mu\nu\rho} + b_2 C_{\lambda\mu\nu\rho}C^{*}_{\lambda\mu\nu\rho}] \\ & + \phi^2[\gamma_1(F^r_{\mu\nu})^2 + \gamma_2 C^2_{\lambda\mu\nu\rho}] . \end{aligned} \tag{6.21}$$

Here ϕ stands for one of the real components (scalar or pseudoscalar) of the complex scalar $\varphi = \varphi_1 + \mathrm{i}\varphi_2$ and a, b, d and γ are numerical constants. It is easy to find (integrating first over $h_{\mu\nu}$ and V_μ and then over ϕ) the additional terms in the infinities (6.8) which are due to the non-minimal terms in (6.21) [107]

$$(\Delta b_4)_\phi = \left(\frac{a_1^2 + b_1^2}{2d_1} - \gamma_1\right)(F^r_{\mu\nu})^2 + \left(\frac{a_2^2 + b_2^2}{4d_2} - \gamma_2\right) C^2_{\lambda\mu\nu\rho} . \tag{6.22}$$

In analogy with the structure of the action of O(4) Poincaré supergravity (3.66), one may expect [107] that the values of the constants in (6.21) should be

[3] It can be explicitly checked that no R^2-infinities are produced by the conformally invariant actions for $T_{\mu\nu}$, Λ and φ.

$$d_1 = -\tfrac{1}{4}, \qquad d_2 = \tfrac{1}{4}, \qquad \gamma_1 = (\tfrac{1}{2}, -\tfrac{1}{2}), \qquad a_1 = (\tfrac{1}{2}, 0),$$
$$b_1 = (0, -\tfrac{1}{2}), \qquad a_2 = (1, 0), \qquad b_2 = 0, \qquad \gamma_2 = (+1, -1)$$

(the values in brackets corresponds to φ_1 and φ_2). In this case (6.22) gives

$$(\Delta b_4)_\varphi = 2[\tfrac{1}{2}C^2_{\lambda\mu\nu\rho} - \tfrac{1}{2}(F^r_{\mu\nu})^2] = R^*R^* + 2W - (F^r_{\mu\nu})^2 . \tag{6.23}$$

We stress that at present eq. (6.23) remains as a guess, which provides the finiteness of a "non-minimal" $N=4$ CSG. Adding (6.23) to the contribution of φ in the "minimal" case, we find the values of β_φ's presented in the second φ-column in table 6.1 ($\beta_{0\varphi}$ is defined as follows: $(b_4)_\varphi = \beta_{0\varphi}[-\tfrac{1}{2}(F^r_{\mu\nu})^2]$.

Now all is prepared for the computation of the total expressions for the β-functions in $N \le 4$ conformal supergravities. Starting with (6.10) and (6.11), and using the information contained in tables 3.3 and 6.1, we find in the gravitational, U(1) and SU(N) sectors [100, 101, 107]

$$\begin{aligned}
&N=1: && \beta_G = \beta_{2h} + \beta_{2\psi} + \beta_{2A} = \tfrac{17}{2},\\
& && \beta_{U(1)} = \tfrac{3}{4}\beta_{0\psi} = \tfrac{17}{2},\\
&N=2: && \beta_G = \beta_{2h} + 2\beta_{2\psi} + \beta_{2A} + 3\beta_{2V} + \beta_{2T} + 2\beta_{2\chi} = \tfrac{13}{3},\\
& && \beta_{U(1)} = \tfrac{1}{4}(2\beta_{0\psi} + 4\beta_{0T} + 2\beta_{0\chi}) = \tfrac{13}{3},\\
& && \beta_{SU(2)} = 2(\tfrac{1}{2}\beta_{0\psi} + 2\beta_{0V} + \tfrac{1}{2}\beta_{0\chi}) = \tfrac{13}{3},\\
&N=3: && \beta_G = \beta_{2h} + 3\beta_{2\psi} + \beta_{2A} + 8\beta_{2V} + 3\beta_{2T} + 9\beta_{2\chi} + \beta_{2\Lambda} + 3\beta_{2E} = 1,\\
& && \beta_{U(1)} = \tfrac{1}{12}(3\beta_{0\psi} + 3\times 4\beta_{0T} + 9\beta_{0\chi} + 9\beta_{0\Lambda} + 3\times 4\beta_{0E}) = 1,\\
& && \beta_{SU(3)} = 2(\tfrac{1}{2}\beta_{0\psi} + 3\beta_{0V} + \tfrac{1}{2}\beta_{0T} + \tfrac{1}{2}\beta_{0\chi} + \tfrac{1}{2}\beta_{0E}) = 1,\\
&N=4: && \beta_G = \beta_{2h} + 4\beta_{2\psi} + 15\beta_{2V} + 6\beta_{2T} + 20\beta_{2\chi} + 4\beta_{2\Lambda} + 10\beta_{2E} + \beta_{2\varphi} = -2,\\
&\text{"minimal"} && \beta_{SU(4)} = 2(\tfrac{1}{2}\beta_{0\psi} + 4\beta_{0V} + \beta_{0T} + \tfrac{13}{2}\beta_{0\chi} + \tfrac{1}{2}\beta_{0A} + 3\beta_{0E}) + \beta_{0\varphi} = -2,\\
&N=4: && \beta_G = (\beta_G)_{\text{minimal}} + \Delta\beta_{2\varphi} = 0,\\
&\text{"non-minimal"} && \beta_{SU(4)} = (\beta_{SU(4)})_{\text{minimal}} + \Delta\beta_{0\varphi} = 0.
\end{aligned} \tag{6.24}$$

An analogous computation in the G-sector gives the values of β_1. The final results [101, 107] are presented in table 6.2 ($N=0$ corresponds to Weyl theory).

Table 6.2
One-loop β-functions in $N \le 4$ conformal supergravities

	N					
	0	1	2	3	4_{minimal}	$4_{\text{non-minimal}}$
β	199/15	17/2	13/3	1	−2	0
β_1	137/60	5/4	11/24	0	0	1

It is gratifying to see that the values of the β-functions computed in different background sectors, agree with one another, thus demonstrating the supersymmetry of the theories under consideration. We also get an implicit check of the value of the β-function in Weyl theory.[4] From a technical point of view we note that the calculation in the SU(N)-sector is the most "economical" one (for example, to find β for $N = 2$, we need only to recall the well-known values of β_{0V} and $\beta_{0\chi}$ and to compute $\beta_{0\psi}$ in the U(1) sector of $N = 1$ CSG).

The values of β for $N = 0, 1, 2, 3, 4$ form a *decreasing* sequence (note that $N \leq 3$ CSG's are asymptotically free as Weyl theory). This suggests that some $N = 4$ theory may be finite. Though the "minimal" $N = 4$ CSG appears to be non-finite ($\beta \neq 0$), it seems likely (see sections 3.3 and 6.1) that there exists a "non-minimal" version of $N = 4$ CSG which satisfies the condition (6.23) and thus has a zero β-function. Another interesting conclusion is the absence of topological infinities ($\beta_1 = 0$) in $N = 3$ and "minimal" $N = 4$ theories. This result is reminiscent of the cancellation of these infinities in $N \geq 3$ Poincaré supergravities with spectra containing special combinations of the antisymmetric gauge tensors (see, e.g. refs. [199] and [68] for references and a review). It is worth stressing that the non-gauge antisymmetric tensors $T_{\mu\nu}$ are automatically present in $N \geq 2$ conformal supergravities and cannot be "rotated away" by a duality transformation. The presence of topological infinity in the "non-minimal" $N = 4$ CSG is probably harmless (it corresponds to a breakdown of scale invariance on topologically non-trivial backgrounds and does not imply any inconsistency at higher loop levels). There is an abstract possibility to make this theory free also of topological infinity by trading one of the 20 auxiliary scalars for a three-index antisymmetric gauge tensor (which is known to contribute $\Delta\beta_1 = -1$ [69]).

Though the actual existence of a "non-minimal" $N = 4$ CSG is still an open question, it seems rather appealing that a version of $N = 4$ conformal supergravity may turn out to be *completely finite* (cf. the previous section 6.1) thus providing the first example of a *locally supersymmetric* power counting renormalizable finite theory (the already known power counting renormalizable finite theories [20, 189, 160, 161, 265] like $N = 4$ super-Yang–Mills, possess only *rigid* supersymmetry, and hence do not incorporate gravity). The analogy with supersymmetric theories in flat space suggests that a class of finite locally superconformal theories may include not only a particular "pure" $N = 4$ conformal supergravity but also conformal supergravities interacting with appropriate superconformal matter multiplets (for instance, the "minimal" $N = 4$ CSG plus $N = 4$ super-Yang–Mills with a particular gauge group). The addition of matter multiplets may in principle turn the anomalous CSG's into anomaly free theories which may also be finite at higher loop levels (for $N \geq 2$) if the superspace non-renormalization theorem of section 6.1 can be applied. One-loop infinities and the finiteness of "CSG plus matter" theories will be studied in section 6.3.

At the end of this section we shall discuss the CSG effective action for the special (instanton) backgrounds [107, 250]. Specification of the different backgrounds (6.9) makes it possible to check in a simple way the results contained in tables 6.1 and 6.2, and to clarify the structure of the finite part of the one-loop effective action. It also makes it possible to determine the higher loop behavior of β-functions using the method of ref. [202]. Let us start with the gravitational sector and suppose that the background metric satisfies the Einstein condition $R_{\mu\nu} = 0$. All such backgrounds are solutions of the CSG field equations. Substituting $R_{\mu\nu} = 0$ in the general expressions (6.15), (6.17), (6.19), etc., for the contributions of the fields in the one-loop effective action (see also (5.30) for the contribution of the Weyl graviton) it is easy to check that all differential operators reduce to products of operators present

[4] It is interesting to note that the mere existence of $N = 1$ conformal supergravity makes it possible to establish the value of the Weyl's theory β-function by computing the contribution of the conformal gravitino to the U(1) gauge field infinities in flat space.

in the effective action for Poincaré supergravity, computed on the $R_{\mu\nu}=0$ background [155, 32]. Explicitly, we get [107] (cf. (5.30))

$$
\begin{aligned}
&Z_h = Z_2^2 Z_1, \qquad Z_\psi = Z_{3/2}^3 Z_{1/2}, \qquad Z_V = Z_A = Z_1, \\
&Z_T = Z_1^2 Z_0 Z_{\tilde{0}}, \qquad Z_\Lambda = Z_{1/2}^3, \qquad Z_E = Z_{\tilde{0}}^2, \qquad Z_\varphi = Z_0^4 .
\end{aligned} \tag{6.25}
$$

(Z_φ is given for the "minimal" $N=4$ CSG). Here Z_2, $Z_{3/2}$, Z_1, $Z_{1/2}$, Z_0, $Z_{\tilde{0}}$ are the standard partition functions for the Einstein graviton, gravitino, gauge vector, Majorana spinor, real scalar and second rank antisymmetric gauge tensor, i.e.

$$
\begin{aligned}
&Z_2 = \det \Delta_1 (\det \Delta_2 \det \Delta_0)^{-1/2} , \\
&Z_{3/2} = [\det \Delta_{3/2}/(\det \Delta_{1/2})^3]^{1/4}, \\
&Z_1 = \det \Delta_0/(\det \Delta_1)^{1/2}, \qquad Z_{1/2} = (\det \Delta_{1/2})^{1/4} \\
&Z_0 = (\det \Delta_0)^{-1/2}, \qquad Z_{\tilde{0}} = (\det \tilde{\Delta}_2)^{-1/2} \det \Delta_1 (\det \Delta_0)^{-3/2} = C Z_0 ,
\end{aligned} \tag{6.26}
$$

where for $R_{\mu\nu}=0$

$$
\begin{aligned}
&\Delta_0 = -\mathscr{D}^2, \qquad \Delta_{1\mu\nu} = -\mathscr{D}^2_{\mu\nu}, \qquad \Delta_{1/2} = -\mathscr{D}^2, \\
&(\Delta_2 \bar{h})_{\mu\nu} = -\mathscr{D}^2 \bar{h}_{\mu\nu} + 2C_{\alpha\mu\nu\beta} \bar{h}_{\alpha\beta} , \\
&\Delta_{3/2\,\mu\nu} = -\mathscr{D}^2_{\mu\nu} - \sigma_{\lambda\rho} C^{\lambda\rho}{}_{\mu\nu}, \qquad (\tilde{\Delta}_2 A)_{\mu\nu} = -\mathscr{D}^2 A_{\mu\nu} - 2C^{\alpha\beta}_{\mu\nu} A_{\alpha\beta}
\end{aligned} \tag{6.27}
$$

and C stands for the contribution of harmonic zero modes. As a consequence, the one-loop CSG effective actions $\Gamma^{(N)}_{\rm CSG}$, computed on the Einstein background, are given by combinations of the effective actions for two Poincaré supergravities (spin 2 multiplets) and several spin 3/2 and spin 1 multiplets $\Gamma^{(N)}_{(s)}$

$$
\begin{aligned}
&\Gamma^{(1)}_{\rm CSG} = \Gamma^{(1)}_{(2)} + \Gamma^{(2)}_{(2)} + \Gamma^{(1)}_{(1)}; \qquad \Gamma^{(2)}_{\rm CSG} = \Gamma^{(2)}_{(2)} + \Gamma^{(4)}_{(2)}; \\
&\Gamma^{(3)}_{\rm CSG} = \Gamma^{(3)}_{(2)} + \Gamma^{(4)}_{(2)} + 2\Gamma^{(3)}_{(3/2)} + \Gamma^{(3)}_{(1)}; \qquad \Gamma^{(4)}_{\rm CSG} = 2\Gamma^{(4)}_{(2)} + 4\Gamma^{(4)}_{(3/2)}
\end{aligned} \tag{6.28}
$$

(the spin content of the ordinary supermultiplets $\{s\}_N$ is:

$$
\begin{aligned}
&\{2\}_1 = (2, \tfrac{3}{2}), \ldots, \qquad \{2\}_3 = (2, 3\times\tfrac{3}{2}, 3\times 1, \tfrac{1}{2}), \\
&\{\tfrac{3}{2}\}_3 = (\tfrac{3}{2}, 3\times 1, 3\times\tfrac{1}{2}, 0, \tilde{0}), \qquad \{1\}_4 = \{1\}_3 = (1, 4\times\tfrac{1}{2}, 6\times 0), \\
&\{\tfrac{3}{2}\}_4 = \{\tfrac{3}{2}\}_3 + \{1\}_3, \text{ etc.}) .
\end{aligned}
$$

Using (6.26)–(6.28), it is easy to check the results for the topological β-functions β_1 in table 6.1, by expressing them in terms of the values of β_1 for the ordinary $s \le 2$ fields (see table 5.1). At the same time, eq. (6.28) makes it possible to relate the vanishing of β_1 in $N=3,4$ CSG's (table 6.2) to the absence of topological infinities (or trace anomalies) in $N\ge 3$ Poincaré supergravities that can be constructed in terms of $N=3$ multiplets (with $\{\tfrac{3}{2}\}_3$ containing the antisymmetric tensor $\tilde{0}$) [199]. In fact, the infinite part of the effective action on the $R_{\mu\nu}=0$ background is governed by β_1 (see (6.8)) but, according to (6.28), $\Gamma_{1\infty}$, for $N=3$ and "minimal" $N=4$ CSG, is given by a sum of $\Gamma_{1\infty}$'s for several $N=3$ or $N=4$ conformal anomaly free ordinary multiplets.

