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RECENT DEVELOPMENTS IN CONFORMAL INVARIANT QUANTUM FIELD THEORY

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Abstract:

A review of the recent results concerning the kinematics of conformal fields, the analysis of dynamical equations and dynamical derivation of the operator product expansion is given.

The classification and transformational properties of fields which are transformed according to the representations of the universal covering group of the conformal group have been considered. A derivation of the partial wave expansion of Wightman functions is given. The analytical continuation to the Euclidean domain of coordinates is discussed. As shown, in the Euclidean space the partial wave expansion can be applied either to one-particle irreducible vertices or to the Green functions, depending on the dimensions of the fields.

The structure of Green functions, which contain a conserved current and the energy-momentum tensor, has been studied. Their partial wave expansion has been obtained. A solution of the Ward identity has been found. Special cases are discussed.

The program of the construction of exact solution of dynamical equations is discussed. It is shown, that integral dynamical equations for vertices (or Green's functions) can be diagonalized by means of the partial wave expansion. The general solution of these equations is obtained. The equations of motion for renormalized fields are considered. The way to define the product of renormalized fields at coinciding points (arising on the right-hand side) is discussed. A recipe for calculating this product is presented. It is shown, that this recipe necessarily follows from the renormalized equations.

The role of bare term and of canonical commutation relations (for unrenormalized fields) is discussed in connection with the problem of the field product determination at coinciding points. As a result an exact relation between fundamental field dimensions is found for various three-linear interactions (section 16 and Appendix 6). The problem of closing the infinite system of dynamical equations is discussed.

Al above said results are demonstrated using Thirring model as an example. A new approach to its solving is developed.

The program of closing the infinite system of dynamical equations is discussed. The Thirring model is considered as an example. A new approach to the solution of this model is discussed.

Methods are developed for the approximate calculation of dimensions and coupling constants in the 3-vertex and 5-vertex approximations. The dimensions are calculated in the $\lambda\varphi^3$ theory in 6-dimensional space.

The problem of calculating the critical indices in statistics (3-dimensional Euclidean space) is considered. The calculation of the dimension is carried out in the framework of the $\lambda\varphi^4$ model. The value of the dimension and the critical indices thus obtained coincide with the experimental ones.

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1. Introduction

Recent investigations have shown that there exists a whole class of Lagrangian quantum field theories, where we are not faced with the "charge-zero" difficulty for renormalized interactions. This primarily refers to Yang–Mills gauge fields interacting with spinor fields and to certain models of unified interactions. In this case it is possible, using a particular choice of the coupling constants (and representation types), to provide the required mass dressing of intermediate vector fields (due to the Higgs' mechanism), on the one hand, and to ensure asymptotic freedom on the other hand. The characteristic feature of theories with asymptotic freedom is that the interaction at small distances decreases with decreasing distance and goes to zero. Another version of the theory (mathematically self-consistent as well) is known, where the effective charge at small distances tends to a constant value (fixed point). In this case, and near the light cone, a conformal invariant solution is realized.

The concepts of scaling and conformal symmetry of interactions at small distances are based on the assumption of structureless particles. It is assumed that at distances much smaller than the particle Compton wavelength there are no characteristic sizes. Small distances correspond to a range of events characterized not only by large squares of external momenta, but also by large momentum transfers, hence rest masses may be neglected. In this range of events the theory is believed to be invariant under scale transformations

$$x_\mu \to \lambda x_\mu \tag{1.1}$$

and the special conformal transformations

$$x_\mu \to \frac{x_\mu - b_\mu x^2}{1 - 2(bx) + b^2 x^2}. \tag{1.2}$$

With the transformations of the Poincaré group

$$x_\mu \to x_\mu + a_\mu, \tag{1.3}$$

$$x_\mu \to \omega_{\mu\nu} x_\nu, \qquad \omega_{\mu\nu} = -\omega_{\nu\mu} \tag{1.4}$$

they form the 15-parametric group of conformal transformations. Instead of the special conformal transformations (1.2), it is more convenient to make use of the (conformal) inversion

$$x_\mu \to R x_\mu = x_\mu / x^2. \tag{1.5}$$

Then (1.2) can be represented as a product of three transformations: $x_\mu \to (RP_b R)x_\mu$, where P_b is the operator of translation by the vector $\boldsymbol{b}$. From (1.5) we have

$$(Rx)^2 = 1/x^2, \qquad (Rx - Ry)^2 = (x - y)^2 / x^2 y^2. \tag{1.6}$$

It is essential that conformal symmetry is acknowledged to be approximate, since real particles have a nonvanishing mass. The conformal invariant theory describes only the asymptotic behaviour of Green functions near the light cone, where the rest mass may be neglected. Hence the step following the construction of the conformal-invariant theory consists in passing over to the mass shell, i.e. to taking account of the terms breaking the conformal symmetry. It should be noted, however, that some experimental predictions can even be made in the case of exact symmetry. This primarily concerns deep-inelastic lepton–hadron reactions, the annihilation of e^+e^- into hadrons, e^-p and e^-e^+ scattering, etc. Their responsible strong interaction is localized at distances much smaller than the hadron dimension, i.e. in the range where it is assumed to be conformal invariant. In this case the problem may be reduced to the calculation of vertices including conserved currents, the external momenta lying off the mass shell.

Another important application of the conformal symmetry hypothesis is met in statistical mechanics when treating the problem of phase transitions. There is a qualitative as well as a formal analogy between statistics and quantum field theory. In particular, the correlation functions correspond to the Euclidean Green functions of the field theory, and the correlation length is the analog of the Compton particle size. Both theories differ only in the dimension D of the space where they are formulated.

Hence we will consider the conformal-invariant field theory for the general case of D-dimensional space. For $D = 4$ and $D = 3$ it describes the interaction of elementary particles and critical phenomena in statistical mechanics, respectively. A large body of literature is devoted to their description in terms of the scaling and conformal symmetry hypothesis (see, e.g. [1] and the references given there). It is essential that in this case the consequences of the conformal symmetry hypothesis can be experimentally verified.

In recent years significant achievements have been made in studying the dynamics of the conformal-invariant theory [2–8], as well as on several problems of conformal kinematics in Minkowski space, for the operator product expansion of fields, etc. [8–22]. In the present review primary emphasis is placed on recent results obtained by the authors for the solution of exact dynamical equations, the Ward identities [4–6, 8] and the derivation of a partial wave expansion [9, 8, 22] in Minkowski and Euclidean spaces. We will also discuss new results (obtained by Schroer and Swieca [13] and by one of the authors (M.Ya.P.) [9, 12, 21, 22]) concerning the analysis of the transformational properties of fields and of their classification. For those interested in a more detailed list of references (including problems untouched in the present review) we may recommend the reviews [23–25]. Another interesting trend untouched in the present review should be particularly mentioned. It refers to the theories with spontaneously-broken conformal symmetries [26].

The review consists of two parts. *In the first part* (sections 2–7) the *general kinematical and axiomatic restrictions on the Green functions are treated and a mathematical technique is developed that enables us to efficiently investigate the dynamics of the theory.* Armed with this technique one may not only hope to achieve a deeper insight into the general principles of quantum field theory, but also to find its exact solution in the vicinity of the light cone. This problem is discussed in the second part of the present review (sections 8–16) where the *detailed analysis of the dynamical equations* is given *beyond the scope of perturbation theory.* In section 15 we discuss *methods for approximate calculations of the dimensions and applications to statistical physics.*

The plan of presentation is the following. In section 3 we discuss the classification of fields in the Minkowski space and their transformation properties. An essential point here is that the fields transform via representations of the universal covering group of the conformal group. This result was first obtained for states obtained by applying the field to the vacuum by one of the present authors (M.Ya.P.) in [9, 12] and independently by Rühl [10] and then generally for fields by Schroer and Swieca [13] (see also [16–22]): We demonstrate that, apart from the scale dimension, an extra quantum number λ is needed for the classification of fields. This is due to the complicated structure of the covering group [9, 13, 17a, 18, 22]. The physical fields $\varphi_d(x)$ are superpositions of irreducible fields with the same value of the dimension d and different values of λ. We also demonstrate that the physical fields are defined [53, 25] on the infinite-sheeted universal covering of the Minkowski space. This is implied by causality and follows from the study of finite conformal transformations of the fields. In sections 3–5 the derivation and properties of partial wave expansions in Minkowski and Euclidean spaces, obtained independently by several authors [2, 9, 27, 28, 31] is discussed (see also [3, 5–8, 12, 18, 21, 22, 32, 33]).

In order not to overload the presentation with mathematical details, a review of the main properties of the conformal group is given in Appendices 1 and 2. Appendix 1 discusses the conformal group for the cases of one (group SO(2, 1)) and two dimensions (group $SO(2,1) \otimes SO(2,1)$). This discussion is based on the results of refs. [27, 12, 22] (see also [32, 34, 34a]). Despite the relative simplicity of the conformal group in one-dimensional space, it has analogs for all important properties of the conformal group in four-dimensional space-time which will be reviewed in Appendix 2 (see also [35]). Hence, we recommend the reader to look through the results of Appendix 1 before studying the real conformal group of space-time.

In sections 6 and 7 our results [5, 36, 37, 8, 58] concerning the *Green functions containing conserved currents and the energy-momentum tensor* are given. A general solution of the Ward identities for a four-point function is considered [5, 37, 8]. A partial wave expansion has been found for the particular choice of Green functions with one or two currents [8].

In sections 8–16 the analysis of the dynamical equations is given. In section 8 we discuss the exact set of renormalized integral equations for vertices that was obtained by one of the present authors (E.S.F.) in ref. [60] and by Symanzik in ref. [61]. In sections 9, 10 this set of equations is shown to be diagonalized by the partial wave expansion. In section 11 the solution of the integral equations is discussed. In section 12 a simple special solution of the Ward identity is studied and the coupling constants of the tensor fields related to the special solution under consideration are found. In section 13 we consider the equations of motion for renormalized fields and discuss the question of how the product of two fields in coinciding points should be defined. This product occurs in the right-hand side of the equations of motion. We show in section 13 that the way to make the arguments of two renormalized fields coincide is uniquely determined by the integral equations and may be found by explicitly solving them. With the use of this result we further formulate a version of the derivation of the closed equations for the Green functions. In section 14 the main results are illustrated by considering the Thirring model. The new approach to solving this model is formulated basing upon the results of sections 6, 7 and 9–13.

In section 16 we discuss the role of bare term (and also of canonical commutation relations for unrenormalized fields) in connection with the problem of the fields product determination at coinciding points. As a result we get an exact relation between dimensions of fundamental (scalar) fields. Analogous relation for fields dimensions of the Yukawa model is given in Appendix 6.

In section 15 the calculations of the dimensions and coupling constants are presented performed within the framework of the bootstrap programme [38–42]. A method is developed for calculating integrals which are met in the three-vertex and five-vertex approximations with the $\lambda\varphi^3$-theory as example.

Applications of the developed methods are considered for calculations in the phase transition theory. Anomalous dimensions and critical indexes are found in the frame of $\lambda\varphi^4$ model. This method allowed to perform calculations in any space-time dimension D without ε-expansion. Two cases are considered: $D = 4 - \varepsilon$ and $D = 3$. It is shown that at $D = 4 - \varepsilon$ conformal invariant equations lead to the known results of ε-expansion. Approximate calculations at $D = 3$ in 5-vertex approximation are performed. The following values are obtained for the dimensions d and Δ of the fields $\varphi(x)$ and $\varphi^2(x)$, respectively: $d = 0.510$, $\Delta = 1.34$.

2. Conformal invariant Green functions in Euclidean space

As is known [44–46], the conformal symmetry fixes the form of two- and three-point Green functions up to constants and imposes some restrictions on higher Green functions. In this section their general expressions will be given for the most important cases. Since in what follows the Euclidean formulation of field theory will often be used [47], we consider Green functions in the D-dimensional Euclidean space whose conformal group is $SO(D + 1, 1)$. The complete analysis is given in [3, 32]. It can readily be seen that for $D = 4$ its Casimir operators coincide with (A2.2). In the general case ($D \neq 4$) the classification of the irreducible representations is involved due to the complicated structure of the Euclidean Lorentz group (group $SO(D)$). Representations of the group $SO(D + 1, 1)$ correspond [17] to analytical representations (or to those of the discrete series) of the universal covering group of the conformal group.

We will restrict ourselves to the consideration of tensor fields, being the symmetric traceless s-rank tensors. Their associated irreducible representations are classified by the values of two numbers: dimension l and spin s. In this case the representations (l, s) and $(D - l, s)$ are equivalent. Introduce the notations

$$\sigma = (l, s), \qquad \tilde{\sigma} = (D - l, s). \tag{2.1}$$

Let $O_\sigma(x) = O^l_{\mu_1 \dots \mu_s}(x)$ be the tensor field with quantum numbers σ. Its transformational properties with respect to the transformations (1.1–3) are as follows:

$$U(a, \omega) O_\sigma(x) U^{-1}(a, \omega) = T(\omega^{-1}) O_\sigma(\omega x + a), \tag{2.2}$$

where $T(\omega^{-1})$ is the matrix of the finite-dimensional representation of the Euclidean Lorentz group,

$$U(\lambda) O_\sigma(x) U^{-1}(\lambda) = \lambda^l O_\sigma(\lambda x). \tag{2.3}$$

Instead of special conformal transformations it will be more convenient to consider the R-transformation [48] of fields, see (1.5). For the fields $O_\sigma(x)$ it has the form

$$U(R) O^l_{\mu_1 \dots \mu_s}(x) U^{-1}(R) = \frac{1}{(x^2)^l} g_{\mu_1 \nu_1}(x) \dots g_{\mu_s \nu_s}(x) O^l_{\nu_1 \dots \nu_s}(Rx), \tag{2.4}$$

where

$$g_{\mu\nu}(x) = \delta_{\mu\nu} - 2x_\mu x_\nu / x^2. \tag{2.4a}$$

Let the two-point Green functions be designated as $\Delta_\sigma(x_1 - x_2) = \langle 0 | T O_\sigma(x_1) O_\sigma(x_2) | 0 \rangle$. As it follows from (2.2), the function $\Delta_\sigma(x_1 - x_2)$ depends on the variable $(x_1 - x_2)^2$. Using (2.3) and (2.4, 4a) and the vacuum invariance, we find

$$\Delta_\sigma(x) = \frac{1}{(2\pi)^h} n(\sigma) \{ g_{\mu_1 \nu_1}(x) \dots g_{\mu_s \nu_s}(x) \} \frac{1}{(\frac{1}{2} x^2)^l}, \tag{2.5}$$

where

$$h = D/2, \tag{2.5a}$$

$n(\sigma)$ is the normalization factor, braces denote trace subtraction and symmetrization in the indices $\mu_1 \dots \mu_s$ and independently in $\nu_1 \dots \nu_s$. Note the important property of two-point functions [29] resulting from (2.4):

$$\langle 0 | O_{\sigma_1}(x_1) O_{\sigma_2}(x_2) | 0 \rangle = 0, \qquad \text{if} \qquad \sigma_2 \neq \sigma_1. \tag{2.6}$$

This property has a simple mathematical meaning: the states $O_\sigma(x)|0\rangle = |\sigma, x\rangle$ form the space of the irreducible representation σ (see Appendices 1 and 2). Equation (2.6) means that the vectors belonging to different representations σ_1 and σ_2 are orthogonal: $\langle x_1, \sigma_1 | \sigma_2, x_2 \rangle = 0$. This property is changed only for $\sigma_2 = \tilde{\sigma}_1$. In this case we have $\langle x_1, \sigma | \tilde{\sigma}, x_2 \rangle \sim \delta(x_1 - x_2)$. In what follows scalar fields will mainly be considered. Introduce the notation $\varphi_d(x)$ for them, where d is the dimension. Let the two-point function of scalar fields be designated as*

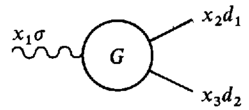
$$G_d(x_{12}) = \langle 0 | \varphi_d(x_1) \varphi_d(x_2) | 0 \rangle = \frac{1}{(2\pi)^h} \frac{\Gamma(d)}{\Gamma(h - d)} \frac{1}{(\frac{1}{2} x_{12}^2)^d}. \tag{2.5b}$$

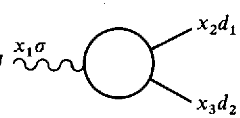
* Where $x_{ij} = x_i - x_j$.

Consider the three-point Green function

$$G(x_1x_2x_3) = \langle 0|\, TO_\sigma(x_1)\varphi_{d_1}(x_2)\varphi_{d_2}(x_3)\,|0\rangle = gC^{\sigma d_1 d_2}(x_1x_2x_3), \tag{2.7}$$

where g is the coupling constant and $C^{\sigma d_1 d_2}(x_1x_2x_3)$ is the properly normalized invariant function. Introduce graphical notations for the functions $G(x_1x_2x_3)$ and $C^{\sigma d_1 d_2}(x_1x_2x_3)$

$G(x_1x_2x_3) =$ 

$= g$ 

. (2.8)

From (2.2–4) one can readily obtain

$$C^{\sigma d_1 d_2}_{\mu_1\ldots\mu_s}(x_1x_2x_3) = \frac{2^{s/2}}{(2\pi)^h} N(\sigma d_1 d_2) \frac{1}{(\frac{1}{2}x_{12}^2)^{(l+d_1-d_2-s)/2}} \frac{1}{(\frac{1}{2}x_{13}^2)^{(l-d_1+d_2-s)/2}}$$

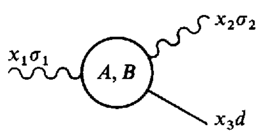
$$\times \frac{1}{(\frac{1}{2}x_{23}^2)^{(d_1+d_2-l+s)/2}} [\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_2x_3) - \text{traces}], \tag{2.9}$$

where

$$\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_2x_3) = \lambda^{x_1}_{\mu_1}(x_2x_3)\ldots\lambda^{x_1}_{\mu_s}(x_2x_3), \qquad \lambda^{x_1}_{\mu}(x_2x_3) = \frac{(x_{21})_\mu}{x_{12}^2} - \frac{(x_{31})_\mu}{x_{13}^2}, \tag{2.9a}$$

$N(\sigma d_1 d_2)$ is the normalization constant. Along with $n(\sigma)$ it will be given in section 5.

Let an expression be given for a three-point function of scalar, vector and tensor of the rank s with the dimensions d_3, d_2 and d_1, respectively. It depends on two arbitrary constants A and B:

$C^{AB,\sigma_1\sigma_2 d_3}_{\mu,\mu_1\ldots\mu_s}(x_1x_2x_3) \equiv \{A, B\} \equiv$ 

$$= \frac{1}{(2\pi)^h} \frac{1}{(\frac{1}{2}x_{12}^2)^{(d_1+d_2-d_3-s-1)/2}} \frac{1}{(\frac{1}{2}x_{13}^2)^{(d_1+d_3-d_2-s+1)/2}} \frac{1}{(\frac{1}{2}x_{23}^2)^{(d_2+d_3-d_1+s-1)/2}}$$

$$\times \{A\lambda^{x_2}_{\mu}(x_1x_3)[\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_2x_3) - \text{traces}]$$

$$+ B\frac{1}{x_{12}^2}[\sum_k g_{\mu\mu_k}(x_{12})\lambda^{x_1}_{\mu_1\ldots\hat{\mu}_k\ldots\mu_s}(x_2x_3) - \text{traces}]\}. \tag{2.10}$$

In order to derive the expressions (2.9) and (2.10) we have used the relationships (1.6) and

$$g_{\mu\mu'}(x)\lambda^{Rx}_{\mu'}(Rx_1, Rx_2) = x^2\lambda^{x}_{\mu}(x_1x_2),$$

$$g_{\mu\tau}(x)g_{\nu\tau}(x) = \delta_{\mu\nu}, \qquad g_{\mu\mu'}(x_1)g_{\nu\nu'}(x_2)g_{\mu'\nu'}(Rx_1 - Rx_2) = g_{\mu\nu}(x_{12}). \tag{2.10a}$$

Consider the higher Green functions $G(x_1\ldots x_n)$. When the number of variables is $n \geqslant 4$, the function $G(x_1\ldots x_n)$ is not determined unambiguously, since from $x_1\ldots x_n$ arguments dimensionless combinations can be composed which are invariant under the R-transformation. In particular, for $n = 4$ we have two such combinations:

$$\xi = x_{12}^2 x_{34}^2 / x_{13}^2 x_{24}^2, \qquad \eta = x_{12}^2 x_{34}^2 / x_{14}^2 x_{23}^2. \tag{2.11}$$

Their invariance can readily be verified with the help of (1.6). Thus, the most general expression for the Green function of four scalar fields $\varphi_d(x)$ may be written as

$$G(x_1 x_2 x_3 x_4) = (x_{12}^2 x_{13}^2 x_{23}^2 x_{14}^2 x_{24}^2 x_{34}^2)^{-d/3} F(\xi, \eta) \tag{2.12}$$

where $F(\xi, \eta)$ is an arbitrary function. The higher Green functions $G(x_1 \ldots x_n)$ depend on $n(n-3)/2$ variables of type (2.11).

Some additional limitations on the function $F(\xi, \eta)$ arise in the case of identical fields from the requirement of locality. Locality implies that the function $G(x_1 x_2 x_3 x_4)$ is invariant under any permutations of the coordinates. As a result we find for $F(\xi, \eta)$:

$$F(\xi, \eta) = F(\eta, \xi) = F(\xi/\eta, 1/\eta) = F(1/\eta, \eta/\xi). \tag{2.13}$$

3. Fields in Minkowski space

3.1. Classification of fields

It is common practice to classify relativistic fields by quantum number of the kinematical group. In a usual relativistic theory this is the Poincaré group, whose Casimir generators are expressed via mass and spin. Irreducible relativistic fields with the specific values of mass and spin satisfy the relations

$$[\varphi(x), P_\mu] = \mathrm{i}\partial_\mu \varphi(x), \qquad [\varphi(x), M_{\mu\nu}] = \mathrm{i}(x_\mu \partial_\nu - x_\nu \partial_\mu + \Sigma_{\mu\nu})\varphi(x) \tag{3.1}$$

which determine their transformational properties. Note that transformation law (3.1) depends only on the spin structure rather than on the field mass. This property holds for finite transformations as well, see (2.2). From this it follows that *reducible fields* (i.e. the superposition of fields with different masses) *are also transformed similar to irreducible ones.*

The situation is markedly complicated in the case of conformal fields, which in addition to (3.1) satisfy the relations

$$\begin{aligned} [\varphi_d(x), D] &= -\mathrm{i}(d + x_\mu \partial_\mu)\varphi_d(x), \\ [\varphi_d(x), K_\mu] &= -\mathrm{i}(2d x_\mu + 2x_\mu x_\nu \partial_\nu + 2x_\nu \Sigma_{\mu\nu})\varphi_d(x) \end{aligned} \tag{3.2}$$

where d is the scale dimension. The operators of mass and spin are no more invariant. Indeed, from the relation $\mathrm{e}^{\mathrm{i}\lambda D} P^2 \mathrm{e}^{-\mathrm{i}\lambda D} = \mathrm{e}^{-2\lambda} P^2$ it follows that P^2 has an infinite continuous spectrum of values in each irreducible representation. There is also a smaller class of representations where $P^2 = 0$. They coincide with massless representations of the Poincaré group and describe free massless fields. For all physically meaningful representations which correspond to interacting fields the spectrum of P^2 is continuous. In addition, representations describing field states meet the spectrality axiom

$$P^2 > 0, \qquad P_0 > 0. \tag{3.3}$$

Spin is not invariant either. In particular, there exist such representations of the conformal group where it takes an infinite spectrum of values. It is essential, however, that every representation satisfying (3.3) has the spin structure*

$$(j_1, j_2) \oplus (j_2, j_1) \tag{3.3a}$$

where j_1, j_2 are the Lorentz group quantum numbers. The values of Casimir operators for these representations are given in Appendix 2. They depend on three numbers: j_1, j_2 and the scale dimension d. In what follows we will consider only those tensor fields where $j_1 = j_2 = s/2$ (symmetric-traceless tensors). They are classified by the value of the dimension and the spin (l and s, respectively).

These numbers, however, are insufficient for a complete classification of the conformal fields. In addition to l and s *another quantum number should be introduced* which will be referred to as λ. *It is related to the properties of the representations of the universal covering group of the conformal group, which should be considered as the true symmetry group of the theory.* The necessity of such an extension of the conformal group may be argued in different ways [9, 10, 12, 13]. Consider, e.g., the following [9, 12] states $\varphi_d(x)|0\rangle$, where d is the dimension. As it will be shown below, they form the space of irreducible representation [27, 12] and, hence, their momentum spectrum may be judged by the spectrum of the Poincaré group representations involved in the given irreducible representation of the conformal group. If one restricts oneself to single-valued representations of the conformal group or to its covering SU(2, 2), then condition (3.3) results in the quantization [27, 49, 50, 51] of the field dimension

$$d = D - 2 + s + n, \qquad n \text{ is a positive integer.} \tag{3.4}$$

To overcome this difficulty, one should consider infinite valued representations of the conformal group which are the representations of the (infinite-sheeted) universal covering group. As shown in Appendix 2, its irreducible representations are characterized by three numbers: l, s and λ, and the condition (3.3) allows any dimension values. The additional number λ characterizes the degree of "ambiguity" of the representation as a representation of the conformal group (and its associated irreducible field).

3.2. *Transformational properties*

The appearance of the additional quantum number markedly complicates the transformational properties of the fields for finite transformations. Note, first, that *finite transformations* (1.2) *disturb causality*. They connect the space-time points falling inside and outside the light cone This leads to the disturbance of local (anti) commutativity of fields. The only exceptions occur in the case of free massless fields in D-dimensional space-time (where D is even [51], $d = h - 1$) and for generalized free fields with dimensions (3.4). The commutator (anti-commutator) of these fields differs from zero only on the light cone [52, 50]. (Notice that dimensions (3.4) correspond to single-valued representations of the group SU(2, 2).)

The above-said is related to the non-local transformation character of fields in the Minkowski space for finite conformal transformations. Indeed, let $\varphi_{ds\lambda}(x)$ be an irreducible conformal field.

* This follows [9, 52] from the analysis of irreducible representations given in ref. [35b]. The rigorous form of this result is presented in a recent work by Mack [35c].

Physical fields with fixed dimension are superposition [13, 22] (see also point 4 of this section)

$$\varphi_{ds}(x) = \sum_{\lambda} \varphi_{ds\lambda}(x), \tag{3.5}$$

i.e. they are in general reducible. The fields φ_{ds} as well as $\varphi_{ds\lambda}$ have the same infinitesimal transformation properties as those given by relation (3.2). This property, however, does not hold when passing to finite transformations. It will be shown below that *finite transformations of the irreducible fields depend explicitly on the number* λ, and this results in a non-local transformation of the fields (3.5), see also (3.18), (3.19) and (3.22), (3.23). E.g., special conformal transformations of scalar fields are of the form [13]

$$U(b)\varphi_{d,\lambda}(x)U^{-1}(b) = [\sigma_+(b,\lambda)]^{-(d+\lambda)/2}[\sigma_-(b,\lambda)]^{-(d-\lambda)/2}\varphi_{d,\lambda}(x_b),$$

where

$$[\sigma_\pm(b,x)]^\nu = (-b^2 \mp \mathrm{i}\varepsilon b_0)^\nu[-(x-b/b^2)^2 \mp \mathrm{i}\varepsilon(x_0 - b_0/b^2)]^\nu, \qquad x_b = (x - bx^2)/(1 - 2bx + b^2x^2).$$

To restore locality one is forced to formulate the field theory in the (infinite-sheeted) universal covering of the compactified Minkowski space. In this space *it proves possible to introduce a causal order and, as a result finite transformations of the fields* $\varphi_{ds\lambda}(x)$ *do not explicitly depend on the* λ *number* and the representations of the universal covering group acting there are (determined as) single-valued. This approach was first formulated by Segal in [53] and then in more detail in [16, 20]. The construction of the universal covering of the compactified space and its causal order are described in detail by Todorov in review [25]. Hence, we will restrict ourselves to a simple illustration of this approach by an example (of the one-dimensional space).

Let $\varphi_{l\lambda}(x)$ be a field with dimension l in a one-dimensional space (with the x-coordinate). The description of the conformal group acting in one-dimensional space is given in Appendix 1. Consider the transformation of the maximal compact subgroup (A1.4a)

$$x' = \frac{x\cos\frac{1}{2}\psi + \sin\frac{1}{2}\psi}{-x\sin\frac{1}{2}\psi + \cos\frac{1}{2}\psi} \tag{3.6}$$

for the states $\varphi_{l\lambda}(x)|0\rangle$, forming the irreducible representation (l,λ) space. These states contradict to the spectrality axiom (see Appendix 1) if $l \neq \lambda$. To be more precise, for $l \neq \lambda$ one should take $\varphi_{l\lambda}|0\rangle = 0$. However, we will tentatively disregard the spectrality axiom and consider these states as different from zero. This method simplifies the analysis of the transformation properties of the $\varphi_{l\lambda}(x)$ irreducible field. Represent the state $\varphi_{l\lambda}(x)|0\rangle$ as the decomposition (A1.6)

$$\varphi_{l\lambda}(x)|0\rangle = \sum_{m=-\infty}^{\infty} f_{lm}^{\lambda}(x)\,|l,\lambda;m\rangle,$$

where

$$f_{lm}^{\lambda}(x) = \langle m;\lambda, 1-l|l,\lambda;x\rangle \sim \frac{1}{(1+x^2)^l}\left(\frac{1-\mathrm{i}x}{1+\mathrm{i}x}\right)^{\lambda+m}.$$

It can be shown [22] that finite transformations in the one-sheeted x-space are of the form

$$\mathrm{e}^{\mathrm{i}\psi\Lambda}\varphi_{l\lambda}(x)|0\rangle = \mu_{l,\lambda}(x,\psi)\varphi_{l\lambda}(x')|0\rangle \tag{3.7}$$

$$\langle m;\lambda,1-l|\mathrm{e}^{\mathrm{i}\psi\Lambda}|l,\lambda;x\rangle = \mu_{l,\lambda}(x,\psi)\,f_{lm}^{\lambda}(x'), \tag{3.7a}$$

where $\Lambda = \frac{1}{2}(P + K)$ is the generator of the transformations (3.6),

$$\mu_{l,\lambda}(x, \psi) = [-(x + i\varepsilon)\sin\tfrac{1}{2}\psi + \cos\tfrac{1}{2}\psi]^{-l-\lambda}\cdot[-(x - i\varepsilon)\sin\tfrac{1}{2}\psi + \cos\tfrac{1}{2}\psi]^{-l+\lambda}.$$

It is essential that these transformations are explicitly dependent on the number λ. According to (A1.5) the transformation (3.7) at $\psi = 2\pi$ should take the form:

$$e^{2\pi i\Lambda}\varphi_{l\lambda}(x)\,|0\rangle = e^{2\pi i\lambda}\varphi_{l\lambda}(x)\,|0\rangle. \tag{3.8}$$

It is this condition that determines the dependence of the function $\mu_{l,\lambda}(x, \psi)$ on λ. Indeed, assuming that $\psi \to 2\pi$ in (3.7) we obtain (3.8). The value $\psi = 2\pi$ corresponds, according to (3.6), to the identity transformation in the one-sheeted space x. Hence, relation (3.8) implies that the fields $\varphi_{l\lambda}$ cannot be single-valued in a usual x-space. The same holds for the functions $f_{lm}^{\lambda}(x)$. If the region of the ψ-parameter is extended to the total real axis, from (3.7a) we will obtain an infinite set of branches $f_{lm}^{(n)\lambda}(x) = e^{2\pi i\lambda n}f_{lm}^{\lambda}(x)$ of the function $f_{lm}^{\lambda}(x)$.

Now introduce a compactified space. For this it suffices to change to the realization of the group SO(2, 1) on the circumference: $e^{i\alpha} = (1 - ix)/(1 + ix)$ or $x = \operatorname{tg}\frac{1}{2}\alpha$. In this case the circumference is the analog of the compactified Minkowski space of refs. [53, 25]. When varying the angle α in the interval $-\pi < \alpha < \pi$, the x-coordinate runs through the overall space: $-\infty < x < \infty$. Now the compact transformations (3.6) take the form $\alpha \to \alpha + \psi$ and the eigenfunctions of the generator $\Lambda = -i\partial/\partial\alpha$ are $f_{lm}^{\lambda}(\alpha) = e^{i\alpha(\lambda + m)}$. One gets the infinite-sheeted covering space by extending the range of α (definition) to the whole real axis $-\infty < \alpha < \infty$. The single sheets correspond to the intervals $-\pi + 2\pi n < \alpha < \pi + 2\pi n$, where n is an integer. Now the compact transformations of functions $f_{lm}^{\lambda}(\alpha)$ are of the form $e^{i\psi\Lambda}f_{lm}^{\lambda}(\alpha) = f_{lm}^{\lambda}(\alpha + \psi)$, any values $-\infty < \psi < \infty$ being allowed for the parameter ψ. The functions $f_{lm}^{\lambda}(\alpha)$ and hence *the states $\varphi_{l\lambda}(\alpha)\,|0\rangle$ are now defined as single-valued in the infinite-sheeted circumference covering.*

Thus we have shown that in the covering space the fields $\varphi_{l\lambda}(x)$ corresponding to various values of λ are transformed in a similar way. As a result, transformations of reducible fields (3.5) prove to be local provided that they are defined in the covering space. An analogous construction in a four-dimensional space enables us to introduce a causal order. The role of time by which the causal order is introduced is played by the analog of the above-given coordinate α, and time translations are generated by the compact operator Λ, see also [26, 34].

Now let the usual D-dimensional Minkowski space be considered again, where finite transformations are non-local and explicitly depend on λ. It is essential that this dependence arises only in those transformations which include the transformation of the SO(2) group entering into the maximal compact subgroup, since the λ-number is related just to the group SO(2), see Appendix 2, eq. (A2.5). A special conformal transformation refers in particular, to this type. It is however more advantageous, to consider directly the transformation $e^{i\alpha\Lambda}$. In the analysis given below this transformation will be used for $\alpha = 2\pi n$. For later use let us introduce the operator

$$V_n = e^{2\pi i n\Lambda}. \tag{3.9}$$

3.3. *Partial wave expansion of field states*

For further analysis of the transformation properties of interacting fields, it is necessary to consider the field states

$$\varphi(x)\,|0\rangle, \qquad \varphi(x_1)\varphi(x_2)\,|0\rangle, \qquad \varphi(x_1)\varphi(x_2)\varphi(x_3)\,|0\rangle, \ldots \tag{3.10}$$

and to decompose them to irreducible representations. This decomposition is closely related to the vacuum operator product expansion. In this case the dynamical spectrum of dimensions in the operator product expansion is determined by the spectrum of the λ-values contributing to (3.5), see point 4 of this section.

Let $\varphi_{d_i}(x)$ be the interacting scalar fields satisfying all axioms of the field theory. Consider the states $\varphi_d(x)|0\rangle$. From the comparison of (3.1, 2) with (A2.7) it may be concluded that these states can be represented as $\varphi_d(x)|0\rangle = \sum_\lambda |\sigma, \lambda; x\rangle$. Then from the spectrality axiom it follows that only the representation of the discrete series D_+, where $\lambda = d$ can contribute to the sum on the right-hand side (any other representations are inconsistent with spectrality, see Appendix 2). Hence, we have [27, 12, 50]

$$|\sigma_{d+}, x\rangle = \varphi_d(x)|0\rangle \tag{3.11}$$

where σ_+ is the representation of the D_+ series and $\sigma_d = (d, 0)$. Thus, the *states* $\varphi_d(x)|0\rangle$ *span the space for the irreducible representation* $(d, 0, \lambda = d)$. From this it follows that *they are completely determined by kinematics* and do not depend on any specific dynamics.

Consider now other states (3.10). They are transformed over reducible representations, each being decomposed into an infinite direct sum of the irreducible representations (l, s) *whose spectrum is determined by dynamics.* The spectrality axiom as applied to these states implies, that every representation contributing to the direct sum belongs to the D_+ series. Hence, with regard to (A2.13), the completeness relation in the space of states (3.10) is of the form:

$$\sum_\sigma \int \mathrm{d}x\, |\sigma_+, x\rangle \langle x, \tilde{\sigma}_+| = I \tag{3.12}$$

where I is the positive-frequency "unit" and $\sum_\sigma$ is taken over all σ_+ representations contributing to the states (3.10). In particular, for the states $\varphi_{d_1}(x_1)\varphi_{d_2}(x_2)|0\rangle$ we have [30, 27, 9, 8]

$$\varphi_{d_1}(x_1)\varphi_{d_2}(x_2)|0\rangle = \sum_\sigma A_\sigma \int \mathrm{d}x Q^{\tilde{\sigma} d_1 d_2}(x|x_1 x_2)\, |\sigma_+, x\rangle, \tag{3.13}$$

where $A_\sigma Q^{\tilde{\sigma} d_1 d_2}(x|x_1 x_2) = \langle x, \tilde{\sigma}_+ |\, \varphi_{d_1}(x_1)\varphi_{d_2}(x_2)|0\rangle$, the value A_σ is determined by the dynamics and $Q^{\tilde{\sigma} d_1 d_2}(x|x_1 x_2)$ is the properly normalized invariant three-point function. It differs from a usual three-point function* and is expressed in terms of hypergeometric functions ${}_2F_1$. This results from the non-local nature [22] of the generator Λ in the basis of vectors $|\tilde{\sigma}_+, x\rangle$, see Appendices 1, 2. The explicit form of the function $Q^{\tilde{\sigma} d_1 d_2}(x|x_1 x_2)$ has been found by one of the authors (M.Ya.P.) [22], and its analog in a one-dimensional space is given in Appendix 1. Later on we will need the Fourier transform of the function $Q^{\tilde{\sigma} d_1 d_2}(x|x_1 x_2)$ in x-coordinate. For the D-dimensional space it is of the form [7, 28]

* Another form of the expansion (3.13) using the invariant three-point functions Q' which obey the local transformation law but not the spectrality axiom is discussed in refs. [18] and [17a]. The spectrality axiom, however, is valid, as before, for the Wightman functions (3.29), since the expansion of them includes the quantity $\int \mathrm{d}y \Delta_\sigma(x-y) Q'(y|x_3 x_4)$, where the Fourier transform of the two-point function $\Delta_\sigma(x-y)$ is nonzero only at $P^2 > 0$, $P_0 > 0$. The most general form for the functions Q' and arbitrariness contained in them have been investigated by Mack in ref. [17a].

$$Q^{\tilde{\sigma}d_1d_2}(p|x_1x_2) = \int \mathrm{d}x\, \mathrm{e}^{+ipx} Q^{\tilde{\sigma}d_1d_2}(x|x_1x_2) = -\frac{2^{-s/2}N(\sigma d_1 d_2)}{\Gamma((l+d_1-d_2+s)/2)\Gamma((l-d_1+d_2+s)/2)}$$
$$\times \left(-\frac{2}{x_{12}^2}\right)^{(l+d_1+d_2+s)/2-h} D_s(\partial_{x_1}\partial_{x_2}) \left(\frac{x_{12}^2}{p^2}\right)^{(l-h+s)/2} \tag{3.14}$$
$$\times \int_0^1 \mathrm{d}u [u(1-u)]^{h/2-1} \left(\frac{1-u}{u}\right)^{(d_2-d_1)/2} \mathrm{e}^{+ip(ux_{12}+x_2)} I_{l-h+s}\{[u(1-u)x_{12}^2 p^2]^{1/2}\},$$

where $D_s(\partial_{x_1}\partial_{x_2})$ is a differential operator of rank s, see [7]. The normalization factor $N(\sigma d_1 d_2)$ can be chosen arbitrarily, since from the dynamics the product $A_\sigma N(\sigma d_1 d_2)$ is determined. For further purposes (see section 4) it is advisable to limit it by the condition

$$\frac{N(\sigma d_1 d_2)}{\Gamma((l+d_1-d_2+s)/2)\Gamma((l-d_1+d_2+s)/2)} = \frac{N(\tilde{\sigma} d_1 d_2)}{\Gamma((D-l+d_1-d_2+s)/2)\Gamma((D-l-d_1+d_2+s)/2)}. \tag{3.15}$$

Let expansion (3.13) meet the positivity condition. In this case all constants A_σ should be real and the representations σ unitary. The unitarity condition leads (with regard to (A2.11)) to the known [9, 52, 54, 55] limitations on the dimension, which in the D-dimensional space are of the form

$$l > \tfrac{1}{2}D - 1, \quad \text{if} \quad s = 0; \qquad l \geqslant D - 2 + s, \quad \text{if} \quad s \geqslant 1. \tag{3.16}$$

Note that these limitations agree with the asymptotic behaviour of the functions $Q^{\tilde{\sigma}d_1d_2}$, since these functions decrease fairly quickly [7] for $l \to \infty$ and increase for $l \to -\infty$, so that (3.16) ensures convergence of the expansion (3.13). On the contrary, the Fourier transform in x-coordinate of a usual three-point function $C^{\tilde{\sigma}d_1d_2}$, where $\tilde{\sigma} = (D-l, s)$, increases for $l \to \infty$ and it cannot enter into (3.13).

Now introduce an additional assumption on the dimension spectrum in (3.13). This spectrum should be determined from dynamics and may be both continuous and discrete. In what follows we will assume that only the discrete spectrum (referred to as σ_α) is realized. This assumption is equivalent to the hypothesis of the field algebra. From (3.11) any vector $|\sigma_\alpha, x\rangle$ may be represented as a result of the action of some tensor field O_{σ_α} with quantum numbers σ_α on the vacuum: $|\sigma_\alpha, x\rangle = O_{\sigma_\alpha}(x)|0\rangle$. As a consequence the expansion (3.13) takes the form of the vacuum operator product expansion

$$\varphi_{d_1}(x_1)\varphi_{d_2}(x_2)|0\rangle = \sum_{\sigma_\alpha} A_{\sigma_\alpha} \int \mathrm{d}x Q^{\tilde{\sigma}_\alpha d_1 d_2}(x|x_1x_2) O_{\sigma_\alpha}(x)|0\rangle. \tag{3.17}$$

It should be stressed that the assumption of the l-spectrum being discrete is not essential. All the results obtained below are readily generalized for the case of a continuous spectrum, see [22].

Finally, consider the transformation (3.9) applied to the states (3.10). From (A2.14) and (3.11) (see also (3.8)) we have

$$V_n \varphi_d(x)|0\rangle = \mathrm{e}^{2\pi i d n} \varphi_d(x)|0\rangle. \tag{3.18}$$

If the transformation V_n is applied to the both parts of the expansion (3.17), we find [9, 21, 22]

$$V_n\varphi_{d_1}(x_1)\varphi_{d_2}(x_2)|0\rangle = \sum_{\sigma_\alpha} e^{2\pi i l_\alpha n} A_{\sigma_\alpha} \int dx Q^{\tilde{\sigma}_\alpha d_1 d_2}(x|x_1x_2)|\sigma_\alpha, x\rangle. \tag{3.19}$$

Note that the phase factors in (3.18) and (3.19) do not appear if these states are defined in the universal covering of the compactified space.

3.4. Decomposition of interacting fields into irreducible ones

Generalized free fields, to be more precise their positive- and negative-frequency parts, are the example of irreducible fields. In the case of generalized free fields states (3.10) are determined as direct products of one-particle states. For example, we have

$$\varphi_{d_1}^{(\pm)}(x_1)\varphi_{d_2}^{(+)}(x_2)|0\rangle = |d_1, x_1\rangle_\pm \otimes |d_2, x_2\rangle_+, \tag{3.20}$$

where $\varphi_d^{(\pm)}$ are the negative- and positive-frequency parts of the field $\varphi_d(x)$, $|d, x\rangle_\pm$ is the vector from the space of irreducible representation of the $D_\pm$ discrete series, see Appendix 2. (The states $\varphi_{d_1}^{(-)}\varphi_{d_2}^{(+)}|0\rangle$ differ from zero only at $d_1 = d_2$.) From (3.18) and (3.20) we find

$$V_n\varphi_{d_1}^{(\pm)}(x_1)\varphi_{d_2}^{(+)}(x_2)|0\rangle = e^{2\pi i(d_2 \pm d_1)}\varphi_{d_1}^{(\pm)}(x_1)\varphi_{d_2}^{(+)}(x_2)|0\rangle. \tag{3.21}$$

A similar result is obtained for other states (3.10). From this it follows that the transformation law for generalized free fields is [9, 13, 8, 22]

$$V_n\varphi_d^{(\pm)}(x)V_n^{-1} = e^{\pm 2\pi i d n}\varphi_d^{(\pm)}(x). \tag{3.22}$$

Consider now the irreducible scalar fields $\varphi_{d,\lambda}(x)$ which are transformed according to the representations (d, λ), where $\lambda \neq \pm d$. For these fields we have $\varphi_{d,\lambda}(x)|0\rangle = 0$, since the states $|d, \lambda; x\rangle$ do not satisfy spectrality (see Appendix 2). At the same time we will define their transformation law according to (A2.8), see also (3.8)

$$V_n\varphi_{d,\lambda}(x)V_n^{-1} = e^{2\pi i\lambda n}\varphi_{d,\lambda}(x). \tag{3.23}$$

Such fields were considered by Schroer and Swieca in ref.* [13]. Note that in contrast to generalized free fields the Fourier transform $\varphi_{d,\lambda}(p) \neq 0$ for $p^2 < 0$.

Any interacting field $\varphi_d(x)$ is a superposition (3.5) of the fields $\varphi_{d,\lambda}(x)$. Indeed, even in the case of generalized free fields we have: $\varphi_d(x) = \varphi_{d,\lambda=d}(x) + \varphi_{d,\lambda=-d}(x)$. For interacting fields the decomposition (3.5) should contain at least one more field $\varphi_{d,\lambda}(x)$, with $\lambda \neq \pm d$, since due to interaction $\varphi_{d,\lambda}(p) \neq 0$ at $p^2 < 0$. It is natural to expect that in this case $\sum_\lambda$ in (3.5) comprises an infinite number of terms. To find the decomposition of the interacting field $\varphi_d(x)$ into irreducible ones consider its "branches": $\varphi_d^{(n)}(x) = V_n\varphi_d(x)V_n^{-1}$ (in the compactified space this transformation would correspond to the transition onto the nth sheet of the covering space, see subsection 2). Now the set of $\varphi_{d,\lambda}$ fields entering into (3.5) may be represented in the form (see also [13]):

$$\varphi_{d,\lambda}(x) = \sum_{n=-\infty}^{\infty} \varphi_d^{(n)}(x)\, e^{-2\pi i\lambda n} = \sum_{n=-\infty}^{\infty} V_n\varphi_d(x)V_n^{-1}\, e^{-2\pi i\lambda n}. \tag{3.24}$$

* The definition of λ adopted here and in refs. [8, 9, 21, 22] is related to the parameter ξ introduced in [13, 18, 19] by the relation $\xi = (d - \lambda)/2$. The operator V_n also differs from the operator Z used in [13, 18, 19].

The inverse transformation is of the form:

$$\varphi_d^{(n)}(x) \sim \int \mathrm{d}\lambda \varphi_{d,\lambda}(x)\, \mathrm{e}^{2\pi i \lambda n}. \tag{3.25}$$

Decomposition (3.5) is a special case of this transformation at $n = 0$. Note that λ takes values in the interval $-\frac{1}{2} < \lambda < \frac{1}{2}$, since the representations (d, λ) and $(d, \lambda + 1)$ are equivalent, see Appendix 2. For the discrete series we may redefine λ so that $\lambda = \pm(d - [d])$ where $[d]$ is the integral part of d. With this definition the positive- and negative-frequency fields also contribute to (3.5) and (3.25).

Consider some limitations on the spectrum of dimensions contributing to the expansion of the the type (3.17). Comparing (3.21) and (3.19), we find that for the generalized free fields the transformation law allows the following dimension values [9, 5]

$$l_\alpha = d_2 \pm d_1 + k, \qquad k \text{ is an integer} \tag{3.26}$$

for the states $\varphi_{d_1}^{(\pm)}\varphi_{d_2}^{(+)}|0\rangle$. These dimensions form a kinematic spectrum [5]. In the general case the expansion (3.17) may comprise fields whose dimensions differ from (3.26) (dynamic spectrum). Represent the interacting field in the form of (3.5) and consider the states $\varphi_{d_1\lambda_1}(x_1)\varphi_{d_2}(x_2)|0\rangle$. Using (3.23), (3.18) and (3.21), for the dynamic spectrum we find [13]

$$l_\alpha = \lambda_1 + d_2 + m, \qquad m \text{ is an integer,} \tag{3.27}$$

where λ_1 characterizes an irreducible component of the field φ_{d_1}. Thus, *the problem of dynamics consists in elucidation of the spectrum of representations* (d, λ) *in the expansion* (3.5).

In conclusion note that more detailed information on the kinematic spectrum of the dimensions (3.26) may be obtained by the decomposition of the direct product of representations in (3.20) into irreducible ones. It is more appropriate to perform special calculations in the Euclidean space using completeness relations (5.4), for more detail see [8, 22]. It may be shown that [8, 21, 22]

$$l_\alpha = d_1 + d_2 + s + 2m, \qquad m \text{ is an integer} \tag{3.26a}$$

for the states $\varphi_{d_1}^{(+)}\varphi_{d_2}^{(+)}|0\rangle$ and

$$l_\alpha = 0 \tag{3.26b}$$

for the states $\varphi_d^{(-)}\varphi_d^{(+)}|0\rangle$. Dimensions (3.26a) correspond to the tensor fields:

$$O_s^m(x) = \lim_{x' \to x} \mathrm{D}_s(i\partial_x, i\partial_{x'})\{\partial^x_{\mu_1\ldots\mu_m}\varphi_{d_1}^{(+)}(x)\partial^{x'}_{\mu_1\ldots\mu_m}\varphi_{d_2}^{(+)}(x')\},$$

where D_s is a differential operator of rank s, and $\partial^x_{\mu_1\ldots\mu_m}$ means m-fold differentiation. These fields enter into operator product expansion (3.17) for the states $\varphi_{d_1}^{(+)}(x_1)\varphi_{d_2}^{(+)}(x_2)|0\rangle$. The value of dimension (3.26b) corresponds to a unit operator by which the expansion of states $\varphi_d^{(-)}(x_1)\varphi_d^{(+)}(x_2)|0\rangle$ is exhausted.

In the same manner [8, 22] an operator product expansion can be obtained for free massless fields. It includes the conserved tensors with the dimension $D - 2 + s$, where $s \geqslant 1$, see also [7, 18].

3.5. Partial wave expansion of Wightman functions

A complete set of Wightman functions can be obtained, calculating all possible invariant scalar products of states (3.10). Representing these states as an expansion of type (3.13) or (3.17), we obtain a partial wave expansion for the Wightman functions. Consider, e.g. a two-point Wightman function. According to (3.11) it is equal to

$$W_d(x_1 - x_2) = \langle 0|\, \varphi_d(x_1)\varphi_d(x_2)\,|0\rangle = \langle x_1, \sigma_{d+}|\sigma_{d+}, x_2\rangle \sim \frac{1}{(-x_{12}^2 + i\varepsilon x_{12}^0)^d}. \tag{3.27}$$

Analogously, for three-point functions we have from (3.11) and (3.13):

$$W(x_1x_2x_3) = \langle 0|\, \varphi_{d_1}(x_1)\varphi_{d_2}(x_2)\varphi_{d_3}(x_3)\,|0\rangle = \langle x_1, \sigma_{d+}|\,\varphi_{d_2}(x_2)\varphi_{d_3}(x_3)\,|0\rangle$$
$$= \int dx W_{d_1}(x_1 - x)Q^{\tilde{d}_1 d_2 d_3}(x|x_2x_3) = C^{d_1 d_2 d_3}(x_1x_2x_3), \tag{3.28}$$

where $C^{d_1d_2d_3}(x_1x_2x_3)$ is function (2.9) in pseudo-Euclidean coordinates. In derivation of (3.28) it has been taken into account that as a consequence of (2.6), from the total sum in (3.13) only one term with quantum numbers of the fields φ_{d_1} contributes to (3.28).

Consider a four-point Wightman function. From (3.17) we find [9, 12, 22, 28]

$$W(x_1x_2x_3x_4) = \langle 0|\, \varphi_{d_1}(x_1)\varphi_{d_2}(x_2)\varphi_{d_3}(x_3)\varphi_{d_4}(x_4)\,|0\rangle$$
$$= \sum_{\sigma_\alpha} \rho_{\sigma_\alpha} \int dx\, dy [Q^{\tilde{\sigma}_\alpha d_1 d_2}(x|x_1x_2)]^* \Delta_{\sigma_\alpha}(x - y) Q^{\tilde{\sigma}_\alpha d_3 d_4}(y|x_3x_4), \tag{3.29}$$

where $\rho_{\sigma_\alpha} = A^*_{\sigma_\alpha}A_{\sigma_\alpha}$, $\Delta_\sigma(x - y) = \langle x, \sigma_+|\sigma_+, y\rangle$ is the invariant scalar product. Here the axioms of spectrality and positivity have been taken into consideration. *Spectrality means that only the discrete series representations contribute to* (3.29). *This results in the appearance of the* $Q^{\tilde{\sigma}}$*-functions in the expansion, and in addition ensures the reality of dimensions* l_α (see condition (A2.10)) *and the absence of infinite-component (in spin indices) representations* (these representations are also present in other series). *Positivity implies that the dimensions* l_α *are limited by condition* (3.16) *and the expansion* (3.29) *converges* (since $Q^{\tilde{\sigma}}$ diminishes for $l \to \infty$), *in addition the constants* ρ_{σ_α} *are positive.*

In a similar way the higher Wightman functions can be written. For example, for a five-point function we have [9, 5, 8]

$$W(x_1x_2x_3x_4x_5) = \langle 0|\, \varphi_1\varphi_2\varphi_3\varphi_4\varphi_5\,|0\rangle = \sum_{\sigma_1\sigma_2} \langle 0|\, \varphi_1\varphi_2\,|\sigma_1\rangle\langle\tilde{\sigma}_1|\, \varphi_3\,|\sigma_2\rangle\langle\tilde{\sigma}_2|\, \varphi_4\varphi_5\,|0\rangle, \tag{3.30}$$

where σ_1 and σ_2 are the representations of the D_+-series.

4. Analytical continuation to Euclidean region of coordinates

We discuss now the analytical continuation of the expansion (3.29) to the Euclidean coordinates. Consider the Wightman function $W(x_1x_2x_3x_4)$ for relatively space-like arguments $x_1 \ldots x_4$. Its value for any arguments, including the Euclidean ones, can be obtained by the analytical continuation through an extended tube. In the Euclidean region of arguments the left-hand side of (3.29) coincides with the Euclidean Green function $G(x_1x_2x_3x_4)$. In the right-hand side it is necessary, first, to analytically continue each term of the sum over σ_α to the Euclidean values $x_1 \dots x_4$, and, second, to represent the integral in the internal line as an integral over the Euclidean space. This program has been realized by one of the authers (M.Ya.P.) in [22], see also [5, 8]. It is appropriate to do it in two steps: first to change in (3.29) from the summation over σ_α to the contour integral over dimension, and then to perform the analytical continuation of the internal integral to that in the Euclidean space. It should be stressed that the latter is possible since *it are the $Q^{\tilde{\sigma}}$ functions that enter into the integrand rather than the $C^{\tilde{\sigma}}$-ones* (or superposition of Q and C functions).

Introduce the analytical function ($Q^{d_1d_2\tilde{\sigma}}(x_1x_2|x) \equiv [Q^{\tilde{\sigma}d_2d_1}(x|x_2x_1)]^*$)

$$I(\sigma) = \frac{2\pi\rho(\sigma)}{\sin\pi(l-h+s)}\int \mathrm{d}x\,\mathrm{d}x' Q^{d_1d_2\tilde{\sigma}}(x_1x_2|x)\Delta_\sigma(x-x')Q^{\tilde{\sigma}d_3d_4}(x'|x_3x_4) \tag{4.1}$$

in the complex plane of dimension. For $\mathrm{Re}\, l > h$ define it by the conditions:

1. All poles of $I(\sigma)$ are positioned in the real axis at the points $l = l_\alpha$.
2. The residues in these poles are equal to

$$-\operatorname*{res}_{\sigma=\sigma_\alpha}\{\mu(\sigma)I(\sigma)\} = \rho_{\sigma_\alpha}\int \mathrm{d}x\,\mathrm{d}x' Q^{d_1d_2\tilde{\sigma}_\alpha}(x_1x_2|x)\Delta_{\sigma_\alpha}(x-x')Q^{\tilde{\sigma}_\alpha d_3d_4}(x'|x_3x_4) \tag{4.2}$$

where $\mu(\sigma) = \mu(\tilde{\sigma})$ is a weight function introduced as a matter of convenience and having no poles, see (5.8). Now the expansion (3.29) may be written as a contour integral:

$$W(x_1x_2x_3x_4) = \sum_s \int_C \mathrm{d}l I(\sigma) + \dots,$$

where points stand for the contribution of the terms with quantum numbers $\sigma_\alpha = (l_\alpha, 0)$, $h-1 < l_\alpha < h$, and the contour C is shown in fig. 1.

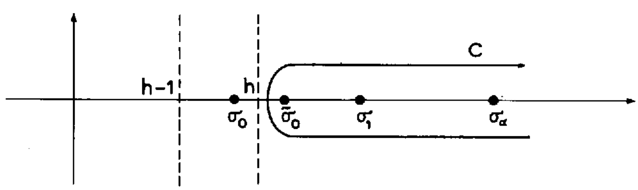


Fig. 1.

The conditions (4.2) do not yet determine the function $I(\sigma)$ unambiguously, since nothing limits its behaviour in the range $\mathrm{Re}\, l < h$. It is evident that this ambiguous definition does not influence the value of the contour integral over l. Note, however, that the analytical continuation to the Euclidean coordinates in each term of the expansion (3.29) is to be realized. As shown in [8, 22], *it proves possible provided that the definition of the function $I(\sigma)$ is completed by the condition*

$$\rho(\sigma) = \rho(\tilde{\sigma}). \tag{4.3}$$

In fact, the condition of type (4.3) should have been imposed on the product $N_Q(\sigma d_1 d_2)N_Q(\sigma d_3 d_4) \times \rho(\sigma)$ where $N_Q(\sigma d_1 d_2)$ is the normalization factor of the function $Q^{\tilde{\sigma}d_1d_2}$, see (3.14). But taking into account (3.15), this condition reduces to (4.3).

It remains to discuss the contribution of terms with the dimensions $l_\alpha < h$. From (3.16) all the poles of $I(\sigma)$ at the points $\sigma_\alpha = (l_\alpha, s_\alpha)$, where $s_\alpha \neq 0$, lie to the right of the point $l = h$ and are embraced by the contour C. For $s_\alpha = 0$ one (or several) term is possible whose dimension is in the interval $h - 1 < l_\alpha < h$. One of these terms is shown in fig. 1 as point $\sigma_0 = (l_0, 0)$. We will make the assumption that at these points the function $\rho(\sigma)$ also has poles whose residues are determined by the condition (4.2). From (4.3) it follows that each pole has its associated shadow pole $\tilde{l}_\alpha = D - l_\alpha$ located to the right of the point h and contributing to the contour integral. Hence additional terms should be introduced to compensate these contributions.

Let the exponential decrease of the functions $Q^{\tilde{\sigma}}$ in the right-hand half-plane be used and the integration contour in fig. 1 be unbended in the complex plane. As a result we obtain [8, 22]

$$W(x_1x_2x_3x_4) = \rho_{\sigma_0} \int \mathrm{d}x\, \mathrm{d}x' \,\{Q^{d_1d_2\tilde{\sigma}_0}(x_1x_2|x)\Delta_{\sigma_0}(x - x')Q^{\sigma_0 d_3d_4}(x'|x_3x_4)$$

$$- Q^{d_1d_2\sigma_0}(x_1x_2|x)\Delta_{\tilde{\sigma}_0}(x - x')Q^{\sigma_0 d_3d_4}(x'|x_3x_4)\}$$

$$+ \frac{1}{2\pi \mathrm{i}} \sum_s \int_{h-\mathrm{i}\infty}^{h+\mathrm{i}\infty} \mu(l)\, \mathrm{d}l \,\frac{2\pi}{\sin \pi(l - h + s)}$$

$$\times \rho(\sigma) \int \mathrm{d}x\, \mathrm{d}x' Q^{d_1d_2\tilde{\sigma}}(x_1x_2|x)\Delta_\sigma(x - x')Q^{\tilde{\sigma}d_3d_4}(x'|x_3x_4). \tag{4.4}$$

The second term in braces has been introduced to reduce the shadow pole contribution at the point $\tilde{\sigma}_0$. Expression (4.4) may be analytically continued to the Euclidean coordinates. The continuation in $x_1^0 \ldots x_4^0$ coordinates causes no difficulty. It remains to discuss the continuation in the internal coordinates x^0 and $x^{0\prime}$. This procedure is given in detail in [22] (see also [7], where an inverse problem was solved). Here we will cite only the main steps. Change in (4.4) to the momentum space in the internal variables and consider the function $Q^{\tilde{\sigma}}(x_1x_2|p)\Delta_\sigma(p)Q^{\tilde{\sigma}}(p|x_3x_4)$. It has a branching point $P^2 = 0$. The momentum integral is taken over the region $P^2 > 0$, $P_0 > 0$. Represent it as a contour integral embracing the cut drawn from the point $P^2 = P_0^2 - |\boldsymbol{P}|^2 = 0$. Then change to the variable $P_D = -\mathrm{i}P_0$ and unbend the integration contour in the complex plane of P_D. As a result we obtain the integral over the Euclidean momentum space.

Note, however, that when unbending the integration contour in the plane of P_D the integrand definition should be extended to the total complex plane from its discontinuity on the cut. It should be done so that first the integrand may decrease in a lower half-plane of P_D (this is necessary for the contour deformation) and, second, may be expressed in terms of the Euclidean three-point Green functions. To meet these conditions, it is sufficient in integral (3.14) to make the substitution

$$\frac{\pi}{\sin \pi\nu} I_\nu(z) \to K_\nu(z) = -\frac{\pi}{2 \sin \pi\nu}[I_\nu(z) - I_{-\nu}(z)]. \tag{4.5}$$

The function $K_\nu(z)$ is responsible for the Euclidean three-point function [28].

It is essential that the above program is possible due to the following properties of the expansion (4.4): a) the function $\rho(\sigma)$ satisfies the condition (4.3) and the normalization of the functions Q meets condition (3.15); b) the range of integration over l in (4.4) is symmetrical with respect to the substitution $l \to D - l$. Owing to these properties, the right-hand part of the expansion (4.4) does not change under the substitution (4.5).

Thus, the final version of the partial wave expansion in the Euclidean region of coordinates is of the form

$$G(x_1x_2x_3x_4) = -2\mu(\sigma_0) \operatorname*{res}_{\sigma=\sigma_0} \rho(\sigma) \int \mathrm{d}x\, \mathrm{d}x' C^{d_1d_2\tilde{\sigma}_0}(x_1x_2x)\Delta_{\sigma_0}(x - x')C^{\tilde{\sigma}_0 d_3d_4}(x'x_3x_4)$$

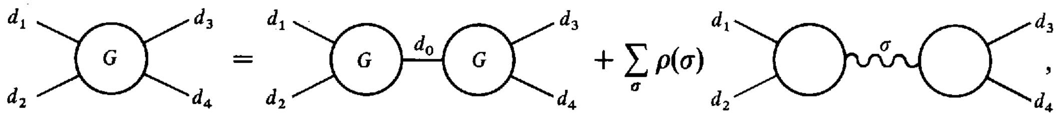
$$+ \frac{1}{2\pi\mathrm{i}} \sum_s \int_{h-\mathrm{i}\infty}^{h+\mathrm{i}\infty} \mathrm{d}l\mu(\sigma)\rho(\sigma) \int \mathrm{d}x\, \mathrm{d}x' C^{d_1d_2\tilde{\sigma}}(x_1x_2x)\Delta_\sigma(x - x')C^{\tilde{\sigma}d_3d_4}(x'x_3x_4), \quad (4.6)$$

where Δ_σ and $C^{\sigma d_1 d_2}$ are the Euclidean conformal-invariant three-point functions (2.4) and (2.9), and the integrals over x and x' are taken in the Euclidean space.

In conclusion, it should be noted that the above derivation of expansions (4.4) and (4.6) is readily generalized for the fields with an arbitrary spin, for more detail see [22].

5. Partial wave expansion in Euclidean space

Rewrite expansion (4.6) in the notations of section 2:



(5.1)

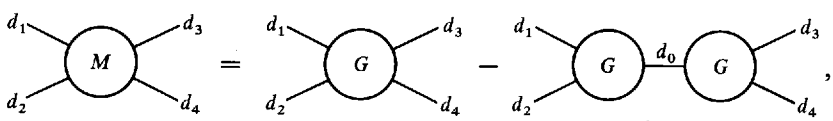
where

$$\sum_\sigma = \frac{1}{2\pi\mathrm{i}} \sum_s \int_{h-\mathrm{i}\infty}^{h+\mathrm{i}\infty} \mathrm{d}l\mu(\sigma) \quad (5.1a)$$

and the following relation is assumed

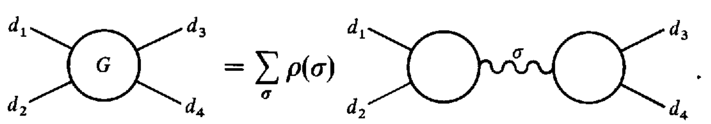
$$-2\mu(\sigma_0) \operatorname*{res}_{\sigma=\sigma_0} \rho(\sigma) = g_{d_1d_2d_0} g_{d_3d_4d_0},$$

see sections 9, 11. As follows from the derivation of the expansion (5.1), it can be applied to the one-particle irreducible vertex



(5.1b)

where $d_0 < h$. If in (3.17) there are no fields with the dimensions $l_\alpha < h$, then the partial wave expansion is applicable directly to the Green function G:



(5.2)

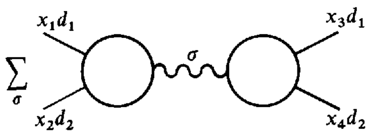
In special theories considered below, either expansion (5.1) or (5.2) should be used, depending on the value of the fundamental field dimension.

In (5.1) the dimensions of fields $\varphi_{d_1} \ldots \varphi_{d_4}$ are assumed to be different. In the case when all fields are identical, i.e. $d_1 = d_2 = d_3 = d_4$, the right-hand side of (5.1) (or (5.2)) should be added by one disconnected graph $G_d(x_{12})G_d(x_{34})$, where $G_d(x)$ is the function (2.5b). This follows directly from (3.17). In the case of identical fields the summation $\sum_{\sigma_\alpha}$ in the right-hand side of (3.17) begins with the term $\Delta_d(x_{12})I$, where I is the unit operator. This term is responsible for the contribution of the trivial representation with zero dimension to (3.13). It is not embraced by the integration contour in fig. 1 and, hence, it does not contribute to $\sum_\sigma$ in (5.1) and (5.2). As a consequence of (2.6), it is absent at $d_1 \neq d_2$. In what follows we will use partial wave expansion (5.1) or (5.2) for the connected Green function which does not contain the term $G_d(x_{12})G_d(x_{34})$.

Note, that as yet the function $\rho(\sigma)$ is not unambiguously defined. It is the function $I(\sigma)$ rather than $\rho(\sigma)$ that is fixed by the conditions (4.2) and (4.3). The function $\rho(\sigma)$ depends on the choice of the normalization factor in (3.14) which on account of (3.15) might be included into $\rho(\sigma)$ without breaking condition (4.3). We will define this factor so that [5, 8, 22] the function $\rho(\sigma)$ contains only dynamic information and goes to unity in the absence of interaction. It implies that this function should have only dynamical poles (3.27), and the kinematical spectrum (3.26a) should be included into the product of normalization factors of the functions (3.14). Thus, for generalized free fields we have

$$\rho_{\sigma_0} = 0, \qquad \rho(\sigma) = 1. \tag{5.3}$$

This condition determines the normalization factors $N(\sigma d_1 d_2)$. For their calculation let the partial wave expansion (3.13) for generalized free fields be considered. From (3.20) it follows that it may be obtained as a result of the decomposition of the direct product of the representations into irreducible components. The functions $Q^{\tilde\sigma}$ entering into this decomposition are the Clebsch–Gordan kernels. The expansion (5.1), using (5.3), takes the form



$= G_{d_1}(x_{13})G_{d_2}(x_{24})$ (5.4)

and acts as a completeness relation in the Euclidean space.

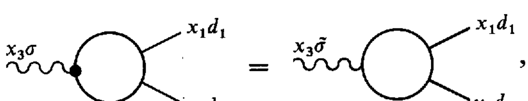
From (5.4) the product of the normalization factors of the vertices and the propagator standing on the internal line is defined. The propagator normalization will be determined by the condition

$$\int dx \Delta_\sigma(x_1 - x)\Delta_{\tilde\sigma}(x - x_2) = I \cdot \delta(x_1 - x_2), \quad \text{or} \quad \Delta_\sigma^{-1}(x) = \Delta_{\tilde\sigma}(x). \tag{5.5}$$

In addition let it be demanded that the functions $C^{\sigma d_1 d_2}(x_3 x_1 x_2)$ meet the condition of amputation

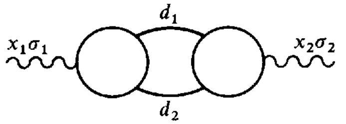
$$\int dx \Delta_\sigma^{-1}(x_3 - x)C^{\sigma d_1 d_2}(x x_1 x_2) = C^{\tilde\sigma d_1 d_2}(x_3 x_1 x_2) \tag{5.6}$$

and analogously in the arguments x_1 and x_2. In graphical notations we have



, (5.6a)

where the point stands for the leg x_3 amputation. The function $C^{\sigma d_1 d_2}$ satisfies the orthogonality relation, see (2.6). Taking into account (5.5) and (5.6) it may be written in the form



$$= \tfrac{1}{2}[\delta_{\sigma_1\sigma_2}\Delta_{\sigma_2}(x_{12}) + \delta_{\sigma_1\tilde{\sigma}_2}\delta(x_1 - x_2)], \tag{5.7}$$

where $\delta_{\sigma_1\sigma_2}$ is defined by the condition $\sum_{\sigma'} \delta_{\sigma\sigma'} f(\sigma') = f(\sigma)$, $\sum_\sigma$ was defined in (5.1a) and the internal lines stand for inverse propagators. It should be noted that an additional coefficient will appear in the second term in the right-hand side of (5.7) provided that the conditions (5.5) and (5.6) are rejected.

The calculation of normalizations from the conditions (5.5)–(5.7) is given in refs. [32, 5], see also [5, 6] and [56], where the normalization factors are calculated in the presence of spinor and vector fields. In the considered case of scalar fields we have (the normalization from refs. [8, 56] is used):

$$\mu(\sigma) = \tfrac{1}{4}2^s(2\pi)^{-h}\frac{\Gamma(h+s)}{\Gamma(s+1)}n(\sigma)n(\tilde{\sigma}), \tag{5.8}$$

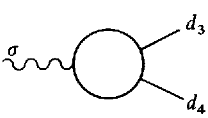
$$n(\sigma) = \frac{\Gamma(l+s)}{\Gamma(h-l)}\,\frac{\Gamma(D-l-1)}{\Gamma(D-l+s-1)}, \tag{5.9}$$

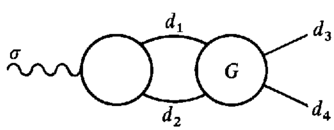
$$N(\sigma d_1 d_2) = \left\{\frac{\Gamma\left(\frac{l+d_1+d_2+s-D}{2}\right)\Gamma\left(\frac{d_1+d_2-l+s}{2}\right)\Gamma\left(\frac{l-d_1+d_2+s}{2}\right)\Gamma\left(\frac{l+d_1-d_2+s}{2}\right)}{\Gamma\left(\frac{2D-l-d_1-d_2+s}{2}\right)\Gamma\left(\frac{D+l-d_1-d_2+s}{2}\right)\Gamma\left(\frac{D-l-d_1+d_2+s}{2}\right)\Gamma\left(\frac{D-l+d_1-d_2+s}{2}\right)}\right\}^{1/2} \tag{5.10}$$

where $\mu(\sigma)$ is the weight function in (5.1a) and $n(\sigma)$ is the normalization factor of the propagator (2.5). The principal relationships required for calculations are given in Appendix 3. This normalization differs from that in refs. [3–7, 32, 37]. The functions $C^{\sigma d_1 d_2}$, Δ_σ and $\mu(\sigma)$ are related to the analogous functions (C', Δ', μ') of refs. [3, 7, 32] and the functions C'', Δ'', μ'' of refs. [5, 6, 37] by the relationships

$$C = 2^{-s/2}C' = (4\pi)^{2h}\cdot 2^{h-(l+d_1+d_2)/2}C'', \qquad \Delta_\sigma = \Delta'_\sigma = 2^{h-l}\Delta''_\sigma, \qquad \mu = 2^{s-1}\mu' = (4\pi)^{-4h}\cdot 2^h\mu''.$$

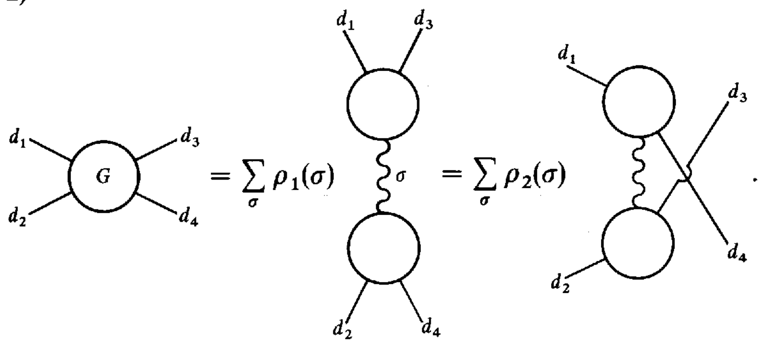
Using orthogonality relation (5.7) as well as (5.6), the function $\rho(\sigma)$ can readily be determined

$\rho(\sigma)$ 

$=$  . (5.11)

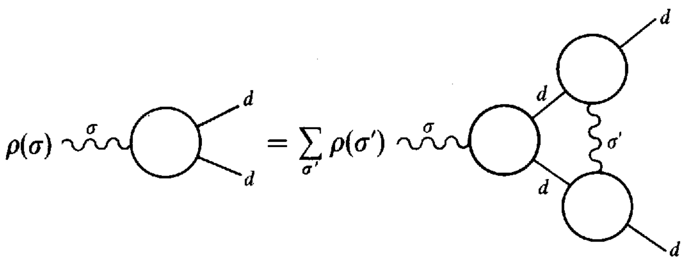
Equations (5.7) and (5.11) are well defined for the values of l lying on the integration contour in (5.1a), i.e. $l = h + i\nu$, $-\infty < \nu < \infty$. One defines them for any complex values of l by the analytical continuation from the integration contour. The possibility of this analytical continuation is ensured by the analytical properties of function $\rho(\sigma)$. Now relations (5.7) and (5.11) may be used in the entire complex plane of the dimension l. Note that with this method of determination of $\rho(\sigma)$ the first graph in (5.1) does not contribute to (5.11) even at $l = d_0$.