Let us now further specify the background by imposing the self-duality condition $R_{\lambda\mu\nu\rho}=R^{*}_{\lambda\mu\nu\rho}$. In this case, it has been proven [155] that the non-zero parts of the spectra of the operators (6.27) can be put into one-to-one correspondence and, hence (up to the contributions of zero modes) $Z_2 \sim Z_{\bar{0}}^2$, $Z_1 \sim Z_{\bar{0}}^2$, $Z_{3/2} \sim Z_{\bar{0}}^{-2}$, $Z_{1/2} \sim Z_{\bar{0}}^{-2}$, $Z_{\bar{0}} \sim Z_0$. Then, because of the zero total number of degrees of freedom, the $N \geq 1$ Poincaré supergravity effective actions, computed on this gravitational instanton background, are determined *only by the zero modes* [155, 32].

In view of (6.25) the same statement is also true for $N=1,2,3$ and "minimal" $N=4$ *conformal* supergravities [107]. As a result the renormalized one-loop effective action takes the general form

$$\Gamma[g]=I[g]-B_4\ln(\rho\mu)-\gamma\ln\Phi\,, \tag{6.29}$$

where I is the classical action, ρ is a "scale" of the instanton, Φ stands for the norm of zero modes, and

$$\begin{aligned} B_4 &\equiv \frac{1}{(4\pi)^2}\int b_4\sqrt{g}\,\mathrm{d}^4x=\sum_k (n_{\mathrm{B}k}-\tfrac{1}{2}n_{\mathrm{F}k})d_k\,,\\ \gamma &= \sum_k (n_{\mathrm{B}k}-n_{\mathrm{F}k})d_k\,. \end{aligned} \tag{6.30}$$

$n_{\mathrm{B}k}$ and $n_{\mathrm{F}k}$ are the numbers of Bose and Fermi zero modes and d_k are the numbers of fields of a given type (cf. (6.10)). Taking a particular compact K3 instanton, and using the results of refs. [155, 32], it is easy to find, from (6.25), that [107]

$$\begin{aligned} &n_h=114, \quad n_\psi=104, \quad n_A=n_V=-2\,,\\ &n_T=22, \quad n_\chi=2, \quad n_\Lambda=6, \quad n_E=2, \quad n_\varphi=4 \end{aligned} \tag{6.31}$$

and thus

$$(B_4)_{R=R^*}=2\beta_1\chi=48\beta_1=60;22;0;0 \text{ for } N=1,2,3,4\,. \tag{6.32}$$

Here we have used the value ($\chi=24$) of the Euler number (5.2) for K3 space. These numbers for β_1 are in agreement with the values given in table 6.2.

Next we turn to the case of the De Sitter background ($R_{\mu\nu}=\Lambda g_{\mu\nu}$, $C_{\lambda\mu\nu\rho}=0$) or that of a "double-self-dual" S^4-instanton. In analogy with the treatment given for Weyl theory (see (5.38)), it is possible to show that [107, 250]

$$\begin{aligned} &Z_\psi=\left[\frac{\det\Delta_{3/2}(0)}{\det\Delta_{1/2}(-\Lambda)}\right]^{1/4}\frac{[\det\Delta_{3/2}(-\frac{1}{3}\Lambda)]^{1/2}}{\det\Delta_{1/2}(-\frac{4}{3}\Lambda)}\,, \qquad Z_V=\frac{\det\Delta_0(0)}{[\det\Delta_1(\Lambda)]^{1/2}}\,,\\ &Z_T=C\frac{\det\Delta_0(0)}{\det\Delta_1(\Lambda)}\,, \qquad Z_\Lambda=[\det\Delta_{1/2}(0)]^{1/4}\det\Delta_{1/2}(-\tfrac{1}{3}\Lambda)\,,\\ &Z_\varphi=[\det\Delta_0(0)\det\Delta_0(\tfrac{2}{3}\Lambda)]^{-1}, \qquad Z_\chi=[\det\Delta_{1/2}(0)]^{1/4}, \qquad Z_E=[\det\Delta_0(\tfrac{2}{3}\Lambda)]^{-1} \end{aligned} \tag{6.33}$$

where $\Delta_{3/2}(X)_{\mu\nu} = -\mathscr{D}^2_{\mu\nu} + (\frac{4}{3}\Lambda + X)g_{\mu\nu}$ is defined on φ_μ (γ_μ-traceless part of ψ_μ, see (6.3)), $\Delta_{1/2}(X) = -\mathscr{D}^2 + \Lambda + X$ and Δ_1 and Δ_0 are the same as in (5.33), (5.37) (all operators act on differentially unconstrained variables). Starting with (6.33), it is easy to check the results for β_{2h} (table 6.1) using only the standard "Δ_2-algorithm" (5.17) and the values of β_{1k} (found, e.g., on the $R_{\mu\nu} = 0$ background).[5] The renormalized effective action on S^4 background is given by

$$\Gamma = I + \tfrac{1}{2}B_4 \ln(\Lambda/3\mu^2) + \text{const.}\,,$$
$$B_4 = 4\beta_1 - 2\beta_2 = -12;\ -41/6;\ -2;\ +4 \text{ for } N = 1,2,3,4\,. \tag{6.34}$$

It turns out [107] that the contributions of all the non-zero modes mutually cancel for $N \geq 3$ and thus for $N = 3$ and $N = 4$ theories B_4 can be obtained by counting only the zero modes, as in (6.30) ($n_h = -29$, $n_\psi = -24$, $n_V = -1 = n_A$, $n_{T,\Lambda,E,\chi,\varphi} = 0$). This is true both for the "minimal" and for the "non-minimal" versions of $N = 4$ CSG, which are equivalent on S^4 ($C_{\lambda\mu\nu\rho} = 0$). Inspection of (6.33) reveals also that the total expression for the U(N) CSG effective action, computed on the S^4 background, contains a part, which is just the effective action for the O(N) gauged supergravity (with $g^2 = -\Lambda k^2/12$) formally continued from the anti De Sitter background to the S^4 background.

Now let us consider the SU(N)-sector and more specifically the case of the SU(N)-instanton background $F_{\mu\nu} = F^*_{\mu\nu}$. It is possible to prove [107] that all mixing terms (3.54) give vanishing contributions in the effective action calculated on this background. Hence, as in the case of the gravitational instanton background, Γ is given by a sum of separate contributions from all the fields. One can consider super-Yang–Mills theories, and conformal supergravity theories, as two supersymmetric systems of fields in which SU(N) Yang–Mills theory can be embedded. This suggests a similarity in the role played by instantons in both types of theories. At the same time, it should be understood that, while SU(N) is an "external" gauge group in the super-Yang–Mills case, it is essentially an "internal" group in the case of conformal supergravities (where different fields belong to different SU(N)-representations). It is known [40] that the one-loop effective action for the N-extended super-Yang–Mills theory, calculated on the gauge instanton background, is determined only by the zero modes for all $N \geq 1$. The analogous cancellation of contributions of non-zero modes takes place for conformal supergravities but only for the $N = 3$ and $N = 4$ ("minimal" version) [107]. This cancellation is related to the existence of the following sum rule: $\Sigma_k\, \nu_k C_{2k} = 0$, which is valid only for $N \geq 3$ (ν_k and C_{2k} are the numbers of dynamical degrees of freedom (table 3.4) and Casimir invariants (table 6.1) of the fields of CSG's). As a result, it is possible to determine the one-loop $\beta(\alpha)$-function for $N \geq 3$ CSG's by counting only zero modes on the SU(N) gauge instanton background [107] (cf. (6.8))

$$B_4 \equiv -\beta q = \sum_k (n_{Bk} - \tfrac{1}{2}n_{Fk}), \qquad n_k = \bar{n}_k C_{2k} q\,,$$
$$q = \frac{1}{32\pi^2}\int d^4x\, F^r_{\mu\nu}F^{r*}_{\mu\nu}\,, \tag{6.35}$$
$$\bar{n}_\psi = 52, \qquad \bar{n}_V = 4, \qquad \bar{n}_T = 6, \qquad \bar{n}_\Lambda = 2, \qquad \bar{n}_\chi = 2, \qquad \bar{n}_{h,A,E} = 0\,.$$

The cancellation of contributions from non-zero modes suggests that, in analogy with the situation in

[5] Thus the use of these particular backgrounds appears to be the simplest method for establishing the β-functions in CSG's [250].

the super-Yang–Mills theory [202], all higher loop corrections in the effective action vanish on the gauge (super) instanton background. Let A_0 be a classical background and

$$\exp\Big(-\frac{1}{\hbar}\Gamma[A_0]\Big) = \int \mathrm{d}A \exp\Big(-\frac{1}{\hbar} I[A+A_0]\Big)$$

be the vacuum–vacuum amplitude on this background. Then

$$\Gamma[A_0] = I(\alpha(L)) + \sum_{n=1}^{\infty} \hbar^n \Gamma_n(\alpha(L), L), \qquad \Gamma_n = \Gamma_{n\infty}(\alpha(L), L) + \Gamma_{h\,\mathrm{fin}}(\alpha(L))$$

where $L \to \infty$ is an ultraviolet cut-off, $\alpha(L)$ is an exact bare coupling constant and $\Gamma_{n\,\mathrm{fin}}$ does not depend explicitly on L. Suppose now that all higher loop corrections vanish for some particular A_0, i.e. $\Gamma_k = 0$, $k \geq 2$. If a theory is renormalizable, all the infinities can be absorbed in $\alpha(L)$ so that $\mathrm{d}\Gamma/\mathrm{d}L = 0$. Thus, we get an equation for the exact β-function

$$\frac{\partial \Gamma_{1\infty}}{\partial t} + \beta_\alpha \frac{\partial(I + \hbar\Gamma_1)}{\partial \alpha} = 0, \qquad \beta_\alpha = \frac{\mathrm{d}\alpha(L)}{\mathrm{d}t}, \qquad t = \ln L$$

which can be solved given the L- and α-dependence of the one-loop effective action. Assuming further that all the non-zero modes mutually cancel in Γ_1, we find that the effective action takes the general form (6.29), (6.30) and hence that β_α depends on two parameters, B_4 and γ [202]. Applying these considerations to conformal supergravities (i.e. making the conjecture[6] that all higher loop corrections vanish in the super-Yang–Mills case), it is possible to establish exact (all-loop) expressions for the gauge β-functions in $N \geq 3$ CSG's [107], namely:

$$\beta_\alpha = \frac{\beta\alpha^3}{(4\pi)^2 - \gamma\alpha^2}, \tag{6.36}$$

where β and γ are given in terms of combinations of zero modes and appear to be non-vanishing for $N = 3$ and "minimal" $N = 4$ conformal supergravities ($\beta = +1$, $\gamma = -17$ for $N = 3$ and $\beta = -2$, $\gamma = -18$ for $N = 4$). An apparent contradiction between (6.36) and the prediction of the non-renormalization theorem of section 6.1 ($\beta_\alpha = \beta\,\alpha^3/(4\pi)^2$, all $L \geq 2$ contributions vanish) may be due to complications connected with the one-loop anomaly. No contradiction would be present in the case of one-loop finiteness ($\beta = 0$), which is probably realized in "non-minimal" $N = 4$ conformal supergravity.

Another method which can be used in principle to determine the exact β-function in super-Yang–Mills theory is based on the observation that the conformal and chiral anomalies should belong to one supermultiplet and also on the Adler–Bardeen theorem [148, 17]. Though the applicability of this method to conformal supergravities is doubtful (here the anomalous currents are coupled to gauge fields and thus $\beta \neq 0$ formally implies the inconsistency of the theory) consideration of multiplets of anomalies may help to clarify the structure of CSG's and eventually may help to give an independent proof of the finiteness of a particular ($N = 4$) theory. The multiplet of anomalies of $N = 1$ CSG includes

[6] The proof of this conjecture, which in the super-Yang–Mills case is based on the rigid superconformal symmetry and chirality–duality connection, has to wait for a deeper understanding of the ($N \geq 1$) superfield formulation of CSG's.

$\theta^\mu_\mu \sim g_{\mu\nu}\, \delta\Gamma/\delta g_{\mu\nu}$, $\partial_\mu j_{\mu 5} \sim \partial_\mu\, \delta\Gamma/\delta A_\mu$, $\gamma_\mu S_\mu \sim \gamma_\mu\, \delta\Gamma/\delta\psi_\mu$, and also two auxiliary spin 0-components (for a general discussion of $N=1$ "superscale" anomalies see [130, 129]). The non-vanishing values of the first three fields indicate a breakdown of the D, U(1) and S invariances of the finite part of the one-loop effective action. The method which we used above to establish the one-loop β-function (see (6.8)) is in fact equivalent to the computation of the conformal anomaly ($\theta^\mu_\mu \sim b_4$, cf. (5.44)). Given that the properly defined anomalies belong to one supermultiplet, it should be possible to establish the value of the one-loop β-function in an alternative way, namely, by computing $\partial_\mu j_{\mu 5} \sim \mathrm{i}\beta(CC^* - \frac{3}{2}FF^*)$ in the gravitational and U(1) sectors. Such a computation of the U(1) chiral anomaly can be carried out also for $N>1$ GSG's (here one must correctly account for the values of the chiral weights and for the contribution of the *non-gauge* antisymmetric tensors $T_{\mu\nu}$, cf. the situation in PSG [174]). The $N>2$ multiplets of anomalies should probably contain also the component $D_\mu J_\mu{}^i{}_j \sim D_\mu\, \delta\Gamma/\delta V_\mu{}^j{}_i$, corresponding to a breakdown of the chiral SU(N) invariance (cf. footnote 8 in section 6.1). It would be very interesting to establish the finiteness of the $N=4$ CSG as a result of the cancellation of the SU(4) chiral non-Abelian anomalies (the fields that contribute in the SU(4) anomaly are ψ^i_μ, $T^{-\,ij}_{\mu\nu}$, Λ^{ijk} and $\chi^{ij}{}_k$ in the "minimal" version and probably also φ in the "non-minimal" version).