Besides (5.1) or (5.2) we may write partial wave expansions by two more methods. As a matter of simplicity, assume that there are no contributions with the dimensions $d_0 < h$. Then in addition to (5.2) we have



(5.2a)

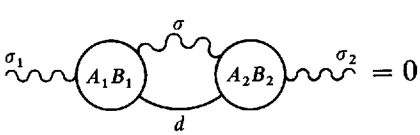
Generally speaking, the functions ρ, ρ_1 and ρ_2 are different. In the case of identical fields function (5.2) is symmetrical in relation to the coordinate permutations, so that one should assume $\rho(\sigma) = \rho_1(\sigma) = \rho_2(\sigma)$. As a result we obtain an additional linear homogeneous integral equation for $\rho(\sigma)$, a so-called crossing-symmetry equation [3–7, 29, 31]



(5.12)

At present the complete analysis of these integral equations has not been fulfilled.

Now let it be assumed that one or several fields entering into the Green function $G(x_1 \ldots x_4)$ are tensors. Then the expansion (5.1) comprises the three-point functions $C^{\sigma_1\sigma_2\sigma_3}(x_1x_2x_3)$ with two or three tensor legs. These functions depend on several arbitrary constants $A, B, C, \ldots$, see, e.g. (2.10), which enables us to determine several mutually-orthogonal sets of the three-point functions [56]. Consider, e.g., the functions (2.10) depending on two constants. Properly choosing the constants A and B, two mutually-orthogonal sets of these functions may be determined

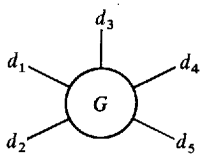


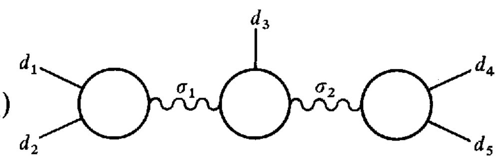
(5.13)

for any values of σ_1, σ_2 and fixed σ. Generally speaking, both sets contribute to the partial wave expansion of vertices. If the functions of each set meet the orthogonality condition, along with (5.13) we obtain [56] three relations between four constants A_1, B_1, A_2, B_2. Hence the functions $\{A_i, B_i\}$ may be chosen in different ways. Using this ambiguity, two orthogonal sets $\{A_1B_1\}$ and

$\{A_2B_2\}$ may be chosen, so that the partial wave expansion of the vertex given may comprise only one of them. In section 7 we will illustrate it by a special example.

In conclusion, it should be noted that the partial wave expansion of type (5.1) or (5.2) may also be obtained for the higher Green functions. In particular, the analog of expansion (3.30) in the Euclidean space is



$= \sum_{\sigma_1\sigma_2} \rho(\sigma_1\sigma_2)$


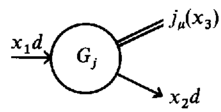
. (5.14)

Here it is assumed that the dimensions $d_1 \dots d_5 > h$.

6. Green functions containing conserved tensors

6.1. Three-point Green functions

Consider the Green function

$$G_\mu(x_1x_2x_3) = \langle 0|\, T\varphi_d(x_1)\varphi_d^+(x_2)j_\mu(x_3)|0\rangle =$$


(6.1)

where $\varphi_d(x)$ is a charged scalar field, $j_\mu(x)$ is a conserved current. This function meets the generalized Ward identity [57]

$$\partial_\mu^{x_3}G_\mu(x_1x_2x_3) = -e[\delta(x_1 - x_3) - \delta(x_2 - x_3)]G_d(x_{12}) \tag{6.2}$$

which fixes the current quantum numbers

$$\sigma_j = (D - 1, 1). \tag{6.3}$$

In accordance to (2.9) the general expression of a three-point function with these quantum numbers is

$$G_\mu(x_1x_2x_3) = g_j(d)\frac{1}{(\frac{1}{2}x_{12}^2)^{d-h+1}}\left\{\frac{1}{(\frac{1}{2}x_{23}^2)^{h-1}}\overleftrightarrow{\partial}_\mu^{x_3}\frac{1}{(\frac{1}{2}x_{13}^2)^{h-1}}\right\} \tag{6.4}$$

where $g_j(d)$ is the coupling constant. Substituting (6.4) to (6.2) we find that in normalization (2.5) and (5.9, 10) it equals

$$g_j(d) = \frac{e}{2}(2\pi)^{-D}\Gamma(h - 1)\frac{\Gamma(d)}{\Gamma(h - d)}. \tag{6.5}$$

If the Green function contains spinor or tensor fields, then it depends on two unknown constants. The Ward identity fixes only one of them, see (6.9, 10). The second constant corresponds to the transverse part of the Green function and remains arbitrary (i.e. it should be found from dynamics).

Let O_{σ_1} and O_{σ_2} be arbitrary tensor fields. One then writes,

$$\langle 0|\, TO_{\sigma_1}(x_1)O_{\sigma_2}(x_2)j_\mu(x_3)\,|0\rangle = 0, \qquad \text{if} \quad \sigma_1 \neq \sigma_2. \tag{6.6}$$

As it follows from the Ward identity and relation (2.6)

$$\partial_\mu^{x_3} \langle 0|\, TO_{\sigma_1}(x_1)O_{\sigma_2}(x_2)j_\mu(x_3)\,|0\rangle = -e[\delta(x_1 - x_3) - \delta(x_2 - x_3)]\, \langle 0|\, TO_{\sigma_1}(x_1)O_{\sigma_2}(x_2)\,|0\rangle. \tag{6.7}$$

It should be stressed that this property is not fulfilled for all fields. It holds only for those Green functions which satisfy the Ward identity without Schwinger terms. There are, however, fields whose equal-time commutator with current contains Schwinger terms. In this case it is possible that the condition for the theorem (6.6) may not be satisfied. As shown in [58], each scalar field $\varphi_d(x)$ may be associated with the set of tensor fields $O^d_{\mu_1\ldots\mu_s}(x)$ with the quantum numbers

$$\sigma_s = (d + s, s). \tag{6.8}$$

The equal-time commutator $[j_0(x), O^d_s(y)]$ of such fields in addition to a usual term $\delta(x - y)O^d_s(y)$ resulting in the right-hand side of (6.7) comprises terms proportional to the field $\varphi_d(x)$ and its derivatives. The general expression for Green function containing these fields has been obtained by us in [58]. It comprises contact terms and is of the form*

$$\begin{aligned} G_{\mu,\mu_1\ldots\mu_s}(x_1x_2x_3) &= \langle 0|\, TO^d_{\mu_1\ldots\mu_s}(x_1)j_\mu(x_2)\varphi_d^+(x_3)\,|0\rangle \\ &= [g^s_j(d) + f_s(d)]2^{s/2}\left\{\frac{1}{(\frac{1}{2}x^2_{23})^{h-1}}\overleftrightarrow{\partial}{}^{x_2}_\mu\left[\frac{1}{(\frac{1}{2}x^2_{12})^{h-1}}(\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_2x_3) - \text{traces})\right]\right\}\frac{1}{(\frac{1}{2}x^2_{13})^{d-h+1}} \\ &+ f_s(d)2^{s/2+1}(2\pi)^h\frac{\Gamma(s)}{\Gamma(h+s-1)}\sum_{k=1}^s\frac{(-2)^{-k}}{\Gamma(k+1)} \\ &\times\left[\delta_{\mu\nu_1}\partial^{x_2}_{\nu_2}\ldots\partial^{x_2}_{\nu_k}\delta(x_1 - x_2)\frac{(x_{13})_{\nu_{k+1}}}{x^2_{13}}\ldots\frac{(x_{13})_{\nu_s}}{x^2_{13}} - \text{traces}\right]\frac{1}{(\frac{1}{2}x^2_{13})^d} \end{aligned} \tag{6.9}$$

where $g^s_j(d)$ and $f_s(d)$ are the coupling constants of the longitudinal and transverse parts. The sum $\sum^s_{k=1}$ includes permutations of the indices $\nu_i \rightleftarrows \nu_l$, $2 \leqslant i \leqslant k$, $k + 1 \leqslant l \leqslant s$ as well as of the index ν_1 with the rest. The Green function (6.9) meets the Ward identity with Schwinger terms

$$\begin{aligned} \partial^{x_2}_\mu G_{\mu,\nu_1\ldots\nu_s}(x_1x_2x_3) &= -g^s_j(d)2^{s/2+1}(2\pi)^D\frac{\Gamma(s)\Gamma(h-d)}{\Gamma(h+s-1)\Gamma(d)} \\ &\times\sum_{k=1}^s\frac{(-2)^{-k}}{\Gamma(k)}\left[\partial^{x_2}_{\nu_1}\ldots\partial^{x_2}_{\nu_k}\delta(x_1 - x_2)\frac{(x_{13})_{\nu_{k+1}}}{x^2_{13}}\ldots\frac{(x_{13})_{\nu_s}}{x^2_{13}} - \text{traces}\right]G_d(x_{13}). \end{aligned} \tag{6.10}$$

The transverse coupling constant does not contribute to (6.10). Calculations required to derive (6.9) and (6.10) are given in [58] and in more detail in [8]. The Wightman function of these fields is transverse

$$\partial^{x_2}_\mu W_{\mu,\nu_1\ldots\nu_s}(x_1x_2x_3) = 0 \tag{6.11}$$

and is equal to

$$\begin{aligned} &W_{\mu,\nu_1\ldots\nu_s}(x_1x_2x_3) \\ &= [g^s_j(d)+f_s(d)]2^{s/2}\left\{\frac{1}{(\frac{1}{2}x^2_{23})^{h-1}}\overleftrightarrow{\partial}{}^{x_2}_\mu\left[\frac{1}{(\frac{1}{2}x^2_{12})^{h-1}}(\lambda^{x_1}_{\nu_1\ldots\nu_s}(x_2x_3) - \text{traces})\right]\right\}\frac{1}{(\frac{1}{2}x^2_{13})^{d-h+1}}. \end{aligned} \tag{6.12}$$

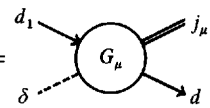
* This expression differs from the analogous one in ref. [58] in the coupling constant redesignation.

In what follows an example of the theory will be given, where the fields $O_s^d(x)$ along with the field $\varphi_d(x)$ appear in the operator product expansion of states $j_\mu(x_1)\varphi_d(x_2)|0\rangle$.

6.2. Four-point functions

Consider the Green function of four fields

$$G_\mu(x_1x_2x_3x_4) = \langle 0|\, T\varphi_{d_1}(x_1)\varphi_d^+(x_2)\chi_\delta(x_3)j_\mu(x_4)\,|0\rangle = \tag{6.13}$$

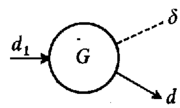


where φ_{d_1} and φ_d are the charged scalar fields, χ_δ is the neutral field with dimension δ. A general conformal-invariant expression for this function may be written similar to (2.12) in the form [37]:

$$G_\mu(x_1x_2x_3x_4) = \{\xi^{h-1}K_\mu^{x_4}(x_1x_3)F(\xi,\eta) - \eta^{h-1}K_\mu^{x_4}(x_2x_3)F(\eta,\xi)\}G^{d_1d\delta}(x_1x_2x_3) \tag{6.14}$$

where

$$G^{d_1d\delta}(x_1x_2x_3) = \langle 0|\, T\varphi_{d_1}(x_1)\varphi_d(x_2)\chi_\delta(x_3)\,|0\rangle = g_{d_1d\delta}C^{d_1d\delta}(x_1x_2x_3) = \quad , \tag{6.14a}$$



$F(\xi,\eta)$ is an arbitrary function of the variables (2.11) and the value K_μ is equal to

$$K_\mu^{x_4}(x_1x_3) = \frac{e}{\Omega_D}\left(\frac{x_{13}^2}{x_{14}^2x_{34}^2}\right)^{h-1}\lambda_\mu^{x_4}(x_1x_3), \qquad \Omega_D = \frac{2\pi^h}{\Gamma(h)}. \tag{6.14b}$$

The Green function (6.14) meets the generalized Ward identity

$$\partial_\mu^{x_4}G_\mu(x_1x_2x_3x_4) = -e[\delta(x_1-x_4) - \delta(x_2-x_4)]G^{d_1d\delta}(x_1x_2x_3) \tag{6.15}$$

from which an additional equation for the function $F(\xi,\eta)$ may be obtained. It is evident, however, that from the Ward identity this function cannot be determined unambiguously, since to G_μ may always be added an arbitrary transverse part G_μ^{tr} such as $\partial_\mu^{x_4}G_\mu^{\text{tr}}(x_1x_2x_3x_4) = 0$. The simplest example of the conformal-invariant transverse function is

$$G_\mu^{\text{tr}}(x_1x_2x_3x_4) \sim \{K_\mu^{x_4}(x_1x_2) + K_\mu^{x_4}(x_2x_3) + K_\mu^{x_4}(x_3x_1)\}G^{d_1d\delta}(x_1x_2x_3). \tag{6.16}$$

To find a general solution of the Ward identity (6.15), one substitutes (6.14) into it and use the relationship

$$\partial_\mu^{x_4}K_\mu^{x_4}(x_1x_2) = -e[\delta(x_1-x_4) - \delta(x_2-x_4)].$$

Under the assumption that the function $F(\xi,\eta)$ is not singular at the points $\xi = 0, \eta = 0$; $\xi = 1, \eta = \infty$ and $\xi = \infty, \eta = 0$, we will obtain the equation

$$(h-1)\left(\frac{1}{\xi}+\frac{1}{\eta}-1\right)[F(\xi,\eta) - F(\eta,\xi)] + \left(\frac{1}{\xi}+\frac{1}{\eta}-1\right)\left[\xi\frac{\partial F(\xi,\eta)}{\partial\xi} - \eta\frac{\partial F(\eta,\xi)}{\partial\eta}\right]$$
$$+ 2\left[\frac{\eta}{\xi}\frac{\partial F(\xi,\eta)}{\partial\eta} - \frac{\xi}{\eta}\frac{\partial F(\eta,\xi)}{\partial\xi}\right] = 0, \tag{6.17}$$

with the boundary conditions

$$F(1,\infty) = F(\infty,1) = 1. \tag{6.18}$$

The general solution of this equation may be presented as [37]

$$F(\xi,\eta)=\xi^{1-h}\left[f_1(A)+(h-1)\int_0^{\xi}\tau^{h-2}f_2(\tau,A(\tau))\,\mathrm{d}\tau\right] \tag{6.19}$$

where

$$A=\eta^{1/2}\left(\frac{1}{\xi}-\frac{1}{\eta}-1\right),\qquad B(\tau)=\left\{\frac{A}{2}+\sqrt{\frac{A^2}{4}+\frac{1}{\tau}-1}\right\}^{-2},$$

$f_1(A)$ and $f_2(\tau, A) = f_2(A, \tau)$ are the arbitrary functions. The symmetric solution is

$$F(\xi,\eta)=F(\eta,\xi)=f(\sqrt{\xi\eta}-\sqrt{\xi/\eta}-\sqrt{\eta/\xi}),$$

f is an arbitrary function.

In a similar way a general expression of the Green function, including an energy-momentum tensor (its quantum numbers are $\sigma_T = (D, 2)$) may be found. Consider, e.g., the Green function $G^T_{\mu\nu}(x_1x_2x_3x_4) = \langle 0|\, T\varphi_d(x_1)\varphi_d(x_2)\chi_\delta(x_3)T_{\mu\nu}(x_4)\,|0\rangle$. Its general expression is [37, 8]

$$\begin{aligned}G^T_{\mu\nu}(x_1x_2x_3x_4)=\{&F(\xi,\eta)K^{x_4}_{\mu\nu}(x_1x_2)+\phi(\xi,\eta)K^{x_4}_{\mu\nu}(x_1x_3)\\&+\phi(\eta,\xi)K^{x_4}_{\mu\nu}(x_2x_3)\}\,\langle 0|\,T\varphi_d(x_1)\varphi_d(x_2)\chi_\delta(x_3)\,|0\rangle\end{aligned} \tag{6.20}$$

where

$$K^{x_4}_{\mu\nu}(x_1x_2)\sim\left(\frac{x_{12}^2}{x_{14}^2x_{24}^2}\right)^{h-1}\{\lambda^{x_4}_{\mu\nu}(x_1x_2)-\text{traces}\}.$$

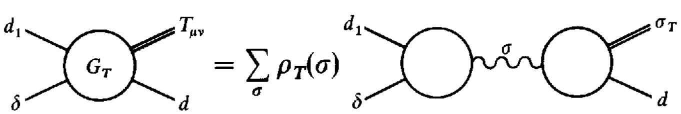
The most interesting case is the one of identical fields when $d = \delta$ and (6.20) is symmetric with respect to any substitution

$$\xi\rightleftarrows\eta;\qquad \xi\to 1/\xi,\ \eta\to\eta/\xi;\qquad \xi\to\xi/\eta,\ \eta\to 1/\eta \tag{6.21}$$

i.e. the functions F and ϕ meet the conditions

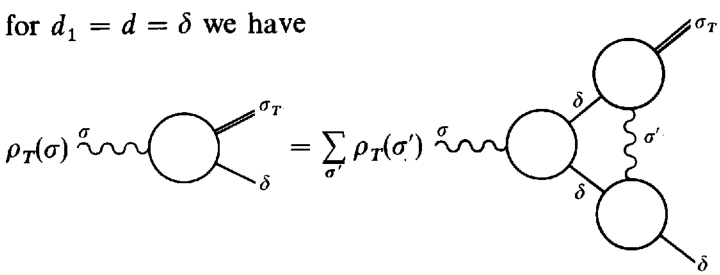
$$F(\xi,\eta)=F(\eta,\xi);\qquad F(\xi,\eta)=\phi(1/\eta,\xi/\eta)=\phi(1/\xi,\eta/\xi). \tag{6.21a}$$

These conditions are equivalent to the relations of crossing symmetry of type (5.12). If the function (6.20) is presented in the form



(6.22)

then for $d_1 = d = \delta$ we have



(6.22a)

This relation is equivalent to the conditions of symmetry (6.21a).

On the other hand, substituting (6.20) to the Ward identity [41]

$$\partial_\mu^{x_4} G^T_{\mu\nu}(x_1x_2x_3x_4) = -\Big[\delta(x_4 - x_1)\partial_\nu^{x_1} + \delta(x_4 - x_2)\partial_\nu^{x_2} + \delta(x_4 - x_3)\partial_\nu^{x_3}$$
$$- \frac{d}{D}\partial_\nu^{x_4}\delta(x_4 - x_1) - \frac{d}{D}\partial_\nu^{x_4}\delta(x_4 - x_2) - \frac{\delta}{D}\partial_\nu^{x_4}\delta(x_4 - x_3)\Big] G(x_1x_2x_3) \qquad (6.22b)$$

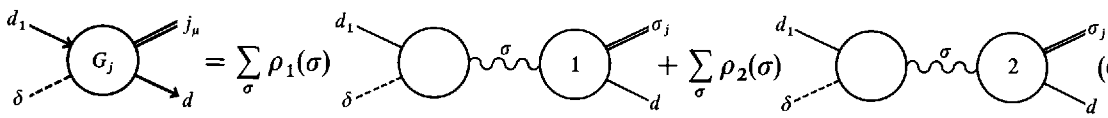
we obtain two differential equations for the functions $F(\xi, \eta)$ and $\phi(\xi, \eta)$ with the boundary conditions

$$F(1, \infty) = F(\infty, 1) = \phi(0, 0) = \phi(1, \infty) = \phi(\infty, 1) = 1.$$

It is essential that in contrast to (6.17) these equations are now completed by the conditions (6.21a). Whether their solution is unique taking into account these conditions, has not yet been clarified. Note, however, that if the symmetry of the functions $\phi(\xi, \eta)$ is also demanded: $\phi(\xi, \eta) = \phi(\eta, \xi)$, they allow the only solution $F(\xi, \eta) = \phi(\xi, \eta) = 1$ and Green function (6.20) is determined unambiguously.

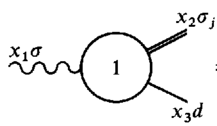
6.3. *Partial wave expansion*

Now represent the Green function (6.14) in the form of a partial wave expansion. Since it has one vector leg, its expansion consists of two terms (see the end of section 6):

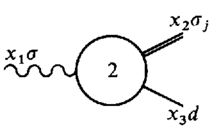


(6.23)

where C_1 and C_2 are the functions belonging to two mutually-orthogonal sets. According to section 5 they may be chosen in different ways. It is appropriate to determine them so that one of them, e.g. C_2 may be transverse. Such functions were found in [56], see also [8]

$C_1^{\sigma\sigma_j d}(x_1x_2x_3) =$ 

$$= \frac{2^{(s+3)/2}}{(2\pi)^h} N_1(\sigma, d)[(D - 2 + l - d - s), 1]\Delta(x_1x_2x_3), \qquad (6.24)$$

$C_2^{\sigma\sigma_j d}(x_1x_2x_3) =$ 

$$= \frac{2^{s/2}}{(2\pi)^h} N_2(\sigma, d)\left[s\frac{(D - 2 - l + d + s)}{(l - d)}, 1\right]\Delta(x_1x_2x_3). \qquad (6.25)$$

Here the notations

$$\Delta(x_1x_2x_3) = (\tfrac{1}{2}x_{12}^2)^{-(D-2+l-d-s)/2}(\tfrac{1}{2}x_{13}^2)^{-(l+d-s-D+2)/2}(\tfrac{1}{2}x_{23}^2)^{-(D-2-l+d+s)/2}, \qquad (6.26)$$

$$[A, B] = A\lambda_\mu^{x_2}(x_1x_3)[\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_2x_3) - \text{traces}]$$
$$+ B\frac{1}{x_{12}^2}\left[\sum_{k=1}^{s} g_{\mu\mu_k}(x_{12})\lambda^{x_1}_{\mu_1\ldots\hat{\mu}_k\ldots\mu_s}(x_2x_3) - \text{traces}\right], \qquad (6.26a)$$

$$N_1(\sigma, d) = \frac{1}{(d - l + s)(D + l - d + s - 2)}$$
$$\times \left\{ \frac{\Gamma((l+d+s-D)/2)\Gamma((D+l-d+s)/2)\Gamma((D-l+d+s)/2)\Gamma((l+d+s)/2)}{\Gamma((d-l+s)/2)\Gamma((l-d+s)/2)\Gamma((2D-l-d+s)/2)\Gamma((D-l-d+s)/2)} \right\}^{1/2} \quad (6.24a)$$

$$N_2(\sigma, d) = \frac{(l - d)(2D - l - d + s - 2)(D - l - d - s)}{(D - l - 1)}$$
$$\times \Gamma\left(\frac{D+l-d+s}{2}\right)\Gamma\left(\frac{l+d+s-D}{2}\right)\Gamma\left(\frac{l+d+s}{2}\right)\Gamma\left(\frac{D-l+d+s}{2}\right) \quad (6.25a)$$

were used. The calculation of normalization factors is discussed below. The function (6.24) satisfies the amputation relation of type (5.6) with respect to a tensor rather than a scalar leg (this proves advantageous for the calculations of section 7), and the function (6.25) meets that with respect to both scalar and tensor legs. Let us point out two useful properties of the function C_2. First, it should be noted that they are transverse at any σ

$$\partial_\mu^{x_2} C^{\sigma\sigma_j d}_{2,\mu,\mu_1\ldots\mu_s}(x_1 x_2 x_3) = 0. \quad (6.27)$$

Owing to this property the second term in (6.23) does not contribute to the left-hand side of the Ward identity, so that from it only the function $\rho_1(\sigma)$ is determined. An explicit expression for $\rho_1(\sigma)$ is given in the next section. The function $\rho_2(\sigma)$ remains arbitrary. It may be related to the arbitrary functions entering into (6.19).

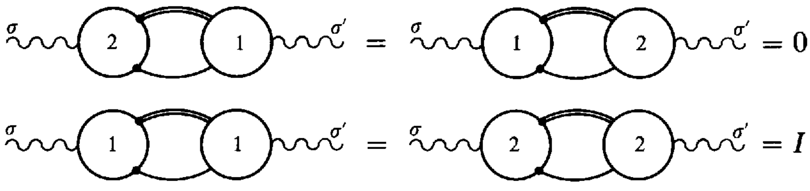
Another important feature of the functions C_2 is the following. They are different from zero at $s \geqslant 1$, and at $s = 0$ there are no transverse functions. Due to this the second term in (6.23) does not comprise a scalar contribution. It is present only in the first term and, hence, is unambiguously determined by the Ward identity. An expression for the function $\rho_1(\sigma)$ at $s = 0$ may be obtained from (7.7). In particular, from this it follows that the function $\rho_1(\sigma)|_{s=0}$ has a pole at the point $l = d$. This pole determines the asymptotic behaviour of the Green function (6.13) at $x_1 \to x_3$ (or $x_2 \to x_4$), provided that the poles of the function $\rho_2(\sigma)$ at $s \neq 0$ are assumed to lie to the right of the point $l = d$. As it follows from (3.16), this is the case if $d < D - 1$. An explicit expression for the asymptotic behaviour may be obtained from (6.14), and the boundary condition (6.18). In particular, in the limit $x_1 \to x_3$ we have [37, 5]

$$G_\mu(x_1 x_2 x_3 x_4)|_{x_1 \to x_3} \simeq \frac{e}{\Omega_D} \lambda_\mu^{x_4}(x_3 x_2) \left[\frac{x_{23}^2}{x_{24}^2 x_{34}^2} \right]^{h-1} G^{d_1 d \delta}(x_1 x_2 x_3).$$

Consider now the calculation of the normalization factors (6.24a) and (6.25a). The difficulties originated here are primarily related to the presence of two different-structure terms in (2.10). When substituting (2.10) into the amputation condition (5.6) functional equations for the coefficients A and B are formed. A general solution of these equations was obtained in [56]. A general expression of functions (2.10) satisfying the amputation condition over both tensor and scalar legs was also found there (see Appendix 4).

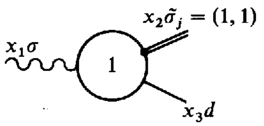
Another difficulty is due to the fact that the tensor leg is of the canonical dimension $l_j = D - 1$. This results in the absurd diverging expressions in the intermediated steps of integral calculations over the internal current line. Hence it is advisable to introduce two additional sets of the functions

$\tilde{C}_1$ and $\tilde{C}_2$ determined by the relations [56]:

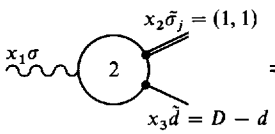


(6.28)

where I is the right-hand side of (5.7). It should be stressed that the functions $\tilde{C}_1$ and $\tilde{C}_2$ cannot be obtained from (6.24), (6.25) by the amputation of a current leg. In more detail these functions are described in Appendix 5. In contrast to (5.6a), points in (6.28) denote the formal substitution $l_j \to D - l_j = 1$ and $d \to D - d$ rather than amputation. The internal lines ending in a point should be correlated with δ-functions. An explicit expression for the functions $\tilde{C}_1$ and $\tilde{C}_2$ has been found in [56]:



$$= \frac{2^{(s-1)/2}}{(2\pi)^h} \tilde{N}_1(\sigma, d)[(l - d - s), 1]\tilde{\Delta}(x_1x_2x_3), \tag{6.29}$$



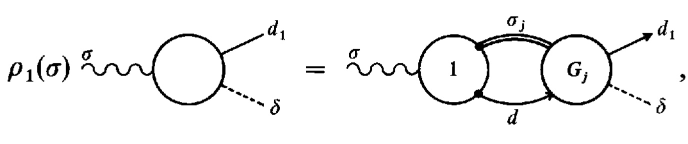
$$= \frac{2^{(s+2)/2}}{(2\pi)^h} \tilde{N}_2(\sigma, d)\left\{-s\frac{3D - l - d + s - 4}{2D - l - d - 2}, 1\right\}\tilde{\Delta}'(x_1x_2x_3) \tag{6.30}$$

where $\tilde{\Delta}(x_1x_2x_3) = (\frac{1}{2}x_{12}^2)^{-(l-d-s)/2}(\frac{1}{2}x_{13}^2)^{-(l+d-s)/2}(\frac{1}{2}x_{23}^2)^{(l-d-s)/2}$ and $\tilde{\Delta}'(x_1x_2x_3)$ is obtained from $\tilde{\Delta}$ by the substitution $d \to D - d$,

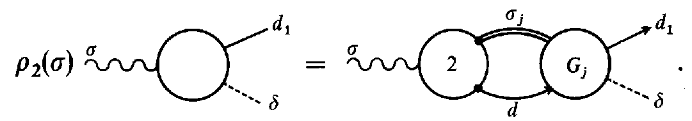
$$\tilde{N}_1(\sigma, d) = \left\{\frac{\Gamma((l+d+s)/2)\Gamma((l-d+s)/2)\Gamma((l+d+s-D)/2)\Gamma((d-l+s)/2)}{\Gamma((D-l+d+s)/2)\Gamma((D-l-d+s)/2)\Gamma((2D-l-d+s)/2)\Gamma((D+l-d+s)/2)}\right\}^{1/2}, \tag{6.29a}$$

$$\tilde{N}_2(\sigma, d) = \frac{1}{s(D + s - 2)} \times \frac{(l - 1)(2D - l - d - 2)}{(D - l - d - s)(2D - l - d + s - 2)(D + l - d + s - 2)(l - d - s)} \times \left\{\Gamma\left(\frac{2D - l - d + s}{2}\right)\Gamma\left(\frac{d - l + s}{2}\right)\Gamma\left(\frac{l + d + s}{2}\right)\Gamma\left(\frac{D - l + d + s}{2}\right)\right\}^{-1}. \tag{6.30a}$$

The function (6.29) meets the amputation condition over both scalar and tensor legs, while (6.30) satisfies only that over a tensor leg. Using the relationships (6.28), for the functions $\rho_1(\sigma)$ and $\rho_2(\sigma)$ we have



(6.31)



(6.32)

The above said is generalized with certain additions for the Green function (6.20) including an energy-momentum tensor. Let its partial wave expansion be briefly discussed. It comprises three-point functions $C^{\sigma\sigma_T d}(x_1x_2x_3)$, where $\sigma_T = (D, 2)$ are the quantum numbers of the energy-momentum tensor. For $\sigma = (l, s)$, where $s \geqslant 2$, these functions have three independent structures and, hence, depend on three coefficients A, B and C. For $s = 1$ there are only two independent coefficients. Similar to the above-said mutual-orthogonal sets of three-point functions may be determined, one of the sets consisting only of transverse functions which do not contribute to the Ward identity (6.21b). For $s = 0, 1$ there exist no transverse functions. It means that from the Ward identity scalar and vector contributions to the partial wave expansion (6.22) can be unambiguously determined. (In the case of identical fields there is no vector contribution due to the anti-symmetry of the function (2.9) for odd s.) The scalar contribution has been found in our work [5], see also [6, 37], and in normalization (5.8–10) it is equal to

$$\rho_T(l) = \frac{8D}{D-1}\,\frac{g}{(2\pi)^D}\,\frac{N(d_1 d\delta)}{\Gamma((d-d_1+\delta)/2)\Gamma((d+d_1-\delta)/2)}\,N^{-1}(ld_1\delta)N^{-1}(\sigma_T l d)$$
$$\times\left\{\Gamma\left(\frac{D-l+d_1-\delta}{2}\right)\Gamma\left(\frac{D-l-d_1+\delta}{2}\right)\right\}^{-1}\frac{\Gamma((l+d-D+2)/2)}{\Gamma((2D-l-d+2)/2)}$$
$$\times\frac{1}{(l-d)(D-l-d)}\left\{\frac{d_1-\delta}{2}(f_1-f_2)(\delta+d_1-D)-\tfrac{1}{2}(f_1+f_2)l(D-l)\right.$$
$$\left.+\frac{1}{2D}(l-d)(D-l-d)[f_1(D-d_1)+f_2(D-\delta)]\right\},$$

where $f_1 = \Gamma((l-d_1+\delta)/2)\Gamma((D-l-d_1+\delta)/2)\Gamma((D+d_1-d-\delta)/2)\Gamma((d_1+d-\delta)/2)$ and f_2 is obtained from f_1 by the substitution $d_1 \rightleftarrows \delta$. The contribution of higher spins can be found from the relationships of ref. [6].

7. Special solution of Ward identities

This section presents a more detailed investigation of two four-point Green functions: with one current, see (6.13), and with two currents

$$\begin{aligned} G_{\mu\nu}(x_1x_2x_3x_4) &= \langle 0|\, T\varphi_d(x_1)\varphi_d^+(x_2)j_\mu(x_3)j_\nu(x_4)\,|0\rangle \\ \partial_\mu^{x_3}G_{\mu\nu}(x_1x_2x_3x_4) &= -e[\delta(x_1-x_3)-\delta(x_2-x_3)]G_\nu(x_1x_2x_4). \end{aligned} \tag{7.1}$$

In both cases the functions $\rho_1(\sigma)$ in (6.23) and their contribution to the operator product expansion of the states $j_\mu(x_1)\varphi_d(x_2)\,|0\rangle$ will be found.

The function $\rho_1(\sigma)$ in refs. [5, 6, 37] was found by the authors directly from the Ward identities (6.15). Here another method of its calculation is used. Note that all special solutions of the equation (6.17) differ only in the transverse part, i.e. in the second term in (6.23). Hence to find the function

$\rho_1(\sigma)$ one may use some special solution and the expansion (5.11) or (6.31). As a special solution it is appropriate to choose the analog of Green functions in two-dimensional space.

As is known, in two-dimensional space there are two conservation laws:

$$\partial_\mu j_\mu(x) = 0 \qquad \text{and} \qquad \varepsilon_{\mu\nu}\partial_\mu j_\nu(x) = 0.$$

Accordingly, there are two charges. In the case of scalar fields, however, the second charge is equal to zero, hence the second Ward identity is of the form $\varepsilon_{\mu\nu}\partial_\mu^{x_4} G_\nu(x_1x_2x_3x_4) = 0$ and is the same as for the function (7.1). From this identity the transverse part is determined. Substituting (6.19) into it, we find [37]

$$F(\xi, \eta) = 1$$

i.e. $G_\mu(x_1x_2x_3x_4)$ is determined unambiguously. Just in the same way the Green functions with an energy-momentum tensor and any number of currents may be found. In particular, for the function (7.1) we have:

$$G_{\mu\nu}(x_1x_2x_3x_4) = \frac{e^2}{(2\pi)^2}\lambda_\mu^{x_3}(x_1x_2)\lambda_\nu^{x_3}(x_1x_2)G_d(x_{12})$$

and in the general case

$$\langle 0|\, T\varphi_{d_1}(x_1)\varphi_d^+(x_2)\chi_\delta(x_3)j_{\mu_1}(z_1)\dots j_{\mu_n}(z_n)\,|0\rangle = \frac{e^n}{(2\pi)^n}\lambda_{\mu_1}^{z_1}(x_1x_2)\dots\lambda_{\mu_n}^{z_n}(x_1x_2)G^{d_1d\delta}(x_1x_2x_3).$$

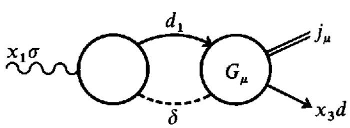
The analog of these solutions in the D-dimensional space is*

$$G_\mu(x_1x_2x_3x_4) = K_\mu^{x_4}(x_1x_2)G^{d_1d\delta}(x_1x_2x_3), \tag{7.2}$$

$$G_{\mu\nu}(x_1x_2x_3x_4) = K_\mu^{x_3}(x_1x_2)K_\nu^{x_4}(x_1x_2)G_d(x_{12}). \tag{7.3}$$

We will assume that $d > \frac{1}{2}D$. Then the partial wave expansion is applicable to Green functions. In what follows it will be shown, that this assumption is valid, see (7.9). In relation to d_1, δ no assumptions are made.

Let the partial wave expansion of the Green function (7.2) be found. Consider the function

$Q_\mu^\sigma(x_1x_2x_3) =$  . (7.4)

Substitute here (2.9), (7.2) and (6.14a). The integral appeared is calculated using (A3.2), (A3.9). After calculation we have

$$Q_\mu^\sigma(x_1x_2x_3) = eg_{d_1d\delta}(2\pi)^{-h}2^{(s+2)/2}\left\{N(d_1d\delta)\frac{\Gamma((D-d+d_1-\delta)/2)}{\Gamma((d-d_1+\delta)/2)}\right\}$$

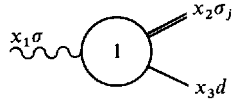
$$\times\,[(d-l+s)(D-2+l-d+s)]^{-1}\frac{\Gamma((l+d+s-D)/2)}{\Gamma((2D-l-d+s)/2)}\left\{\frac{\Gamma((D-l-d_1+\delta+s)/2)}{\Gamma((l+d_1-\delta+s)/2)}N(\sigma, D-d_1, D-\delta\right.$$

$$\times\,[(D-2+l-d-s), 1]\Delta(x_1x_2x_3).$$

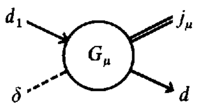
* General solution of Ward identity for Green function was found in ref. [59].

Here notations (2.10) and (6.26) are used. Note that the right-hand side is proportional to the function (6.24). If $\rho_{G_\mu}(\sigma)$ is determined by the equation

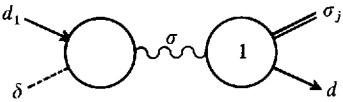
$$Q^{\sigma}_{\mu}(x_1x_2x_3) = \rho_{G_\mu}(\sigma)$$



we obtain [56, 8]



$$= \sum_{\sigma} \rho_{G_\mu}(\sigma) \qquad (7.6)$$



where

$$\rho_{G_\mu}(\sigma) = \frac{1}{\sqrt{2}} e g_{d_1 d\delta} \left\{ N(d_1 d\delta) \frac{\Gamma((D-d+d_1-\delta)/2)}{\Gamma((d-d_1+\delta)/2)} \right\} N^{-1}(\sigma d_1 \delta) N_1^{-1}(\sigma, d)$$
$$\times \frac{1}{(d+s-l)(D-2+l-d+s)} \frac{\Gamma((l+d+s-D)/2)\Gamma((l-d_1+\delta+s)/2)}{\Gamma((2D-l-d+s)/2)\Gamma((D-l+d_1-\delta+s)/2)}. \qquad (7.7)$$

The same result can be obtained using the equations (6.31) and (6.32). From (6.31) we will find the function $\rho_1(\sigma)$ as (7.7) and from (6.32) we will obtain $\rho_2(\sigma) = 0$. Thus, the expansion of Green function (7.2) consists of one term. This always may be achieved by redefinition of the function $C_1 \to C_1 + f(\sigma, d)C_2$ entering into (6.23), and by the proper choice of C_1 so that the second term may be compensated. Proceeding from these considerations we have chosen the function C_1 entering into expansion (7.6) in the form of (6.24). Note that it is possible since the condition for amputation over the scalar leg is not included.

Expansion (7.6) is in agreement with the hypothesis of a discrete-dimensions spectrum, made in section 3, see (3.17). Indeed, when substituting (7.7) into (7.6), all root branch points of the factors $N(\sigma d_1 \delta)$ and $N_1(\sigma, d)$ are cancelled. This implies that the function (7.2) meets the field algebra hypothesis. If expansion (7.6) is analytically continued to the pseudo-Euclidean space, we will find that the dimension spectrum in the expansion of type (3.29) is

$$d_s = d + s \qquad (7.8)$$

i.e. it is contributed only by the fields O^d_s, considered in section 6. It is evident, however, that the operator product expansion of type (3.17) for each of the states $j_\mu(x_1)\varphi_d(x_2)|0\rangle$, $\chi_\delta(x_1)\varphi_{d_1}(x_2)|0\rangle$ may contain a more completed set of the fields O_{σ_α}, and the fields O^d_s form, according to (2.6), only the intersection of these sets. In order to find all the fields contributing to the states $j_\mu(x_1)\varphi_d(x_2)|0\rangle$, an expansion of their scalar product, i.e. the Green function (7.3), should be examined.

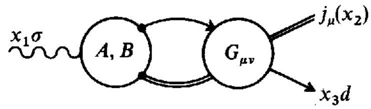
According to (3.16) for the states $j_\mu(x_1)\varphi_d(x_2)|0\rangle$ to be positive the condition $d_s \geqslant D - 2 + s$ should be fulfilled. With the account of (7.8) from this it follows*

$$d > D - 2. \qquad (7.9)$$

* As is shown in section 16 the dimensions of the fundamental fields are limited by the condition $d < h$, so that the results of this section can be applied either to compound fields φ_d for which $d > h$ or in the case $D < 4$. In particular, in the Thirring model ($D = 2$) there exists an analog of the fields O^d_s. They contribute to the operator expansion of the product $j_\mu(x_1)\psi_d(x_2)$, where ψ_d is the spinor field of the Thirring model, see section 14.

In more detail this condition is discussed in section 12, where coupling constants of the fields O_s^d are calculated.

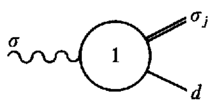
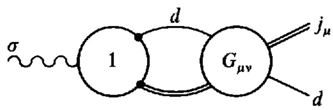
Consider a partial wave expansion of the Green function (7.3). Using relationships (A3.2), (A3.9) we have (for more detail see [56])

$Q_\mu^{AB}(x_1x_2x_3) =$  $=$

$$= (-)^{s+1}\frac{2e^2}{(2\pi)^h}\frac{\Gamma(d)}{\Gamma(h-d)}\{A(2D-l-d-2)+B\cdot s\cdot(3D-l-d+s-4)\}$$
$$\times\,[(l-d-s)(D-l-d-s)(D-2+l-d+s)(2D-l-d+s-2)]^{-1}$$
$$\times\,[(D-2+l-d-s),1]\Delta(x_1x_2x_3) \tag{7.10}$$

where points stand for the substitution $d_j \to D - d_j = 1$, $d \to D - d$ (rather than amputation) and internal lines stand for the δ-function, see (6.28); A and B are the arbitrary constants. When calculating the integral in (7.10), the uncertainty $0 \times \infty$ arises. For its resolving one should perform the substitution $h - 1 \to (d_j - 1)/2$ in the expressions for $K_\mu^{x_3}(x_1x_2)$ and the function $\{A, B\}$ and then pass to the limit $d_j \to D - 1$ in the final result.

From (7.10) it follows that the expansion of the Green function (8.3) comprises, as in the case of (7.6), only functions C_1. Indeed, substituting function (6.30), for which $A/B = -s(3D-l-d+s-4)/(2D-l-d-2)$ to (7.10) we find that Q_μ^{AB} goes to zero. Defining $\rho_{G_{\mu\nu}}(\sigma)$ by the equation

$\rho_{G_{\mu\nu}}(\sigma)$  $=$ 

we find [56, 8]

$$\rho_{G_{\mu\nu}}(\sigma) = \frac{(-)^{s+1}}{8}\frac{e^2}{(2\pi)^h}\frac{\Gamma(d)}{\Gamma(h-d)}N_1^{-2}(\sigma,d)$$
$$\times\frac{\Gamma((l+d+s-D)/2)\Gamma((D-2+l-d+s)/2)}{\Gamma((d-l+s+2)/2)\Gamma((2D-l-d+s)/2)}\cdot\frac{1}{(d+s-l)(D-2+l-d+s)}. \tag{7.11}$$

All the above-said on the expansion (7.6) is also valid for (7.11).

8. Dynamical equations

In the foregoing sections we have discussed all limitations for Green functions resulting from the conformal group kinematics and the axiomatic requirements as well as from the current and energy-momentum conservation. The next problem is to find a solution of dynamical equations in the framework of these limitations. Let this problem be formulated more accurately. As shown above, all n-point Green functions for $n \geqslant 4$ can be represented as partial wave expansions. This form of the Green function representation takes into account all consequences of the conformal kinematics. Further analysis is to be made on unknown functions $\rho(\sigma)$ which should be found from a special dynamic model. Note, however, that these functions are not completely arbitrary: axioms of the field theory impose restrictions on their complex-dimension plane analytical properties. As shown, all poles and branching points of these functions lie on a real dimension axis and in addition $\rho(\sigma) \to 0$ when $l \to \infty$. Everywhere, except for these points, the function $\rho(\sigma)$ is analytic. Hence, for its determination it suffices to calculate all singular points. If the function $\rho(\sigma)$ has simple poles, then according to sections 3 and 4, this is equivalent to a vacuum operator product expansion, see (3.17). Provided that it has branching points, this implies that in (3.17) there is also a continuous spectrum of dimensions. Thus the problem of dynamics consists in the elucidation of the singularity nature of the function $\rho(\sigma)$ and analogous functions entering the partial wave expansion of higher Green functions, see (9.15) and (10.4). As it will be shown in the next section, dynamic equations lead to the algebra of fields in the operator form (in Euclidean space). The next problem is to find the position of poles and their residues. The position of the poles governs the dimensions of tensor fields contributing to the operator product expansion, and the residues are the coupling constants of these fields. In section 12 an example of Green functions (7.2, 3) is considered for which this problem has completely been solved. In the general case its solution is related to the problem of closing a system of equations, see below.

Strictly speaking, the above said refers to the product $C^{\tilde{\sigma}}\Delta_{\sigma}C^{\tilde{\sigma}}\rho(\sigma)$ rather than to $\rho(\sigma)$. The analytical properties of $\rho(\sigma)$ depend on the choice of normalization for the function C^{σ}. In the normalization here accepted, see (5.8–10), the function $\rho(\sigma)$ includes "kinematic" branch points which are cancelled by normalization factors of the function C_{σ}. This is what happens in the examples considered in section 7: both Green functions (7.2) and (7.3) meet the algebra hypothesis, though the function (7.7) comprises branching points due to the factors $N^{-1}(\sigma d_1 \delta)N^{-1}(\sigma, d)$. They are cancelled when substituted in (7.6). In what follows, however, we will, for short, consider only simple poles of the functions $\rho(\sigma)$.

We will proceed from the Lagrangian renormalized field equations [60], see also [61]. There are several physically meaningful Lagrangians, which in the limit of zero mass allow a conformal-invariant solution. This primarily refers to the Lagrangians with a dimensionless coupling constant free of derivative couplings. A formal proof for the unrenormalized or classical field theory is given in refs. [23, 62]. In our approach the conformal-invariant solution is realized as a Gell-Mann–Low limit [63] (see also [64]) of the theory at small distances with a finite charge renormalization. As shown in [65], the scale invariance in this limit results in the conformal invariance of the theory.

The simplest Lagrangians with dimensionless coupling constants are: $L_{\text{int}} = \lambda\bar{\Psi}\gamma_5\Psi\varphi$, where Ψ and φ are the spinor and pseudoscalar fields in four-dimensional space (see Appendix 6); $L_{\text{int}} = \lambda\varphi^{+}\varphi\chi$ and $\lambda\varphi^3$ where φ, χ are the scalar fields in six-dimensional space. The method developed below will be applied just to these interactions. Note, however, that the limitation to trilinear Lagrangians is not restrictive of principal nature, since any Lagrangian with a higher order in the field may be reduced to a set of trilinear interactions. For example, in the $\lambda\varphi^4$ theory one may take $\varphi^2 = \chi$ and consider the interaction $\lambda\varphi^2\chi$. In a similar way a four-fermion interaction can be represented as an interaction of two spinor fields with a scalar or a vector. This can be examplified by the Thirring model, where the interaction of four spinors can be realized via a conserved current. Certainly, this reduction of Lagrangians results in an increased number of initial fields.

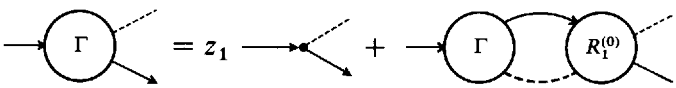
As an initial interaction consider the Lagrangian

$$L_{\text{int}} = \lambda\varphi^{+}\varphi\chi \tag{8.1}$$

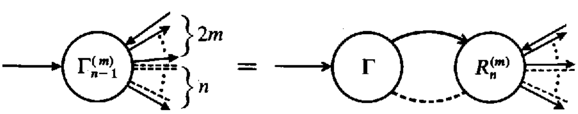
where $\varphi \equiv \varphi_d(x)$, $\chi \equiv \chi_\delta(x)$ are the scalar fields in a D-dimensional space. All the results are generalized for the Yukawa interaction, see Appendix 6. We will use the Euclidean field theory formulation [47]. As to the dimensions d, δ in this and foregoing sections it will be assumed that

$$h - 1 < d, \delta < h. \tag{8.2}$$

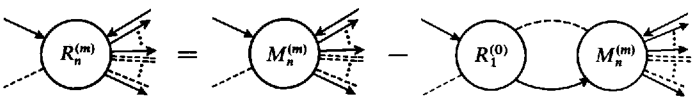
A system of renormalized field equations was obtained by one of the authors (E.S.F.) in ref. [60], and is of the form



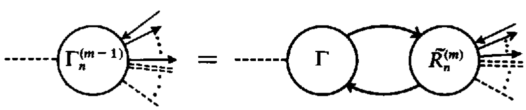
(8.3)



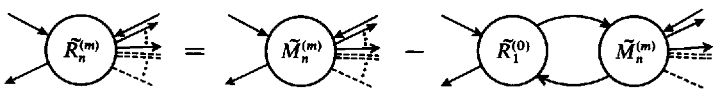
(8.4)



(8.5)

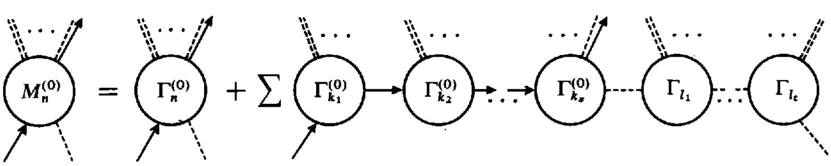


(8.6)



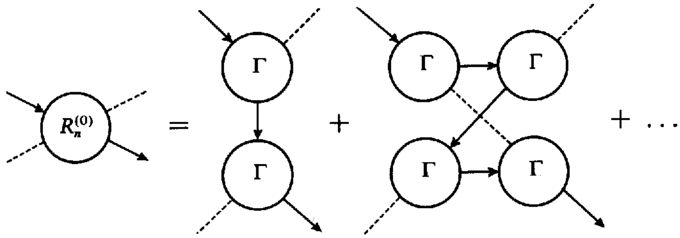
(8.7)

Here solid lines stand for the charged field $\varphi_d(x)$ and dashed lines are given for the neutral field $\chi_\delta(x)$. The $\Gamma_n^{(m)}$'s are the one-particle irreducible vertices. The vertices $M_n^{(m)}$ are obtained from the connected Green functions by subtracting from them all graphs which may be disconnected by a cut made across one line. They may also be expressed via the vertices $\Gamma_n^{(m)}$, if the latter are added by the graphs which may be cut along one line in the longitudinal direction. We have, e.g.



(8.8)

where the sum is taken over all possible divisions of dashed legs into the group $k_1 \ldots k_s$ and $l_1 \ldots l_t$ and their symmetrization. The meaning of the vertices $R_n^{(m)}$ and $\tilde{R}_n^{(m)}$ is seen from the equations (8.5) and (8.7) which serve as their definition. It may be shown that in the perturbation theory these vertices are defined by the sum of graphs uncut along two lines in the transverse direction, e.g.

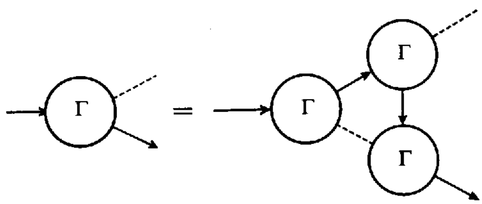


(8.9)

It is essential that the system of equations (8.3)–(8.7) is conformally invariant only in the limit of infinite renormalizations, i.e. at $z_1 = 0$. It should be taken into consideration that for the interaction of scalar fields of type (8.1) this condition is fulfilled in a D-dimensional space for $D \geqslant 6$. Nevertheless, calculations will be performed for an arbitrary value of D.

Let the limit $z_1 = 0$ be taken. Equation (8.3) becomes homogeneous. All infinite renormalizations and the required resolution of the uncertainty $0 \times \infty$ appearing in the limit $z_1 = 0$ have been effectively performed due to the introduction of vertices $R_n^{(m)}$, see [60]. Now it can readily be seen that the resulting system (8.3)–(8.7) is conformal-invariant. For this it suffices to use the transformations (1.1, 2) and the transformation properties (2.3), (2.4) as applied to Green functions.

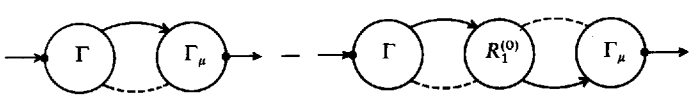
It should be noted that in a skeleton theory the equation (8.3) becomes the condition of self-consistency and can be used for the calculation of coupling constants. Indeed, if one substitutes (8.9) into (8.3) at $z_1 = 0$ and restricts himself to the first term, the bootstrap equation discussed in several works [38–43] is obtained



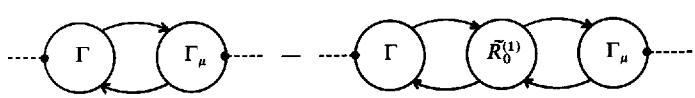
(8.10)

This equation along with equations for the propagators (8.11, 12) may be used to determine the field dimensions and coupling constants. As shown in [42], all integrals over the internal lines converge, provided that the field dimensions lie within interval (8.2). Outside this interval the integrals should be defined by the analytical continuation over the dimensions d and δ.

Equations (8.3)–(8.7) should be added by an equation for the propagators which will be written in the form [60]

$-(G^{-1})_\mu =$ 

(8.11)

$-(D^{-1})_\mu =$ 

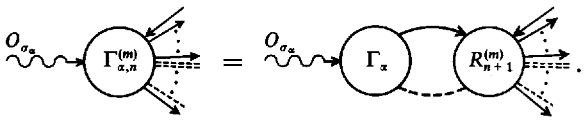
(8.12)

where points stand for amputation, see (5.6a), and the index μ is given for the external momentum p_μ derivative. These equations play the role of normalization conditions, see section 9.

Now let the equations for tensor fields which will be referred to as $O_{\sigma_\alpha}(x)$ be given. They may be determined as a renormalized limit of products of the type $F_\alpha(x, y) = \mathrm{D}_\alpha(\mathrm{i}\partial_x, \mathrm{i}\partial_y)\varphi(x)\chi(y)$, where D_α is the tensor differential operator. Let

$$O_{\sigma_\alpha}(x) = \lim_{y\to x} z_\alpha F_\alpha(x, y),$$

where z_α in the renormalization constant, $z_\alpha \to 0$. Resolving the uncertainty $0 \times \infty$ as in the case of (8.3)–(8.7), we have



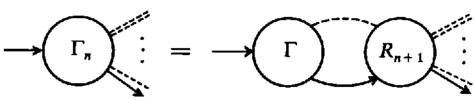
(8.13)

Similar equations may also be written for neutral tensor fields. Conserved currents as well as energy momentum tensors are of special importance among them.

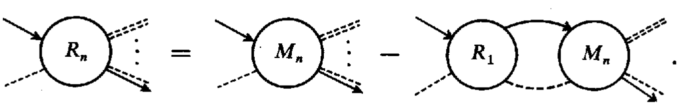
In conclusion it should be noted that the above-given form of equations is most advantageous, if the dimensions d, δ are limited by the interval (8.2). As shown in section 5, in this case the partial wave expansion is applicable just to one-particle irreducible vertices. If the dimensions lie outside this interval, it is more appropriate to represent the system (8.3–7) and (8.13) as equations for connected Green functions, see section 10.

9. Diagonalization of dynamical equations

Consider the equations (8.4, 5) at $m = 0$. The results will be generalized later to the case of arbitrary m. Introducing the notations $R_n^{(0)} = R_n$, $M_n^{(0)} = M_n$, $\Gamma_n^{(0)} = \Gamma_n$, we have

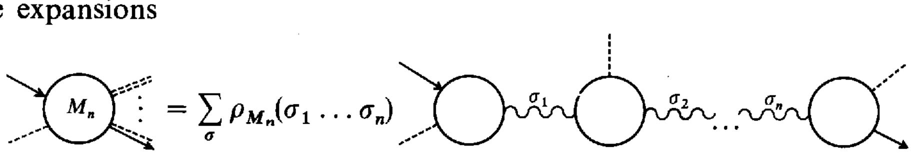


(9.1)

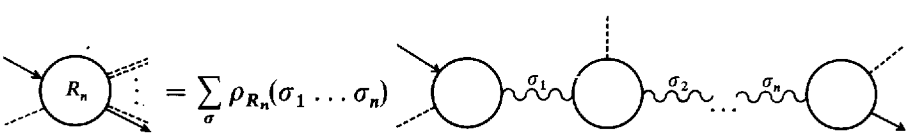


(9.2)

The equations (9.1, 2) are diagonalized [2–7], if their constituent vertices are represented as partial wave expansions



(9.3)

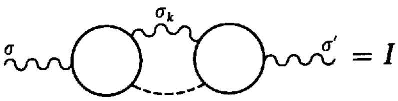


(9.4)

For the functions ρ_{M_n} and ρ_{R_n} we find from (9.2)

$$\rho_{R_n}(\sigma_1 \ldots \sigma_n) = \rho_{M_n}(\sigma_1 \ldots \sigma_n) - \rho_{R_1}(\sigma_1)\rho_{M_n}(\sigma_1 \ldots \sigma_n). \tag{9.5}$$

In the derivation of (9.5) the orthogonality relationship (5.7) and its generalization

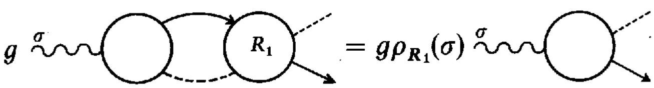


where $2 \leqslant k \leqslant n$ was used. In the special case $n = 1$ we have [2]

$$\rho_{M_1}(\sigma) = \frac{\rho_{R_1}(\sigma)}{1 - \rho_{R_1}(\sigma)}.$$

From this it follows that the function $\rho_{M_1}(\sigma)$ has poles at the points $\sigma = \sigma_\alpha$, where $\rho_{R_1}(\sigma_\alpha) = 1$. According to sections 4, 5 each pole is associated with the field with quantum numbers σ_α. These fields enter into the operator product expansion of the states $\varphi_d(x_1)\chi_\delta(x_2)|0\rangle$. In what follows we will show that *each pole of the functions* $\rho_{M_1}(\sigma)$ *and* $\rho_{M_n}(\sigma_1 \ldots \sigma_n)$ *with respect to the argument* σ_1 *is responsible for the field* O_{σ_α} *determined by equation* (8.13).

First of all let the equation (9.1) be considered for $n = 1$. Its right-hand side can be written as a limit of the quantity

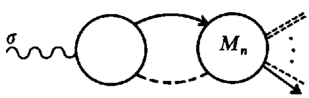


(9.6)

where $s = 0$, $l \to d$. From (9.1) we have

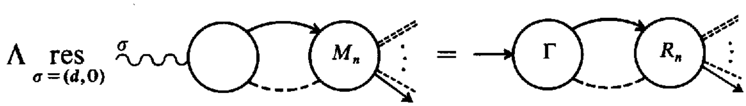
$$\rho_{R_1}(\sigma)\big|_{\sigma=(d,0)} = 1. \tag{9.7}$$

From this it follows that the quantity



(9.8)

has a pole at $\sigma = (d, 0)$. According to equation (9.1) for $n \neq 1$ the residue of this pole is equal to

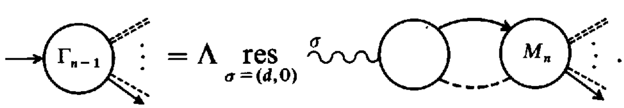


(9.9)

where

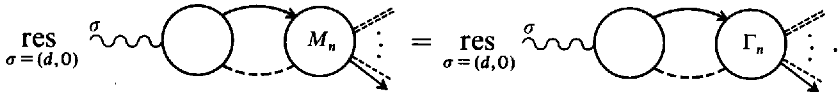
$$\Lambda = -g \frac{\mathrm{d}}{\mathrm{d}l} \rho_{R_1}(\sigma)\bigg|_{\sigma=(d,0)}. \tag{9.9a}$$

Now the equations (9.1) can be written in the form



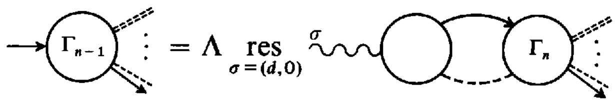
(9.10)

Thus from the system of the equations (9.1, 2) the auxiliary vertices R_n have been eliminated. The next step is to express vertices M_n via Γ_n and to obtain equations connecting only the vertices Γ_n with different values of n. Note that the vertices M_n and Γ_n differ only in the graphs which may be cut in the longitudinal direction, see (8.8). When substituted in (9.8), these graphs provide integrals converging in the limit $\sigma = (d, 0)$, for more detail see [5–6]. As a result we have



(9.10a)

Substituting it into (9.10), we finally find [4–6]

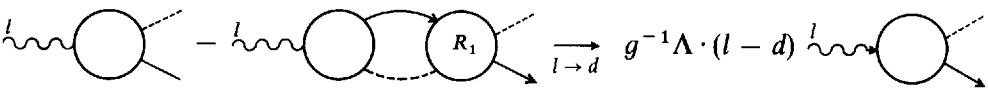


(9.11)

where the constant Λ is determined from the equation (8.11) and is equal to [3–6]

$$g\Lambda = -2\mu(d). \tag{9.12}$$

To calculate Λ it is appropriate to use the results of (9.6) and (9.7):



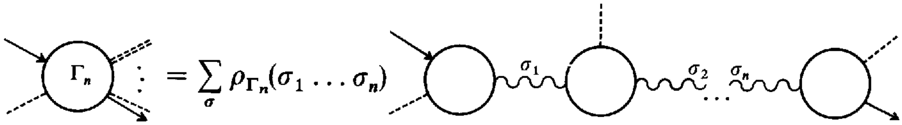
(9.13)

and to represent equation (8.11) in the form

(9.14)

The momentum derivative $\partial/\partial p_\mu$ implies that in the coordinate space the propagator is multiplied by the difference of coordinates. The same holds for a vertex in the right-hand side of (9.14), so that the integral here is finite for $l \neq d$. Note that the initial equation (8.11) includes the difference of two terms, each of them being converging. Relationship (9.13) should be considered as a means of resolution of the $0 \times \infty$ uncertainty entering into (8.11), see also [43].

Equation (9.11) is diagonalized by the expansion similar to (9.3, 4)



(9.15)

where the function ρ_{Γ_n} meets the relation

$$[\rho_{\Gamma_n}(\sigma_1 \ldots \sigma_n)]^* = \rho_{\Gamma_n}(\sigma_n \ldots \sigma_1).$$

As follows from (9.10a), the functions ρ_{M_n} and ρ_{Γ_n} have equal residues

$$\operatorname*{res}_{\sigma_1 = (d,0)} \rho_{\Gamma_n}(\sigma_1 \ldots \sigma_n) = \operatorname*{res}_{\sigma_1 = (d,0)} \rho_{M_n}(\sigma_1 \ldots \sigma_n). \tag{9.16}$$

Substituting (9.15) into (9.11), we find

$$\operatorname*{res}_{\sigma=(d,0)} \rho_{\Gamma_1}(\sigma) = g\Lambda^{-1}; \qquad \rho_{\Gamma_n}(\sigma_1 \ldots \sigma_n) = \Lambda \operatorname*{res}_{\sigma=(d,0)} \rho_{\Gamma_{n+1}}(\sigma\sigma_1 \ldots \sigma_n) \tag{9.17}$$

and an analogous result for the argument σ_n.

Thus *an infinite system of dynamical integral equations* (9.1, 2) *is transformed into a system of algebraic equations* (9.17). Their general solution was found by the authors in refs. [4–6] and is of the form

$$\rho_n(\sigma_1 \ldots \sigma_n) = g\Lambda^{-n}\left\{\prod_{i=1}^{n} \bar{\rho}_1(\sigma_i)\right\}\left\{\prod_{i=1}^{n-1} \bar{\rho}_2(\sigma_i\sigma_{i+1})\right\}\left\{\prod_{i=1}^{n-2} \bar{\rho}_3(\sigma_i\sigma_{i+1}\sigma_{i+2})\right\}\ldots \bar{\rho}_n(\sigma_1 \ldots \sigma_n) \tag{9.18}$$

where $\bar{\rho}_1(\sigma)$ is an arbitrary function having a pole at the point $\sigma = (d, 0)$ with the unit residue

$$\operatorname*{res}_{\sigma=(d,0)} \bar{\rho}_1(\sigma) = 1 \tag{9.19}$$

and $\bar{\rho}_k(\sigma_1 \ldots \sigma_k)|_{k\geq 2}$ are functions satisfying the boundary conditions

$$\bar{\rho}_k(\sigma_1 \ldots \sigma_k)|_{\sigma_1=(d,0)} = \bar{\rho}_k(\sigma_1 \ldots \sigma_k)|_{\sigma_k=(d,0)} = 1. \tag{9.20}$$

Note that the general solution of the dynamical equations (9.12) comprises the infinite set of arbitrary functions

$$\bar{\rho}_1(\sigma_1), \bar{\rho}_2(\sigma_1\sigma_2), \ldots \bar{\rho}_k(\sigma_1 \ldots \sigma_k), \ldots \tag{9.21}$$

which are restricted only by the conditions (9.19, 20). The first n functions of this set enter into each vertex Γ_n. The presence of arbitrary functions in the general solution is due to the fact that the dynamical equations (9.1, 2) should be completed by a set of operator conditions analogous to the canonical commutation relations of the non-renormalized theory. They have not been used in the process of solving equations and should be taken into account additionally. In more detail this is discussed in section 13. It should be emphasized that in this case conserved tensors whose Green functions meet the generalized Ward identities play an important role. They are of particular importance in a conformally-invariant theory. In particular, it may be shown [8] that the general solution (6.14) of the Ward identity for the current-containing Green functions ensures the fulfillment of equal-time commutation relations

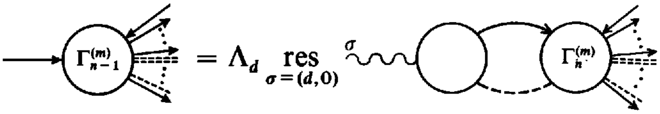
$$\delta(x_0 - y_0)[j_0(x), \Phi_d(y)] = -e\delta(x - y)\Phi_d(y).$$

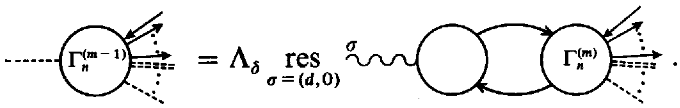
Similar relations for the energy-momentum tensor (they may be obtained from the Ward identity (6.21b) as well) result in the Hamiltonian equations

$$[H\Phi] = \mathrm{i}\partial_\mu\Phi.$$

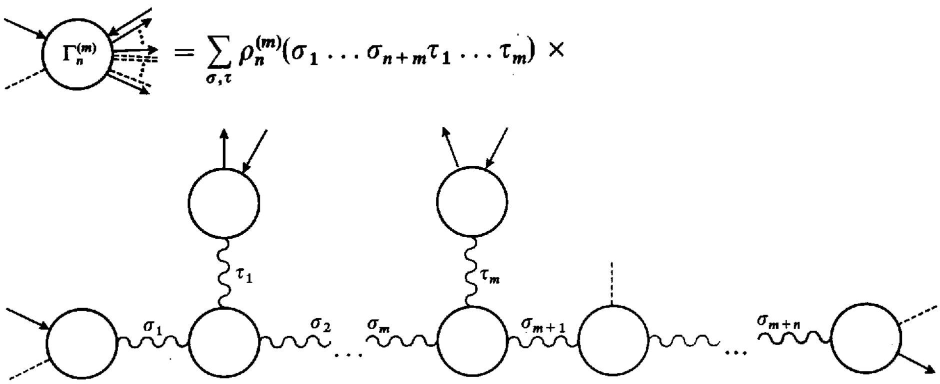
As is known, in the perturbation theory Hamiltonian equations are equivalent to taking into account canonical commutation relations. It is natural to expect that this property holds in an exact theory as well. It becomes particularly important in the cases when canonical commutation relations cannot be taken into account in their explicit form. It is this case that arises in the conformally-invariant theory where field renormalizations are infinite.

The above-said is readily generalized [5, 6] to the case of the equations (8.4–7) at $m \neq 0$. Provided that the auxiliary vertices $R_n^{(m)}$, $\tilde{R}_n^{(m)}$, $M_n^{(m)}$ and $\tilde{M}_n^{(m)}$ are eliminated as in the above case, we obtain equations, connecting only the vertices $\Gamma_n^{(m)}$





These equations are diagonalized by the expansion

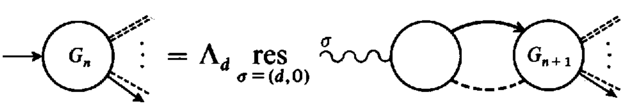


10. Another form of dynamical equations

In sections 8 and 9 the dimensions of fundamental fields were assumed to be in the interval (8.2). Consider now another case. Let the lower limit for the dimension of charged field be defined by the condition*

$$d > h. \tag{10.1}$$

In this case a partial wave expansion can not be applied to one-particle irreducible vertices but to connected Green functions. Hence it is appropriate to use another form of the equations (9.11), representing them as equations for the Green functions. The equivalent form of these equations is

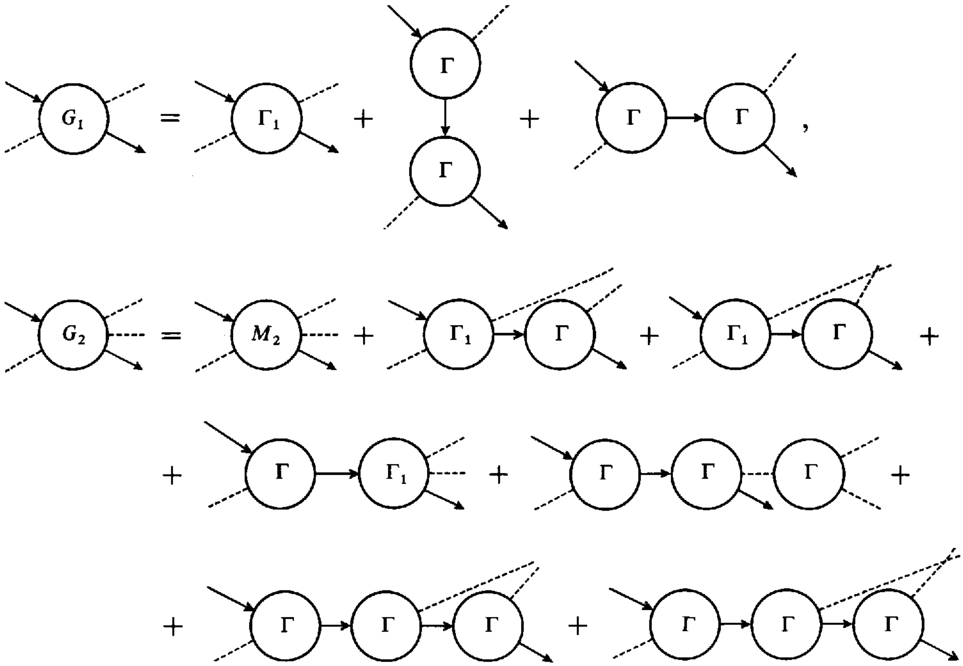


(10.2)

where G_n are the connected Green functions. To verify this statement it is sufficient to represent the Green function as a sum of the one-particle irreducible vertex Γ_n and one-particle reducible

* Strictly speaking the results of this section are applicable only to composite fields since for fundamental fields $d < h$. However in this and the next sections we shall assume that eq. (10.1) is valid also for fundamental fields. This simplifies considerably the calculations of section 11, all the results of which can also be obtained for $d < h$ but in a more cumbersome way, see ref. [8]. Simplifications appearing at $d > h$ are due to the fact that partial expansion is applicable directly to Green functions (see section 5).

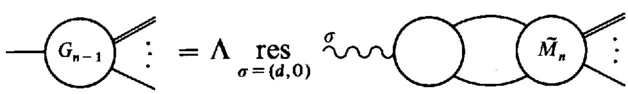
graphs, each containing lower one-particle irreducible vertices Γ_k, $k \leqslant n-1$. The function G_{n+1}, entering into the right-hand side, should be represented as a sum of the vertex M_{n+1} and the graphs which may be cut in the transverse direction. E.g., for $n = 1$ we have



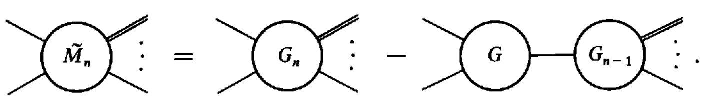
(10.3)

Then it should be taken into consideration that the last four graphs in (10.3) do not contribute to (10.2). As a result the equations (10.2) prove to be equivalent to the equations (9.10). The same case takes place for any m and n.