6.3. *Finite theories of conformal supergravity coupled to matter multiplets*

In this section we shall study the possibility of cancelling the one-loop infinities of conformal supergravities by coupling them to superconformal matter multiplets. In general, conformal supergravity, interacting with renormalizable (in flat space) superconformal matter multiplets, is a power counting renormalizable theory. To avoid dangerous R^2-type one-loop infinities (absent in the one-loop S-matrix), one has to use proper background covariant gauges (for example, the superconformal gauges are to be fixed on the supergravity fields and not on the matter fields). The non-renormalizable R^2-type infinities are generated, however, at the higher loop level and are due to one-loop superconformal anomalies. To make the theory formally consistent (in particular, renormalizable) it is necessary to cancel the one-loop infinities (and hence the anomalies). One-loop infinities are given by a sum of the three terms: a contribution of pure conformal supergravity, a contribution of the matter multiplets interacting with the background conformal supergravity fields, and a contribution produced by mixings between CSG and matter quantum fields. The third contribution is proportional to the action of the matter multiplets, while the first two are proportional to the CSG action and therefore determine the CSG β-function.

Let us first consider the one-loop infinities produced by the $N=1$ chiral multiplet interacting with $N=1$ CSG according to eq. (2.51). Making use of the "Δ_2-algorithm" (5.17), it is easy to obtain the expression for the corresponding infinities in the mixed gravitational U(1)-sector (we shall neglect the total derivative $\mathscr{D}^2R$-terms)

$$b_4 = \beta_1 R^*R^* + \beta(W - \tfrac{3}{4}F^2_{\mu\nu}), \qquad \beta_1 = \tfrac{1}{48}, \qquad \beta = \beta_2 = \tfrac{1}{12}. \tag{6.37}$$

This result is in agreement with supersymmetry (cf. (6.8)). The fact that the Weyl term and the Maxwell term contribute in the $N=1$ CSG action with opposite signs now finds the following "explanation": the standard scalars and spinors are known to give *positive* ("asymptotically free") contributions in the Weyl infinities and *negative* ("non-asymptotically free") contributions in the Maxwell infinities. The values of β_1 and β follow also from the numbers given in table 5.1: $\beta_1 = \frac{1}{90}$ (complex scalar) $+ \frac{7}{720}$ (Majorana spinor), $\beta = \beta_2 = \frac{1}{30}$ (complex scalar) $+ \frac{1}{20}$ (Majorana spinor). Analogous results are found for the $N=1$ vector (2.56) and for the $N=1$ tensor (2.62) multiplets: $\beta_1=-\frac{1}{16}$, $\beta=\frac{1}{4}$ and $\beta_1=\frac{25}{48}$, $\beta=\frac{1}{12}$ (quantizing the tensor multiplet one has to assume that ϕ in (2.62) has a non-zero vacuum value, $\phi=1+\phi_q$ and also one has to note that β_1 and β_2, written for the antisymmetric gauge tensor $E_{\mu\nu}$ are the same as for a real scalar, except for the additional contribution of the harmonic zero modes $\Delta\beta_1=\frac{1}{2}$ [69]). Hence all ordinary superconformal multiplets give *positive* contributions in β and thus cannot save $N=1$ CSG (see table 6.2) from anomalies. The fields that give negative contributions in β_2 are (according to table 6.1) the conformal gravitino, the higher derivative spinor Λ, and the scalar φ (the latter is assumed to have no non-minimal couplings). This is why we have to consider the $N=1$ multiplets which contain these fields (see (4.58)). The simplest is the higher derivative chiral multiplet $\{0'\}_1=(\varphi,\Lambda,E)$, with the Lagrangian (2.53). The corresponding values of the $N=1$ β-functions (β_2 and β_1) follow from table 6.1 and are presented in table 6.3 (we use the universal notation: E is a complex and ϕ is a real conformal scalar, χ is a Majorana spinor, V_μ is a gauge vector, $T_{\mu\nu}$ is a non-gauge antisymmetric tensor, $E_{\mu\nu}$ is a gauge antisymmetric tensor, etc.; we do not indicate the auxiliary fields in the multiplets).

Table 6.3
Contributions of $N=1$ superconformal multiplets in $N=1$ conformal supergravity one-loop β-functions

	$\{0\}_1$ (E,χ)	$\{1\}_1$ (V_μ,χ)	$\{\tilde{0}\}_1$ $(\phi,\chi,E_{\mu\nu})$	$\{0'\}_1$ (φ,Λ,E)	$\{1/2\}_1$ $(\Lambda,T_{\mu\nu},\chi,E)$	$\{3/2\}_1$ $(\psi_\mu,2V_\mu,T_{\mu\nu},\chi)$	$\{2\}_1$
β	1/12	1/4	1/12	−1/4	1/6	−53/12	17/2
β_1	1/48	−1/16	25/48	1/16	5/12	−35/48	5/4

It is easy to check that these numbers are consistent with the numbers given in table 6.2. One has only to remember that, according to (4.59),

$$\begin{aligned}
\{2\}_2 &= \{2\}_1+\{\tfrac{3}{2}\}_1+\{1\}_1\,,\\
\{2\}_3 &= \{2\}_1+2\{\tfrac{3}{2}\}_1+4\{1\}_1+\{\tfrac{1}{2}\}_1+2\{0\}_1\,,\\
\{2\}_{4\,\text{minimal}} &= \{2\}_1+3\{\tfrac{3}{2}\}_1+8\{1\}_1+3\{\tfrac{1}{2}\}_1+6\{0\}_1+\{0'\}_1\,.
\end{aligned}\tag{6.38}$$

We conclude that the only multiplets (among those multiplets that contain fields of non-negative mass dimensions), which give negative contributions in β, are the higher derivative chiral multiplet and the conformal gravitino multiplet. We cannot consider arbitrary couplings of the gravitino multiplets to the matter multiplets and to $N=1$ CSG: in general such theories will be classically inconsistent except for the cases when they coincide with the extended conformal supergravities already discussed. Thus the only possibility to make β equal to zero within the $N=1$ theory is to combine the $N=1$ CSG with 34 chiral $\{0'\}_1$-multiplets [109]. If one also adds the additional "ordinary" matter multiplets, it is then necessary to increase the number of $\{0'\}_1$-multiplets to preserve the condition $\beta=0$. It is interesting to note that the contribution of the vector multiplet (both in β and in β_1) is exactly opposite to that of $\{0'\}_1$. Hence, to have a zero $N=1$ CSG β-function in the $N=1$ theory containing N_0 scalar multiplets $\{0\}_1$ and N_1 vector multiplets $\{1\}_1$ we have to introduce $34+N_1+\frac{1}{3}N_0$ higher derivative multiplets $\{0'\}_1$.

Having found a class of $N=1$ theories which are one-loop finite (up to topological infinities) in the $N=1$ CSG sector, it is natural to study their finiteness in the matter sector. In view of the results of ref.

[99] for the infinities of a system of matter fields interacting with higher derivative gravity, we anticipate in general the necessity of a wave function renormalization (absorbing the infinities produced by the interaction with $N = 1$ CSG) for those matter multiplets, which contain scalar fields. At the same time, it is possible to prove the following statement: the interaction with $N = 1$ conformal supergravity *does not modify* the (one-loop) β-function of the $N = 1$ vector gauge multiplet, i.e. all one-loop infinities, proportional to the action of the $N = 1$ super-Yang–Mills theory, which result from the interaction with $N = 1$ CSG, mutually cancel. The proof is easy to carry out in the vector gauge sector of the vector multiplet: $B_\mu \neq 0$, $\chi = 0$ (see (2.56)). First, we recall that the interaction of B_μ with the Weyl graviton does not produce additional $F^2_{\mu\nu}(B)$-infinities (see (6.13) and refs. [99, 100]). Second, we note that the $\bar\psi_\mu \sigma F \gamma_\mu \chi$ mixing term in (2.56), gives a vanishing contribution in infinities found in the gauge sector[1] (the infinite part of the corresponding one-loop diagram is proportional to $\mathrm{tr}(\sigma F \gamma_\mu \gamma_\alpha \gamma_\mu \sigma F \gamma_\alpha) = 0$). Finally all other terms in (2.56) cannot contribute to the gauge infinities.

To have a manifestly finite theory we have thus to cancel the infinities met in the matter sector (*any* type of infinities spoils the superconformal symmetries and probably makes a theory containing conformal supergravity inconsistent). This suggests going to higher N. Extended $N \geq 2$ supersymmetry is also welcome in order to have the possibility to prove finiteness to higher loop orders, applying the non-renormalization theorem (section 6.1). But here we confront the following difficulty: all $N = 2$ and $N = 3$ matter multiplets, except the gravitino multiplet (which a priori suffers from the problem of higher spin inconsistencies) give *positive* contributions in β. Hence there are no analogs of the $N = 1$ $\{0'\}_1$-multiplet that can cancel the *positive* β's of $N = 2$ and $N = 3$ CSG's (see table 6.2). The results for infinities due to $N = 2$ multiplets (see (4.61)) are given in table 6.4 (cf. table 6.3).

Table 6.4
Contributions of $N = 2$ superconformal multiplets in $N = 2$ CSG β-functions

	$\{0\}_2$ $(2E, 2\chi)$	$\{1\}_2$ $(V_\mu, 2\chi, E)$	$\{\tilde 0\}_2$ $(3\phi, 2\chi, E_{\mu\nu})$	$\{1/2\}_2$ $(2\Lambda, T_{\mu\nu}, 2\chi, 3E, \varphi)$	$\{3/2\}_2$ $(\psi_\mu, 3V_\mu, 2T_{\mu\nu}, 3\chi, \Lambda, E)$	$\{2\}_2$
β	1/6	1/3	1/6	0	−4	13/3
β_1	1/24	−1/24	13/24	1/2	−3/8	11/24

Note that

$$\begin{aligned} \{2\}_3 &= \{2\}_2 + \{\tfrac{3}{2}\}_2 + 2\{1\}_2 \,, \\ \{2\}_4 &= \{2\}_2 + 2\{\tfrac{3}{2}\}_2 + 5\{1\}_2 + \{\tfrac{1}{2}\}_2 \,. \end{aligned} \tag{6.39}$$

Here $\{0\}_2$, $\{1\}_2$ and $\{\tilde 0\}_2$ stand for the ordinary $N = 2$ scalar multiplet (3.84), $N = 2$ gauge vector multiplet (3.71) and $N = 2$ tensor multiplet (3.89). Observing that $\{\tfrac{1}{2}\}_2 = \{\tfrac{1}{2}\}_1 + \{0'\}_1 + \{0\}_1$, we conclude that the negative contribution in β due to $\{0'\}_1$ is here completely "screened" by the positive contributions of $\{\tfrac{1}{2}\}_1$ and $\{0\}_1$. Therefore it appears to be impossible to construct a finite $N = 2$ superconformal theory which contains only "lower spin" matter multiplets. The same conclusion is true also for $N = 3$. The $N = 3, 4$ multiplets and their infinities are collected in table 6.5 (cf. tables 6.2–6.4).

[1] This may be considered as a consequence of the chiral-dual invariance of this coupling.

Table 6.5
Contributions of $N=3,4$ superconformal multiplets in $N=3,4$ CSG β-functions

	$\{1\}_3=\{1\}_4$ $(V_\mu, 4\chi, 3E)$	$\{3/2\}_3$ $(\psi_\mu, 6V_\mu, 3T_{\mu\nu}, 11\chi, 3\Lambda, 7E, \varphi)$	$\{2\}_3$	$\{2\}_{4\,\text{minimal}}$ $=\{2\}_3+\{3/2\}_3$
β	1/2	-3	1	-2
β_1	0	0	0	0

Here $\{1\}_3$ is the $N=3$ vector gauge multiplet which has the same field content as the $N=4$ vector multiplet discussed in sections 3.2 and 3.5 (see eq. (3.94)). In the last column of table 6.5 we repeated the values of β and β_1 for "minimal" $N=4$ conformal supergravity. The important fact is the *negative* sign of the β-function for "minimal" $N=4$ CSG. It implies that, while $N=3$ CSG cannot be made finite by coupling it to $N=3$ vector multiplets, it is possible to *cancel* the one-loop infinities of "minimal" $N=4$ CSG by adding to it *four* $N=4$ vector multiplets ($\beta_2=-2+\frac{1}{2}\times 4=0$). The resulting theory ($N=4\,\text{CSG}+4\,N=4$ vector multiplets) is thus finite in the one-loop approximation in the $N=4$ conformal supergravity sector. All that is left is to check that it is also finite in the matter sector. We start with the observation that, even if some three of the four vector multiplets are taken to be self-interacting, i.e. the vector multiplets form in a $\mathrm{SU}(2)\times\mathrm{U}(1)$ $N=4$ super-Yang–Mills theory, the infinities due to the self-interactions identically vanish, because the $N=4$ super-Yang–Mills theory is known to be finite at the one loop level [88] (as well as at the higher loop level [144, 228, 146, 189, 20, 160]). Next, we shall prove the following statement: *interaction with $N\le 4$ conformal supergravity does not modify the (one-loop) β-function of the $N\le 4$ super-Yang–Mills theory.* We have already discussed the $N=0$ and $N=1$ cases. Let us consider now the $N=2$ case. The Lagrangian for the $N=2$ super-Yang–Mills theory, interacting with $N=2$ CSG, is given by eq. (3.71). Again it is useful to consider the background sector with the gauge vector only having a non-zero background value, and to compute infinities proportional to the Yang–Mills term (in view of the supersymmetry property this is sufficient to establish the super-Yang–Mills β-function). As in the $N=1$ case, the fermionic mixing terms give vanishing contributions to the one-loop gauge infinities. The only potentially dangerous vertices are provided by the ($F^+_{\mu\nu}(B)\,T^+_{\mu\nu}X$ + c.c.)-mixing terms (see eq. (3.72)). Integrating first over the scalar field X, and noting that the free $T_{\mu\nu}$ Lagrangian ($\sim\partial_\mu T^+_{\mu\nu}\,\partial_\rho T^-_{\rho\nu}$, cf. (1.17)) is invariant under the duality rotations, it is easy to understand that the infinite part of the one-loop diagram with one $T_{\mu\nu}$ and one X line, is proportional to $\mathrm{tr}(F^+_{\mu\nu}F^-_{\mu\nu})$ and hence vanishes. Now it is straightforward to generalize the proof to the $N=4$ case (see eq. (3.94)). The only new relevant type of couplings is connected with the dimensionless scalar φ of $N=4$ CSG, $\mathscr{L}_\varphi=(F^+_{\mu\nu}(B))^2\,(a_1\varphi+a_2\varphi^2)+\text{c.c.}$ (cf. (3.95)).[2] A simple analysis shows that these "duality-invariant" couplings give a vanishing contribution to $F^2_{\mu\nu}(B)$ infinities for all a_1 and a_2. This follows, e.g., from the general result (6.22) for the one-loop infinities corresponding to the Lagrangian (6.21) (here $\mathscr{L}_\varphi=2a_2F^2(\varphi_1^2-\varphi_2^2)+2a_1(F^2\varphi_1+\mathrm{i}FF^*\varphi_2)+\cdots$).

The final conclusion is that the "minimal" $N=4$ SU(4) conformal supergravity, coupled to $N=4$ super-Yang–Mills theory with $\mathrm{SU}(2)\times\mathrm{U}(1)$ or $(\mathrm{U}(1))^4$ as a gauge group, is a *completely finite theory* [106] (note that according to table 6.5, this theory is also free from topological infinities). The finiteness

[2] It should be noted that the structure of the $N=4$ super-Yang–Mills-$N=4$ CSG couplings that follows from the ten-dimensional theory (see section 4.1, especially eqs. (4.25), (4.26)) is different from that presented in section 3.5. This is probably due to the fact that the former corresponds to a "non-minimal" $N=4$ CSG while the latter corresponds to a "minimal" $N=4$ CSG. It is likely that the above statement is also true in the "non-minimal" case.

at the higher than one loop level follows from the non-renormalization theorem of section 6.1 (we make a highly plausible assumption that this theory can be formulated in terms of unconstrained $N=2$ superfields; see in this connection ref. [160]). A possible interpretation of this model as a "realistic" theory will be considered in the next section.

We finish this section with some remarks on the possible existence of finite theories among hypothetical $N>4$ conformal supergravities (see section 4.3). It is natural to try to construct higher N theories using lower N supermultiplets as building blocks. An attractive choice for the latter is provided by the $N=3$ multiplets $\{2\}_3$, $\{\frac{3}{2}\}_3$ and $\{1\}_3$ (see table 6.5) all of which have a zero β_1-coefficient. The condition of one-loop finiteness of a theory considered as a candidate for an N-extended CSG with the spectrum $\{2\}_3+(N-3)\{\frac{3}{2}\}_3+N_1\{1\}_3$ (N is the total number of the conformal gravitinos), is $\beta_{\text{tot}}=1-3(N-3)+\frac{1}{2}N_1=0$, i.e. $N_1=6N-20$, $N\geq 4$. The case of $N=4$ corresponds to $\{2\}_4+4\{1\}_4$, i.e. to the finite $N=4$ superconformal theory found above. For $N=5$ we get $N_1=10$ and thus the theory containing e^a_μ, $5\psi_\mu$, $31V_\mu$, $9T_{\mu\nu}$, 71χ, 7Λ, $47E$, 2φ. This spectrum does not coincide with the "natural" SU(5)-spectrum (4.56). The model corresponding to the latter spectrum is not finite: taking e.g., $N_0=4$ and $N_{1/2}=19$ in order to satisfy the condition (4.57) and also to provide $\beta_1=0$, we get $\beta_{\text{tot}}=-6$. Note that this model could be made finite if it were possible to couple it to 12 $N=4$ vector multiplets.

An analogous search for finite theories can be carried out in terms of $N=1$ or $N=2$ multiplets. For example, using the $N=2$ multiplets of table 6.4 we find that the conditions of finiteness ($\beta_{\text{tot}}=0$, $\beta_{1\,\text{tot}}=0$) of the theory $\{2\}_2+(N-2)\{\frac{3}{2}\}_2+N_1\{1\}_2+N_1\{1\}_2+N_{1/2}\{\frac{1}{2}\}_2+N_0\{0\}_2$ imply, that $N_0=14N-44-8N_{1/2}$, $N_1=5N-15+4N_{1/2}$. For $N=4$ these equations have two solutions: $N_{1/2}=0$, $N_0=12$, $N_1=5$ and $N_{1/2}=1$, $N_0=4$, $N_1=9$. The latter corresponds to the finite $N=4$ theory $\{2\}_4+4\{1\}_4$, while the former cannot be represented as a $N=4$ CSG coupled to some matter multiplets and, thus, it is probably inconsistent. For $N=5$, there are four solutions ($N_{1/2}=0$, 1, 2, 3) but only one of them can in principle correspond to a consistent theory (solution with $N_{1/2}=2$, $N_1=18$, $N_0=10$ coincides with the $N=5$ theory found above in terms of $N=3$ multiplets). The point is that the use of conformal gravitino multiplets is legitimate only if the resulting theory has an N-extended supersymmetry. We conclude that, though there are many possible candidates for a finite theory containing $N>4$ conformal gravitinos (as well as the Weyl graviton and a large number of "lower spin" fields), there remain questions about their formal consistency. This is equivalent to the presently open question of the mere existence of $N>4$ extended conformal supergravities. This problem is however absent for the finite $N=4$ superconformal theory as established above.

6.4. *Towards a "realistic" finite locally superconformal theory*

In the previous sections we have found two attractive candidates to play the role of a finite superconformal theory: 1) the "non-minimal" $N=4$ CSG and 2) the "minimal" $N=4$ CSG coupled to the $SU(2)\times U(1)$ (or $(U(1))^4$) $N=4$ super-Yang–Mills theory. To these models we can also add, in principle, hypothetical finite $N>4$ CSG's, discussed at the end of section 6.3. A finite superconformal theory is supposed to describe the fundamental constituents of matter and their interactions at sub-Planck distances, or at energies $E>M_{\text{P}}\sim 10^{19}$ GeV. To decide whether such a theory can be considered as realistic, one has to study its low-energy limit and to compare it with the accepted "standard model". Hence, an effective *low-energy* theory

(i) should incorporate Einstein gravity, i.e. an Einstein term $((1/k^2)R)$ in the Lagrangian;

(ii) should contain at least three generations of quarks and leptons, transforming under the chiral representations of $SU(3)\times SU(2)_L\times U(1)$ (the minimal number of chiral fermions needed is $3\times$

$[2(3+\bar{3})+2+1]=45)$, as well as the $SU(3)\times SU(2)_L\times U(1)$ gauge fields and probably a number of scalar Higgs fields;

(iii) should be unitary in perturbation theory, i.e. should be free of tree-level ghosts which can interact with physical particles.

The first two conditions can be generalized, requiring correspondence with some of the currently popular unification schemes based on $N=1$ Poincaré supergravity interacting with several ordinary $N=1$ scalar and vector multiplets, assigned to the representations of $SU(3)\times SU(2)_L\times U(1)$ or $SU(5)$ (for reviews, see [196, 70, 201]). The models based on $N=1$ Poincaré supergravity are considered as "phenomenological" (valid at energies smaller than the Planck energy) because of the presence of non-renormalizable on-shell infinities present already at the one-loop level (see, e.g. [254]). The final goal is thus to derive a phenomenological $N=1$ supergravity model as the low-energy ($E\lesssim M_P$) approximation of a finite $N\geq 4$ superconformal theory.

One can imagine two possible scenarios to reach this goal:

1) "*Perturbative scenario*". One attempts to directly identify the fields of the "standard model" with the fields of a superconformal theory and try to obtain the Einstein term in the action, assuming a spontaneous breaking of superconformal symmetry.

2) "*Non-perturbative scenario*". One accounts for the quantum dynamics and one proves that the fundamental fields of a finite theory are confined, while their bound states can be identified with the observable particles (graviton, quarks, leptons, etc.) and that the effective low-energy theory, describing the bound states, incorporates the long-range Einstein interaction and is unitary (cf. section 5.2).

The perturbative approach is surely naive but it has the advantage that its consequences are easier to study. A practical realization of the perturbative scenario is based on several dynamical assumptions. Thus some points revealed in the "perturbative" approach may be relevant also to the non-perturbative scenario.

We shall start with the problem of generating the Einstein term in the action. The important property of $N=4$ superconformal models is that their spectra contain the ordinary conformal scalars (the scalars E of $N=4$ CSG or the scalars ϕ of $N=4$ super-Yang–Mills). This opens the possibility to establish the correspondence with the Einstein theory, assuming that Weyl invariance is spontaneously broken by the non-zero vacuum expectation value of one (or several) scalar(s) Φ. To get the correct sign of the Einstein term we have to take a conformal scalar as a *ghost*

$$e^{-1}\mathcal{L}=\tfrac{1}{12}R\Phi^2+\tfrac{1}{2}(\partial_\mu\Phi)^2+\cdots. \tag{6.40}$$

Assuming that Φ has no potential (as it is true for the scalars of $N=4$ CSG and the scalars of U(1) $N=4$ super-Yang–Mills) we conclude that the theory is parametrized by arbitrary vacuum values of the scalars. Supposing that in our world $\langle\Phi\rangle=a$ is of order of the Planck mass, $a^2/12=1/k^2$, we find that (6.40) contains the Einstein term if Φ is expanded near its vacuum value, $\Phi=a+\Phi_q$,

$$e^{-1}\mathcal{L}=\frac{1}{k^2}R+\tfrac{1}{2}(\partial_\mu\Phi_q)^2+\cdots. \tag{6.41}$$

The reexpanded theory (6.41) has a non-linearly realized (super) conformal symmetry (see also [169]). Using this symmetry, we can completely gauge away Φ_q. Equivalently, we could absorb Φ in the metric, making a Weyl transformation: $\Phi'=a$, $g'_{\mu\nu}=a^{-1}\Phi g_{\mu\nu},\dots$. Again a is an a priori arbitrary constant mass, which parametrizes the theory. It is important to stress that the equivalence of the full theories, before and after the Weyl transformation, depends on the absence of the conformal anomaly, i.e. on the *finiteness* of the theory. Hence a finite superconformal theory, written in the "unitary" gauge, where $\Phi = a = \text{const.}$, always contains the Einstein term in the action.[1] This term will dominate over any higher derivative one in the low-energy approximation. The arbitrariness in scale a is the important property of a finite superconformal theory. It seems attractive to assume that the choice of a particular scale is made by the vacuum state, i.e. that the breakdown of the (super)conformal symmetry (necessary to make contact with low-energy physics) is done *spontaneously*.

Having found the Einstein term in the action, it is natural to ask whether or not the "low-energy theory" will coincide with some Poincaré supergravity theory. As discussed in section 3.5, the theory which has a consistent truncation to the $N = 4$ Poincaré supergravity, corresponds to six O(4) $N = 4$ vector multiplets coupled to $N = 4$ CSG. As it follows from table 6.5, this particular theory is not finite at the one loop level. It seems impossible to derive $N = 4$ PSG starting from a theory containing less than six $N = 4$ vector multiplets. For example, the finite $N = 4$ theory with four $N = 4$ vector multiplets can in principle have only $N \leq 3$ low-energy supersymmetry.[2] Suppose that $N = 4$ supersymmetry is somehow broken to $N = 1$ supersymmetry. Then $N = 4$ CSG can be represented as a $N = 1$ CSG interacting with several $N = 1$ "matter" multiplets (see (6.38)). An analogous $N = 1$ splitting is possible for the $N = 4$ vector multiplets: $\{1\}_4 = \{1\}_1 + 3\{0\}_1$. Assuming that one of the scalar multiplets contribute in the action with the ghost sign and that it is possible to go to the Poincaré gauge (2.52) (i.e. that either the scalar E or ϕ has a non-zero vacuum expectation value) we find that the theory can be consistently truncated to a low-energy $N = 1$ supersymmetric model. The latter contains $N = 1$ Poincaré supergravity in its minimal auxiliary field formulation (see (2.51) and the discussion in section 2.3) interacting with several scalar and vector multiplets (higher derivative multiplets can be probably ignored at low energies). Here it is useful to note that the unified models based on "phenomenological" $N = 1$ Poincaré supergravity can also be described (in the framework of the $N = 1$ superconformal tensor calculus), for instance, as a system of a ghost scalar multiplet, N_0 physical scalar multiplets and N_1 physical vector multiplets interacting with the external fields of $N = 1$ CSG (as well as with each other). Such a conformal description (and thus the elimination of the ghost multiplet) is possible at the classical level only. To make these models renormalizable at the one-loop level it is sufficient to add in the action the $N = 1$ CSG term. However to avoid the R^2-type infinities (cf. (6.22)) at higher orders[3] it is necessary to cancel the one-loop infinities. Though we have seen in section 4.3 that it is possible to get rid of the infinities proportional to the $N = 1$ CSG action by adding a sufficient number of higher derivative multiplets $\{0'\}_1$ (see table 6.3) it seems unlikely that one can cancel all the infinities in the matter sector. However, it is possible to construct a finite $N = 1$ theory by taking the particular values of N_0 and N_1 and adding also the higher derivative $\{\frac{3}{2}\}_1$ and $\{\frac{1}{2}\}_2$ multiplets, so that the resulting theory coincides with the $N = 1$ truncation of a finite $N = 4$ (or $N > 4$) theory. The numbers of physical scalar and vector multiplets ($N_0 \leq 8$, $N_1 \leq 9$) that correspond to the two finite $N = 4$ superconformal theories are apparently too small for a phenomenologically acceptable $N = 1$ Poincaré supergravity model (see also below). There remains the interesting possibility that a more realistic $N = 1$ spectrum may result from a finite, $N > 4$, CSG.

[1] The absence of conformal anomalies and thus the possibility to break Weyl symmetry spontaneously, are the important advantages of a finite superconformal theory over previous attempts to generate the Newton's gravitational constant from the vacuum expectation value of a scalar field conformally coupled to gravity (see e.g. refs. [71, 268] and references therein).

[2] Though the full theory is perfectly $N = 4$ supersymmetric, it is impossible to carry out its low-energy truncation in a way preserving $N = 4$ supersymmetry.

[3] The explicit breakdown of superconformal invariance at the quantum level implies that one cannot gauge away the ghost scalar multiplet and thus presents serious problems with unitarity.

Let us now turn to the problem of the identification of fields within the "perturbative" approach. The important phenomenological advantage of extended U(N) conformal supergravity over O(N) Poincaré (or De Sitter) supergravity is that it contains fermions transforming under the *chiral* representations of U(N). This follows directly from the superconformal algebra (3.1). For example, in $N = 4$ CSG, we have a 20 of ordinary spinors $\chi^{ij}{}_k$ and a $\bar{4}$ of higher derivative spinors Λ_i (table 3.1).[4] To these we can add also the fermions of the SU(2)$\times$U(1) $N = 4$ super-Yang–Mills part of the second finite $N = 4$ model ($(1+3)_{SU(2)} \times 4_{SU(4)} = 16$ fermions of definite chirality). Thus the maximal number of available fermions is $20 + 4 + 16 = 40$ (we assume that Λ is also present in the low-energy theory, in analogy with the case of $N = 4$ PSG, see (3.42)). Even leaving aside serious problems which arise in any attempt to identify particular representations with quarks and leptons, we see that the total number of fermions is too small (<45) to allow one to make a correspondence with the standard model. Though the total number of gauge vectors (SU(4) plus SU(2)$\times$U(1)) is formally sufficient, the SU(4) couplings are manifestly chiral and, thus, there seems to be no place for a non-chiral colour SU(3).