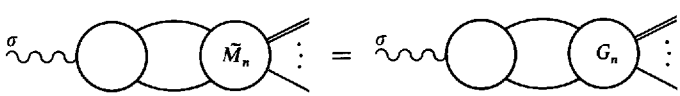
Another way to obtain (10.2) is to use the system of equations for Green functions obtained in [61]. This system of equations is used in ref. [3] for the $\lambda\Phi^3$ theory. We won't present the initial equations [61] (see also [3]) and will give only the result of ref. [3] (solid lines stand here for the neutral field Φ_d)



where

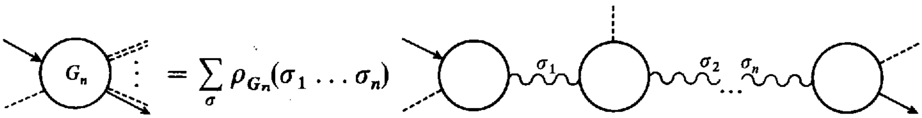


Taking into account that



we again obtain equations in the form (10.2).

Hereafter we will proceed from the assumption that the dimension d is in the interval (10.1) and examine the equations (10.2). All integrals over the internal lines in the cases when they diverge should be determined by the analytical continuation from the convergence region. Now an expansion analogous to (9.15) is valid only for the connected Green function



(10.4)

where the function ρ_{G_n} meets equations (9.17). In what follows it will be assumed that the general expression (9.18) refers to the function ρ_{G_n}.

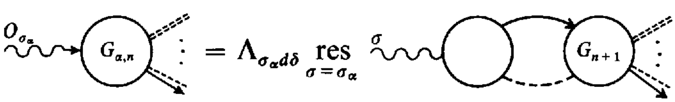
11. Analysis of general solution of equations

11.1. Tensor fields

Consider the equations (8.13). Let the dimensions of all O_{σ_α} fields be assumed to lie in the interval

$$l_\alpha > h. \tag{11.1}$$

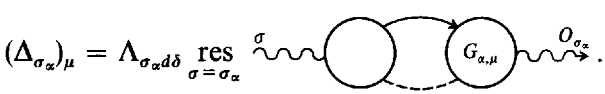
Similar to section 10, represent the equations (8.13) as those for Green functions. Eliminating auxialiary vertices we have



(11.2)

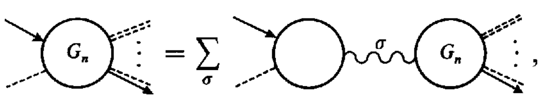
$$g_{\sigma_\alpha d\delta}\Lambda_{\sigma_\alpha d\delta} = -2\mu(\sigma_\alpha), \tag{11.3}$$

where $g_{\sigma_\alpha d\delta}$ is the coupling constant of the field O_{σ_α} with fundamental fields. The relationship (11.3) is the consequence of the propagator equation which may be represented similar to (9.14) in the form



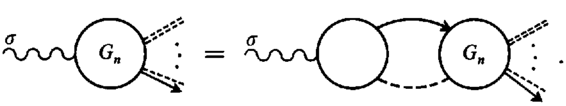
(11.4)

From the equation (11.2) it follows that *the fields O_{σ_α} determine the operator product expansion of the states* $\varphi_d(x_1)\chi_\delta(x_2)\,|0\rangle$. Indeed, substitute the Green function G_n as the expansion



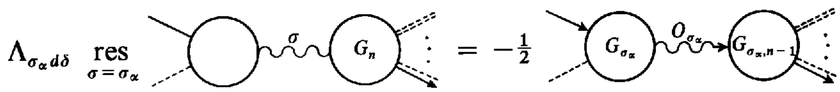
(11.5)

where



(11.6)

Each field O_{σ_α} corresponds to a pole of the amplitude (11.6). Taking account of (9.12) and (11.3) one can rewrite (11.5) in the form*

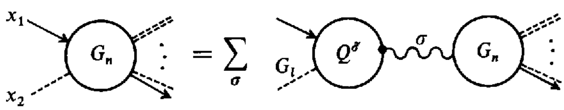


(11.7)

Assume now that the quantity (11.6) has no other singularities but poles at the points $\sigma = \sigma_\alpha$. Then the contribution of these poles into the singularity of the Green function can be obtained as follows. Represent the function $C^{\sigma d\delta}$ in the form (see section 3)

$$C^{\sigma d\delta}(xx_1x_2) = \frac{\pi}{\sin \pi(h - l + s)}\left[Q^{\sigma d\delta}(x|x_1x_2) - \int dx'\Delta_\sigma(x - x')Q^{\tilde{\sigma}d\delta}(x|x_1x_2)\right] \tag{11.5a}$$

and make use of the symmetry of the integration contour over l under substitution $l \to D - l$. We obtain [3]



where

$$G_l = \frac{-2\pi}{\sin \pi(h - l + s)}.$$

The strength of singularity of the function $Q^{\tilde{\sigma}}$ at $x_1 = x_2$ is decreased when l increases. Hence, shifting the integration contour over l to the right we obtain contributions of the poles l_α to the singularity structure of the Green function G_n and strengths of these singularities decreases with l_α increasing. The contribution of each field O_{σ_α} can be written down in an operator form

$$\varphi_d(x_1)\chi_\delta(x_2) = \sum_{\sigma_\alpha} g_{\sigma_\alpha} \frac{\pi}{\sin \pi(l_\alpha - h + s)} \int dx Q^{\tilde{\sigma}_\alpha d\delta}(x|x_1x_2)O_{\sigma_\alpha}(x). \tag{11.8}$$

Note that this is not the operator expansion, since in the Euclidean space the contour of integration over l cannot be closed. Equation (11.8) holds in the limit $x_1 \to x_2$ only.

11.2. Poles of the functions $\bar{\rho}_k$

Represent the Green function G_n in the form of the expansion (10.4) and use expression (9.18) for $\rho_{G_n}(\sigma_1 \ldots \sigma_n)$. Here we want to elucidate analytical properties of the functions $\bar{\rho}_k(\sigma_1 \ldots \sigma_k)$ entering into the general solution (9.18). Below it will be shown that *none of these functions has poles for even k. For odd k each function $\bar{\rho}_k(\sigma_1 \ldots \sigma_k)$ has poles only with respect to the* $((k-1)/2 + 1)$*th argument.* This property will be illustrated by the simplest examples of lower order Green functions

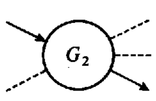
* An analogous relation is valid if the dimension of the fundamental field $d < h$. In this case an essential role in the derivation of eq. (11.7) is played by a one-particle irreducible graph. As has been shown by Mack [3], this graph is responsible for the diminishing of the contribution from the *shadow pole* at the point $d' = D - d$ and for the appearance in its place of the first term in the right-hand side of eq. (11.7) which corresponds to the fundamental field.

which depend on the functions $\bar{\rho}_1$, $\bar{\rho}_2$ and $\bar{\rho}_3$. Generalization to the case of other $\bar{\rho}_k$ functions will be evident.

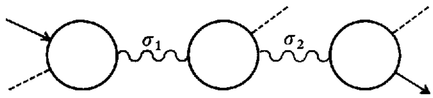
First of all it should be noted, that the above fields are related to the poles of functions $\bar{\rho}_1(\sigma)$. From the equation (11.2) at $n = 1$ and (11.3) we have

$$\operatorname*{res}_{\sigma=\sigma_\alpha} \bar{\rho}_1(\sigma) = \frac{g^2_{\sigma_\alpha d\delta}}{g^2} \frac{\mu(d)}{\mu(\sigma_\alpha)}.$$

Consider now the Green function



$$= \sum_\sigma \rho_{G_2}(\sigma_1\sigma_2)$$

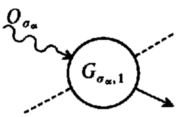


(11.8)

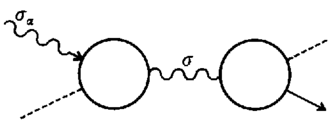
where

$$\rho_{G_2}(\sigma_1\sigma_2) = g\Lambda^{-2}\bar{\rho}_1(\sigma_1)\bar{\rho}_1(\sigma_2)\bar{\rho}_2(\sigma_1\sigma_2). \tag{11.9}$$

Poles of the function $\rho_{G_2}(\sigma_1\sigma_2)$ determine fields entering into the operator product expansion of states $\varphi_d\chi_\delta|0\rangle$. These poles have already been taken into account in $\bar{\rho}_1(\sigma)$. Hence, the function $\bar{\rho}_2(\sigma_1\sigma_2)$ has no poles. However, it has influence on the partial wave expansion of a Green function including any field O_{σ_α}. From (11.2) we have



$$= \sum_\sigma \rho_{G_\alpha,1}(\sigma)$$



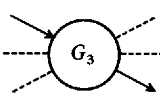
(11.10)

where

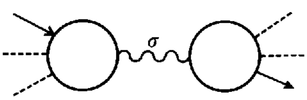
$$\rho_{G_{\sigma_\alpha},1}(\sigma) = g\Lambda^{-2}\Lambda_{\sigma_\alpha d\delta}\bar{\rho}_1(\sigma)\bar{\rho}_2(\sigma_\alpha\sigma). \tag{11.11}$$

The expansion (11.10) has contributions of the poles corresponding only to those fields O_{σ_α} which enter into the expansion of the states $\varphi_d\chi_\delta|0\rangle$ and $O_{\sigma_\alpha}\chi_\delta|0\rangle$. Some of the poles of the function $\bar{\rho}_1(\sigma)$ can be cancelled by zeroes of the function $\bar{\rho}_2(\sigma_1\sigma_2)$. Fields responsible for these poles do not contribute to (11.10). These fields enter into the operator product expansion of states $\varphi_d\chi_\delta|0\rangle$, but do not contribute to the states $O_{\sigma_\alpha}\chi_\delta|0\rangle$.

Consider the Green function G_3. Its partial wave expansion may be written in the form

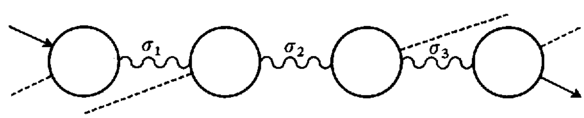


$$= \sum_\sigma$$



(11.12)

$$= \sum_\sigma \rho_{G_3}(\sigma_1\sigma_2\sigma_3)$$



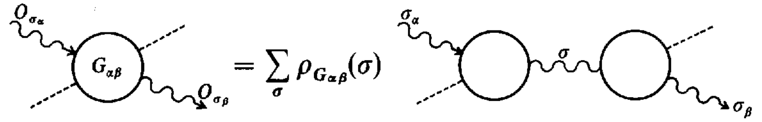
, (11.13)

where

$$\rho_{G_3}(\sigma_1\sigma_2\sigma_3) = g\Lambda^{-3}\bar{\rho}_1(\sigma_1)\bar{\rho}_1(\sigma_2)\bar{\rho}_1(\sigma_3)\bar{\rho}_2(\sigma_1\sigma_2)\bar{\rho}_2(\sigma_2\sigma_3)\bar{\rho}_3(\sigma_1\sigma_2\sigma_3).$$

It is evident that the function $\bar{\rho}_3(\sigma_1\sigma_2\sigma_3)$ has poles neither with respect to the argument σ_1 nor with respect to σ_3. It can have poles only with respect to the argument σ_2. These poles contribute to expansion (11.12). Responsible for them are those fields entering into the operator product expansion of states $\varphi_d\chi_\delta\chi_\delta|0\rangle$. Designate these fields as P_{σ_α}. Such fields do certainly exist in theories with internal symmetry.

It is evident that the fields $P_{\sigma_\alpha}(x)$ also appear in the operator product expansion of the states $O_{\sigma_\alpha}(x_1)\chi_\delta(x_2)|0\rangle$. Indeed, from (11.2) we have



(11.14)

where

$$\rho_{G_{\alpha\beta}}(\sigma) = g\Lambda^{-3}\Lambda_{\sigma_\alpha d\delta}\Lambda_{\sigma_\beta d\delta}\bar{\rho}_1(\sigma)\bar{\rho}_2(\sigma_\alpha\sigma)\bar{\rho}_2(\sigma\sigma_\beta)\bar{\rho}_3(\sigma_\alpha\sigma\sigma_\beta).$$

From this it follows that expansion (11.14) is contributed by both the poles of the function $\bar{\rho}_1(\sigma)$ and those of $\bar{\rho}_3(\sigma_\alpha\sigma\sigma_\beta)$ with respect to the argument σ.

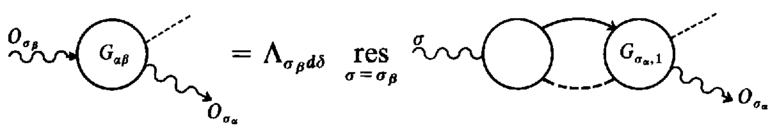
11.3. The universality of the conformal invariant theory

Thus we have an infinite set of the fields

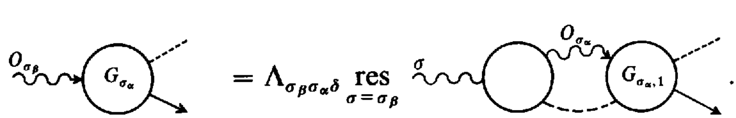
$$\varphi_d(x),\ \chi_\delta(x),\ O_{\sigma_\alpha}(x),\ P_{\sigma_\alpha}(x), \ldots \tag{11.15}$$

where the fields O_{σ_α} are responsible for the poles of $\bar{\rho}_1(\sigma)$, the fields P_{σ_α} correspond to the poles of $\rho_3(\sigma_1\sigma_2\sigma_3)$ with respect to the argument σ_2, etc. The fields O_{σ_α}, P_{σ_α} may be considered as composite, since they are the product of two and more fundamental fields taken at one space-time point. Note that the internal lines in the equations (8.3–7) and (11.2) stand for the propagators which correspond to the fundamental fields φ_d and χ_δ. This distinguishes them from other fields of the infinite set (11.15). Let it be shown now that in the conformally-invariant theory the equations (11.2) may be completed by other equations where the internal lines stand for the propagators of composite fields. As a result *we obtain a complete system of equations of type* (11.2) *into which all fields enter on equal grounds, and fundamental fields have no privileges over the others.* This property of the conformal-invariant theory is closely connected with the orthogonality relation (2.6) which is valid for any conformal fields.

Consider first the Green function (11.10). Let O_{σ_β} be the field responsible for one of the poles of the function $\rho_{G_{\alpha},1}(\sigma)$. This field enters into the operator product expansions of both states $\varphi_d\chi_\delta|0\rangle$ and $O_{\sigma_\alpha}\chi_\delta|0\rangle$. Hence an equation for this field can be written in two forms:



(11.16)

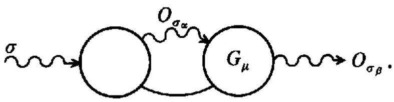


(11.17)

For the compatibility of these equations it is necessary to meet the conditions

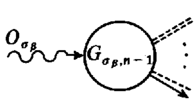
$$\Lambda_{\sigma_\beta d\delta} g_{\sigma_\beta d\delta} = \Lambda_{\sigma_\beta \sigma_\alpha \delta} g_{\sigma_\beta \sigma_\alpha \delta}. \tag{11.18}$$

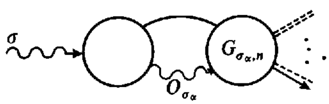
It can be obtained after the substitution of the expansion (11.10) to (11.16) and (11.17) and elimination of the value $\operatorname{res}_{\sigma=\sigma_\beta} \rho_{G_{\sigma_\alpha,1}}(\sigma)$. The condition (11.18) implies that the value $\Lambda_{\sigma_\beta\sigma_\alpha\delta} g_{\sigma_\beta\sigma_\alpha\delta}$ is independent of the quantum numbers σ_α and δ, since the internal lines in (11.17) may correspond to any fields from the set (11.15). It can readily be seen that (11.18) is always fulfilled. This follows from the equation for the field O_{σ_β} propagator which may be written either in the form (11.4) or similar to (11.17) in the form

$$(\Delta_{\sigma_\beta})_\mu = \Lambda_{\sigma_\beta\sigma_\alpha\delta} \operatorname*{res}_{\sigma=\sigma_\beta}$$ 

(11.19)

It is natural to assume that for higher Green functions equations analogous to (11.17) and (11.19) exist



$$= \Lambda_{\sigma_\beta\sigma_\alpha\delta} \operatorname*{res}_{\sigma=\sigma_\beta}$$ 

(11.20)

In contrast to (11.17) these equations are not identically fulfilled for $n \geqslant 2$. They lead to additional relationships between the functions $\bar{\rho}_k$ for $k \geqslant 2$. It may be shown, e.g., that

$$\bar{\rho}_3(\sigma_\alpha\sigma_\beta\sigma_\gamma) = 1, \qquad \bar{\rho}_4(\sigma_\alpha\sigma_\alpha\sigma\sigma_\alpha) = [\bar{\rho}_3(\sigma_\alpha\sigma_\alpha\sigma)]^{-1}, \text{ etc.}$$

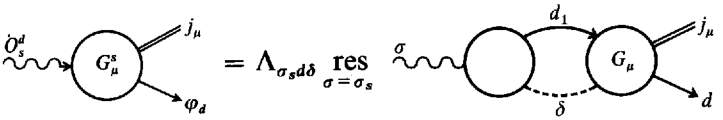
Equations analogous to (11.20) may be written for the Green functions which comprise only the composite fields O_{σ_α}, P_{σ_α} etc.

Thus we have an infinite system of equations for the set of the fields (11.15). *This system structure is such that none of the fields* (11.15) (*including the fundamental fields* φ_d, χ_δ) *is in any way privileged over the others.* In its derivation we proceed from quite definite Lagrangian (8.1). It is clear, however, that if some other Lagrangian which includes some pair of tensor fields (one of them should be charged) from the set (11.15) is chosen as the initial one, one would obtain the same system of equations once again.

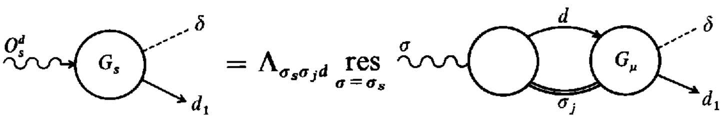
12. Calculation of coupling constants of tensor fields with current

Proceeding from the above equations for the Green functions, we may continue now the analysis of special solution of the Ward identities found in section 7. We will use specially-chosen current-containing Green functions in the form (7.2) and (7.3). In this case the operator product expansion of the states $\varphi_d\chi_\delta|0\rangle, j_\mu\varphi_d|0\rangle$ includes (see section 7) tensor fields $O_s^d(x)$ described in section 6. In what follows the coupling constants of these fields with current and fundamental fields will be found.

Consider the Green function (7.2). According to (11.16) and (11.17) the following equations



(12.1)

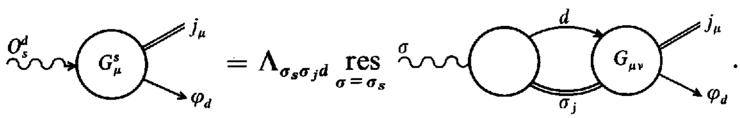


(12.2)

hold, where G_μ^s is the Green function (6.9), σ_s are quantum numbers (6.8) and

$$G_s(x_1x_2x_3) = \langle 0|\, TO_s^d(x_1)\varphi_{d_1}^+(x_2)\chi_\delta(x_3)\,|0\rangle = g_s(dd_1\delta)C^{\sigma_s d_1\delta}(x_1x_2x_3). \tag{12.3}$$

Note that in the equation (12.2) the internal line is associated with the current propagator. There is a similar equation for the Green function (7.3)



(12.4)

Substitute expression (7.5) into (12.1). Calculating the residue, we find

$$G_\mu^s(x_1x_2x_3) = g_j^s(d)2^{s/2}\{(\tfrac{1}{2}x_{23}^2)^{-h+1}\overleftrightarrow{\partial}_\mu^{x_2}[(\tfrac{1}{2}x_{12}^2)^{-h+1}(\lambda_{\mu_1\ldots\mu_s}^{x_1}(x_2x_3) - \text{traces})]\}(\tfrac{1}{2}x_{13}^2)^{h-1-d}, \tag{12.5}$$

where $g_j^s(d)$ is the coupling constant. Another coupling constant $f_s(d)$ entering into the general expression (6.9) for these Green functions equals to zero

$$f_s(d) = 0. \tag{12.5a}$$

This is due to the special choice of the Green function (6.13) in the form (7.2). It is interesting that in (12.5) there are no contact terms which are present in the general expression (6.9).

Let the coupling constants $g_j^s(d)$ and $g_s(d_1d\delta)$ be found. In principle they may be obtained from the equations (12.2) and (12.4). In practical calculations, however, it should be taken into account that in contrast to (2.8) the constant $g_j^s(d)$ is determined by the relation (6.9) and the normalization factor of the function $C^{\sigma\sigma_j d}$ is singular at $\sigma = \sigma_s$, see (6.24, 24a). Hence it is appropriate to use instead of the equations (12.2) and (12.4) the equations of type (11.6), which do not comprise explicitly normalization factors of the functions $C^{\sigma\sigma_j d}$. In more details these calculations are given in our work [8]. The result is

$$[g_j^s(d)]^2 = g_j^2(d)\frac{\Gamma^2(h+s-1)}{\Gamma^2(h-1)}A_s(d) \tag{12.6}$$

$$[g_s(d_1d\delta)]^2 = g_{d_1d\delta}^2\left\{\frac{\Gamma((d-d_1+\delta)/2+s)}{\Gamma((d-d_1+\delta)/2)}\,\frac{N(dd_1\delta)}{N(\sigma_s d_1\delta)}\right\}^2 A_s(d) \tag{12.7}$$

$$A_s(d) = \frac{2^s}{\Gamma(s+1)}\,\frac{(d+2s-1)\Gamma(d+s-1)}{\Gamma(d)}\left\{\frac{\Gamma(h-d)\Gamma(D-d-s-1)}{\Gamma(h-d-s)\Gamma(D-d-1)}\right\}. \tag{12.8}$$

The field $O_s^d(x)$ with $s = 0$ coincides with the field $\varphi_d(x)$. According to (12.6–8) at $s = 0$ we have $A_s(d)|_{s=0} = 1$ and $g_j^s(d)|_{s=0} = g_j(d)$, $g_s(dd_1\delta)|_{s=0} = g_{dd_1\delta}$, so that equations (12.1, 2) and (12.4) for $s = 0$ are fulfilled identically. Note that this is the consequence of the Ward identities, which unambiguously determine a scalar contribution to the partial wave expansion of the Green functions G_μ and $G_{\mu\nu}$.

As it follows from the expression (12.6), the constant $g_j^s(d)$ is real if $A_s(d) > 0$. The quantity $A_s(d)$ is an oscillating function in the interval $h - 1 < d < D - 2$ and is strictly positive at $d > D - 2$. As shown in section 7, the latter inequality may be obtained from the requirement of positivity. Thus the reality of the constant $g_j^s(d)$ has been ensured.

In conclusion note that the expressions obtained for the coupling constants $g_j^s(d)$ and $g_s(dd_1\delta)$, as well as the condition (12.5a) take place only for the proper choice of the four-point Green functions (7.2, 3). In a general case if arbitrary transverse parts are added to (7.2, 3), it is evident that the coupling constants g_j^s and g_s change. In particular, it is possible to choose a transverse part so that the total coupling constant $\tilde{g}_j(d) = g_j^s(d) + f_s(d)$, entering into (6.9), would be zero. Then Wightman function (6.10) will also be equal to zero and the fields O_s^d are absent in this theory. Thus the existence of the fields O_s^d cannot be granted without further analysis of the dynamics.* It is essential, however, that if these fields do exist in the theory, *the dependence of the coupling constant $g_s(dd_1\delta)$ on the dimensions d_1 and δ is universal*, since the addition of arbitrary transverse parts to (7.2, 3) changes only the value $A_s(d)$. This dependence is given by the expression in braces in (12.7). Note that in two-dimensional space, where the transverse part of the Green function is fixed, the fields O_s^d do exist.

13. Differential equations for Green functions

As it was already noted the values of dimensions and coupling constants of the O_{σ_α}-fields and the basic fields φ and χ cannot be found dealing with the integral equations of section 8 alone. For determining them it is necessary to find a way to close the equations. At the present time this problem has not been solved yet. In the present section and the next one we discuss a possible version of solving it basing upon a fuller exploitation of the equations of motion for the bare fields

$$\Box\varphi(x) = \lambda : \varphi(x)\chi(x) :, \qquad \Box\chi(x) = \lambda : \varphi^+(x)\varphi(x) :. \tag{13.1}$$

It was already emphasized in the previous sections that some part of information contained in the bare Lagrangian is lost when neglecting the bare term $z_1\gamma$ in the equations (8.3)–(8.7) and the term $z_2\Box$ in the equations for the Green functions. It proves that the equations of motion can be written in such a form as to effectively include the limitations on the solutions of the integral equations coming from these small corrections (for more detailed discussion see section 16) that were omitted from the integral equations. To this end we carry out the renormalizations (see also ref. [66]) directly in the equations (13.1)

$$\varphi_R(x) = z_2^{-1/2}\varphi(x), \qquad \chi_R(x) = z_3^{-1/2}\chi(x), \qquad \lambda_R = z_1^{-1}z_2z_3^{1/2}\lambda.$$

For the renormalized fields we have (with account of mass counterterms)

$$\Box\varphi_R(x) = \lambda_R\{z_\varphi\varphi_R(x)\chi_R(x)\} - \delta m_0^2\varphi_R(x); \qquad \Box\chi_R(x) = \lambda_R\{z_\chi\varphi_R^+(x)\varphi_R(x)\} - \delta\mu_0^2\chi_R(x)$$

* See also the footnote before eq. (7.9) and the end of section 16.4.

where $z_\varphi = z_1 z_2^{-1}, z_\chi = z_1 z_3^{-1}$. With the designation

$$\underline{\varphi_R(x)\chi_R(x)} \equiv \{z_\varphi \varphi_R(x)\chi_R(x)\} - \frac{\delta m^2}{\lambda_R} \varphi_R(x) \tag{13.2}$$

and the analogous designation $\underline{\varphi_R^+(x)\varphi_R(x)}$ one writes the equations in the form

$$\Box \varphi_R(x) = \lambda_R \underline{\varphi_R(x)\chi_R(x)}, \tag{13.3}$$

$$\Box \chi_R(x) = \lambda_R \underline{\varphi_R^+(x)\varphi_R(x)}. \tag{13.4}$$

The subscript R will be hereafter omitted.

To make the equations (13.3, 4) meaningful one must point out a way to define their right-hand sides, i.e. to define products of renormalized fields at coinciding points. It will be shown below that this may be done in an explicit form if *one uses the solution of the integral equations* (see section 8) obtained in sections 9–11. We shall demonstrate further, with the use of the Thirring model as an example, how the equations (13.3, 4) can be exploited for building the unique conformally invariant theory.

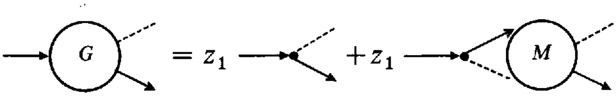
13.1. *Product of renormalized fields in coinciding points*

The right-hand side of the equations (13.3) and (13.4) contains the $0 \times \infty$-uncertainty. Indeed, let us write (13.2) in the form of the limit

$$\underline{\varphi_d(x)\chi_\delta(x)} = \lim_{\varepsilon \to 0} z_\varphi(\varepsilon)\{\varphi_d(x)\chi_\delta(x+\varepsilon) - \ldots\}, \tag{13.5}$$

where the dots stand for subtractions. The operator product $\varphi_d(x)\chi_\delta(x+\varepsilon)$ is singular as $\varepsilon \to 0$ while the constant $z_\varphi(\varepsilon)$ tends to zero. The task is to determine the constant $z_\varphi(\varepsilon)$ and the explicit form of all necessary subtractions.

It is therefore natural to exploit the solutions of integral equations (8.3–8.7) for finding the limit (13.5). They contain the whole information necessary for the limiting procedure (13.5). Indeed, consider the equations (8.3), (8.4) and (8.6). Before passing to the limit $z_1 = 0$ one has [60]



(13.6)

where the dot designates the bare vertex γ. The equations for higher vertices have an analogous form. The $0 \times \infty$-uncertainty in the second term of eq. (13.6) is effectively resolved due to the introduction of auxiliary vertices $R_n^{(m)}$ defined by the equations (8.5), (8.7). The solving of these equations (see section 9) results in a simple way of resolving the uncertainty directly for the Green functions (see eqs. (9.11), (10.2) and (11.2)). The role of the bare term $z_1\gamma$ is played by the right-hand side of (9.13). It tends to zero as $l \to d$, while the quantity (11.6) has a pole in this point. According to the equations (9.11) we must obtain the field $\varphi_d(x)$ again as a result of such resolving. Symbolically one may write $\varphi_d(x) = \{z_\varphi^{(0)}\varphi_d(x)\chi_\delta(x)\}$. Our task now is to represent this result in the limiting form (13.5). Note, that owing to the conformal invariance one may represent in this form not only the field $\varphi_d(x)$ but also its derivatives of any order.

For fulfilling this task it is convenient to represent the solution of equations (10.2) and (11.2) in the form (11.7a). Assume that the dimension of the basic field $\varphi_d(x)$ is less than that of any of the charged composite fields. Then as $x_{12} \to 0$ the first term of (11.8) is the most singular one. Consequently we have

$$\varphi_d(x_1)\chi_\delta(x_2)|_{x_{12}\to 0} = g\frac{\pi}{\sin \pi(d-h+s)}\int dx Q^{d\delta\tilde{d}}(x_1x_2|x)\varphi_d(x). \tag{13.7}$$

Let us expand the function $Q^{d\delta\tilde{d}}(x_1x_2|x)$ in powers of x_{12}. To this end the relation

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\nu+k+1)}\left(\frac{z^2}{4}\right)^k \tag{13.8}$$

may be used. Within the leading order in x_{12} one has $Q^{d\delta\tilde{d}}(x_1x_2|x) \sim (x_{12}^2)^{-\delta/2}\delta(x_1-x)$. By substituting this relation into (13.7) and going to the limit $x_{12}\to 0$ we find

$$\varphi_d(x) = \lim_{\varepsilon\to 0} z_\varphi^{(0)}(\varepsilon)\varphi_d(x)\chi_\delta(x+\varepsilon) \tag{13.9}$$

where

$$z_\varphi^{(0)}(\varepsilon) = g^{-1}N^{-1}(d\delta d)\frac{\Gamma(d)}{\Gamma(h-d)}(\tfrac{1}{2}\varepsilon^2)^{\delta/2}. \tag{13.10}$$

In a similar way any field O_{σ_α} may be represented in the form

$$O_{\sigma_\alpha}(x) = \lim_{\varepsilon\to 0} z_\alpha^{(0)}(\varepsilon)\{\varphi_d(x)\chi_\delta(x+\varepsilon) - \dots\}$$

where $z_\alpha^{(0)}(\varepsilon) \sim (\frac{1}{2}\varepsilon^2)^{(d-l_\alpha+\delta)/2}$ and the dots designate the subtractions of the contributions of all fields (and their derivatives) whose dimensions are less than l_α. Any derivative of the fields φ_d and O_{σ_α} may also be represented in analogous form. It is essential that (13.7) determines not only the renormalization constant but also the explicit form of all necessary subtractions in (13.5).

Consider, for example, the contribution of the first and the second derivatives of the field $\varphi_d(x)$ into (13.7). With the use of (13.8) one finds:

$$Q^{d\delta\tilde{l}}(x_1x_2|x) = A(l)(\tfrac{1}{2}x_{12}^2)^{(l-d-\delta)/2}\left\{1 - \tfrac{1}{2}\frac{l-d+\delta}{l}(x_{12})_\mu\partial_\mu^{x_1}\right.$$
$$\left. + \tfrac{1}{8}\frac{(l-d+\delta)(l-d+\delta+2)}{l(l+1)}(x_{12})_\mu(x_{12})_\nu\partial_\mu^{x_1}\partial_\nu^{x_1} - \tfrac{1}{16}\frac{(l-d+\delta)(l+d-\delta)}{l(l+1)(l-h+1)}x_{12}^2\Box_{x_1} + \dots\right\}\delta(x_1-x)$$

where

$$A(l) = \frac{\sin \pi(l-h)}{\pi}N(ld\delta)\frac{\Gamma(h-l)}{\Gamma(l)}.$$

The substitution of this expression into (13.7) leads to

$$\varphi_d(x)\chi_\delta(x+\varepsilon) = A\frac{1}{(\frac{1}{2}\varepsilon^2)^{\delta/2}}\left\{\varphi_d(x) - \tfrac{1}{2}\frac{\delta}{d}\varepsilon_\mu\partial_\mu^x\varphi_d(x)\right.$$
$$\left. + \tfrac{1}{8}\frac{\delta(\delta+2)}{d(d+1)}\varepsilon_\mu\varepsilon_\nu\partial_\mu\partial_\nu\varphi_d(x) - \tfrac{1}{16}\frac{(2d-\delta)\delta}{d(d+1)(d-h+1)}\varepsilon^2\Box\varphi_d(x)\right\} + \dots \tag{13.11}$$

where

$$A = gN(dd\delta)\Gamma(h-d)/\Gamma(d). \tag{13.11a}$$

It follows that $\Box\varphi_d(x)$ may be represented as the limit

$$\Box\varphi_d(x) = 4A^{-1}\frac{d(d-h+1)D}{\delta(\delta-D+2)}\lim_{\varepsilon\to 0}(\tfrac{1}{2}\varepsilon^2)^{\delta/2-1}\int d\Omega_\varepsilon \times\left\{\varphi_d(x)\chi_\delta(x+\varepsilon) - \frac{A}{(\frac{1}{2}\varepsilon^2)^{\delta/2}}\left[\varphi_d(x) - \frac{\delta}{2d}\varepsilon_\mu\partial_\mu\varphi_d(x)\right]\right\}, \tag{13.12}$$

where $\int d\Omega_\varepsilon$ stands for the averaging over directions of the vector ε_μ. The comparison with (13.3) yields

$$\underbrace{\varphi_d(x)\chi_\delta(x)} = \lim_{\varepsilon\to 0} z_\varphi^{(1)}(\varepsilon)\int d\Omega_\varepsilon\left[\varphi_d(x)\chi_\delta(x+\varepsilon) - \frac{A}{(\frac{1}{2}\varepsilon^2)^{\delta/2}}\varphi_d(x) + \frac{\delta A}{2d}\frac{1}{(\frac{1}{2}\varepsilon^2)^{\delta/2}}\varepsilon_\mu\partial_\mu\varphi_d(x)\right], \tag{13.13}$$

where

$$z_\varphi^{(1)} \sim (\tfrac{1}{2}\varepsilon^2)^{\delta/2-1} \tag{13.14}$$

and the constant A is given by eq. (13.11a). The relation $\int d\Omega_\varepsilon \varepsilon_\mu\varepsilon_\nu/\varepsilon^2 = \delta_{\mu\nu}/D$ was used in deriving (13.12) and (13.13).

In an analogous manner one can find the operator product $\underbrace{\varphi_d^+(x)\varphi_d(x)}$. The result is

$$\underbrace{\varphi_d^+(x)\varphi_d(x)} = \lim_{\varepsilon\to 0} z_\chi(\varepsilon)\int d\Omega_\varepsilon\left\{\varphi_d^+(x)\varphi_d(x+\varepsilon) - \frac{2}{(\frac{1}{2}\varepsilon^2)^{d-\delta/2-1}}\left[\frac{B}{\varepsilon^2}\chi_\delta - \frac{B}{2}\frac{\varepsilon_\mu}{\varepsilon^2}\partial_\mu\chi_\delta(x)\right]\right\} \tag{13.15}$$

where

$$z_\chi(\varepsilon) \sim (\tfrac{1}{2}\varepsilon^2)^{d-\delta/2-1}. \tag{13.16}$$

Relations (13.13)–(13.16) solve the task formulated above.

13.2. Equations for renormalized Green functions

Note, however, that the derivation of the relation (13.12) cannot be referred to as that of the equation (13.3). Indeed, let us assume that, apart from $\varphi_d(x)$, some fields O_1 and O_2 whose dimensions are $d_1 = d+1$ and $d_2 = d+2$, respectively, enter into the right-hand side of the equation (11.7a). In this case equation (13.9) remains unaffected since the contribution of the field $\varphi_d(x)$ is still most singular. However, *the contribution of the derivatives of the field $\varphi_d(x)$ cannot now be distinguished from that of the fields O_1 and O_2*, since they are of equal singularity. In particular one has instead of (13.12)

$$\Box\varphi_d(x) + (\partial_x O_1(x) + O_2(x)) = \lambda\underbrace{\varphi_d(x)\chi_\delta(x)}. \tag{13.17}$$

By comparing (13.17) with the equations of motion we conclude that any Green functions containing the field $(\partial_x O_1(x) + O_2(x))$ must vanish. This condition leads to additional equations that determine the Green functions of the fields φ and χ.

To make this programme sufficiently meaningful it is necessary to justify the existence of the fields O_1 and O_2 in the theory. This problem is far from being simple and is yet awaiting its solution. Here we confine ourselves to an example in two-dimensional space, where the existence of such fields follows from the conservation laws of the current and we succeed in carrying out the above programme completely, see the next section. Namely, we show that the solution of the Thirring model is totally determined by the requirement that these additional fields should not contribute into the equations of motion.

Let us now discuss the general restrictions on the asymptotic behavior of the Green function that follow from the equation of motion. To this end it is convenient to exploit the representation of the Green functions that was described in section 2. In particular, one has

$$G_1(x_1x_2x_3x_4) = \langle 0|\, T\varphi_d(x_1)\varphi_d^+(x_2)\chi_\delta(x_3)\chi_\delta(x_4)\,|0\rangle$$
$$= (\tfrac{1}{2}x_{12}^2)^{-d+\delta}[(\tfrac{1}{2}x_{13}^2)(\tfrac{1}{2}x_{23}^2)(\tfrac{1}{2}x_{14}^2)(\tfrac{1}{2}x_{24}^2)]^{-\delta/2}F(\xi,\eta), \tag{13.18}$$

where ξ and η are the variables (2.11) and $F(\xi,\eta)$ is an arbitrary function. Since $G(x_1x_2x_3x_4)$ is symmetric in the arguments x_3 and x_4 one has

$$F(\xi,\eta) = F(\eta,\xi).$$

The leading contribution to the asymptotic behaviour of G_1, as $x_{14}\to 0$ comes from the field $\varphi_d(x)$. It follows from equation (13.9) that

$$\lim_{\varepsilon\to 0}(\tfrac{1}{2}\varepsilon^2)^{\delta/2}G(x_1x_2x_3x_1+\varepsilon) = AG(x_1x_2x_3). \tag{13.19}$$

The substitution of (13.18) into it gives

$$F(1,\infty) = F(\infty,1) = g^2N^2(dd\delta)/(2\pi)^h\frac{\Gamma(h-d)}{\Gamma(d)}. \tag{13.20}$$

One finds analogously the leading contribution to the asymptotic behaviour in the other channel, when $x_{12}\to 0$. The equation, which is the analog of (13.9), leads to

$$\lim_{x_2\to x_1}(\tfrac{1}{2}x_{12}^2)^{d-\delta/2}G_1(x_1x_2x_3x_4) \sim \langle 0|\, T\chi_\delta(x_1)\chi_\delta(x_3)\chi_\delta(x_4)\,|0\rangle$$

whence it follows that

$$F(\xi,\eta)|_{\xi=\eta\to 0} \sim \xi^{-\delta/2}. \tag{13.21}$$

More detailed information about the asymptotic behavior can be obtained from the differential equations (13.3) and (13.4) which are complemented by the equations (13.13)–(13.16). Note first of all, that equation (13.3) for the propagator holds identically (to be more precise, it determines the coefficient of proportionality in (13.14)). One has

$$\Box G(x_1x_2) = \lambda G(\underbrace{x_1x_2x_1}). \tag{13.22}$$

The quantity

$$G(\underbrace{x_1x_2x_1}) = \langle 0|\, T\underbrace{\varphi_d(x_1)\chi_\delta(x_1)}\varphi_d(x_2)\,|0\rangle$$

in the right-hand side is defined according to (13.13). It may be easily verified, that the following relation holds

$$G(\underbrace{x_1x_2x_1}) \sim \frac{1}{(x_{12}^2)^{d+1}}\, G_d(x_{12})$$

and that it turns the equation of motion (13.22) into an identity when being substituted into it.

Consider now the equation for the Green function

$$\Box_{x_1} G(x_1x_2x_3) = \lambda G_1(\underbrace{x_1x_2x_3x_1}) \tag{13.23}$$

where

$$G_1(\underbrace{x_1x_2x_3x_1}) = \langle 0|\, T\underbrace{\varphi_d(x_1)\chi_\delta(x_1)}\varphi_d^+(x_2)\chi_\delta(x_3)\,|0\rangle .$$

The calculation of the right-hand side of (13.23) with the help of (13.13) leads to

$$G_1(\underbrace{x_1x_2x_3x_1}) = \lim_{x_4\to x_1} \int \mathrm{d}\Omega_\varepsilon (x_{14}^2)^{\delta/2-1} \Big\{ G_1(x_1x_4x_2x_3)$$

$$- A\, \frac{1}{(\frac{1}{2}x_{14}^2)^{\delta/2}}\, G(x_1x_2x_3) + \frac{A\delta}{2d}\, \frac{(x_{14})_\mu}{(\frac{1}{2}x_{14}^2)^{\delta/2}}\, \partial_\mu^{x_1} G(x_1x_2x_3)\Big\}. \tag{13.24}$$

For further calculations it is useful to substitute the representation (13.18) into (13.24) and take the condition (13.20) into account. Let us expand $F(\xi,\eta)$ in the vicinity of the point $\xi = 1$, $\eta = \infty$ into a series in powers of x_{14} up to the second order terms. This expansion is possible due to the fact that $F(1,\infty) \neq 0$, see (13.20). This expansion contains three unknown constants $\partial F/\partial\xi|_{\xi=1,\eta=\infty}$, $\partial^2 F/\partial\xi\partial\xi|_{\xi=1,\eta=\infty}$, and $\partial F/\partial(1/\eta)|_{\xi=1,\eta=\infty}$. By substituting the expansion thus obtained into (13.23) and (13.22) we find three equations for these constants. In this way the next three terms in the asymptotic expansion of the function $G_1(x_1x_2x_3x_4)$ at $x_{14}\to 0$ will be found. We do not present the corresponding results here, since they will not be needed in what follows. We only point out the connection of these results with the above described programme of closing the equations. Indeed, the fields $\partial_x O_1$ and O_2 give the same (in what concerns the strength of singularity) contribution into the asymptotic behavior of the function $G_1(x_1x_2x_3x_4)$ as the fields $\Box\varphi_d(x)$ and $\partial_\mu\partial_\nu\varphi_d(x)$. On the other hand the total contribution is fixed by the equations for higher Green functions, analogous to (13.22), while the contributions of the derivatives of the field $\varphi_d(x)$ is known. Thus the contribution of the field $\partial_x O_1 + O_2$ becomes known, too. We show in the next section that in the Thirring model this is sufficient for closing the equations.

We conclude this section by noting that the statement made in section 11 about the universality of the interaction at small distances concerns only to the leading contributions in the asymptotic behavior of the Green functions. As it is seen from the above analysis if one keeps less singular contributions the basic fields φ_d and χ_δ prove to be singled out as compared with the composite ones.

14. Thirring model

In this section all the results of sections 5–13 will be demonstrated on the example of (exactly solved) Thirring model, describing four-fermion interaction in two-dimensional space. It is convenient to write the Lagrangian of this model as a three-linear interaction $\lambda\bar{\psi}\gamma_\mu\psi j_\mu$, introducing conserved current $j_\mu = \bar{\psi}\gamma_\mu\psi$. Note, that due to the current conservation the renormalization constants of the vertex (z_1) and spinor field (z_2) are equal: $z_1 = z_2$, and the constant z_3 is finite. This means that the definition of the operator product (see section 13) $\underline{j_\mu(x)\psi(x)}$ comprises only one subtraction (mass counterterm), and the multiplicative renormalization is absent ($z_\psi = z_1 z_2^{-1} = 1$). Note also, that the dimension dependence on the coupling constant, see (14.8), can be calculated using the general methods of section 16. Evaluating the constants z_1 and z_2 and assuming $z_1 = z_2$, one obtains the known [68–70] dependence of the dimension on the coupling constant (and charges a and $\bar{a}$). Here, however, we use another method of calculation based on the results of the previous section.

14.1. Preliminary remarks

The equation of motion is

$$-\hat{\partial}\psi(x) = \lambda\gamma_\mu j_\mu(x)\psi(x) \tag{14.1}$$

where ψ is a spinor field and j_μ is a conserved current. In the two-dimensional space it satisfies two conservation laws (see section 7): $\partial_\mu j_\mu = 0$ and $\varepsilon_{\mu\nu}\partial_\mu j_\nu = 0$. Hereafter we shall consider the fields and the Green functions in the Euclidean space. We adopt the following relations for the γ-matrices

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}, \qquad \gamma_\mu^+ = \gamma_\mu \tag{14.2}$$

$$\gamma_5^2 = -1, \qquad \gamma_5^+ = -\gamma_5, \qquad \gamma_\mu\gamma_5 = -\varepsilon_{\mu\nu}\gamma_\nu \tag{14.3}$$

where $\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$, $\varepsilon_{01} = -1$. The idea of the present approach is as follows. We shall show that the conservation laws imply that the expansion of the operator product $j_\mu(x_1)\psi_d(x_2)$ should include a spinor field ψ_1 of the dimension $d_1 = d + 1$. This field is analogous to the fields O_s^d considered in section 6. The Ward identity for the Green functions of this field contains the Schwinger terms, the same as eq. (6.10) did. According to the previous section, this field, along with the derivative $\partial_\mu\psi$, contributes into the operator product $\underline{j_\mu(x)\psi_d(x)}$, cf. (13.27). Further on, we shall demand, in accord with (14.1), that the product $\gamma_\mu \underline{j_\mu(x)\psi(x)}$ should include only the field $\hat{\partial}\psi$, and not $\psi_1(x)$, and show that this demand uniquely determines all the Green functions and also the dependence of the dimension d upon the coupling constant λ. For example we shall consider the two Green functions

$$G_{\mu\nu}(x_1x_2|x_3x_4) = \langle 0|\, T\psi_d(x_1)\psi_d^+(x_2)j_\mu(x_3)j_\nu(x_4)\,|0\rangle, \tag{14.4}$$

$$G_1(x_1x_2x_3x_4) = \langle 0|\, T\psi_d(x_1)\psi_d^+(x_2)\psi_d(x_3)\psi_d^+(x_4)\,|0\rangle. \tag{14.5}$$

Before starting this programme we list now some known [68–70] results concerning the solution of this model for the sake of comparison. Consider the lowest Green functions

$$G_d(x_1x_2) = \langle 0|\, T\psi_d(x_1)\psi_d^+(x_2)\,|0\rangle, \tag{14.6}$$

$$G_{j_\mu}(x_1x_2x_3) = \langle 0|\, T\psi_d(x_1)\psi_d^+(x_2)j_\mu(x_3)\,|0\rangle. \tag{14.7}$$

The equation of motion is

$$-\hat{\partial}G(x_1x_2) = \lambda\gamma_\mu \langle 0|\, Tj_\mu(x_1)\psi_d(x_1)\psi_d^+(x_2)\,|0\rangle. \tag{14.8}$$

The Green function $G_{j_\mu}(x_1x_2x_3)$ can be uniquely found from the two Ward identities

$$\begin{aligned} \partial_\mu^{x_3}G_{j_\mu}(x_1x_2x_3) &= -a[\delta(x_{13}) - \delta(x_{23})]G_d(x_{12}) \\ \varepsilon_{\mu\nu}\partial_\mu^{x_3}G_{j_\nu}(x_1x_2x_3) &= \bar{a}[\delta(x_{13}) - \delta(x_{23})]\gamma_5 G_d(x_{12}) \end{aligned} \tag{14.9}$$

whence it follows that

$$G_{j_\mu}(x_1x_2x_3) = \frac{1}{4\pi}[a\partial_\mu^{x_3} + \bar{a}\gamma_5\varepsilon_{\mu\tau}\partial_\tau^{x_3}]\ln\frac{x_{23}^2}{x_{13}^2}\cdot G_d(x_{12}) \tag{14.10}$$

$$\gamma_\mu G_{j_\mu}(x_1x_2x_3) = \frac{a-\bar{a}}{2\pi}\left[\frac{(x_{23})_\mu}{x_{23}^2} - \frac{(x_{13})_\mu}{x_{13}^2}\right]\gamma_\mu G_d(x_{12}). \tag{14.11}$$

Note now that the equations of motion contain the quantity (14.11) at $x_{13} = 0$. If we formally put $x_{13} = 0$ in it, however, we obtain infinity. *It is therefore necessary to find such a way for making the arguments coincide which would prevent the appearance of the infinity.* In the original work by Johnson [68] the symmetrization of (14.10) with respect to x_{13} prior to the limiting transition

$$\gamma_\mu j_\mu(x)\psi(x) \equiv \tfrac{1}{2}\lim_{\varepsilon\to 0}\{j_\mu(x+\varepsilon)\gamma_\mu\psi(x) + \gamma_\mu\psi(x)j_\mu(x-\varepsilon)\} \tag{14.12}$$

was used. With this definition of the operator product the second term in (14.11) vanishes and we are lead to the closed equation for the propagator

$$-\hat{\partial}_{x_1}G(x_{12}) = \frac{\lambda(\bar{a}-a)}{2\pi}\frac{\hat{x}_{21}}{(x_{12}^2)}G(x_{12}). \tag{14.13}$$

We must emphasize that the definition (14.12) is a postulate complementary to the equation of motion. The derivation of closed equation for the Green function (14.5) may be performed by analogy with that of (14.13). It has the form

$$\begin{aligned} -\hat{\partial}G(x_1x_2x_3x_4) = \frac{\lambda}{4\pi}\{&a[\hat{\partial}_{x_1}\ln x_{12}^2 - \hat{\partial}_{x_1}\ln x_{13}^2 + \hat{\partial}_{x_1}\ln x_{14}^2] \\ &- \bar{a}\gamma_\mu\varepsilon_{\mu\rho}\partial_\rho^{x_1}[\gamma_5^{x_2}\ln x_{12}^2 + \gamma_5^{x_3}\ln x_{13}^2 + \gamma_5^{x_4}\ln x_{14}^2]\}\, G(x_1x_2x_3x_4). \end{aligned} \tag{14.14}$$

The Green function

$$G_{j_\mu}(x_1x_2x_3x_4|x_5) = \langle 0|\, T\psi_d(x_1)\psi_d^+(x_2)\psi_d(x_3)\psi_d^+(x_4)j_\mu(x_5)\,|0\rangle, \tag{14.15}$$

which is necessary for deriving (14.14) is uniquely determined by the two Ward identities

$$\partial_\mu^{x_5}G_{j_\mu}(x_1x_2x_3x_4|x_5) = -a[\delta(x_{15}) - \delta(x_{25}) + \delta(x_{35}) - \delta(x_{45})]G(x_1x_2x_3x_4) \tag{14.16}$$

$$\varepsilon_{\mu\nu}\partial_\mu^{x_5}G_{j_\nu}(x_1x_2x_3x_4|x_5) = \bar{a}[\gamma_5^{x_1}\delta(x_{15}) + \gamma_5^{x_2}\delta(x_{25}) + \gamma_5^{x_3}\delta(x_{35}) + \gamma_5^{x_4}\delta(x_{45})]G(x_1x_2x_3x_4). \tag{14.17}$$

Within our approach the definition of the operator product is a consequence of the solution of the integral equations. One easily sees that the equations (14.13) and (14.14) are obtained within our approach from the general relations of the previous section. In the 2-dimensional case it may be shown that there is no multiplicative renormalization of the operator product $\underline{j_\mu(x)\psi_d}(x)$ and the resolving of the $0 \times \infty$ ambiguity is reduced to one subtraction that removes the leading singularity $\varepsilon_\mu/\varepsilon^2$. By using this definition for $\underline{j_\mu(x)\psi_d}(x)$ we also obtain the equations of motion for the Green function in the form (14.13) and (14.14). Equation (14.13) results in the known [68–70] relationship between the dimension and the coupling constant

$$d = \tfrac{1}{2} + (\lambda/4\pi)(a - \bar{a}) \tag{14.18}$$

and equation (14.14) enables one to calculate unambiguously the Green function (14.5).

Let us present explicit expressions for the propagator (14.6) and the Green function (14.7)

$$G_d(x) = \frac{\mathrm{i}}{2\pi}\frac{\Gamma(d+\frac{1}{2})}{\Gamma(\frac{3}{2}-d)}\frac{\hat{x}}{(\frac{1}{2}x^2)^d}, \tag{14.19}$$

$$G_d(x) = G^{-1}_{2-d}(x). \tag{14.20}$$

We present also the expression for the propagator of the current. There are two conformally invariant propagators:

$$\Delta_{\mu\nu}(x) = \frac{1}{4\pi}\partial_\mu\partial_\nu \ln x^2, \tag{14.21}$$

and the transversal one

$$\bar{\Delta}_{\mu\nu}(x) = \frac{1}{4\pi}\varepsilon_{\mu\sigma}\varepsilon_{\nu\tau}\partial_\sigma\partial_\tau \ln x^2 = \delta_{\mu\nu}\delta(x) - \frac{1}{4\pi}\partial_\mu\partial_\nu \ln x^2. \tag{14.22}$$

Their sum is the unit operator

$$\Delta_{\mu\nu}(x) + \bar{\Delta}_{\mu\nu}(x) = \delta_{\mu\nu}\delta(x).$$

It can be easily seen that each of these propagators acts as a projective operator. Let $f_\mu(x)$ be an arbitrary function. Present it in the form $f_\mu(x) = f_{1,\mu}(x) + f_{2,\mu}(x)$, where $f_{1,\mu}(x) = \partial_\mu f_1(x)$, $f_{2,\mu}(x) = \varepsilon_{\mu\nu}\partial_\nu f_2(x)$. Then we have

$$\int \mathrm{d}y\Delta_{\mu\nu}(x-y)f_{1,\nu}(y) = f_{1,\mu}(x), \qquad \int \mathrm{d}y\bar{\Delta}_{\mu\nu}(x-y)f_{2,\nu}(y) = f_{2,\mu}(x). \tag{14.23}$$

This property of the propagator of the current significantly simplifies the amputation of the current line: the amputated 3-point function coincides with the primary one.

14.2. The Green function $G_{\mu\nu}(x_1x_2|x_3x_4)$

Consider the Green function (14.4). It contains two conserved currents and, consequently, the Ward identities for it should, generally, include the Schwinger terms coming from the commutation of the currents. We have

$$\partial_\mu^{x_4} G_{\mu\nu}(x_1x_2|x_3x_4) = -a[\delta(x_{14}) - \delta(x_{24})]G_{j_\nu}(x_1x_2x_3) + \{A_1\partial_\nu\delta(x_3 - x_4) + A\gamma_5\varepsilon_{\nu\sigma}\partial_\sigma\delta(x_3 - x_4)\}\, G_d(x_{12}), \tag{14.24}$$

$$\varepsilon_{\mu\rho}\partial_\mu^{x_4} G_{\rho\nu}(x_1x_2|x_3x_4) = \bar{a}\gamma_5[\delta(x_{14}) - \delta(x_{24})]G_{j_\nu}(x_1x_2x_3) - A\gamma_5\partial_\nu\delta(x_3 - x_4)G_d(x_{12}) - A_2\partial_\tau\varepsilon_{\nu\tau}\delta(x_{34})G_d(x_{12}). \tag{14.25}$$

The solution of these identities is

$$G_{\mu\nu}(x_1x_2|x_3x_4) = G_{\mu\nu}^{(0)}(x_1x_2|x_3x_4) + G_{\mu\nu}^{(1)}(x_1x_2|x_3x_4), \tag{14.26}$$

where

$$G_{\mu\nu}^{(0)}(x_1x_2|x_3x_4) = \frac{1}{(4\pi)^2}[a\partial_\mu^{x_3} + \bar{a}\gamma_5\varepsilon_{\mu\sigma}\partial_\sigma^{x_3}]\ln\frac{x_{23}^2}{x_{13}^2}[a\partial_\nu^{x_4} + \bar{a}\gamma_5\varepsilon_{\nu\tau}\partial_\tau^{x_4}]\ln\frac{x_{24}^2}{x_{14}^2}\, G_d(x_{12}), \tag{14.27}$$

$$G_{\mu\nu}^{(1)}(x_1x_2|x_3x_4) = A_1\Delta_{\mu\nu}(x_{34})G_d(x_{12}) + A[\varepsilon_{\mu\sigma}\Delta_{\sigma\nu}(x_{34})\gamma_5 G_d(x_{12}) + \varepsilon_{\nu\sigma}\Delta_{\mu\sigma}(x_{34})\gamma_5 G_d(x_{12})] + A_2\varepsilon_{\mu\sigma}\varepsilon_{\nu\tau}\Delta_{\sigma\tau}(x_{34})G_d(x_{12}). \tag{14.28}$$

The Schwinger terms are due to the function $G_{\mu\nu}^{(1)}$. Thus, the Green function $G_{\mu\nu}(x_1x_2|x_3x_4)$ is determined by conformal invariance and conservation laws up to the three constants: A_1, A_2 and A. These constants must be found from dynamics.

14.3. Partial wave expansion of $G_{\mu\nu}$

Consider the partial wave expansion of the Green function $G_{\mu\nu}$. To this end it is necessary to find normalized 3-point functions. We confine ourselves to the calculation of the spinor contribution to the partial wave expansion. Denote the 3-point function of two spinors (x_1, l) and (x_2, d) and a conserved current $(x_3\sigma_j)$ as $C^{ld\sigma_j}(x_1x_2x_3)$. There are four types of such functions

$$C_{1,\mu}^{ld\sigma_j}(x_1x_2x_3) = N_1\frac{\hat{x}_{12}}{(\frac{1}{2}x_{12}^2)^{(l+d)/2}}\partial_\mu^{x_3}\left\{\frac{1}{(\frac{1}{2}x_{13}^2)^{(l-d)/2}}\frac{1}{(\frac{1}{2}x_{23}^2)^{(d-l)/2}}\right\} \tag{14.28}$$

$$C_{2,\mu}^{ld\sigma_j}(x_1x_2x_3) = N_1\varepsilon_{\mu\nu}\gamma_5\frac{\hat{x}_{12}}{(\frac{1}{2}x_{12}^2)^{(l+d)/2}}\partial_\nu^{x_3}\left\{\frac{1}{(\frac{1}{2}x_{13}^2)^{(l-d)/2}}\frac{1}{(\frac{1}{2}x_{23}^2)^{(d-l)/2}}\right\} \tag{14.29}$$

$$\bar{C}_{1,\mu}^{ld\sigma_j}(x_1x_2x_3) = N_2\frac{1}{(\frac{1}{2}x_{12}^2)^{(l+d)/2}}\partial_\mu^{x_3}\left\{\frac{\hat{x}_{13}}{(\frac{1}{2}x_{13}^2)^{(l-d)/2}}\frac{\hat{x}_{32}}{(\frac{1}{2}x_{23}^2)^{(d-l)/2}}\right\}, \tag{14.30}$$

$$\bar{C}_{2,\mu}^{ld\sigma_j}(x_1x_2x_3) = N_2\frac{1}{(\frac{1}{2}x_{12}^2)^{(l+d)/2}}\varepsilon_{\mu\nu}\gamma_5\partial_\nu^{x_3}\left\{\frac{\hat{x}_{13}}{(\frac{1}{2}x_{13}^2)^{(l-d)/2}}\frac{\hat{x}_{32}}{(\frac{1}{2}x_{23}^2)^{(d-l)/2}}\right\} \tag{14.31}$$

where $\hat{x} = \gamma_\mu x_\mu/\sqrt{x^2}$

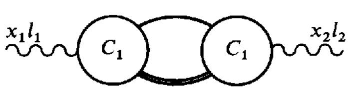
$$N_1 = \frac{1}{4\pi}\frac{\mathrm{i}}{2}\frac{\sqrt{(3-l-d)(l+d-1)}}{(l-d)}\frac{\Gamma((l+d-1)/2)}{\Gamma((5-l-d)/2)} \tag{14.32}$$

$$N_2 = -\frac{1}{4\pi}\frac{1}{\sqrt{(l-d+1)(d+1-l)}}\frac{\Gamma((l+d)/2)}{\Gamma((4-l-d)/2)}. \tag{14.33}$$

It is easy to check, that these functions obey the amputation relations

$$\begin{aligned}
&\int \mathrm{d}x_4 G_l^{-1}(x_{14}) C_{1,\mu}^{ld\sigma_j}(x_4 x_2 x_3) = \bar{C}_{1,\mu}^{2-l,d,\sigma_j}(x_1 x_2 x_3)\\
&\int \mathrm{d}x_4 C_{1,\mu}^{ld\sigma_j}(x_1 x_4 x_3) G_d^{-1}(x_4 - x_2) = -\bar{C}_{1,\mu}^{l,2-d,\sigma_j}(x_1 x_2 x_3)\\
&\int \mathrm{d}x_4 C_{2,\mu}^{ld\sigma_j}(x_1 x_4 x_3) G_d^{-1}(x_4 - x_2) = -\bar{C}_{2,\mu}^{l,2-d,\sigma_j}(x_1 x_2 x_3)\\
&\int \mathrm{d}x_4 G_l^{-1}(x_{14}) C_{2,\mu}^{ld\sigma_j}(x_4 x_2 x_3) = -\bar{C}_{2,\mu}^{2-l,d,\sigma_j}(x_1 x_2 x_3)\\
&\int \mathrm{d}x_4 \Delta_{\mu\nu}(x_{34}) C_{1,\nu}^{ld\sigma_j}(x_1 x_2 x_4) = C_{1,\mu}^{ld\sigma_j}(x_1 x_2 x_3)\\
&\int \mathrm{d}x_4 \bar{\Delta}_{\mu\nu}(x_{34}) \bar{C}_{1,\nu}^{ld\sigma_j}(x_1 x_2 x_4) = \bar{C}_{1,\mu}^{ld\sigma_j}(x_1 x_2 x_3)
\end{aligned} \tag{14.34}$$

and the orthgonality properties



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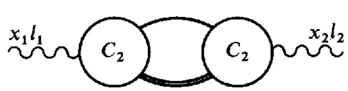
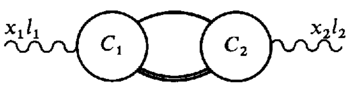


$= \frac{1}{2} G_l(x_{12}) 2\pi \mathrm{i} \mu^{-1}(l) \delta(l_1 - l_2)$ (14.35)



$= 0$ (14.36)

where

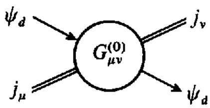
$$\mu(l) = \frac{2}{2\pi}(l - \tfrac{1}{2})(\tfrac{3}{2} - l). \tag{14.37}$$

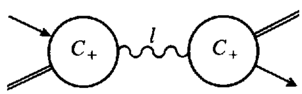
Therefore there are two mutually orthogonal sets of functions. They may both participate in the partial wave expansion. The following relation has been used in the calculations (in (14.34)–(14.37)).

$$\int \mathrm{d}x_4 \frac{\hat{x}_{14}}{(\frac{1}{2}x_{14}^2)^{d_1}} \frac{\hat{x}_{42}}{(\frac{1}{2}x_{24}^2)^{d_2}} \frac{1}{(\frac{1}{2}x_{34}^2)^{\delta}} = 2\pi \frac{\Gamma(\frac{3}{2}-d_1)\Gamma(\frac{3}{2}-d_2)\Gamma(1-\delta)}{\Gamma(d_1+\frac{1}{2})\Gamma(d_2+\frac{1}{2})\Gamma(\delta)} \frac{\hat{x}_{13}}{(\frac{1}{2}x_{13}^2)^{2-d_2}} \frac{\hat{x}_{32}}{(\frac{1}{2}x_{23}^2)^{2-d_1}} \frac{1}{(\frac{1}{2}x_{12}^2)^{2-\delta}}.$$

When deriving the fifth and the sixth relations of amputation (with respect to the current line) eq. (14.26) has been used along with the fact that $\tilde{\sigma}_j = \sigma_j$ in the two-dimensional space.

Consider the partial wave expansion of the function $G^{(0)}_{\mu\nu}(x_1x_2|x_3x_4)$. Some simple calculations give for the *spinor contribution*



$= \sum_l \rho(l)$


$+ \ldots$ (14.38)

where

$$\sum_l = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} dl\,\mu(l)$$

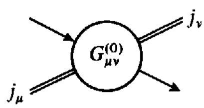
and the dots stand for the contribution of higher spins. Here the function C_+ is

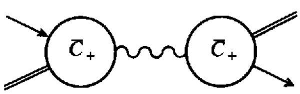
$$C_+^{ld\sigma_j} = aC^{ld\sigma_j}_{1,\mu}(x_1x_2x_3) + \bar{a}C^{ld\sigma_j}_{2,\mu}(x_1x_2x_3) \tag{14.39}$$

and the function $\rho(l)$ is equal to

$$\rho(l) = -\frac{1}{4\pi}\frac{\Gamma(d+\frac{1}{2})}{\Gamma(\frac{3}{2}-d)}\frac{\Gamma((l-d)/2)}{\Gamma((2-l+d)/2)}\frac{\Gamma((3-l-d)/2)}{\Gamma((l+d+1)/2)}. \tag{14.40}$$

With the use of the amplutation relation (14.34) the expansion (14.38) can be equivalently represented in the other form

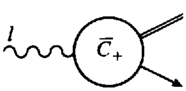


$= -\sum_l \bar{\rho}(l)$


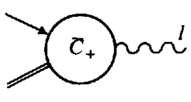
$+ \ldots$ (14.40a)

where

$$\bar{C}^{ld\sigma_j}_{+,\mu}(x_1x_2x_3) = \int dx_4 G_l(x_1-x_4)C_+^{2-l,d\sigma_j}(x_4x_2x_3) = a\bar{C}_1^{ld\sigma_j}(x_1x_2x_3) - \bar{a}\bar{C}_2^{ld\sigma_j}(x_1x_2x_3)$$

$=$


(14.41)

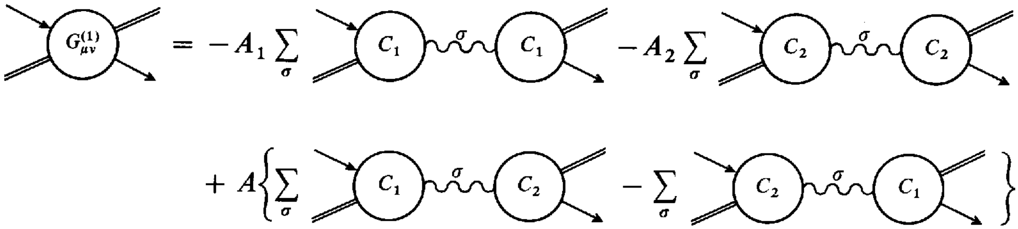
$[\bar{C}_+^{ld\sigma_j}(x_1x_2x_3)]^+ =$


$= (a\bar{C}_1 - \bar{a}\bar{C}_2)^+ = a\bar{C}_1^{dl\sigma_j}(x_2x_1x_3) + \bar{a}\bar{C}_2^{dl\sigma_j}(x_2x_1x_3)$

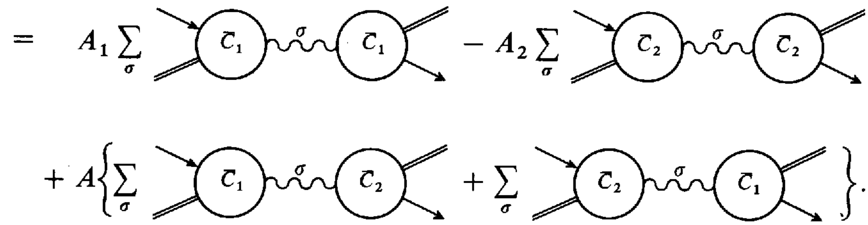
and the function $\bar{\rho}(l)$ is connected with $\rho(l)$ by the relation

$$\bar{\rho}(l) = \rho(2-l). \tag{14.42}$$

Consider the partial wave expansion of the function $G^{(1)}_{\mu\nu}(x_1x_2|x_3x_4)$. Since it is expressed in terms of the product of the propagators $\Delta_{\mu\nu}$ and G_d one readily finds, with the aid of (14.34), the two equivalent forms of the expansion



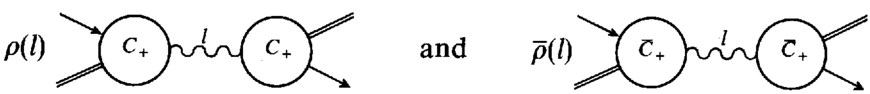
(14.43)



(14.44)

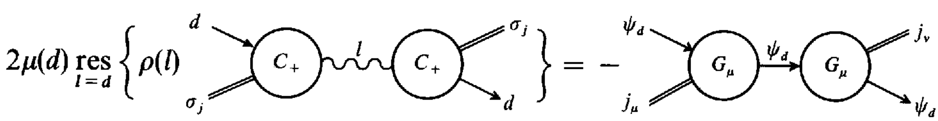
14.4. Expansion of the operator product $j_\mu(x_1)\psi_d(x_2)$. Contribution of the field ψ_1

To find out which fields contribute into the expansion of the operator product $j_\mu(x_1)\psi_d(x_2)$ one needs to know the poles in the l-plane of the functions



(14.45)

which determine the expansion of $G^{(0)}_{\mu\nu}$, and analogous poles in the expansions (14.43) and (14.44). With the use of the eqs. (14.28), (14.29) and (14.40), one makes sure that the expansion (14.38) contains a pole in the point $l = d$. This pole corresponds to the basic field



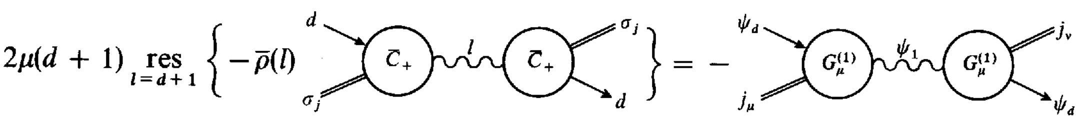
(14.46)

where G_μ is the Green function (14.10). The substitution of the explicit expressions for the quantities involved here leads to the conclusion that this equation holds identically.

Consider now the second function out of (14.45). It has a pole at $l = d + 1$, coming from the normalization factors (14.33). The spinor field ψ_1, with the dimension

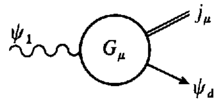
$$d_1 = d + 1 \tag{14.47}$$

may be associated with this pole. The residue in this pole gives the coupling constant between this field and the basic field ψ_d and the current. We have



(14.48)

where

$$G_\mu^{(1)}(x_1x_2x_3) = \langle 0|\, T\psi_1(x_1)\psi_d^+(x_2)j_\mu(x_3)\,|0\rangle =$$  . (14.49)

Now we show that the field ψ_1 is analogous to the fields O_s^d treated in section 6. Define the Green function (14.49) by the relation

$$G_\mu^{(1)}(x_1x_2x_3) = g_j^{(0)} \lim_{l\to d+1} N_2^{-1}(l,d)\bar{C}_+^{ld\sigma_j}(x_1x_2x_3) \tag{14.50}$$

where $g_j^{(0)}$ is the coupling constant. By calculating the limit (14.50) we find

$$G_\mu^{(1)}(x_1x_2x_3) = -\tfrac{1}{2}g_j^{(0)}[a\delta_{\mu\nu} - \bar{a}\varepsilon_{\mu\nu}\gamma_5]\partial_\nu^{x_3}\{\hat{\partial}_{x_3}\ln x_{13}^2\gamma_\rho(x_{32})_\rho\}\frac{1}{(\frac{1}{2}x_{12}^2)^{d+1/2}}. \tag{14.51}$$

It can be easily verified that this Green function satisfies Ward identities, analogous to (6.10)

$$\partial_\mu^{x_3}G_\mu^{(1)}(x_1x_2x_3) = -\frac{\sqrt{2}}{2}g_j^{(0)}a\hat{\partial}_{x_3}\delta(x_1 - x_3)\frac{\hat{x}_{12}}{(\frac{1}{2}x_{12}^2)^d} \tag{14.52}$$

$$\varepsilon_{\mu\nu}\partial_\mu^{x_3}G_\nu^{(1)}(x_1x_2x_3) = -\frac{\sqrt{2}}{2}g_j^{(0)}\bar{a}\gamma_5\hat{\partial}_{x_3}\delta(x_1 - x_3)\frac{\hat{x}_{12}}{(\frac{1}{2}x_{12}^2)^d}. \tag{14.53}$$

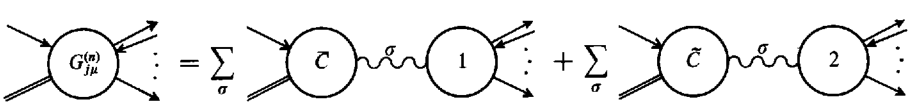
The coupling constant $g_j^{(0)}$ is calculated with the help of equation (14.48). The substitution of (14.51) and (14.30, 31) into it yields

$$[g_j^{(0)}(d)]^2 = \frac{4}{(4\pi)^4}\frac{\Gamma(d+\frac{3}{2})}{\Gamma(\frac{3}{2}-d)}. \tag{14.51a}$$

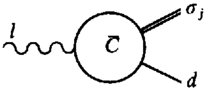
Thus, we have shown that *if one uses $G_{\mu\nu}^{(0)}$ determined by* (14.27) *for the Green function $G_{\mu\nu}$ then, in such a theory, the operator expansion of the product $j_\mu(x_1)\psi_d(x_2)$ contains the field ψ_1 with the dimension $d+1$ apart from the field ψ_d and its derivative $\partial_\mu\psi_d(x)$*. The general expression (14.26) for $G_{\mu\nu}$, however, contains the extra term $G_{\mu\nu}^{(1)}$ that includes the unknown constants A_1, A_2 and A. By choosing these constants in such a way that the field ψ_1 should not participate in the expansion of the operator product $\gamma_\mu j_\mu(x_1)\psi_d(x_2)$ we just achieve the fulfillment of the equations of motion, see subsection 14.6.