Another problem is connected with our obtaining the correct physical signs for the "observable" fields present in the action. A possible choice for the signs is the following:

(i) the $N = 4$ CSG (Minkowski) action is taken with the same sign as in (3.30) and (3.53),

$$e^{-1}\mathcal{L} = \frac{1}{2\alpha^2}(C^2 - (F'_{\mu\nu})^2 + \cdots),$$

so that all non-higher derivative terms in it (corresponding to V_μ, χ and E) have the proper physical signs;

(ii) one U(1) $N = 4$ vector multiplet is taken with the ghost sign, in order to obtain the physical sign for the Einstein term (and the gravitino, Λ and φ terms) in the low-energy action;

(iii) the SU(2) $N = 4$ super-Yang–Mills action is taken with the physical sign. Then, gauging away one of the ghost scalars, we find the following gravitational terms in the Lagrangian

$$e^{-1}\mathcal{L} = \frac{a^2}{12}R + \frac{1}{2\alpha^2}C^2_{\lambda\mu\nu\rho}. \tag{6.42}$$

Comparing this with (3.40) and (3.41), we conclude that the linearized spectrum of this theory contains a spin 2 tachyon instead of a spin 2 ghost. The presence of the Weyl term with the "wrong" sign, as in (6.42), is usually considered as undesirable (see e.g. ref. [232]). Though the arguments (like the oscillations of the static potential, etc.) may not be very convincing in view of the fact that the value of α (which is a fixed "non-running" constant in a finite theory) is completely arbitrary, it would be more satisfying to change the sign of the Weyl term in (6.42). However, this is impossible without making at the same time the SU(4) gauge vectors (as well as the spinors χ and scalars E of the $N = 4$ CSG) ghost particles.

The above problems present in the "perturbative" approach may be possibly avoided in the "non-perturbative" approach. A spectrum of bound states may be richer than the spectrum of the fundamental fields and may be sufficient to obtain the proper correspondence with a phenomenological $N = 1$ PSG model. The sign of the Weyl term in the low-energy effective action may not be correlated with the sign of the gauge vector term (the Einstein term describing a bound state graviton will appear

[4] The SU(4)$\to$SU(3)$\times$U(1) branching rules are [222]: $4 = 1(1) + 3(-1/3)$, $20 = 3(-1/3) + \bar{3}(-5/3) + \bar{6}(-1/3) + 8(1)$.

in the *effective* action and not as an addition to the fundamental CSG action). Ghost states may be absent in a low-energy effective action so that the theory may be unitary. To prove these conjectures one has to develop a systematic understanding of the dynamics of locally superconformal theories (probably, by devising and studying their lattice formulations, cf. refs. [167, 168, 246] and section 5.2). The crucial property that can help to provide an exact solution of the corresponding continuum quantum theories is their finiteness and hence their exact superconformal symmetry at the quantum level. Useful insights can be gained by solving first a "prototype conformal supergravity", the finite $N = 4$ super-Yang–Mills theory. Possessing rigid superconformal symmetry, both the $N = 4$ super-Yang–Mills theory, and the finite $N = 4$ locally superconformal theories, quantized in supersymmetrical gauges, have 2-point and 3-point exact Green's functions which coincide with their tree level values (this follows from the non-renormalization theorem, see ref. [146] and section 6.1). This fact may have important implications in the attempt to solve these theories using the methods of conformal quantum field theory (see e.g. refs. [96, 97, 242]) properly generalized to a supersymmetrical case (cf. ref. [82, 228]). As for the 4-point amplitudes, they should give information about the bound states spectrum. Though the off-shell amplitudes are completely finite, the S-matrix elements are infrared divergent in perturbation theory [141, 147]. Thus the masses of the bound states will in general be proportional to an infrared cut-off, which it is natural to take to be of the order of the Planck mass (in addition, one hopes of course to find a sufficient number of *massless* bound states that can be identified with the observable particles). For a sufficiently small value of the coupling constant α, the theory should be governed by perturbation theory. Though even an unnaturally small value of α (e.g. $\alpha \sim 10^{-20}$) does not contradict observations in a theory like (6.42) [232], a sufficiently large value ($\alpha \sim 1$) seems necessary in order to have confinement of all fundamental fields (and hence of ghosts) and, hence, to provide a realization of the "non-perturbative" scenario. The strong coupling phase of the theory can probably be studied with the help of a lattice formulation. In contrast with QCD, there is here no point in checking that confinement persists in the weak coupling phase, because the coupling constant is the same for all scales in the finite theory. The methods of the strong coupling expansion in finite supersymmetrical gauge theories remain to be developed.

To finish with an optimistic note, we would like to stress that, in spite of all the problems connected with the low-energy interpretation of conformal supergravities, they remain attractive candidates for a fundamental unified quantum field theory, providing the first examples of power counting renormalizable finite locally supersymmetric theories, incorporating quantized gravity, and also invariant under the highest possible (superconformal) symmetry at the quantum level. Even if they have nothing to do with the description of the real world, a deeper understanding of their properties would undoubtedly be important for future progress in supersymmetric quantum field theory and quantum (super) gravity.

Appendix A. Notation and conventions

We use a flat metric with Euclidean signature

$$\delta_{ab} = \mathrm{diag}(++++) \tag{A.1}$$

assuming that, in Minkowski space theory, the time components of vectors and coordinates are imaginary. Thus the physical signs of a scalar and a gauge vector terms in the Minkowski space theory Lagrangian are

$$\mathscr{L}_{(M)} = -\tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{4}(F_{\mu\nu})^2 + \cdots . \tag{A.2}$$

Continuation to the Euclidean theory is carried out by "rotating" the time components to make them real and changing the over-all sign of the Lagrangian, $\mathscr{L}_{(E)} = -\mathscr{L}_{(M)}$.

The totally antisymmetric symbol $\varepsilon^{\mu\nu\lambda\rho}$ satisfies the relations

$$\varepsilon_{1234} = \varepsilon^{1234} = +1, \qquad \varepsilon_{\mu\nu\lambda\rho}\varepsilon^{\mu\nu\lambda\rho} = 4! \tag{A.3}$$

$a, b, \ldots$ are the local Lorentz indices; Greek indices $\mu, \nu, \ldots$ are used for coordinate indices. The dual tensors are defined as follows

$$F^{*}_{\mu\nu} = \tfrac{1}{2} e \varepsilon_{\mu\nu\lambda\rho} F^{\lambda\rho} \tag{A.4}$$

where $e = \det e^a_\mu$ and e^a_μ is the vierbein matrix. We shall also use a star to denote present complex conjugation. To avoid possible confusion a star present on *antisymmetric tensors* will be understood as a duality transformation only. For example,

$$(F^{\pm}_{\mu\nu})^* = F^{\pm}_{\mu\nu}, \qquad F^{\pm}_{\mu\nu} \equiv \tfrac{1}{2}(F_{\mu\nu} \pm F^{*}_{\mu\nu}) \tag{A.5}$$

while the complex conjugation of $F^{\pm}_{\mu\nu}$ is $F^{\mp}_{\mu\nu}$.

Our convention for the (anti)symmetrization is "weighted"

$$(\mu\nu) = \tfrac{1}{2}(\mu\nu + \nu\mu), \qquad [\mu\nu] = \tfrac{1}{2}(\mu\nu - \nu\mu) . \tag{A.6}$$

The Dirac γ-matrix algebra is generated by

$$\gamma_{(a}\gamma_{b)} = \delta_{ab} . \tag{A.7}$$

We choose γ_a to be Hermitean so that

$$\gamma^\dagger_a \equiv (\gamma^{\mathrm{T}}_a)^* = \gamma_a, \qquad \gamma_5 \equiv \gamma_1\gamma_2\gamma_3\gamma_4 = \gamma^\dagger_5, \qquad \sigma_{ab} \equiv \tfrac{1}{2}\gamma_{[a}\gamma_{b]} = -\sigma^\dagger_{ab} . \tag{A.8}$$

The charge conjugation matrix C is defined by (see e.g. [254])

$$C\gamma_a C^{-1} = -\gamma^{\mathrm{T}}_a, \qquad C^{\mathrm{T}} = -C, \qquad C^{-1} = -C . \tag{A.9}$$

The Majorana conjugation of a Dirac spinor ψ is

$$\hat{\psi} \equiv \psi^{\mathrm{T}} C, \qquad \hat{\gamma}_a = C^{-1}\gamma_a C, \qquad \hat{\gamma}_5 = \gamma_5 . \tag{A.10}$$

For Majorana spinors it coincides with the Dirac conjugation

$$\hat{\psi} = \bar{\psi}, \qquad \bar{\psi} \equiv \psi^\dagger \gamma_4 . \tag{A.11}$$

Dealing with spinors which transform under the complex representations of SU(N), we use chiral notation [51, 47, 7]. We define the complex conjugation in $\mathbb{C}^N$ as $(V^i)^* = V_i$, $i = 1, \ldots, N$. If the V^i transforms according to the N-representation of SU(N), $V^{i\prime} = U^i{}_j V^j$, $UU^\dagger = I$, then V_i transforms according to the $\bar{N}$-representation ($U_i^j \equiv U^i{}_j{}^*$). Let $\psi^i_{(M)}$ be a set of N Majorana spinors, such that $\Pi_+\psi^i_{(M)}$,

$$\Pi_\pm \equiv (1 \pm \gamma_5)/2\,, \tag{A.12}$$

transforms under the N-representation of SU(N). Then we put

$$\psi^i \equiv \Pi_+\psi^i_{(M)}, \qquad \psi_i \equiv \Pi_-\psi^i_{(M)}, \qquad \hat{\psi}_i = \bar{\psi}^i \tag{A.13}$$

(in the Majorana representation of the γ-matrices $\psi_i = \psi^{i*}$). Then

$$\bar{\psi}^i = \bar{\psi}^i_{(M)}\Pi_+, \qquad \bar{\psi}_i = \bar{\psi}^i_{(M)}\Pi_-, \qquad \overline{\psi^i} = \bar{\psi}_i\,. \tag{A.14}$$

Analogously, if $\Pi_+\psi^i_{(M)}$ transforms under the $\bar{N}$-representation,

$$\begin{aligned} &\psi_i \equiv \Pi_+\psi^i_{(M)}, \qquad \psi^i \equiv \Pi_-\psi^i_{(M)}\,,\\ &\bar{\psi}^i = \bar{\psi}^i_{(M)}\Pi_-, \qquad \bar{\psi}_i = \bar{\psi}^i_{(M)}\Pi_+\,. \end{aligned} \tag{A.15}$$

The index M is always dropped in the text, where we explicitly indicate the chirality of spinors (i.e. $\psi^i = \gamma_5\psi^i$ or $\psi^i = -\gamma_5\psi^i$).

Our curvature conventions are

$$R^\lambda{}_{\mu\nu\rho} = \partial_\nu \Gamma^\lambda_{\mu\rho} - \cdots, \qquad R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}, \qquad R = R^\mu_\mu\,. \tag{A.16}$$

Hence, the Einstein term in the Lagrangian of the Minkowski variant of the theory is

$$\mathscr{L}_E = \frac{1}{k^2} R\sqrt{g}, \qquad k^2 = 16\pi G\,. \tag{A.17}$$

We often use the following quantities

$$R^*R^* \equiv \tfrac{1}{4}\varepsilon_{\mu\nu\lambda\rho}\varepsilon^{\alpha\beta\gamma\delta} R^{\mu\nu}{}_{\alpha\beta} R^{\lambda\rho}{}_{\gamma\delta}\,, \tag{A.18}$$

$$W \equiv R^2_{\mu\nu} - \tfrac{1}{3}R^2\,. \tag{A.19}$$

Defining the traceless Weyl tensor $C^\lambda{}_{\mu\nu\rho}$

$$C^{\lambda\mu}{}_{\nu\rho} = R^{\lambda\mu}{}_{\nu\rho} - 2\delta^{[\lambda}_{[\nu} R^{\mu]}_{\rho]} + \tfrac{1}{3}\delta^{\lambda}_{[\nu}\delta^{\mu}_{\rho]} R \tag{A.20}$$

and using the algebraic identity

$$R^*R^* = R^2_{\lambda\mu\nu\rho} - 4R^2_{\mu\nu} + R^2 \tag{A.21}$$

it is easy to prove that

$$W=\tfrac{1}{2}C^2-\tfrac{1}{2}R^*R^*,\qquad C^2=C_{\lambda\mu\nu\rho}C^{\lambda\mu\nu\rho}. \tag{A.22}$$

Thus W differs from the square of the Weyl tensor only in the total (covariant) derivative term R^*R^*.

Appendix B. Weyl transformation

The Weyl, or conformal, transformation of the metric is given by

$$g'_{\mu\nu}=\mathrm{e}^{-2\lambda(x)}g_{\mu\nu}. \tag{B.1}$$

The transformations of the Christoffel connection and of the curvature tensor induced by (B.1) are ($\Gamma^{\lambda}_{\mu\nu}\equiv\{{}^{\lambda}_{\mu\nu}\}$)

$$\begin{aligned}\Gamma'^{\lambda}_{\mu\nu}&=\Gamma^{\lambda}_{\mu\nu}-\delta^{\lambda}_{\mu}\xi_\nu-\delta^{\lambda}_{\nu}\xi_\mu+g_{\mu\nu}\xi^\lambda,\\ R'^{\lambda}{}_{\mu\nu\rho}&=R^{\lambda}{}_{\mu\nu\rho}+2\delta^{\lambda}_{[\nu}\sigma_{\rho]\mu}+2\sigma^{\lambda}_{[\nu}g_{\rho]\mu}-2\xi_\sigma\xi^\sigma\delta^{\lambda}_{[\nu}g_{\rho]\mu},\end{aligned} \tag{B.2}$$

$$\xi_\mu\equiv\partial_\mu\lambda,\qquad\sigma_{\mu\nu}\equiv\mathscr{D}_\mu\xi_\nu+\xi_\mu\xi_\nu,\qquad\mathscr{D}_\mu=\partial_\mu+\{{}^{\lambda}_{\mu\rho}\}. \tag{B.3}$$

Hence

$$R'_{\mu\nu}=R_{\mu\nu}+(d-2)\sigma_{\mu\nu}+[\mathscr{D}_\rho\xi_\rho-(d-2)\xi_\rho\xi^\rho]g_{\mu\nu}, \tag{B.4}$$

$$R'=\mathrm{e}^{2\lambda}[R+2(d-1)\mathscr{D}_\mu\mathscr{D}^\mu\lambda-(d-1)(d-2)\mathscr{D}_\mu\lambda\mathscr{D}^\mu\lambda], \tag{B.5}$$

where d is a space–time dimension.