14.5. Derivation of closed equations for the Green functions

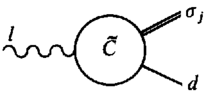
Now we formulate the general receipt for obtaining closed equations for all Green functions. In accord with the programme presented above (see previous section and subsection 14.1) the new dynamical principle is the requirement that *the field ψ_1 should not appear in the expansion of the operator product $\gamma_\mu j_\mu(x_1)\psi(x_2)$*. This requirement may be stated in the following form. Let G_n be any Green function that includes n fields. Let us associate with it the Green function $G_{j_\mu}^{(n)}$, that is obtained from G_n by adding one current j_μ. Represent the function $G_{j_\mu}^{(n)}$ as the expansion

 (14.54)

where

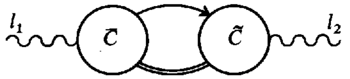
$\bar{C}^{ld\sigma_j}_{\mu}(x_1 x_2 x_3) =$ 

$= \bar{C}^{ld\sigma_j}_{1,\mu} - \bar{C}^{ld\sigma_j}_{2,\mu} = (\delta_{\mu\nu} - \varepsilon_{\mu\nu}\gamma_5)\bar{C}^{ld\sigma_j}_{1,\nu},$ (14.55)

$\tilde{C}^{ld\sigma_j}_{\mu}(x_1 x_2 x_3) =$ 

$= \bar{C}^{ld\sigma_j}_{1,\mu} + \bar{C}^{ld\sigma_j}_{2,\mu} = (\delta_{\mu\nu} + \varepsilon_{\mu\nu}\gamma_5)\bar{C}^{ld\sigma_j}_{1,\nu}.$ (14.56)

The functions $\bar{C}$ and $\tilde{C}$ form mutually orthogonal sets

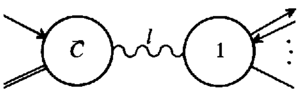


$= 0$

and satisfy the conditions

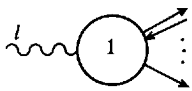
$$\gamma_\mu \bar{C}_\mu = 2\gamma_\mu \bar{C}_{1,\mu}; \qquad \gamma_\mu \tilde{C}_\mu = 0. \tag{14.57}$$

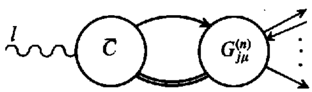
Consider equation (14.54). The requirement that the first term of (14.54) be free of the pole in the point $d+1$ provides the fulfilment of the equations of motion (14.1) and may be written as

$\operatorname*{res}_{l=d+1}$ 

$= 0,$ (14.58)

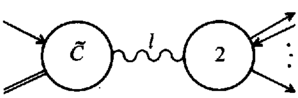
where



$=$ 

$.$ (14.59)

Indeed, with the condition (14.57), equation (14.58) makes the field ψ_1 disappear from the expansion of the operator product $\gamma_\mu j_\mu(x_1)\psi_d(x_2)$. Note, that the residue of the second term in (14.54) is different from zero

$\operatorname*{res}_{l=d+1}$ 

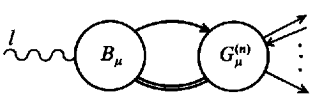
$\neq 0$

and determines the contribution of the field ψ_1 into the expansion of the operator product $j_\mu(x_1)\psi_d(x_2)$ (and not of $\gamma_\mu j_\mu(x_1)\psi_d(x_2)$).

We rewrite now the equation (14.58) in a different form, which is more convenient for calculations. The normalizing factor of the function $\bar{C}_\mu$ involved in (14.58) is singular at the point $l = d+1$, see (14.33). Instead of $\bar{C}$, let us consider the function

$$B_\mu(x_1 x_2 x_3) = N_2^{-1}(ld\sigma_j)\bar{C}^{ld\sigma_j}_\mu(x_1 x_2 x_3) = (\delta_{\mu\nu} - \varepsilon_{\mu\nu}\gamma_5)\frac{1}{(\frac{1}{2}x_{12}^2)^{(l+d)/2}}\partial^{x_3}_\mu\left\{\frac{\hat{x}_{13}}{(\frac{1}{2}x_{13}^2)^{(l-d)/2}}\frac{\hat{x}_{32}}{(\frac{1}{2}x_{23}^2)^{(d-l)/2}}\right\}, \tag{14.60}$$

which is regular in this point. Now equation (14.58) may be written as

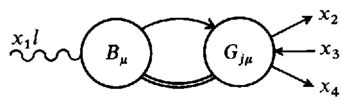
$\operatorname*{res}_{l=d+1}$ 

$= 0.$ (14.61)

This equation leads to closed equations for the Green functions G_n since the function $G_\mu^{(n)}$ is expressed in a unique way in term of G_n with the aid of the Ward identities. We illustrate this below for the functions $G_1(x_1x_2x_3x_4)$ and $G_{\mu\nu}(x_1x_2|x_3x_4)$ taken as examples.

14.6. *Equations for the Green functions* $G_1(x_1x_2x_3x_4)$ *and* $G_{\mu\nu}(x_1x_2|x_3x_4)$

We now consider the Green function $G_1(x_1x_2x_3x_4)$ as defined by (14.5). It is convenient to use the equation from which it must be determined in the form (14.61). We have

$\operatorname*{res}_{l=d+1}$ 

$= 0$ (14.62)

or

$$\operatorname*{res}_{l=d+1} \int \mathrm{d}x_5\, \mathrm{d}x_6\, \mathrm{d}x_6' B_\mu^{ld\sigma_j}(x_1x_6'x_5)G_d^{-1}(x_6' - x_6)G_{j_\mu}(x_6x_2x_3x_4|x_5) = 0 \tag{14.63}$$

where $B_\mu^{ld\sigma_j}$ is the function (14.60). The x_6'-integration is performed with the help of relations (14.34) to give

$$\int \mathrm{d}x_6' B_\mu^{ld\sigma_j}(x_1x_6'x_5)G_d^{-1}(x_6' - x_6)$$
$$\sim (\delta_{\mu\nu} - \varepsilon_{\mu\nu}\gamma_5)\frac{\hat{x}_{16}}{(\frac{1}{2}x_{16}^2)^{(l-d+2)/2}}\partial_\nu^{x_5}\left\{\frac{1}{(\frac{1}{2}x_{15}^2)^{(l+d-2)/2}}\;\frac{1}{(\frac{1}{2}x_{56}^2)^{(2-l-d)/2}}\right\} \tag{14.64}$$

After the substitution of (14.64) into (14.63), and integration by parts we find

$$\operatorname*{res}_{l=d+1}\int \mathrm{d}x_5\,\mathrm{d}x_6\frac{\hat{x}_{16}}{(\frac{1}{2}x_{16}^2)^{(l-d+2)/2}}\;\frac{1}{(\frac{1}{2}x_{15}^2)^{(l+d-2)/2}}\;\frac{1}{(\frac{1}{2}x_{56}^2)^{(2-l-d)/2}}$$
$$\times\,(\delta_{\mu\nu} - \varepsilon_{\mu\nu}\gamma_5)\partial_\mu^{x_5}G_{j_\nu}(x_6x_2x_3x_4|x_5) = 0.$$

We substitute here the derivatives $\partial_\mu G_{j_\mu}$ and $\varepsilon_{\mu\nu}\partial_\mu G_{j_\nu}$, as expressed from the Ward identities (14.16), (14.17). This gives us the possibility to perform the x_5-integration, the remaining integrand containing only the function $G_1(x_1x_2x_3x_4)$.

To carry out the last integration over x_6 we take the $\operatorname{res}_{l=d+1}$ before the integration by using the relation

$$\operatorname*{res}_{l=d+1}\frac{\hat{x}_{16}}{(\frac{1}{2}x_{16}^2)^{(l-d+2)/2}} \sim \hat{\partial}_{x_1}\operatorname*{res}_{l=d+1}\frac{1}{(x_{16}^2)^{(l-d+1)/2}} \sim \hat{\partial}_{x_1}\delta(x_1 - x_6).$$

After some simple calculations we find that equation (14.62) reduces to the form

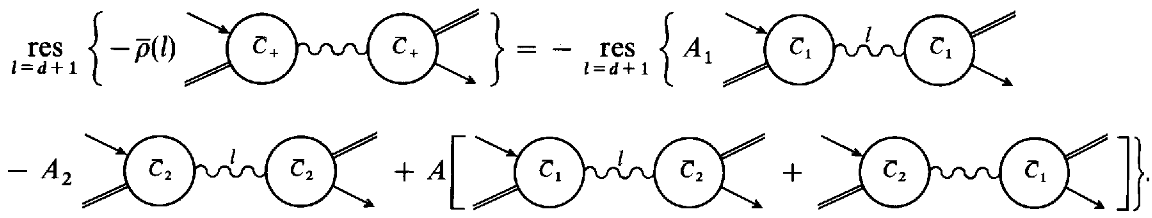
$$-\hat{\partial}G_1(x_1x_2x_3x_4) = \frac{d-\frac{1}{2}}{a-\bar{a}}\{a[\hat{\partial}_{x_1}\ln x_{12}^2 - \hat{\partial}_{x_1}\ln x_{13}^2 + \hat{\partial}_{x_1}\ln x_{14}^2]$$
$$-\bar{a}\gamma_\mu\varepsilon_{\mu\rho}\partial_\rho^{x_1}[\gamma_5^{x_2}\ln x_{12}^2 + \gamma_5^{x_3}\ln x_{13}^2 + \gamma_5^{x_4}\ln x_{14}^2]\}\,G_1(x_1x_2x_3x_4). \tag{14.65}$$

If, further, one introduces

$$\lambda = 4\pi \frac{d - \frac{1}{2}}{(a - \bar{a})}, \tag{14.66}$$

we obtain the known equation (14.14) *and also the relation between the coupling constant* λ *and the dimension* d. It may be shown, that eq. (14.65), when supplemented by the requirement of antisymmetricity with respect to the variables x_1, x_3 and by the condition $[G_1(x_1 \dots x_4)]^+ = G_1(x_4 x_3 x_2 x_1)$, uniquely determines the Green function $G_1(x_1 x_2 x_3 x_4)$ and results in the well-known expression for the latter.

Consider now the function $G_{\mu\nu}(x_1 x_2 | x_3 x_4)$. For its calculation it is convenient to use the equations in the form (14.58). By the substitution of (14.26) into it and the use of (14.40a) and (14.44) we get



From (14.48) and (14.51a) we find

$$-\frac{a(a - \bar{a})}{4\pi(d - \frac{1}{2})} = A_1 - A, \qquad -\frac{\bar{a}(a - \bar{a})}{4\pi(d - \frac{1}{2})} = A_2 - A$$

or, after the use of relation (14.66) also

$$A - A_1 = a/\lambda, \qquad A - A_2 = \bar{a}/\lambda. \tag{14.67}$$

We have obtained only two equations for the three constants. It may be shown that the relations (14.67) are sufficient for the equation of the one-current Green function, which connects it with $G_{\mu\nu}$ to be satisfied. In order to find a third equation for the constants A_1, A_2 and A, one needs to consider the Green function with three currents. This consideration can be carried out analogously.

Thus, equations (14.58) or (14.61) enable us to uniquely find all the Green functions of the fields. When currents are involved these equations also determine the structure of the Schwinger terms (i.e. one can find the coefficients A_1, A_2 and A in the Ward identities (14.24) and (14.25)), which *necessarily appear in the Thirring model.*

15. Calculation of dimensions and critical indices

1. For the calculation of critical indices we shall use the simplest model

$$L_{\text{int}} = \lambda \varphi^4(x) \tag{15.1}$$

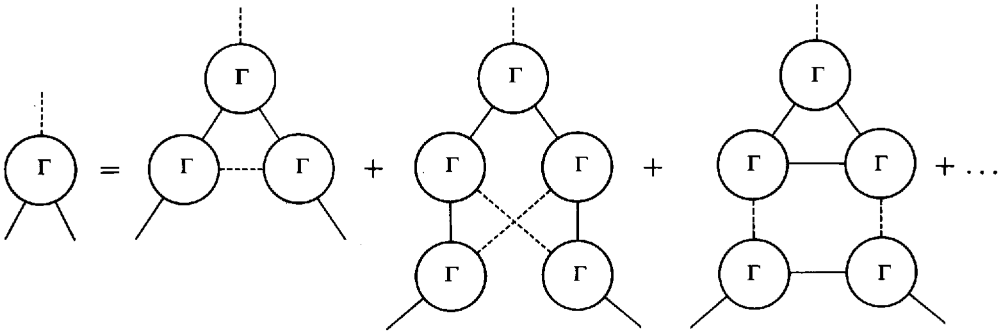
where $\varphi(x)$ is a field in the 3-dimensional Euclidean space. Let us suppose, that the theory described by the interaction (15.1) is conformally invariant. To exploit this assumption effectively, it is useful to formulate the interaction (15.1) in terms of an intermediate field $\chi(x) = \sqrt{\lambda}\varphi^2(x)$. The new Lagrangian is of the form:

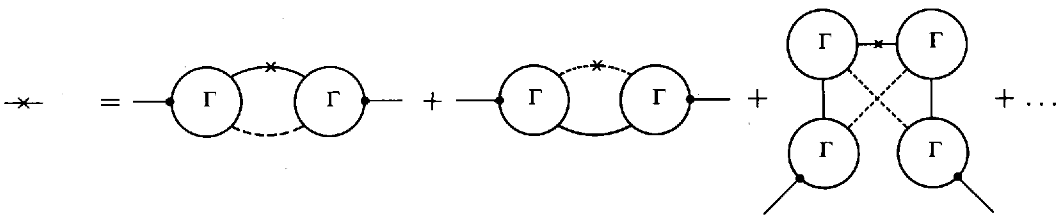
$$L_{\text{int}} = \tfrac{1}{2}\lambda_1\varphi^2(x)\chi(x). \tag{15.2}$$

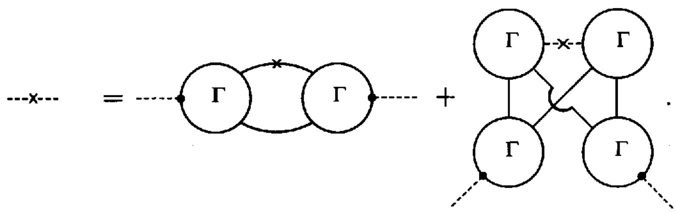
It can be shown, that theories with L_{int} of the type (15.1) and (15.2) are completely equivalent if

$$\lambda_1^2 = -8\lambda. \tag{15.3}$$

Let d be the dimension of the fundamental field $\varphi(x) \equiv \varphi_d(x)$, and Δ the dimension of the "composite" field $\chi(x) \equiv \chi_\Delta(x)$. These dimensions can be evaluated in the framework of the bootstrap program [38–43]. We shall take the renormalized skeleton equations for vertices and propagators as a starting point. In the present case these equations are of the form:


(15.4)


(15.5)


(15.6)

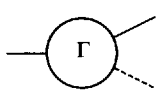
The solid and dashed lines describe the inverse propagators $G_d^{-1}(x)$ and D_Δ^{-1} respectively, where

$$G_d(x_{12}) = \langle 0|\, T\varphi_d(x_1)\varphi_d(x_2)\,|0\rangle = \pi^{-D/2}\frac{\Gamma(d)}{\Gamma(D/2-d)}(x_{12}^2)^{-d}, \tag{15.7}$$

and D_Δ being defined by an analogous expression. The meaning of the symbol $\times$ is:

$$x_1 \text{ —×— } x_2 = (x_{12})_\mu \partial_\nu^{x_1} G_d^{-1}(x_{12}).$$

The dot at the origin of a line means that this line is amputated. For Green's functions in vertices of the above equations the following conformally invariant expression will be used:

$$\Gamma^{dd\Delta}(x_1x_2x_3) = g(x_{12}^2)^{-(2d-\Delta)/2}(x_{13}^2x_{23}^2)^{-\Delta/2} =$$  (15.8)

where g is the coupling constant. Inserting (15.7) and (15.8) into (15.4)–(15.6), we shall obtain three algebraic equations for the determination of the coupling constant g and two dimensions d and Δ:

$$1 = g^2 f_i^{(1)}(d, \Delta) + g^4 f_i^{(2)}(d, \Delta) + \dots, \qquad i = 1, 2, 3. \tag{15.9}$$

The functions $f_i^{(1,2)}$, entering these equations, will be explicitely evaluated below (point 3). It will be shown, that eq. (15.9) possess the solution:

$$d = 0.510, \qquad \Delta = 1.34.$$

These values of the dimensions correspond to the following values of critical indices η, ν and α (for $D = 3$)

$$\eta = 2d + 2 - D = 0.02; \qquad \nu = \frac{1}{D - \Delta} = 0.60. \tag{15.10}$$

The latter result is in a good agreement with other theoretical calculations and experimental data [1].

2. The evaluation of integrals entering the right-hand side of eqs. (15.4)–(15.6) is a difficult problem. Partially this problem was solved by Symanzik in ref. [40], where the integral with power functions was evaluated, and by the authors in refs. [36], (see Appendix 7) where the exact evaluation was carried out in terms of 3-vertex contributions arising in equations for Green's functions with the current and energy-momentum tensor (the latter equations are equivalent to eqs. (15.5) and (15.6)). However the integrals arising in the vertex equation (15.4) have not been evaluated, and this was the main obstacle for realization of the bootstrap program described above. A method of approximate evaluation of indicated integrals will be developed below. This method gives the possibility to evaluate integrals in eqs. (15.4)–(15.6) with the sufficient accuracy in that range of dimensions which is of practical interest.

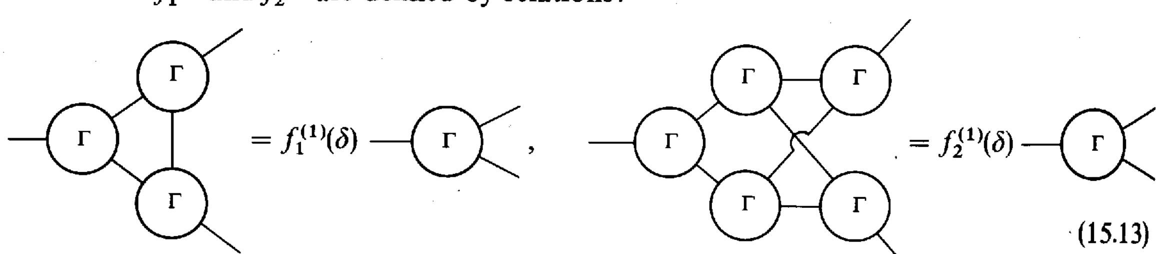
We shall illustrate the method in the case of a simpler model (more detailed calculations for this model have been published in ref. [36a]):

$$L_{\text{int}} \sim \lambda \varphi^3, \tag{15.11}$$

where $\varphi \equiv \varphi_\delta(x)$ is a field in D-dimensional Euclidean space, and then briefly discuss calculations connected with eqs. (15.4)–(15.6). The integral equations for the vertex and the propagator, corresponding to the interaction (15.11), can be obtained from equations (15.4) and (15.5) by means of identification of solid and dashed lines. Correspondingly one must put $d = \Delta = \delta$ in (15.7) and (15.8), where δ is the dimension of the field $\varphi(x)$ in the case of the interaction (15.11). Eqs. (15.4) and (15.5) at $d = \Delta = \delta$ give two algebraic equations:

$$g^2 f_1^{(1)}(\delta) + g^4 f_2^{(1)}(\delta) = 1; \qquad g^2 f_1^{(2)}(\delta) + g^4 f_2^{(2)}(\delta) = 1 \tag{15.12}$$

where functions $f_1^{(1)}$ and $f_2^{(1)}$ are defined by relations:



(15.13)

There are analogous relations for $f_{1,2}^{(2)}(\delta)$. We kept only two terms on the right-hand side of each of equations (15.4) and (15.5). Consequently we are interested only in the solutions with a small coupling constant. It must be emphasized that the latter does not mean that the dimension will be close to its canonical value $\frac{1}{2}D - 1$. The only restriction on the dimension is:

$$\delta > \tfrac{1}{2}D - 1. \tag{15.14}$$

So, let the coupling constant be small. However, according to (15.12), it is the 3-vertex term which must be close to unity in this case. Consequently the values of functions $f_{1,2}^{(i)}(\delta)$ satisfying (15.12) must be large. This means in its turn that one must look for a solution near the values of the dimension δ, for which the integrals (15.13) diverge. Therefore we shall be interested in the behaviour of functions $f_{1,2}^{(i)}(\delta)$ near their poles in δ.

Let us consider the 3-vertex term first of all. Three of six internal integrations can be easily accomplished with the aid of relations of ref. [40]. One has:

$$\int \Gamma^{\delta\delta\delta}(x_1x_4x_6)G_\delta^{-1}(x_4 - x_a)\Gamma^{\delta\delta\delta}(x_ax_bx_3)G_\delta^{-1}(x_b - x_5)$$
$$\times \Gamma^{\delta\delta\delta}(x_5x_2x_c)G_\delta^{-1}(x_c - x_6)\,dx_a\,dx_b\,dx_c\,dx_4\,dx_5\,dx_6$$
$$= \int \Gamma^{\delta\delta\delta}(x_1x_4x_6)\Gamma^{\tilde\delta\tilde\delta\delta}(x_4x_5x_3)\Gamma^{\delta\delta\tilde\delta}(x_5x_2x_6)\,dx_4\,dx_5\,dx_6, \tag{15.15}$$

where $\tilde\delta$ denotes the amputation of a corresponding leg. Normalizing according to (15.5), (15.6), we find:

$$\Gamma^{\tilde\delta\tilde\delta\tilde\delta}(x_4x_5x_3) = g\frac{\Gamma^3((D-\delta)/2)\Gamma((2D-3\delta)/2)}{\Gamma^3(\delta/2)\Gamma((3\delta-D)/2)}(x_{35}^2x_{34}^2x_{54}^2)^{(-D+\delta)/2}, \tag{15.16}$$

$$\Gamma^{\delta\delta\tilde\delta}(x_5x_2x_6) = g\frac{\Gamma^2((D-\delta)/2)}{\Gamma^2(\delta/2)}(x_{25}^2)^{-(3\delta-D)/2}(x_{26}^2x_{56}^2)^{-(D-\delta)/2}. \tag{15.17}$$

Besides that, it is convenient to integrate both sides of (15.17) over x_3:

$$f_1^{(1)}(\delta)(x_{12}^2)^{-\delta/2} = g^2\frac{\Gamma^3((D-\delta)/2)\Gamma((2D-3\delta)/2)}{\Gamma^3(\delta/2)\Gamma((3\delta-D)/2)}\int (x_{14}^2\cdot x_{16}^2\cdot x_{46}^2)^{-\delta/2}$$
$$\times (x_{26}^2x_{56}^2)^{-(D-\delta)/2}(x_{25}^2)^{-(3\delta-D)/2}(x_{45}^2)^{-(2D-3\delta)/2}\,dx_6\,dx_5\,dx_4. \tag{15.18}$$

The right-hand side of this equation possesses two types of poles in δ: firstly there are poles in the factor in front of the integral:

$$\delta_n = \tfrac{2}{3}D + \tfrac{2}{3}n, \qquad \delta_m = D + 2m \tag{15.19}$$

where n and m are integers, and secondly there are poles at

$$\delta_k = \tfrac{1}{3}D + \tfrac{2}{3}k, \tag{15.20}$$

where k is a positive integer, which arise because of the singularity of power functions of the integrand. Putting $\delta = \delta_k + \frac{2}{3}\varepsilon$ we find in these points:

$$\left.\frac{1}{(x^2)^{h+k+\varepsilon}}\right|_{\varepsilon\to 0} = -\frac{1}{\varepsilon}\frac{\pi^h}{\Gamma(h+k)}\frac{4^{-k}}{\Gamma(k+1)}\Box^k\delta(x). \tag{15.21}$$

Note that the integral (15.18) can be evaluted at small ε. Thus the function $f_1^{(1)}(\delta)$ can be found in the neighbourhood of any of the points (15.19) or (15.20).

The functions $f_2^{(1)}(\delta)$ and $f_{1,2}^{(2)}(\delta)$ can be calculated near the points (15.19) and (15.20) in an analogous manner.

Let us find these functions in the neighbourhood of the point $\frac{2}{3}D$. We put:

$$\delta = \tfrac{2}{3}D + \tfrac{2}{3}\varepsilon. \tag{15.22}$$

Using eq. (15.21) at $k = 0$ and the relation:

$$(x_{12}^2 x_{23}^2 x_{31}^2)^{-\delta/2} = -\frac{1}{\varepsilon}\frac{\pi^D}{\Gamma(D/2)}\frac{\Gamma^3(D/6)}{\Gamma^3(D/3)}\delta(x_{12})\delta(x_{13}) \tag{15.23}$$

we find (at $D = 6$):

$$1 = -\frac{1}{\varepsilon^3}\bar{g}^2\{1 - \varepsilon[\tfrac{7}{2} + 4\psi(1)]\} - \bar{g}^4\frac{2}{\varepsilon^5}, \tag{15.24}$$

$$1 = -\frac{1}{\varepsilon^3}\bar{g}^2\tfrac{1}{4}\{1 - \varepsilon[4\psi(1) + \tfrac{55}{18}]\} - \bar{g}^4\frac{1}{12\varepsilon^5} \tag{15.25}$$

where

$$\bar{g}^2 = g^2\left(\frac{\pi^{3/2}}{2}\right)^3, \qquad \psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x).$$

These equations are correct when $\varepsilon \ll 1$. It can be easily seen, that in the 3-vertex approximation (i.e. when confining ourselves only to the leading terms on the right-hand side) there are no solutions. However in the 5-vertex approximation there is a solution with small ε:

$$\bar{g}^2 \simeq -1.7\times 10^{-3}; \qquad \varepsilon = 7\times 10^{-2}; \qquad \delta = 4 + \tfrac{14}{3}10^{-2}. \tag{15.26}$$

Note that the coupling constant turned out purely imaginary. It can be shown with the aid of another method, that $g^2 < 0$ near any of the points (15.19) or (15.20) when $m, k \neq 6r, r$ being an integer.

It follows already from the 3-vertex approximation that $g^2 \sim \varepsilon^3$, and thus the true expansion parameter is ε – the deviation of the dimension δ from $\delta = 4$. For this reason we kept the next order in ε in the 3-vertex term and only the leading order in the 5-vertex term.

3. Let us consider now the interaction (15.2). Clearly, the calculational technique, developed above, is applicable in this theory.

Let us introduce the new variables for convenience of calculations

$$x = \tfrac{1}{2}(\Delta - 2d), \qquad y = \tfrac{1}{2}(D - 2d - \Delta) \tag{15.27}$$

and expand the integrands at the point $x = y = 0$. Confirming ourselves to the leading order in x and y we obtain, after some cumbersome calculations that will be published elsewhere, the algebraic equations (15.9) in the form:

$$1 = -u_0\frac{1}{xy}(x + y) + u_0^2\frac{2}{xy}, \tag{15.28}$$

$$1 = -u_0 \cdot \frac{1}{\varepsilon_1} \cdot \frac{\Gamma(h) \cdot (3x - y)}{3h} + \frac{1}{\varepsilon_1} u_0^2 \cdot \frac{\Gamma(h)}{3h}, \tag{15.29}$$

$$1 = u_0 \cdot \frac{1}{2\varepsilon_2} \tag{15.30}$$

where $u_0 = g^2\pi^{3h}/\Gamma(h)$, $\varepsilon_1 = d - h + 1 = 1 - \frac{1}{4}D - \frac{1}{2}(x + y)$, $\varepsilon_2 = h - \Delta = y - x$.

For the derivation of the second and third equations we used the exact expressions for the 3-vertex terms [5, 36], see Appendix 7, and for the derivation of the first equation we used the formulae

$$(x_{12}^2)^{-(h+x)}(x_{13}^2 x_{23}^2)^{-(D/4+(y+x)/2)}\Big|_{\substack{x\to 0\\ y\to 0}} = \frac{\pi^D}{\Gamma^2(h)} \frac{1}{(x-y)} \delta(x_{12}) \cdot \left\{ \frac{1}{y} \delta(x_{23}) - (x_{23}^2)^{-(h-x+y)} \frac{\Gamma(h)}{x\Gamma(x-y)} \frac{1}{\pi^h} \right\}.$$

These equations are true in any D-dimensional space-time. Putting here $D = 4 - \varepsilon$ and expanding in ε, we obtain:

$$1 = -\left(\frac{u_0}{\varepsilon}\right) \frac{2}{(\frac{1}{2} - c_2)(\frac{1}{2} + c_2)} + \left(\frac{u_0}{\varepsilon}\right)^2 \frac{8}{(\frac{1}{2} - c_2)(\frac{1}{2} + c_2)}$$

$$1 = -\left(\frac{u_0}{\varepsilon}\right) \frac{1}{12c_1} + \left(\frac{u_0}{\varepsilon}\right) \frac{c_2}{2c_1} + \left(\frac{u_0}{\varepsilon}\right)^2 \frac{1}{6c_1}$$

$$1 = \left(\frac{u_0}{\varepsilon}\right) \frac{1}{2c_2}$$

where $d = h - 1 + c_1\varepsilon^2$ and $\Delta = h - c_2\varepsilon$. The solution of these equations leads to the known results [1] of the ε-expansion:

$$c_1 = 1/108, \qquad c_2 = 1/6.$$

Note, however, that the method, described here, essentially differs from the ε-expansion since it allows us to work directly in the 3-dimensional space.

Moreover, the use of the conformally-invariant vertices and propagators is equivalent to the summation of all corrections to the bare vertexes and lines, which appear in higher orders in ε. Solving the system (15.28–15.30) at $D = 3$, we obtain:

$$d = 0.510, \qquad \Delta = 1.34. \tag{15.31}$$

The following values of the critical indices correspond to these values of dimensions:

$$\eta = 2d - D + 2 = 0.02, \qquad \nu = 0.60. \tag{15.32}$$

These results are in a good agreement with other theoretical calculations and experimental data [1, 67].

Note in conclusion that the set of equations (15.28)–(15.30) is obtained as a result of a rather rough approximation made while estimating the integrals involved in (15.4)–(15.6) (the expansion parameters x and y are of order of 1/4). One can essentially improve the accuracy of calculations within the same method. This leads, however, to a very complicated set of algebraic equations whose solution requires computer calculations. The results of them will be published elsewhere.

Note, however, that it is not very difficult to evaluate the first correction to the 3-vertex diagram in the leading order for the 5-vertex diagram in the equation for the "composite" field $\chi(x)$. Calculations with these corrections lead to the results: $\eta = 0.004$, $\nu = 0.625$.

16. Conclusion. Contribution of bare terms

Instead of listing the results reviewed in this paper and consequences of them we would like here, in the concluding section, to discuss once again completely the contribution coming from the terms $z_1\gamma$, $z_2\Box$ and $z_3\Box$ which is an essential problem for the approach considered in this paper. This problem is partially considered in sections 13 and 14. However, it has not yet been finally solved. In this section we present some preliminary results. At the same time we shall reconsider the results of section 13 from a somewhat different point of view.

Section 13 utilizes practically only a part of the information contained in the bare terms. In fact, the differential equations of section 13 keep the information coming from the terms of the order of $z_1 z_2^{-1}/\delta m_0^2$ and $z_1 z_3^{-1}/\delta\mu_0^2$ (where δm_0^2 and $\delta\mu_0^2$ are the mass counterterms, see below). To gain all the information contained in the initial renormalized equations it is necessary that *the terms proportional to z_1, z_2 and z_3 should be separated.* In the present section we shall discuss how to do this.

16.1. Formulation of the problem

Now let us consider again the renormalized equations describing the interaction (8.1) of two scalar fields φ and χ and keep the terms proportional to z_1, z_2 and z_3. Introduce the designation:

$$\langle 0|\, T\varphi(x_1)\varphi^+(x_2)\ldots\chi(x)\ldots|0\rangle = \langle\varphi(x_1)\varphi^+(x_2)\ldots\chi(x)\ldots\rangle.$$

First we consider the equation for the propagator

$$z_2\Box\,\langle\varphi(x_1)\varphi^+(x_2)\rangle + z_2\delta m_0^2\,\langle\varphi(x_1)\varphi^+(x_2)\rangle = \delta(x_1 - x_2) + \lambda z_1\,\langle\varphi(x_1)\chi(x_1)\varphi^+(x_2)\rangle, \tag{16.1}$$

where δm^2 is the mass counterterm, z_1 and z_2 are the renormalization constants. If in this equation one formally takes the limit $z_1 = 0$, $z_2 = 0$, $\delta m_0^2 = \infty$, one obtains (considering that $z_2\delta m_0^2 \to \infty$):

$$\lambda z_1\,\{\langle\varphi(x_1)\chi(x_1)\varphi^+(x_2)\rangle - z_2 z_1^{-1}\lambda^{-1}\delta m_0^2\,\langle\varphi(x_1)\varphi^+(x_2)\rangle\} + \delta(x_1 - x_2) = 0. \tag{16.2}$$

However, the initial equation (16.1) *is more informative since when passing over to the limit* $z_2 = 0$ *we loose the term* $z_2\Box\,\langle\varphi(x_1)\varphi^+(x_2)\rangle$.

This term may, indeed, be neglected as long as we consider quantities of the type of $z_1\,\langle\varphi(x_1)\chi(x_1)\varphi^+(x_2)\rangle$. Note, that in deriving the integral equations in section 2 we met just with this type of uncertainty, i.e. we found the limit $z_1\varphi(x_1)\chi(x_1)$ at $z_1 \to 0$.

Let us multiply now the left-hand side of eq. (16.2) by the infinitely large quantity z_2^{-1}. Since the left-hand side of eq. (16.2) is tending to zero, we obtain the uncertainty $0 \times \infty$. To resolve it one should again turn to eq. (16.1). By multiplying it by z_2^{-1} and considering the limit $z_1, z_2 \to 0$, we have

$$z_2^{-1}[\lambda z_1\{\langle\varphi(x_1)\chi(x_1)\varphi^+(x_2)\rangle - z_2 z_1^{-1}\lambda^{-1}\delta m_0^2\langle\varphi(x_1)\varphi^+(x_2)\rangle\} + \delta(x_1 - x_2)] = \Box\langle\varphi(x_1)\varphi^+(x_2)\rangle. \quad (16.3)$$

This equation cannot be obtained from (16.2) and must be considered independently. Equation (16.2) should be considered *as an equation determining the most singular part of the product* $\varphi(x_1)\chi(x_1+\varepsilon)|\varepsilon\to 0$, whereas eq. (16.3) fixes *the next to the leading singularity of this product.*

The analogous situation arises when the higher Green functions are considered. Remember that from the integral equations of section 8 it follows that there exists only the pole of the functions ρ_n at the point $\sigma = (d, 0)$, the residue at this pole being expressed through the coupling constant. It is this pole that determines the leading asymptotical term in the Green functions when any two of its arguments are close to one another (see below). The structure of the functions $\rho_n(\sigma)$ is otherwise arbitrary. By supplementing the initial integral equations with those for tensor fields we establish a correspondence between the poles of the functions ρ_n and various fields. As a result we come to a universal scheme in which all the fields are treated on equal grounds. As has been shown in section 11 this scheme does not depend on the initial Lagrangian. Confirming ourselves to this universal scheme alone, however, we are unable to obtain any information on the quantum numbers of the poles of the functions ρ_n. To gain the necessary information one should consider the renormalized equations with the terms of the order of z_1, z_2, z_3 kept and fulfill then various limiting transitions like the ones we did while deriving eqs. (16.2) and (16.3).

Solution of integral equations of section 8 *should be regarded as only the first step towards this direction* (*they are analogous to eq.* (16.2). The terms $z_2\Box$ and $z_1\gamma$ where γ is the bare vertex are omitted in the derivation of these equations.

According to what has been said above (see the paragraph after eq. (16.3)) *integral equations determine only the leading part of the asymptotical* behaviour of the Green function when two of its arguments are close to one another. Therefore, the poles of the functions ρ_n at the point $l = d$ must be connected with the most singular part of the asymptotical behaviour. Hence it follows that *the dimension d of the fundamental field is the smallest of all the dimensions* l_α. Note that this statement cannot be treated as a consequence of integral equations within which none of the fields (11.15) is anyhow singled out.

The next step is *to take into account the discarded terms and to obtain additional equations for higher Green functions* analogous to eq. (16.3). As is shown, below these terms destroy the democracy of the fields in the set (11.5) and distinguish the fundamental fields. The discussion of the term $z_2\Box$ was started in section 13 and is continued below. However, first we consider (in more detail than in section 13) the resolving of the uncertainties in eqs. (16.2), (16.3) and find the constants z_1, z_2 and z_3.

16.2. Equations for propagators. Dynamical restrictions on dimensions of fundamental fields

Represent the left-hand sides of eqs. (16.2) and (16.3) as limits of the type*

$$\int d\Omega_\varepsilon z_1(\varepsilon)\varphi(x)\chi(x+\varepsilon)|_{\varepsilon\to 0}, \qquad \int d\Omega_\varepsilon z_2^{-1}(\varepsilon) z_1(\varepsilon)\varphi(x)\chi(x+\varepsilon)|_{\varepsilon\to 0}.$$

* Another way to define the renormalization constants has been done in ref. [66].