The conformally invariant part of the curvature is the Weyl tensor

$$C^{\lambda}{}_{\mu\nu\rho}=R^{\lambda}{}_{\mu\nu\rho}-\frac{2}{(d-2)}(\delta^{\lambda}_{[\nu}R_{\rho]\mu}+R^{\lambda}_{[\nu}g_{\rho]\mu})-\frac{2}{(d-1)(d-2)}\delta^{\lambda}_{[\nu}g_{\rho]\mu}R. \tag{B.6}$$

In four dimensions

$$(C^2\sqrt{g})'=C^2\sqrt{g}. \tag{B.7}$$

In view of (A.22) and (B.7) the integral $\int W\sqrt{g}\,\mathrm{d}^4x$ is invariant under regular Weyl transformations (which do not change the Euler number). $W\sqrt{g}$ itself changes by a total derivative term

$$\begin{aligned}(W\sqrt{g})'&=W\sqrt{g}+\partial_\mu(\sqrt{g}\,J_\mu),\\ J_\mu&=4\xi_\nu\mathscr{D}_\mu\xi_\nu-\xi_\mu\mathscr{D}_\rho\xi_\rho+(R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R)\xi_\nu+\xi_\nu\xi^\nu\xi_\mu.\end{aligned} \tag{B.8}$$

References

[1] S.L. Adler, Einstein gravity as a symmetry breaking effect in quantum field theory, Rev. Mod. Phys. 54 (1982) 729–766.

[2] J.L. Anderson, Scale invariance of the second kind and the Brans–Dicke scalar-tensor theory, Phys. Rev. D3 (1970) 1689–1691.

[3] N.H. Barth and S.M. Christensen, Quantizing fourth order gravity theories: the functional integral, Phys. Rev. D28 (1983) 1876–1893.

[4] I.A. Batalin and G.A. Vilkovisky, Relativistic S-matrix of dynamical fields with boson and fermion constraints, Phys. Lett. 69B (1977) 309–312.

[5] F. Bayen, Conformal invariance in physics, in: Differential geometry and relativity, eds. M. Cohen and C. Flato (Reidel, Dordrecht, 1976), pp. 171–195.

[6] F.A. Berends, J.W. van Holten, P. van Nieuwenhuizen and B. de Wit, On field theory for massive and massless spin 5/2 particles, Nucl. Phys. B154 (1979) 261–282.

[7] E. Bergshoeff, Conformal invariance in supergravity, Ph.D. thesis, Leiden University, 1983, 169 pp.

[8] E. Bergshoeff, M. de Roo and B. de Wit, Extended conformal supergravity, Nucl. Phys. B182 (1981) 173–204.

[9] E. Bergshoeff, M. de Roo and B. de Wit, Conformal supergravity in ten dimensions, Nucl. Phys. B217 (1983) 489–530.

[10] E. Bergshoeff, M. de Roo, J.W. van Holten, B. de Wit and A. Van Proeyen, Extended conformal supergravity and its applications, in: Proc. Nuffield Supergravity Workshop, eds. S.W. Hawking and M. Roček (Cambridge Univ. Press, 1981) pp. 237–256.

[11] E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuizen, Ten-dimensional Maxwell–Einstein supergravity, its currents, and the issue of its auxiliary fields, Nucl. Phys. B195 (1982) 97–136.

[12] J. Birell and P.C.W. Davies, Quantum fields in curved space–time (Cambridge Univ. Press, Cambridge, 1982).

[13] D.G. Boulware, Quantization of higher derivative theories of gravity, in: Quantum theory of gravity, ed. S.M. Christensen (Adam Hilger, Bristol, 1984).

[14] D.G. Boulware and S. Deser, Canonical analysis of the fermion sector in higher derivative supergravity, Phys. Rev. D30 (1984) 707–711.

[15] D.G. Boulware and D.J. Gross, Lee–Wick indefinite metric quantization: a functional integral approach, Nucl. Phys. B233 (1984) 1–23.

[16] D.G. Boulware, G.T. Horowitz and S. Strominger, Zero energy theorem for scale invariant gravity, Phys. Rev. Lett. 50 (1983) 1726–1729.

[17] P. Breitenlohner, D. Maison and K.S. Stelle, Anomalous dimensions and the Adler–Bardeen theorem in supersymmetric Yang–Mills theories, Phys. Lett. 134B (1984) 63–66.

[18] P. Breitenlohner and M.F. Sohnius, Superfields, auxiliary fields and tensor calculus for $N = 2$ extended supergravity, Nucl. Phys. B165 (1980) 483–510.

[19] P. Breitenlohner and M.F. Sohnius, An almost simple off-shell version of SU(2) Poincaré supergravity, Nucl. Phys. B178 (1981) 151–176.

[20] L. Brink, O. Lindgren and B.E.W. Nilsson, The ultraviolet finiteness of $N = 4$ Yang–Mills theory, Phys. Lett. 123B (1983) 323–326.

[21] L. Brink, J. Scherk and J.H. Schwarz, Supersymmetric Yang–Mills theories, Nucl. Phys. B121 (1977) 77–92.

[22] L.S. Brown and J.C. Collins, Dimensional renormalization of scalar field theory, Ann. Phys. (N.Y.) 130 (1980) 215–248.

[23] D.M. Capper and M.J. Duff, Conformal anomalies and the renormalization problem in quantum gravity, Phys. Lett. 53A (1975) 361–362.

[24] L. Castellani, P. Fre and P. van Nieuwenhuizen, A review of the group manifold approach and its application to conformal supergravity, Ann. Phys. (N.Y.) 136 (1981) 398–434.

[25] L.N. Chang and F. Mansouri, Non-linear gauge fields and structure of gravity and supergravity theories, Phys. Rev. D17 (1978) 3168–3178.

[26] A.H. Chamseddine, $N = 4$ supergravity coupled to $N = 4$ matter and hidden symmetries, Nucl. Phys. B185 (1981) 403–415.

[27] A.H. Chamseddine, Interacting supergravity in ten dimensions: the role of the six-index gauge field, Phys. Rev. D24 (1981) 3065–3072.

[28] A.H. Chamseddine and P.C. West, Supergravity as a gauge theory of supersymmetry, Nucl. Phys. B129 (1977) 39–44.

[29] J. Crispim-Romao, Conformal and superconformal gravity and non-linear realizations, Nucl. Phys. B145 (1978) 535–546.

[30] J. Crispim-Romao, A. Ferber and P.G.O. Freund, Unified superconformal gauge theories, Nucl. Phys. B126 (1977) 429–435.

[31] S.M. Christensen, Quantizing fourth order gravity theories, in: Quantum structure of space and time, eds. M.J. Duff and C.J. Isham (Cambridge Univ. Press, 1982).

[32] S.M. Christensen and M.J. Duff, New gravitational index theorems and supertheorems, Nucl. Phys. B154 (1979) 301–342.

[33] S. Coleman, Acausality, in: Subnuclear Phenomena, ed. A. Zichichi (Academic, N.Y., 1970) pp. 282–327.

[34] E. Cremmer and B. Julia, SO(8) supergravity, Nucl. Phys. B159 (1979) 141–212.

[35] E. Cremmer and J. Scherk, Algebraic simplifications of supergravity, Nucl. Phys. B127 (1977) 259–268.

[36] E. Cremmer, J. Scherk and S. Ferrara, SU(4)-invariant supergravity theory, Phys. Lett. 74B (1978) 61–64.

[37] R. Critchley, Trace anomaly for gravitons, Phys. Rev. D18 (1978) 1849–1855.

[38] T.L. Curtright, Massless field supermultiplets with arbitrary spin, Phys. Lett. 85B (1979) 219–224.

[39] T.L. Curtright and D.Z. Freedman, Non-linear sigma-model with extended supersymmetry in four dimensions, Phys. Lett. 90B (1980) 71–75 (Err: 91B (1980) 487).

[40] A. D'Adda and P. di Vecchia, Supersymmetry and instantons, Phys. Lett. 73B (1978) 162–166.

[41] A. Das, SO(4) invariant extended supergravity, Phys. Rev. D15 (1977) 2805–2809.

[42] A. Das, M. Fischler and M. Roček, Super Higgs effect in a class of scalar models and a model of super QED, Phys. Rev. D16 (1977) 3427–3436.

[43] A. Das, M. Kaku and P.K. Townsend, A unified approach to matter coupling in Weyl and Einstein supergravity, Phys. Rev. Lett. 40 (1978) 1215–1218.

[44] S. Deser, Scale invariance and gravitational coupling, Ann. Phys. (N.Y.) 59 (1970) 248–253.
[45] S. Deser and R.I. Nepomechie, Anomalous propagation of gauge fields in conformally flat spaces, Phys. Lett. 132B (1983) 321–324.
[46] S. Deser and R.I. Nepomechie, Electric–magnetic duality of conformal gravitation, Phys. Lett. 97A (1983) 329–332.
[47] M. de Roo, J.W. van Holten, B. de Wit and A. Van Proeyen, Chiral superfields in $N = 2$ supergravity, Nucl. Phys. B173 (1980) 175–188.
[48] B. de Wit, Conformal invariance in gravity and supergravity, in: Proc. 18th Winter school in Karpacz (Gordon and Breach, 1982).
[49] B. de Wit, Conformal invariance in extended supergravity, in: Supergravity '81, eds. S. Ferrara and J.G. Taylor (Cambridge Univ. Press, 1982) pp. 267–314.
[50] B. de Wit, Multiplet calculus, in: Supersymmetry and supergravity '82, eds. S. Ferrara, J.G. Taylor and P. van Nieuwenhuizen (World Scientific, 1983) pp. 85–124.
[51] B. de Wit, Properties of SO(8) extended supergravity, Nucl. Phys. B158 (1979) 189–212.
[52] B. de Wit and S. Ferrara, On higher-order invariants in extended supersymmetry, Phys. Lett. 81B (1979) 317–320.
[53] B. de Wit and D.Z. Freedman, Systematics of higher-spin gauge fields, Phys. Rev. D21 (1980) 358–367.
[54] B. de Wit and M.T. Grisaru, On-shell counter-terms and nonlinear invariances, Phys. Rev. D20 (1979) 2082–2085.
[55] B. de Wit, P.G. Lauwers, R. Philippe, S.Q. Su and A. Van Proeyen, Gauge and matter fields coupled to $N = 2$ supergravity, Phys. Lett. 134B (1984) 37–43.
[56] B. de Wit, P.G. Lauwers, R. Phillippe and A. Van Proeyen, Noncompact $N = 2$ supergravity, Phys. Lett. 135B (1984) 295–301.
[57] B. de Wit, R. Philippe and A. Van Proeyen, The improved tensor multiplet in $N = 2$ supergravity, Nucl. Phys. B219 (1983) 143–166.
[58] B. de Wit and M. Roček, Improved tensor multiplet, Phys. Lett. 109B (1982) 439–442.
[59] B. de Wit and J.W. van Holten, Multiplets of linearized SO(2) supergravity, Nucl. Phys. B155 (1979) 530–542.
[60] B. de Wit, J.W. van Holten and A. Van Proeyen, Transformation rules of $N = 2$ supergravity multiplets, Nucl. Phys. B167 (1980) 186–204 (Err: B172 (1980) 543–544).
[61] B. de Wit, J.W. van Holten and A. Van Proeyen, Central charges and conformal supergravity, Phys. Lett. 95B (1980) 51–55.
[62] B. de Wit, J.W. van Holten and A. Van Proeyen, Structure of $N = 2$ supergravity, Nucl. Phys. B184 (1981) 77–108 (Err: B222 (1983) 516).
[63] B.S. De Witt, Dynamical Theory of Groups and Fields (Gordon and Breach, N.Y., 1965).
[64] B.S. De Witt, A gauge invariant effective action, in: Quantum gravity II, eds. C. Isham, R. Penrose and D. Sciama (Cambridge Univ. Press, 1982).
[65] P.H.Dondi and M. Sohnius, Supersymmetry transformations with isospin, Nucl. Phys. B81 (1974) 317–329.
[66] N. Dragon, Torsion and curvature in extended supergravity, Z. Phys. C2 (1979) 29–32.
[67] M.J. Duff, Observations on conformal anomalies, Nucl. Phys. B125 (1977) 334–348.
[68] M.J. Duff, Ultraviolet divergences in extended supergravity in: Supergravity '81, eds. S. Ferrara and J.G. Taylor (Cambridge Univ. Press, 1982).
[69] M.J. Duff and P. van Nieuwenhuizen, Quantum inequivalence of different field representations, Phys. Lett. 94B (1980) 179–182.
[70] J. Ellis, Supersymmetry and Grand Unified Theories, CERN prepr. TH3718 (1983).
[71] F. Englert, C. Truffin and R. Gastmans, Conformal invariance in quantum gravity, Nucl. Phys. B117 (1976) 407–432.
[72] J. Fang and C. Fronsdal, Massless fields with half integer spin, Phys. Rev. D18 (1978) 3630–3633.
[73] P. Fayet, Fermi–Bose hypersymmetry, Nucl. Phys. B113 (1976) 135–155.
[74] P. Fayet, Spontaneous generation of massive multiplet and central charge in extended supersymmetrical theory, Nucl. Phys. B149 (1979) 137–169.
[75] P. Fayet and S. Ferrara, Supersymmetry, Phys. Reports 32 (1977) 249–334.
[76] A. Ferber, Supertwistors and conformal supersymmetry, Nucl. Phys. B132 (1978) 55–64.
[77] A. Ferber and P.G.O. Freund, Superconformal supergravity and internal symmetry, Nucl. Phys. B122 (1977) 170–176.
[78] S. Ferrara, Supergauge transformations on the six-dimensional hypercone, Nucl. Phys. B77 (1974) 73–90.
[79] S. Ferrara, R. Gatto and A.F. Grillo, Conformal algebra in space–time, Springer Tracts in Modern Physics, vol. 67 (1973).
[80] S. Ferrara, L. Girardello. T. Kugo and A. Van Proeyen, Relation between different auxiliary field formulations of $N = 1$ supergravity coupled to matter, Nucl. Phys. B223 (1983) 191–217.
[81] S. Ferrara, M.T. Grisaru and P. van Nieuwenhuizen, Poincaré and conformal supergravity models with closed algebras, Nucl. Phys. B138 (1978) 430–444.
[82] S. Ferrara, J. Illiopoulos and B. Zumino, Supergauge invariance and the Gell-Mann–Low eigenvalue, Nucl. Phys. B77 (1974) 413–419.
[83] S. Ferrara, M. Kaku, P. Townsend and P. van Nieuwenhuizen, Unified field theories with U(N) internal symmetries: gauging the superconformal group, Nucl. Phys. B129 (1977) 125–134.
[84] S. Ferrara, J. Scherk and B. Zumino, Algebraic properties of extended supergravity theories, Nucl. Phys. B121 (1977) 393–402.
[85] S. Ferrara and P. van Nieuwenhuizen, The auxiliary fields of supergravity, Phys. Lett. 74B (1978) 333–335.
[86] S. Ferrara and P. van Nieuwenhuizen, Tensor calculus for supergravity, Phys. Lett. 76B (1978) 404–408.
[87] S. Ferrara and P. van Nieuwenhuizen, Structure of supergravity, Phys. Lett. 78B (1978) 573–576.
[88] S. Ferrara and B. Zumino, Supergauge invariant Yang–Mills theories, Nucl. Phys. B79 (1974) 413–421.
[89] S. Ferrara and B. Zumino, Transformation properties of the supercurrent, Nucl. Phys. B87 (1975) 207–214.
[90] S. Ferrara and B. Zumino, Structure of linearized supergravity and conformal supergravity, Nucl. Phys. B134 (1978) 301–326.
[91] S. Ferrara, B. Zumino and C.A. Savoy, General massive multiplets in extended supersymmetry, Phys. Lett. 100B (1981) 393–398.
[92] A.W. Fischer, Supercurrents in extended supersymmetry, Nucl. Phys. B241 (1984) 243–252.

[93] E.S. Fradkin, S-matrix for non-linear Lagrangians of a general type, in: Problems of Theoretical Physics, I.E. Tamm memorial volume (Moscow, Nauka, 1972) pp. 146–176 (in Russian).
[94] E.S. Fradkin, S-matrix for the general interaction, Proc. Xth Wratisl. school of Theor. Phys. in Karpacz, Acta Univ. Wratislav. N207 (1973) 111–115.
[95] E.S. Fradkin and T.E. Fradkina, Quantization of relativistic systems with Boson–Fermion first and second class constraints, Phys. Lett. 72B (1978) 343–347.
[96] E.S. Fradkin and M.Ya. Palchik, Recent developments in conformal quantum field theory, Phys. Reports 44 (1978) 249–349.
[97] E.S. Fradkin and M.Ya. Palchik, New results in conformal invariant quantum field theory, in: Group Theoretical Methods in Physics (Moscow, Nauka, 1983) vol. 1, pp. 84–98; Conformal invariance in Yang–Mills theory, Phys. Lett. 147B (1984) 86–90.
[98] E.S. Fradkin and A.A. Tseytlin, Higher derivative quantum gravity: one-loop counterterms and asymptotic freedom, Lebedev Inst. prepr. N70 (1981) 59 pp.
[99] E.S. Fradkin and A.A. Tseytlin, Renormalizable asymptotically free quantum theory of gravity, Phys. Lett. 104B (1981) 377–381; Nucl. Phys. B201 (1982) 469–491.
[100] E.S. Fradkin and A.A. Tseytlin, One-loop β-function in conformal supergravity, Nucl. Phys. B203 (1982) 157–178.
[101] E.S. Fradkin and A.A. Tseytlin, Asymptotic freedom in extended conformal supergravities, Phys. Lett. 110B (1982) 117–122 (Err: 126B (1983) 506).
[102] E.S. Fradkin and A.A. Tseytlin, Asymptotic freedom in renormalizable gravity and supergravity, in: Quantum Gravity, Proc. IId Quantum Gravity Seminar, Oct. 13–15, 1981, Moscow, eds. M.A. Markov and P.C. West (Plenum, N.Y., 1984) pp. 29–46.
[103] E.S. Fradkin and A.A. Tseytlin, Quantization of two-dimensional supergravity and critical dimensions for string models, Phys. Lett. 106B (1981) 63–68.
[104] E.S. Fradkin and A.A. Tseytlin, Quantized strings and QCD, in: Proc. Nara Conf., Lect. Notes Phys. 176 (1983) 284–292.
[105] E.S. Fradkin and A.A. Tseytlin, Off-shell one-loop divergences in gauged O(N) supergravities, Phys. Lett. 117B (1982) 303–308.
[106] E.S. Fradkin and A.A. Tseytlin, Conformal anomaly in Weyl theory and anomaly free superconformal theories, Phys. Lett. 134B (1984) 187–193.
[107] E.S. Fradkin and A.A. Tseytlin, Instanton zero modes and β-functions in conformal supergravity, Phys. Lett. 134B (1984) 307–312.
[108] E.S. Fradkin and A.A. Tseytlin, One-loop effective potential in gauged O(4) supergravity and the problem of Λ-term, Nucl. Phys. B234 (1984) 472–508.
[109] E.S. Fradkin and A.A. Tseytlin, New results in quantum supergravity, Budapest preprint KFKI-116 (1983) 60 pp.
[110] E.S. Fradkin and A.A. Tseytlin, On the new definition of off-shell effective action, Nucl. Phys. B234 (1984) 509–523.
[111] E.S. Fradkin and A.A. Tseytlin, Quantum equivalence of dual field theories, Lebedev Inst. prepr. N67 (1984) 28 pp.
[112] E.S. Fradkin and M.A. Vasiliev, Model of supergravity with minimal electromagnetic interaction, Lebedev Inst. prepr. N197 (1976) 23 pp.
[113] E.S. Fradkin and M.A. Vasiliev, Minimal set of auxiliary fields in SO(2) extended supergravity, Phys. Lett. 85B (1979) 47–51.
[114] E.S. Fradkin and G.A. Vilkovisky, Quantization of relativistic systems with constraints, Phys. Lett. 55B (1975) 224–227.
[115] E.S. Fradkin and G.A. Vilkovisky, Quantization of relativistic systems with constraints. Equivalence of canonical and covariant formalisms in quantum theory of gravitational field, CERN preprint TH.2332 (1977).
[116] E.S. Fradkin and G.A. Vilkovisky, Conformal off-mass-shell extension and elimination of conformal anomalies in quantum gravity, Phys. Lett. 73B (1978) 209–213.
[117] E.S. Fradkin and G.A. Vilkovisky, Conformal invariance and asymptotic freedom in quantum gravity, Phys. Lett. 77B (1978) 262–266.
[118] E.Z. Freedman and A. Das, Gauge internal symmetry in extended supergravity, Nucl. Phys. B120 (1977) 221–230.
[119] D.Z. Freedman and J.H. Schwarz, $N = 4$ supergravity theory with local SU(2) × SU(2) invariance, Nucl. Phys. B137 (1978) 333–339.
[120] P.G.O. Freund, Local scale invariance and gravitation, Ann. Phys. (N.Y.) 84 (1974) 440–454.
[121] C. Fronsdal, Massless fields with integer spins, Phys. Rev. D18 (1978) 3624–3629.
[122] S. Fubini and E. Rabinovici, Superconformal quantum mechanics, Nucl. Phys. B245 (1984) 17–45.
[123] T. Fulton, F. Rohrlich and L. Witten, Conformal invariance in physics, Rev. Mod. Phys. 34 (1962) 442–457.
[124] M.K. Gaillard and B. Zumino, Duality rotations for interacting fields, Nucl. Phys. B193 (1981) 221–244.
[125] S.J. Gates Jr., Superconformal transformations and six-dimensional space–time, Nucl. Phys. B162 (1980) 79–92.
[126] S.J. Gates Jr., Supercovariant derivatives, super-Weyl group and $N = 2$ supergravity, Nucl. Phys. B176 (1981) 397–418.
[127] S.J. Gates Jr., On shell and conformal $N = 4$ supergravity in superspace, Nucl. Phys. B213 (1983) 409–444.
[128] S.J. Gates Jr. and R. Grimm, Consequences of conformal constraints for $N > 4$ superspace, Phys. Lett. 133B (1983) 192–196.
[129] S.J. Gates Jr., M.T. Grisaru, M. Roček and W. Siegel, Superspace: one thousand and one lessons in supersymmetry (Benjamin/Cummings, Mass., 1983).
[130] S.J. Gates Jr., M.T. Grisaru and W. Siegel, Auxiliary field anomalies, Nucl. Phys. B203 (1982) 189–204.
[131] S.J. Gates Jr. and W. Siegel, unpublished (1981).
[132] S.J. Gates Jr. and W. Siegel, Linearized $N = 2$ superfield supergravity, Nucl. Phys. B195 (1982) 39–60.
[133] S.J. Gates Jr., K.S. Stelle and P.C. West, Algebraic origin of superspace contraints in supergravity, Nucl. Phys. B169 (1980) 347–361.
[134] S.J. Gates Jr. and B. Zwiebach, Gauged $N = 4$ supergravity theory with a new scalar potential, Phys. Lett. 123B (1983) 200–204.
[135] A. Gavrielides, T.K. Kuo and S.Y. Lee, Ghost problem of quantum field theories with higher derivatives, Phys. Rev. D13 (1976) 2912–2915.
[136] G.W. Gibbons, S.W. Hawking and M.J. Perry, Path integrals and the indefiniteness of the gravitational action, Nucl. Phys. B138 (1978) 141–150.

[137] P. Gilkey, The spectral geometry of a Riemannian manifold, J. Diff. Geom. 10 (1975) 601–618.
[138] J. Gilkey, The spectral geometry of the higher order laplacian, Duke Math. J. 47 (1980) 511–528 (Err: 48 (1981) 887).
[139] A.M. Gleeson, R.J. Moore, H. Rechenberg and E.C.G. Sudarshan, Analyticity, covariance and unitarity in indefinite-metric quantum field theories, Phys. Rev. D4 (1971) 2242–2254.
[140] F. Gliozzi, J. Scherk and D. Olive, Supersymmetry, supergravity theories and the dual spinor model, Nucl. Phys. B122 (1977) 253–290.
[141] M.B. Green, J.H. Schwarz and L. Brink, $N = 4$ Yang–Mills and $N = 8$ supergravity as limits of string theories, Nucl. Phys. B198 (1982) 474–492.
[142] R. Grimm, J. Wess and B. Zumino, A complete solution of the Bianchi identities in superspace with supergravity constraints, Nucl. Phys. B152 (1979) 255–265.
[143] M.T. Grisaru, Anomalies, field transformations and the relation between SU(4) and SO(4) supergravity, Phys. Lett. 79B (1978) 225–230.
[144] M.T. Grisaru, M. Roček and W. Siegel, Zero 3-loop β-function in $N = 4$ super-Yang–Mills theory, Phys. Rev. Lett. 45 (1980) 1063–1066.
[145] M.T. Grisaru and W. Siegel, Supergraphity, Nucl. Phys. B187 (1981) 149–173.
[146] M.T. Grisaru and W. Siegel, Supergraphity II, Manifestly covariant rules and higher-loop finiteness, Nucl. Phys. B201 (1982) 292–314.
[147] M.T. Grisaru and W. Siegel, The one-loop four particle S-matrix in extended supergravity, Phys. Lett. 110B (1982) 49–53.
[148] M.T. Grisaru and P.C. West, Supersymmetry and Adler–Bardeen theorem, Brandeis preprint (1983).
[149] F. Gürsey and L. Marchildon, Spontaneous symmetry breaking and non-linear invariant lagrangians: applications to $SU(2) \times U(1)$ and OSp(1, 4), Phys. Rev. D17 (1978) 2038–2047.
[150] R. Haag, J.T. Lopuszanski and M. Sohnius, All possible generators of supersymmetries of the S-matrix, Nucl. Phys. B88 (1975) 257–274.
[151] B. Hasslacher and E. Mottola, Gauge field model of induced classical gravity, Phys. Lett. 95B (1980) 237–240.
[152] B. Hasslacher and E. Mottola, Asymptotically free quantum gravity and black holes, Phys. Lett. 99B (1981) 221–224.
[153] J.B. Hartle and G.T. Horowitz, Ground-state expectation value of the metric in the $1/N$ or semiclassical approximation to quantum gravity, Phys. Rev. D24 (1981) 257–274.
[154] S.J. Hathrell, Trace anomaly and QED in curved space, Ann. Phys. (N.Y.) 142 (1982) 34–63.
[155] S.W. Hawking and C.N. Pope, Symmetry breaking by instantons in supergravity, Nucl. Phys. B146 (1978) 381–392.
[156] C.F. Hayes, Quantization of generalized Hamiltonian, J. Math. Phys. 10 (1969) 1555–1664.
[157] P.S. Howe, A superspace approach to extended conformal supergravity, Phys. Lett. 100B (1981) 389–392.
[158] P.S. Howe, Supergravity in superspace, Nucl. Phys. B199 (1982) 309–364.
[159] P.S. Howe, K.S. Stelle and P.K. Townsend, Supercurrents, Nucl. Phys. B192 (1981) 332–352.
[160] P.S. Howe, K.S. Stelle and P.K. Townsend, Miraculous ultraviolet cancellations in supersymmetry made manifest, Nucl. Phys. B236 (1984) 125–166.
[161] P.S. Howe, K.S. Stelle and P.C. West, A class of finite four dimensional models, Phys. Lett. 124B (1983) 55–58.
[162] P.S. Howe and R.W. Tucker, Scale invariance in superspace, Phys. Lett. 80B (1978) 138–140.
[163] G. Horowitz and R. Wald, Dynamics of Einstein's equation modified by a higher order derivative term, Phys. Rev. D17 (1978) 414–416.
[164] J. Illiopulos and B. Zumino, Broken supergauge symmetry and renormalization, Nucl. Phys. B76 (1974) 310–332.
[165] J. Julve and M. Tonin, Quantum gravity with higher derivative terms, Nuov. Cim. 46B (1978) 137–158.
[166] M. Kaku, Quantization of conformal gravity: another approach to the renormalization of gravity, Nucl. Phys. B203 (1982) 285–296.
[167] M. Kaku, Superconformal gravity in Hamiltonian form; another approach to the renormalization of gravitation, Phys. Rev. D27 (1983) 2809–2818.
[168] M. Kaku, Strong-coupling approach to the quantization of conformal gravity, Phys. Rev. D27 (1983) 2819–2834.
[169] M. Kaku and P.K. Townsend, Poincaré supergravity as broken superconformal gravity, Phys. Lett. 76B (1978) 54–58.
[170] M. Kaku, P.K. Townsend and P. van Nieuwenhuizen, Gauge theory of the conformal and graded conformal group, Phys. Lett. 69B (1977) 304–308.
[171] M. Kaku, P.K. Townsend and P, van Nieuwenhuizen, Superconformal unified theory, Phys. Rev. Lett. 39 (1977) 1109–1112.
[172] M. Kaku, P.K. Townsend and P. van Nieuwenhuizen, Properties of conformal supergravity, Phys. Rev. D17 (1978) 3175–3187.
[173] R.E. Kallosh, Modified Feynman rules in supergravity, Nucl. Phys. B141 (1978) 141–152.
[174] R.E. Kallosh, Axial and conformal anomalies in supergravity, JETP Lett. 37 (1983) 509–511.
[175] R.E. Kallosh, O.V. Tarasov and I.V. Tyutin, One-loop finiteness of quantum gravity off mass shell, Nucl. Phys. B137 (1978) 145–163.
[176] S. Kawasaki and T. Kimura, A possible mechanism of ghost confinement in a renormalizable quantum gravity, Progr. Theor. Phys. 65 (1981) 1767–1770.
[177] S. Kawasaki and T. Kimura, Canonical quantum theory of gravitational field with higher derivatives II, Asymptotic fields and unitarity, Progr. Theor. Phys. 68 (1982) 1116–1129.
[178] S. Kawasaki, T. Kimura and K. Kitago, Canonical quantum theory of gravitational field with higher derivatives, Progr. Theor. Phys. 66 (1981) 2085–2102.
[179] B.S. Kay, In–out semi-classical gravity and $1/N$ quantum gravity, Phys. Lett. 101B (1981) 241–245.
[180] T.W.B. Kibble, Lorentz invariance and the gravitational field, J. Math. Phys. 2 (1961) 212–221.
[181] J. Kiskis, Modified field theory for quark binding, Phys. Rev. D11 (1975) 2178–2202.
[182] T. Kugo and S. Uehara, Improved superconformal gauge condition in the $N = 1$ supergravity Yang–Mills matter system, Nucl. Phys. B222 (1983) 125–138.
[183] T. Kugo and S. Uehara, Conformal and Poincaré tensor calculus in $N = 1$ supergravity, Nucl. Phys. B226 (1983) 49–92.

[184] T. Kugo and S. Uehara, $N = 1$ superconformal tensor calculus: multiplets with external Lorentz indices and spinor derivative operations, CERN prepr. TH.3672 (1983) 63 pp.
[185] S.-C. Lee and P. van Nieuwenhuizen, Counting of states in higher-derivative field theories, Phys. Rev. D26 (1982) 934–937.
[186] T.D. Lee and G.C. Wick, Negative metric and unitarity of the S-matrix, Nucl. Phys. B9 (1969) 209–243.
[187] T.D. Lee and G.C. Wick, Questions of Lorentz invariance in field theories with indefinite metric, Phys. Rev. D3 (1970) 1046–1047.
[188] S.W. MacDowell and F. Mansouri, Unified geometrical theory of gravity and supergravity, Phys. Rev. Lett. 38 (1977) 739–741 (Err. 38 (1977) 1376).
[189] S. Mandelstam, Light cone superspace and ultraviolet finiteness of $N = 4$ model, Nucl. Phys. B213 (1983) 149–168.
[190] F. Mansouri, Conformal gravity as a gauge field, Phys. Rev. Lett. 42 (1979) 1021–1024.
[191] F. Mansouri and C. Schaer, SU(N) extended conformal invariant actions in superspace, Phys. Lett. 101B (1981) 51–54.
[192] F. Mansouri and C. Schaer, Local symmetry in extended conformal gravity and conformal supergravity in superspace, J. Math. Phys. 24 (1983) 890–896.
[193] L. Marchildon, Non-linear realization of the superconformal group and conformal supergravity, Phys. Rev. D18 (1978) 2804–2809.
[194] W. Nahm, Supersymmetries and their representations, Nucl. Phys. B135 (1978) 149–166.
[195] P. Nath and R. Arnowitt, Prototype models for particle structure in gauged supersymmetry, Nucl. Phys. B179 (1981) 226–238.
[196] P. Nath, R. Arnowitt and A.H. Chamseddine, Applied $N = 1$ supergravity, Northeastern Univ. prepr. NUB-2613 (1988) 110 pp.
[197] R.I. Nepomechie, Einstein gravity as the low energy effective theory of Weyl gravity, Phys. Lett. 136B (1984) 33–37.
[198] H. Nicolai, A possible constructive approach to (super $\psi^3)_4$, I. Euclidean formulation of the model, Nucl. Phys. B140 (1978) 294–300.
[199] H. Nicolai and P.K. Townsend, $N = 3$ supersymmetry multiplets with vanishing trace anomaly: building blocks of the $N > 3$ supergravities. Phys. Lett. 98B (1981) 257–260.
[200] N.K. Nielsen, Ghost counting in supergravity, Nucl. Phys. B140 (1978) 499–509.
[201] H.P. Nilles, Supersymmetry, supergravity and particle physics, Phys. Reports 110 (1984) 1–162.
[202] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Exact Gell-Mann–Low function of supersymmetric Yang–Mills theories from instanton calculus, Nucl. Phys. B229 (1983) 381–393.
[203] V. Ogievetsky and E. Sokatchev, Structure of the supergravity group, Phys. Lett. 79B (1978) 222–224.
[204] V. Ogievetsky and E. Sokatchev, Supercurrent, Yad. Fiz. 28 (1978) 825–836.
[205] V. Ogievetsky and E. Sokatchev, The simplest Einstein supergravity group, Yad. Fiz. 31 (1980) 264–279.
[206] R.D. Pisarski, A soluble model with massive ghosts, Phys. Rev. D28 (1983) 2547–2566.
[207] A.M. Polyakov, Quantum geometry of fermionic strings, Phys. Lett. 103B (1981) 211–213.
[208] R.J. Riegert, A non-local action for the trace anomaly, Phys. Lett. 134B (1984) 56–60.
[209] V.O. Rivelles and J.G. Taylor, Off-shell no-go theorems for higher dimensional supersymmetries and supergravities, Phys. Lett. 121B (1983) 37–41.
[210] A. Salam, Gauge interactions, elementarity and superunification, Phil. Trans. Roy. Soc. London A304 (1982) 135–153.
[211] A. Salam and J. Strathdee, Supersymmetry and superfields, Fortschr. Phys. 26 (1978) 57–142.
[212] A. Salam and J. Strathdee, Remarks on high-energy stability and renormalizability of gravity theory, Phys. Rev. D18 (1978) 4480–4489.
[213] J.H. Schwarz, Superstring theory, Phys. Reports 89 (1982) 223–321.
[214] J. Schwinger, Particles, sources and fields (Addison-Wesley, Reading, MA 1970).
[215] W. Siegel, Solution to constraints in Wess–Zumino formalism, Nucl. Phys. B142 (1978) 301–305.
[216] W. Siegel, Superconformal invariance of superspace with non-minimal auxiliary fields, Phys. Lett. 80B (1978) 224–228.
[217] W. Siegel, Gauge spinor superfield as scalar multiplet, Phys. Lett. 85B (1979) 333–339.
[218] W. Siegel, On-shell O(N) supergravity in superspace, Nucl. Phys. B177 (1981) 325–332.
[219] W. Siegel and S.J. Gates Jr., Superfield supergravity, Nucl. Phys. B147 (1979) 77–104.
[220] W. Siegel and S.J. Gates Jr., Superprojectors, Nucl. Phys. B189 (1981) 295–316.
[221] W. Siegel and M. Roček, On off-shell supermultiplets, Phys. Lett. 105B (1981) 275–277.
[222] R. Slansky, Group theory for model builders, Phys. Reports 79 (1981) 1–128.
[223] A.V. Smilga, Spontaneous generation of the Newton constant in the renormalizable gravity theory, ITEP prepr. N63 (1982) 8 pp.
[224] A.V. Smilga, Vacuum structure in quantum gravity, Nucl. Phys. B234 (1984) 402–412.
[225] L. Smolin, A fixed point for quantum gravity, Nucl. Phys. B208 (1982) 439–466.
[226] M.F. Sohnius, Supersymmetry and central charges, Nucl. Phys. B138 (1978) 109–121.
[227] M.F. Sohnius, Soft gauge algebras, Z. Phys. C18 (1983) 229–237.
[228] M.F. Sohnius and P.C. West, Conformal invariance in $N = 4$ supersymmetric Yang–Mills theory, Phys. Lett. 100B (1981) 245–249.
[229] M.F. Sohnius and P.C. West, An alternative minimal off-shell version of $N = 1$ supergravity, Phys. Lett. 105B (1981) 353–356.
[230] E. Sokatchev, A superspace action for $N = 2$ supergravity, Phys. Lett. 100B (1981) 466–468.
[231] K.S. Stelle, Renormalization of higher derivative quantum gravity, Phys. Rev. D16 (1977) 953–969.
[232] K.S. Stelle, Classical gravity with higher derivatives, Gen. Relat. Grav. 9 (1978) 353–371.
[233] K.S. Stelle and P.K. Townsend, Vanishing β-functions in extended supergravities, Phys. Lett. 113B (1981) 25–29.
[234] K.S. Stelle and P.C. West, Minimal auxiliary fields for supergravity, Phys. Lett. 74B (1978) 330–332.
[235] K.S. Stelle and P.C. West, Tensor calculus for the vector multiplet coupled to supergravity, Phys. Lett. 77B (1978) 376–378.
[236] K.S. Stelle and P.C. West, Spontaneously broken De Sitter symmetry and the gravitational holonomy group, Phys. Rev. D21 (1980) 1466–1488.

[237] A. Strominger, Vacuum topology and incoherence in quantum gravity, Phys. Rev. Lett. 52 (1984) 1733–1736.
[238] A. Strominger, G.T. Horowitz and M.J. Perry, Instantons in conformal supergravity, Nucl. Phys. B238 (1984) 653–664.
[239] A. Strominger, Is there a quantum theory of gravity?, in: Quantum Theory of Gravity, ed. S.M. Christensen (Adam Hilger, Bristol, 1984).
[240] G. 't Hooft, An algorithm for the poles at dimension four in the dimensional regularization procedure, Nucl. Phys. B62 (1973) 444–460.
[241] G. 't Hooft and M. Veltman, One-loop divergences in the theory of gravitation, Ann. Inst. H. Poincaré 20 (1974) 69–94.
[242] I.T. Todorov, M.C. Mintchev and V.B. Petkova, Conformal invariance in quantum field theory (SNS, Pisa, 1978).
[243] E. Tomboulis, $1/N$ expansion and renormalization in quantum gravity, Phys. Lett. 70B (1977) 361–364.
[244] E. Tomboulis, Renormalizability and asymptotic freedom in quantum gravity, Phys. Lett. 97B (1980) 77–80.
[245] E. Tomboulis, Renormalization and asymptotic freedom in quantum gravity, in: Quantum Theory of Gravity, ed. S.M. Christensen (Adam Hilger, Bristol, 1984).
[246] E. Tomboulis, Unitarity in higher-derivative quantum gravity, Phys. Rev. Lett. 52 (1984) 1173–1176.
[247] P.K. Townsend and P. van Nieuwenhuizen, Geometrical interpretation of extended supergravity, Phys. Lett. 67B (1977) 439–442.
[248] P.K. Townsend and P. van Nieuwenhuizen, Simplification of conformal supergravity, Phys. Rev. D19 (1979) 3166–3169.
[249] A.A. Tseytlin, Poincaré and De Sitter gauge theories of gravity with propagating torsion, Phys. Rev. D26 (1982) 3327–3341.
[250] A.A. Tseytlin, Effective action in De Sitter space and conformal supergravity, Yad. Fiz. 39 (1984) 1598–1607.
[251] R. Utiyama and B. De Witt, Renormalization of a classical gravitational field interacting with quantized matter fields, J. Math. Phys. 3 (1962) 608–618.
[252] J.W. van Holten and A. Van Proeyen, $N = 1$ supersymmetry algebras in 2, 3, 4 mod 8, J. Phys. A15 (1982) 3763–3783.
[253] P. van Nieuwenhuizen, On ghost free tensor lagrangians and linearized gravitation, Nucl. Phys. B60 (1973) 478–492.
[254] P. van Nieuwenhuizen, Supergravity, Phys. Reports 68 (1981) 189–398.
[255] P. van Nieuwenhuizen and P.C. West, From conformal supergravity in ordinary space to its superspace constraints, Nucl. Phys. B169 (1980) 501–514.
[256] A. Van Proeyen, Superconformal tensor calculus in $N = 1$ and $N = 2$ supergravity, in: Proc. Winter school in Karpacz, 14–26 Feb. 1983, ed. B. Milewski (World Scientific, 1983); CERN prepr. TH.3579 (1983) 71 pp.
[257] M.A. Vasiliev, "Gauge" form of description of free massless fields with arbitrary spin, Yad. Fiz. 32 (1980) 855–861.
[258] G.A. Vilkovisky, A gospel according to De Witt, in: Quantum Theory of Gravity, ed. S.M. Christensen (Adam Hilger, Bristol, 1984).
[259] S. Weinberg, Ultraviolet divergences in quantum theories of gravity, in: General Relativity, eds. S.W. Hawking and W. Israel (Cambridge Univ. Press, 1979) pp. 790–832.
[260] J. Wess, The conformal invariance in quantum field theory, Nuov. Cim. 18 (1960) 1086–1107.
[261] J. Wess and J. Bagger, Supersymmetry and supergravity (Princeton Univ. Press, 1983).
[262] J. Wess and B. Zumino, A lagrangian model invariant under supergauge transformations, Phys. Lett. 49B (1974) 52–54.
[263] J. Wess and B. Zumino, Supergauge transformations in four dimensions, Nucl. Phys. B70 (1974) 39–50.
[264] J. Wess and B. Zumino, Superspace formulation of supergravity, Phys. Lett. 66B (1977) 361–364.
[265] P.C. West, Supersymmetry and finiteness, in: Proc. Shelter Island II, June 1983 (Plenum, 1984); in: Proc. of La Jolla Conf., January 1983 (to be published).
[266] H. Weyl, Gravitation und Elektrizität, Sitz. Preuss. Akad. Wiss. (1918) p. 465.
[267] H. Weyl, Space–Time–Matter (Dover, N.Y., 1950).
[268] A. Zee, Broken symmetric theory of gravity, Phys. Rev. Lett. 42 (1979) 417–420.
[269] A. Zee, A theory of gravity based on the Weyl–Eddington action, Phys. Lett. 109B (1982) 183–186.
[270] A. Zee, Einstein gravity emerging from quantum Weyl gravity, Ann. Phys. (N.Y.) 151 (1983) 431–443.
[271] B.M. Zupnik, The equation of conformal supergravity, Phys. Lett. 105B (1981) 153–154.

Note added in proof (December 1984)

We would like to add several recent references dealing with questions relevant to the topics discussed in this review [272–281]. In particular, let us mention ref. [274] where the complete action for an arbitrary number of Abelian vector multiplets coupled to $N = 4$ ("minimal") conformal supergravity was found.

Added references

[272] B. Binegar, On the unitarity of conformal supergravity, Los Angeles preprint, UCLA/84/TEP/6.
[273] F. David and A. Strominger, On the calculability of Newton's constant and the renormalizability of scale invariant quantum gravity, Phys. Lett. 143B (1984) 125–129.

[274] M. de Roo, Matter couplings in $N = 4$ supergravity, Groningen preprint, October 1984, 26 pp.
[275] B. de Wit and A. Van Proeyen, Potentials and symmetries of general gauged $N = 2$ supergravity – Yang–Mills models, Nucl. Phys. B245 (1984) 89–118.
[276] E.S. Fradkin and M.Ya. Palchik, Euclidean linear conformal gravity, Class. Quant. Grav. 1 (1984) 131–147.
[277] C. Fronsdal, Ghost-free non-linear spin 2 conformal gauge theory, Phys. Rev. D30 (1984) 2081–2096.
[278] M. Günaydin and N. Marcus, The unitary supermultiplet of $N = 8$ conformal superalgebra involving fields of spin ≤ 2, Cal Tech preprint, CALT-68-1208 (1984).
[279] R.J. Riegert, Birkhoff's theorem in conformal gravity, Phys. Rev. Lett. 53 (1984) 315–318.
[280] A. Strominger, Positive energy theorem for $R + R^2$ gravity, Phys. Rev. D30 (1984) 2257–2259.
[281] A. Strominger and V.P. Nair, (Super) Conformal symmetry breaking, Princeton preprint, IAS (1984).