Here $\int \mathrm{d}\Omega_\varepsilon$ stands for averaging with respect to the angles of the vector ε_μ and

$$z_1(\varepsilon) = a_1 \cdot (\varepsilon^2)^{\alpha_1}, \qquad z_2(\varepsilon) = a_2 \cdot (\varepsilon^2)^{\alpha_2}, \qquad \delta m_0^2 = a_m/\varepsilon^2 \tag{16.4}$$

where a_1, a_2 and a_m are constants.

Substituting these equations in (16.1) we find

$$\lambda \int \mathrm{d}\Omega_\varepsilon \langle \varphi(x_1)\chi(x_1 + \varepsilon)\varphi^+(x_2)\rangle =$$

$$= \left\{ \frac{a_2 \cdot a_m}{a_1} (\varepsilon^2)^{-\alpha_1 + \alpha_2 - 1} + \frac{a_2}{a_1} (\varepsilon^2)^{-\alpha_1 + \alpha_2} \Box \right\} \langle \varphi(x_1)\varphi^+(x_2)\rangle - a_1^{-1} \cdot (\varepsilon^2)^{-\alpha_1} \delta(x_1 - x_2). \tag{16.5}$$

This equation is equivalent to eqs. (16.2), (16.3) taken together. If we multiply both its parts by $(\varepsilon^2)^{\alpha_1}$ and calculate the limit $\varepsilon \to 0$, we obtain eq. (16.2). Multiplying both its parts by $(\varepsilon^2)^{\alpha_1 - \alpha_2}$ we get in the limit eq. (16.3).

On the other hand the left-hand side of eq. (16.5) can be found from the explicit form of the function $\langle \varphi(x_1)\chi(x_3)\varphi^+(x_2)\rangle$ at $x_3 = x_1 + \varepsilon$, $\varepsilon \to 0$

$$\langle \varphi(x_1)\chi(x_3)\varphi^+(x_2)\rangle|_{x_3 \to x_1} \sim \int \mathrm{d}p\, \mathrm{e}^{\mathrm{i}px_2} \{ Q^d(x_1 x_3|p) - (p^2)^{d-h} Q^{D-d}(x_1 x_3|p)\} \tag{16.6}$$

$$= (\varepsilon^2)^{-\delta/2} \{\alpha_0 + a_1 \varepsilon_\mu \partial_\mu + \alpha_2' \varepsilon^2 \Box + \alpha_2'' \varepsilon_\mu \varepsilon_\nu \partial_\mu \partial_\nu + \ldots\} \langle \varphi(x_1)\varphi^+(x_2)\rangle$$
$$+ (\varepsilon^2)^{-d-\delta/2+h} \{\beta_0 + \beta_1 \varepsilon_\mu \partial_\mu + \beta_2' \varepsilon^2 \Box + \beta_2'' \varepsilon_\mu \varepsilon_\nu \partial_\mu \partial_\nu \cdot \ldots\} \delta(x_1 - x_2). \tag{16.7}$$

Here α_i and β_i are known constants depending on d and δ. We do not need their explicit form. For obtaining (16.7) from (16.6) relations (4.5) and (13.8) were used. The terms of the first line present the expansion of the function Q^{D-d} in the powers of ε_μ while the terms of the second line present that of the function Q^d. If we average eq. (16.7) over the angles of the vector ε_μ, in the limit $\varepsilon^2 \to 0$ the former turn into (16.5). We have

$$\int \mathrm{d}\Omega_\varepsilon \langle \varphi(x_1)\chi(x_1 + \varepsilon)\varphi^+(x_2)\rangle = (\varepsilon^2)^{-\delta/2} \{\alpha_0 + \alpha_2 \varepsilon^2 \Box + \cdots\} \langle \varphi(x_1)\varphi^+(x_2)\rangle$$
$$+ (\varepsilon^2)^{-d-\delta/2+h} \{\beta_0 \delta(x_1 - x_2) + \beta_2 \varepsilon^2 \Box \delta(x_1 - x_2) + \ldots\}. \tag{16.8}$$

By comparing this expression with (16.5) we find

$$\alpha_2 - \alpha_1 - 1 = -\tfrac{1}{2}\delta, \qquad \alpha_1 = d + \tfrac{1}{2}\delta - h$$

or

$$z_1 \sim (\varepsilon^2)^{d+\delta/2-h}, \qquad z_2 \sim (\varepsilon^2)^{d-h+1}, \qquad z_1 z_2^{-1} \sim (\varepsilon^2)^{\delta/2-1}. \tag{16.9}$$

The coefficients of proportionality in (16.9) can also be calculated if one uses explicit expressions for the constants α_i and β_i, see e.g. eq. (16.14b).

Relations (16.9) guarantees the equivalence of eqs. (16.5) and (16.8) only if the term $\beta_2 \cdot (\varepsilon^2)^{-d-\delta/2+h+1} \Box \delta(x_1 - x_2)$ in the right-hand side of (16.8) is less singular than the other terms. Indeed, this term is absent from eq. (16.5) which fixes three most singular terms and, therefore, it is less singular than any of them (if eq. (16.8) is multiplied by z_1, or $z_1 \cdot z_2^{-1}$, we obtain eq. (16.2) or (16.3) from which this term is absent). As a matter of fact it is sufficient to require that the term $\beta_2(\varepsilon^2)^{-d-\delta/2+h+1}\Box\delta(x_{12})$ be less singular than $\alpha_i(\varepsilon^2)^{-\delta/2+1}$. This gives: $d < h$. This condition is a *dynamical restriction* on the dimension of the field $\varphi_d(x)$. Taking into account the kinematical limitations of section 3 we have

$$h - 1 < d < h. \tag{16.10}$$

It is essential that this condition distinguishes the fundamental field as compared with the composite fields, since the dimension of the latter is beyond this range.

Note that according to (16.9) the constant $z_2 \to 0$ for all $d > h - 1$. The constant z_1 tends to zero only if $2d + \delta \geqslant D$. This condition is fulfilled if $D \geqslant 6$. Finally, the term

$$z_2 \delta m_0^2 \to \infty$$

if the dimension d is within the range of (16.10).

The equation for the propagator of the field χ can be treated in the same fashion

$$z_3 \Box \langle\chi(x_1)\chi(x_2)\rangle + z_3\delta m_0^2 \langle\chi(x_1)\chi(x_2)\rangle = \delta(x_1 - x_2) + \lambda z_1 \langle\varphi(x_1)\varphi^+(x_1)\chi(x_2)\rangle. \tag{16.11}$$

In the limit $z_1 = 0$, $z_3 = 0$ eq. (16.11) is equivalent to the two equations

$$\lambda z_1 \{\langle\varphi(x_1)\varphi^+(x_1)\chi(x_2)\rangle - z_3 z_1^{-1}\delta\mu_0^2\lambda^{-1}\langle\chi(x_1)\chi(x_2)\rangle\} + \delta(x_1 - x_2) = 0, \tag{16.11a}$$

$$\lambda z_1 z_3^{-1}\{\langle\varphi(x_1)\varphi^+(x_1)\chi(x_2)\rangle - z_3 z_1^{-1}\delta\mu_0^2\lambda^{-1}\langle\chi(x_1)\chi(x_2)\rangle\} + z_3^{-1}\delta(x_1 - x_2) = \Box\langle\chi(x_1)\chi(x_2)\rangle. \tag{16.11b}$$

The same as above we shall define the left-hand sides of these equations as the limits of the quantities

$$\int \mathrm{d}\Omega_\varepsilon z_1(\varepsilon)\varphi(x)\varphi^+(x + \varepsilon) \quad \text{and} \quad \int \mathrm{d}\Omega_\varepsilon z_1(\varepsilon) z_3^{-1}(\varepsilon)\varphi(x)\varphi^+(x + \varepsilon).$$

In this case for the constant z_3 we find

$$z_3(\varepsilon) \sim (\varepsilon^2)^{\delta - h + 1} \tag{16.12}$$

and for the dimension δ of the field χ we obtain by analogy with (16.10)

$$h - 1 < \delta < h. \tag{16.13}$$

Using (16.9) and (16.12) it is easy to find the expression for the renormalized coupling constant. Assuming that

$$\lambda_R = z_1^{-1} z_2 z_3^{1/2} \lambda_0$$

we find

$$\lambda_0/\lambda_R \sim (\varepsilon^2)^{(h-3)/2}.$$

This relation is seen to be finite in the six-dimensional space only, where the interaction (8.1) is renormalizable. At $D > 6$ this ratio is equal to zero (the interaction is unrenormalizable).

Now we present another restriction on the dimensions d and δ that results from the joint consideration of eqs. (16.2) and (16.11a). Let us assume that

$$\lambda z_1 = a \cdot (\tfrac{1}{2}\varepsilon^2)^{d+\delta/2-h}$$

where a is some constant. Then we have

$$\lambda z_1 \langle \varphi(x_1)\chi(x_1)\varphi^+(x_2)\rangle - z_2\delta m_0^2 \langle \varphi(x_1)\varphi^+(x_2)\rangle \tag{16.14}$$

$$= -ag\frac{\pi}{\sin \pi(d-h)}(\tfrac{1}{2}\varepsilon^2)^{d+\delta/2-h}Q^{d\delta d}(x_1, x_1+\varepsilon|x_2)\Big|_{\varepsilon\to 0} \tag{16.14a}$$

$$= ag\frac{\pi}{\sin \pi(d-h)}\frac{N(dd\delta)}{\Gamma(h-d+1)\,\Gamma(D-d)}\frac{\Gamma((D-\delta)/2)\Gamma((D+\delta-2d)/2)}{\Gamma(\delta/2)\Gamma((2d-\delta)/2)}\delta(x_1-x_2). \tag{16.14b}$$

Here the normalization (2.8) is used: g is the coupling constant and N is the factor (5.10). When passing from (16.14) to (16.14a) we used relations (16.6)–(16.9). The second term in (16.14) is canceled by the term proportional to α_0. The contribution next in the degree of singularity is given by the function $Q^{d\delta d}$ (the term $\beta_0\cdot(\varepsilon^2)^{-d-\delta/2+h}\delta(x_1-x_2)$). Calculating the factor β_0 we obtain eq. (16.14b). Substituting now (16.14b) into (16.2) we find

$$ag = \frac{\pi\cdot N(dd\delta)\cdot\Gamma((D-\delta)/2)\Gamma((D+\delta-2d)/2)}{\sin \pi(d-h)\Gamma(h-d+1)\Gamma(D-d)\Gamma(\delta/2)\Gamma((2d-\delta)/2)} = -1. \tag{16.14c}$$

One can similarly show that

$$\lambda z_1 \langle \varphi(x_1)\varphi^+(x_1)\chi(x_2)\rangle + z_2\delta\mu_0^2 \langle \chi(x_1)\chi(x_2)\rangle$$

$$= ag\frac{\pi\cdot N(dd\delta)}{\sin \pi(\delta-h)\Gamma(h-\delta+1)\Gamma(D-\delta)}\cdot\frac{\Gamma^2((D-\delta)/2)}{\Gamma^2(\delta/2)}\delta(x_1-x_2) \tag{16.14d}$$

where a is the same constant as in (16.14b). From eq. (16.11a) we have

$$ag\frac{\pi}{\sin \pi(\delta-h)}\frac{N(dd\delta)\Gamma^2((D-\delta)/2)}{\Gamma(h-\delta+1)\Gamma(D-\delta)\Gamma^2(\delta/2)} = -1.$$

Comparing this equation with eq. (16.14c) we find

$$\frac{\Gamma((D-\delta)/2)\Gamma(\delta-h)}{\Gamma(\delta/2)\Gamma(D-\delta)} = \frac{\Gamma((D+\delta-2d)/2)\Gamma(d-h)}{\Gamma((2d-\delta)/2)\Gamma(D-d)}. \tag{16.14e}$$

We should emphasize that this relation between d and δ is a consequence of the fact that eqs. (16.2) and (16.11a) and, therefore, (16.14), (16.14d) include the same renormalization constant.

It is worth noting in conclusion that a more detailed utilization of the initial set of renormalized integral equations with the bare term (8.3)–(8.7) kept allows to obtain the relations presented. Indeed, by calculating the bare term with the use of the equations for the vertex Green function of the fundamental fields in the two channels we come again to equation (16.14e). The analogous program may be formulated for the composite fields out of which we believe the relations for the dimensions of the composite fields can be obtained.

16.3. Equations for higher Green functions

Now we discuss how one can take consistently into account the terms $\sim z_1, z_2$ and z_3 in the equations for higher Green functions. Consider the equation

$$z_2\Box\langle\varphi(x_1)\varphi^+(x_2)\chi(x_3)\rangle + z_2\delta m_0^2\langle\varphi(x_1)\varphi^+(x_2)\chi(x_3)\rangle \tag{16.15}$$

$$= \lambda z_1\langle\varphi(x_1)\chi(x_1)\varphi^+(x_2)\chi(x_3)\rangle$$

$$\equiv \lambda z_1 G(x_{12})D(x_{13}) + \lambda z_1\langle\varphi(x_1)\chi(x_1)\varphi^+(x_2)\chi(x_3)\rangle_{\text{conn.}} \tag{16.16}$$

where $\langle\ldots\rangle_{\text{conn.}}$ is the connected part of the Green function, $G(x)$ and $D(x)$ are propagators of the fields φ and χ. The first term in (16.16) is the bare term designated above by γ. In the limit $z_1 \to 0$, $z_2 \to 0$ eqs. (16.15)–(16.16) are equivalent to the set of three equations

$$\lambda z_1\langle\varphi(x_1)\chi(x_1)\varphi^+(x_2)\chi(x_3)\rangle_{\text{conn.}} - z_2\delta m_0^2\langle\varphi(x_1)\varphi^+(x_2)\chi(x_3)\rangle = 0, \tag{16.17}$$

$$\Box\langle\varphi(x_1)\varphi^+(x_2)\chi(x_3)\rangle$$

$$= \lambda z_1 z_2^{-1}\langle\varphi(x_1)\chi(x_1)\varphi^+(x_2)\chi(x_3)\rangle_{\text{conn.}} - \delta m_0^2\langle\varphi(x_1)\varphi^+(x_2)\chi(x_3)\rangle, \tag{16.18}$$

$$\lambda\langle\varphi(x_1)\chi(x_1)\varphi^+(x_2)\chi(x_3)\rangle_{\text{conn.}} + z_1^{-1}z_2\delta m_0^2\langle\varphi(x_1)\varphi^+(x_2)\chi(x_3)\rangle$$

$$- z_1^{-1}z_2\Box\langle\varphi(x_1)\varphi^+(x_2)\chi(x_3)\rangle = -\lambda G(x_{12})D(x_{13}). \tag{16.19}$$

Equations of the type (16.17), (16.18) *and* (16.19) *are essentially independent and contain different information.* Such equations can be written for any connected Green function. To make them meaningful it is necessary to resolve all the uncertainties of the type $z_1\varphi(x_1)\chi(x_1)$, $z_1 z_2^{-1}\varphi(x_1)\chi(x_1) - \ldots$, $\varphi(x_1)\chi(x_1) - \ldots$, where the dots imply subtractions. To this end it is sufficient to substitute the product $\int d\Omega_\varepsilon \varphi(x)\chi(x+\varepsilon)$ for $\varphi(x)\chi(x)$ and to use expressions (16.9) for the renormalization constants. This has been carried out partially in section 13 for the example of equations of the type (16.18).

16.4. Conclusion

In conclusion we would like to mention some problems which have not been discussed in the review and which should be solved in the first place.

This concerns first of all the development of a formalism which would allow gauge fields to be included into the scheme. In this way some results have been attained (concerning mostly electrodynamics), but the question has not yet been solved completely.

Another problem concerns the investigation of theories with internal symmetry and establishment of all possible types of the latter. Some results in this direction now can be obtained by using the above relation between the dimensions d and δ and the dynamical restrictions on them, see (16.10), (16.13). Indeed, in the theory with internal symmetry the equation of the type of (16.14e) also exists, but in this case it depends on the type of this symmetry (the dependence is easily found). It is obvious that the solution of eqs. (16.10), (16.13) and (16.14e) does not exist for all types of symmetry.

One of the most important problems is a further investigation of vertices that include concerned currents, since one must know such vertices to be able to formulate experimental predictions. Note that a number of additional restrictions can be found for the vertices discussed in sections 6 and 7. It has been shown above that the dimension of the fundamental fields d is limited by the inequality $d < \frac{1}{2}D$. This means that the tensor fields O_d^s with the dimension $d + s$ considered in sections 6, 7, 12 do not contribute into the operator expansion of the product $\varphi_d(x_1)j_\mu(x_2)$, otherwise $d > D - 2$ (this is the consequence of positivity, see section 7). However, the inequalities

$d < \frac{1}{2}D$ and $d > D - 2$ are compatible at $D = 2, 3$ only. Thus at $D \geqslant 4$ the appearance of the fields O_d^s in the partial wave expansion of vertices including the product $\varphi_d j_\mu$ should be forbidden. According to sections 11, 12 this leads to an infinite set of additional conditions which guarantee the absence of poles of the functions $\rho_n(\sigma)$ at the points $\sigma_s = (d + s, s)$.

These conditions must be imposed on the general solution of the Ward identities (see section 6), i.e. to a rather narrow class of conformally invariant functions. Therefore, they may prove to be rather informative. Note, that the above said does not concern the vertices including the product $j_\mu(x_1)O_\alpha(x_2)$, where O_α is some compound field (scalar or tensor), since the dimensions of compound fields lie beyond the interval (16.10) and the operator expansion of this product could, in principle, include the fields O_d^s.

It is worth noting that the final solution of the pointed problems may become possible no sooner than all the questions about the closing of the equations are answered. In conclusion we emphasize once again that to clarify the physics underlying this theory one must break the conformal invariance by keeping the mass terms and consider the mass shell.

Appendix 1. Conformal group in one- and two-dimensional space

A1. General properties of representations

Conformal transformations of the one-dimensional space x from a three-parametric group of fractional-linear transformations

$$x' = \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad \alpha\delta - \beta\gamma = 1, \tag{A1.1}$$

which is locally isomorphic to the group SO (2.1) [72]. These transformations comprise the analog of translations ($\gamma = 0, \alpha = \delta = 1$), scale transformations ($\gamma = \beta = 0$) and special conformal transformations ($\alpha = \delta = 1$, $\beta = 0$). Their generators (P, D and K) satisfy the commutational relations

$$[D, P] = \mathrm{i}P, \qquad [D, K] = -\mathrm{i}K, \qquad [P, K] = -2\mathrm{i}D.$$

In a coordinate realization (in x-axis) the generators are given by the equations

$$P\,|x\rangle = \mathrm{i}\partial_x\,|x\rangle, \qquad D\,|x\rangle = -\mathrm{i}(l + x\partial_x)\,|x\rangle, \qquad K\,|x\rangle = \mathrm{i}(x^2\partial_x + 2lx)\,|x\rangle, \tag{A1.2}$$

where l is the analog of the scale dimension. Generators (A1.2) act in the space of functions $f(x) = \langle f|x\rangle$.

Irreducible representations of the group SO(2, 1) are well known [72]. They are characterized by the scale dimension l, which determines the Casimir operator values equal to:

$$Q = D(\mathrm{i} - D) - PK = l(l - 1). \tag{A1.3}$$

Note that in ref. [72] the analysis of single-valued representations of the group SO(2, 1) is given, while for our purpose the infinite-valued representations [12, 16] are of more interest. They are the representations of the universal covering group of the SO(2, 1) group.

To describe the latter it is appropriate to introduce a basis consisting of eigen vectors of the generator Λ of the group SO(2) (maximal compact subgroup of SO(2, 1) group). This generator is equal to

$$\Lambda = \tfrac{1}{2}(P + K). \tag{A1.4}$$

Transformations of the subgroup SO(2) correspond to the values of the parameters in (A1.1) obeying

$$\alpha^2 + \beta^2 = 1, \qquad \delta = \alpha. \tag{A1.4a}$$

It may be shown that [72] its eigen values in the irreducible representation space have the discrete spectrum: $\Lambda = \lambda + m$ where λ is a fixed parameter and $m = 0, \pm 1, \ldots$. The number λ may be restricted to the interval

$$-\tfrac{1}{2} < \lambda < \tfrac{1}{2}.$$

If $\lambda = 0, \frac{1}{2}$, we have one and two-valued representations considered in [72]. Designate the eigen vectors of the generator Λ as $|l, \lambda; m\rangle$. We have

$$\Lambda\,|l, \lambda; m\rangle = (\lambda + m)\,|l, \lambda; m\rangle. \tag{A1.5}$$

It is evident that representations with different parameters λ are non-equivalent.

Thus *the irreducible representations of the universal covering group are classified* [12] *by the values of two parameters l and λ*. According to (A1.3) two representations (l, λ) and (l', λ') are equivalent if and only if

$$l' = 1 - l, \qquad \lambda = \lambda'.$$

It is evident that *the universal covering group is infinite-sheeted*, because the maximal compact subgroup SO(2) has this property. Representations where λ has rational values are multivalued. For irrational values of λ we have *infinite-valued representations*. This can readily be seen when passing to the realization [72] of the SO(2, 1) group on circumference, see subsection 3.2, where this realization has been comprehensively considered for the analysis of field transformation properties for compact transformations (see eq. (3.8)).

Introduce the coordinate basis

$$|l, \lambda; x\rangle = \sum_{m=-\infty}^{\infty} f_{lm}^{\lambda}(x)\,|l, \lambda; m\rangle \tag{A1.6}$$

where $f_{lm}^{\lambda}(x) = \langle m, \lambda; 1 - l|l, \lambda; x\rangle$ are the eigen functions of generator Λ in realization (A1.2). From (A1.2) and (A1.4, 5) we have

$$f_{lm}^{\lambda}(x) \sim \frac{1}{(1 + x^2)^l}\left(\frac{1 - \mathrm{i}x}{1 + \mathrm{i}x}\right)^{\lambda + m}. \tag{A1.7}$$

Consider the invariant scalar product of vectors (A1.6). According to (A1.2) it is the power function of the difference $(x_1 - x_2)$. Note, however, that the exact form of this function can only be found after the analysis of the transformation properties of vectors (A1.6) under finite transformations. Using the results of section 3, see (3.7), we find [22]

$$\Delta_{l,\lambda}(x) \sim (x - \mathrm{i}\varepsilon)^{-l-\lambda}(x + \mathrm{i}\varepsilon)^{-l+\lambda}. \tag{A1.8}$$

It may be shown that the function (A1.7) satisfies the equation

$$\int \mathrm{d}y \Delta_{1-l,\lambda}(y-x) f^{\lambda}_{lm}(y) \sim f^{\lambda}_{1-l,m}(x) \tag{A1.9}$$

(in calculations it is appropriate to change to a momentum representation, see [27]). This equation implies the equivalence of representations (l, λ) and $(1-l, \lambda)$ and can be rewritten in a more general form

$$|1-l, \lambda; x\rangle = \int \mathrm{d}y \Delta_{1-l,\lambda}(y-x)\, |l, \lambda; y\rangle. \tag{A1.9a}$$

From this it follows in particular that the relation of completeness in the irreducible representation space may be written in the form [22]

$$\int \mathrm{d}x\, |l, \lambda; x\rangle \langle x; \lambda, 1-l| = I. \tag{A1.10}$$

A2. Discrete series

Representations of the discrete series require a spacial consideration, since only they can contribute to the field states. There are two series of these representations: D_+ and D_-, where the parameter λ takes a fixed value $\lambda = \pm l$ for the $D_\pm$ series and the spectrum of the generator Λ is limited by the condition $m \gtrless 0$ for $D_\pm$ series. In the case of integer and half-integer l values they are described in [72]. In representations of the universal covering group λ may take any real values. This results in certain distinctions (in particular, if λ is not equal to an integer or a half-integer, there are no finite-dimensional representations, described in [72]) which are of a purely mathematical interest and won't be considered here. For our purposes *it is essential that only the representations of the discrete series satisfy the spectrality condition. This property holds when passing to non-integer* l. To see this, it is sufficient to consider the eigen functions of the generator Λ which form a basis in the irreducible representations space. Assuming $\lambda = \pm l$ in (A1.7), for the $D_\pm$ series we have

$$f_{l_\pm, m}(x) \sim (1 \pm \mathrm{i}x)^{-2l-m} (1 \mp \mathrm{i}x)^m \tag{A1.11}$$

where $l_\pm \equiv (l, \lambda = \pm l)$ are the quantum numbers of the $D_\pm$ series representation. Calculating the Fourier transform of (A1.11), we find [12, 27] (remember that $m > 0$) for D_+ series

$$f_{l_+, m}(p) \sim \theta(p) \cdot p^{l-1} W_{l+m, 1/2-l}(2p) \sim \theta(p) \cdot p^{2l-1}\, \mathrm{e}^{-p} L^{2l-1}_m(p).$$

Analogously it can be seen that $f_{l_-, m}(p) = 0$ at $p > 0$. Thus the spectrum of momenta is limited by the condition $\pm p > 0$ (for the $D_\pm$ series).

Consider the transformation of vectors $|l_+, x\rangle$ analogous to (A1.9a). In this case the scalar product (A1.8) takes the form $\Delta_{l_+}(x) \sim (x - \mathrm{i}\varepsilon)^{-2l}$. It can readily be seen that the transformation (A1.9) is not reduced to the formal substitution $l \to 1 - l$, which corresponds, due to $\lambda = l$, to a change to the non-equivalent representation $(1-l, 1-\lambda)$. Instead of (A1.9) we have

$$\int \mathrm{d}y \Delta^{-1}_{l_+}(y-x) f_{l_+, m}(y) = \tilde{f}_{l_+, m}(x) \sim (1 + \mathrm{i}x)^{-1}\, {}_2F_1\left(-m, 1; 2l; \frac{2}{1 + \mathrm{i}x}\right), \tag{A1.12}$$

where the function $\Delta_{l_+}^{-1}(x)$ is determined by the equation

$$\int \Delta_{l_+}^{-1}(x-y)\Delta_{l_+}(y)\,\mathrm{d}y = \langle x, \tilde{l}_+ | l_+, 0\rangle = \frac{1}{2\pi \mathrm{i}}\,\frac{1}{x - \mathrm{i}\varepsilon}.$$

The functions $f_{l_+,m}$ and $\tilde{f}_{l_+,m}$ may be responsible for two sets of basis vectors

$$|l_+, x\rangle = \sum_{m=0}^{\infty} f_{l_+,m}(x)\,|l_+, m\rangle, \qquad |\tilde{l}_+, x\rangle = \sum_{m=0}^{\infty} \tilde{f}_{l_+,m}(x)\,|l_+, m\rangle \tag{A1.13}$$

which are related by the integral transformation

$$|\tilde{l}_+, x\rangle = \int \Delta_{l_+}^{-1}(y-x)\,|l_+, y\rangle\,\mathrm{d}y. \tag{A1.14}$$

This enables us to write the completeness relation in the form

$$\int \mathrm{d}x\,|l_+, x\rangle\,\langle x, \tilde{l}_+| = I. \tag{A1.15}$$

This completeness relation form is advantageous for the analysis of the partial wave expansion of field states, see section 3. For (A1.15) to be effectively used, certain features of the basis vectors $|\tilde{l}_+, x\rangle$ should be considered.

Note first, that *the generators* Λ *and* K *acting as differential operators in the basis* $|l_+, x\rangle$, *become nonlocal operators when passing to the basis* $|\tilde{l}_+, x\rangle$. Indeed, let $|f\rangle$ be any positive frequency state. In accordance with (A1.13) it may be associated with two types of projections

$$f(x) = \langle f | l_+, x\rangle \qquad \text{and} \qquad \tilde{f}(x) = \langle f | \tilde{l}_+, x\rangle \tag{A1.16}$$

which, due to (A1.14) are related by the transformation

$$\tilde{f}(x) = \int \mathrm{d}y\,\Delta_{l_+}^{-1}(y-x) f(y). \tag{A1.17}$$

The action of generators in the basis $|l_+, x\rangle$ is determined by the relations (A1.2). In particular, the K-transformation is

$$\langle f|\,K\,|l_+, x\rangle = \mathrm{i}(2lx + x^2\partial_x)\langle f|l_+, x\rangle = \mathrm{i}(2lx + x^2\partial_x) f(x)$$

$$\langle f|\,K\,|l_+, x\rangle = \int \mathrm{d}y f(y) K(y, x), \tag{A1.18}$$

where

$$K(y, x) = \frac{1}{2\pi}(2lx + x^2\partial_x)\frac{1}{y - x - \mathrm{i}0}.$$

In a similar way in the basis $|\tilde{l}_+, x\rangle$ we find

$$\langle f|\,K\,|\tilde{l}_+, x\rangle = \int \mathrm{d}y \langle f|\tilde{l}_+, y\rangle\,\tilde{K}(y, x) \tag{A1.18a}$$

where

$$\tilde{K}(y, x) = \int \mathrm{d}x'\,\mathrm{d}y' \Delta_{l_+}(y - y') K(y, x') \Delta_{l_+}^{-1}(x' - x) = \frac{1}{2\pi}(2ly + y^2 \partial_y) \frac{1}{y - x - \mathrm{i}\varepsilon}.$$

In contrast to (A1.18) this transformation cannot be written in a differential form. The above said is true for the generator Λ, which also proves to be an integral operator in basis (A1.14) and its eigen functions $\tilde{f}_{l_+,m}(x)$, see (A1.12), are the solutions of the integral equation [22]

$$\Lambda \tilde{f}_{l_+,m}(x) \equiv \int \mathrm{d}y \Lambda(y, x) \tilde{f}_{l_+,m}(y) = (l + m) \tilde{f}_{l_+,m}(x),$$

where

$$\Lambda(y, x) = \frac{1}{2\pi}[(1 + y^2)\partial_y + 2ly] \frac{1}{y - x - \mathrm{i}\varepsilon}$$

is the kernel of the generator Λ.

Let $|f\rangle$ be the set of field states. Assume that, e.g., $|f\rangle = \varphi_{l_1}(x_1)\varphi_{l_2}(x_2)|0\rangle$. Its projection on the basis vector of the irreducible representation is an invariant three-point function. According to (A1.16) *there are two different three-point functions*

$$C^{ll_1l_2}(x|x_1x_2) = \langle x, l_+|\varphi_{l_1}(x_1)\varphi_{l_2}(x_2)|0\rangle \quad \text{and} \quad Q^{\tilde{l}l_1l_2}(x|x_1x_2) = \langle x, \tilde{l}_+|\varphi_{l_1}(x_1)\varphi_{l_2}(x_2)|0\rangle.$$

As it follows from (A1.17), they are related by the transformation

$$Q^{\tilde{l}l_1l_2}(xx_1x_2) = \int \mathrm{d}y \Delta_{l_+}^{-1}(x - y) C^{ll_1l_2}(yx_1x_2). \tag{A1.19}$$

If for the function $C^{ll_1l_2}(xx_1x_2)$ the usual expression

$$C^{ll_1l_2}(xx_1x_2) \sim (x_{12} - \mathrm{i}\varepsilon)^{-l_1 - l_2 + l}(x_1 - x - \mathrm{i}\varepsilon)^{-l_1 + l_2 - l}(x_2 - x - \mathrm{i}\varepsilon)^{l_1 - l_2 - l}$$

is chosen, then we find for the Q function

$$Q^{\tilde{l}l_1l_2}(x_3x_1x_2) \sim \frac{1}{(x_{12})^{l_1 + l_2 - l}} \frac{1}{x_{32}} {}_2F_1\left(1, l + l_1 - l_2; 2l; 1 - \frac{x_{13}}{x_{23}}\right). \tag{A1.20}$$

In order to obtain (A1.20), it is necessary to calculate the integral arising in (A1.19)

$$\mathscr{I} \sim \int \mathrm{d}u \frac{1}{(-u - \mathrm{i}\varepsilon)^{\delta_3}} \frac{1}{(-u + \mathrm{i}\varepsilon + \xi)^{\delta_1}} \frac{1}{(-u + \mathrm{i}\varepsilon + \lambda)^{\delta_2}},$$

where

$$\delta_1 + \delta_2 + \delta_3 = 2, \qquad \delta_1 = l_1 - l_2 + l, \qquad \delta_2 = l_2 - l_1 + l, \qquad \delta_3 = 2 - 2l,$$

$$\xi = (x - x_1), \qquad \lambda = (x - x_2).$$

For the calculation of this integral (for more detail see [22]) it is necessary to close the integration contour via the lower half-plane (at $\lambda > 0$, $\xi > 0$) and to evaluate an integral over the cut discontinuity along the negative half-axis from the point $u = 0$. As a result we obtain the table integral [73]

$$\mathscr{I} \sim \int_0^\infty \mathrm{d}u \cdot u^{-\delta_3}(u + \xi)^{-\delta_1}(u + \lambda)^{-\delta_2} \sim \frac{1}{\xi}\, {}_2F_1\left(1, \delta_2; 2 - \delta_3; 1 - \frac{\lambda}{\xi}\right).$$

This result is valid for any values of ξ, λ if it is taken with the prescription $\xi \to \xi + i\varepsilon$, $\lambda \to \lambda + i\varepsilon$. Substituting it to (A1.19), we obtain (A1.20).

In conclusion note that the above-said is readily generalized to the case of two-dimensional time space. In order to pass to a two-dimensional space, two variables $x_\pm = \frac{1}{2}(x_0 \pm x_1)$ should be introduced, where x_0 is the time, x_1 is the space coordinate. The conformal group of a two-dimensional space consists of the transformations of type (A1.1),

$$x'_\pm = \frac{\alpha_\pm x_\pm + \beta_\pm}{\gamma_\pm x_\pm + \delta_\pm}, \qquad \alpha_\pm \delta_\pm - \beta_\pm \gamma_\pm = 1.$$

This group is locally isomorphic to the group SO(2, 1) ⊗ SO(2, 1). Changing to the coordinates x_μ, $\mu = 0, 1$, we obtain usual conformal transformations of a two-dimensional space. Irreducible representations are characterized by four numbers [12]: the dimension $d = \frac{1}{2}(l_+ + l_-)$, the spin $s = \frac{1}{2}(l_+ - l_-)$ and two additional numbers $\lambda_\pm$. The appearance of the two numbers $\lambda_\pm$ is due to the fact that in this case the maximal compact subgroup is SO(2) ⊗ SO(2) and the universal covering group of each group SO(2) is infinite-sheeted. An explicit expression for Q-functions analogous to (A1.20) has been found in [22].

Appendix 2. The conformal group in four-dimensional space-time

The generators of infinitesimal conformal transformations form the SO(4, 2) group algebra

$$\begin{aligned}
&[M_{\mu\nu}, M_{\rho\tau}] = i(g_{\mu\tau}M_{\nu\rho} + g_{\nu\rho}M_{\mu\tau} - g_{\mu\rho}M_{\nu\tau} - g_{\nu\tau}M_{\mu\rho}), \\
&[M_{\mu\nu}, P_\rho] = i(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu), \\
&[M_{\mu\nu}, K_\rho] = i(g_{\nu\rho}K_\mu - g_{\mu\rho}K_\nu), \\
&[P_\mu, D] = iP_\mu, \qquad [K_\mu, D] = -iK_\mu, \\
&[P_\mu, K_\nu] = 2i(g_{\mu\nu}D + M_{\mu\nu}), \\
&[P_\mu, P_\nu] = [K_\mu, K_\nu] = [M_{\mu\nu}, D] = 0.
\end{aligned} \tag{A2.1}$$

The group SO(4,2) and its covering group SU(2,2) were examined in several works [35]. Irreducible representations of the universal covering group of SO(4,2) were considered in [9, 35c, 52, 54].

To describe these representations we will use the analogy between the groups SO(4,2) and SO(2,1) comprehensively considered in Appendix 1. The main complications arising in the case of the group SO(4,2) are due to the spin structure of its representations. Taking into consideration the above analogy we will restrict ourselves to a brief review of the main properties of the representations.

Of greatest interest are the representations of type (3.3a), namely $(j_1, j_2) \oplus (j_2, j_1)$ since it is this class of representations to which the representations of the discrete series satisfying spectrality condition (3.3) belong. Casimir operators for the representations (3.3a) take the values [9, 52] (see also [22, 74])

$$\begin{aligned}
&C_2 = (l-2)^2 - 4 + 2j_1(j_1+1) + 2j_2(j_2+1) \\
&C_3 = -(l-2)[j_1(j_1+1) - j_2(j_2+1)] \\
&C_4 = \tfrac{1}{4}(l-2)^4 - (l-2)^2[j_1(j_1+1) + j_2(j_2+1) + 1] + 4j_1j_2(j_1+1)(j_2+1).
\end{aligned} \tag{A2.2}$$

For derivation of (A2.2) the definition of Casimir operators adopted in [35b] was used. As it follows from (A2.2), Casimir operators are invariant under the substitution

$$l \to 4 - l, \qquad j_1 \rightleftarrows j_2. \tag{A2.3}$$

Consider the universal covering group of the conformal group. The maximal compact subgroup of SO(4,2) is

$$\mathrm{SO}(4) \otimes \mathrm{SO}(2). \tag{A2.4}$$

It is infinitesheeted, since it includes the group SO(2). Hence, *the universal covering of the conformal group is also infinitesheeted.* Let Λ be the generator of the group SO(2). As in the above case of the conformal group of a one-dimensional space, the spectrum of the generator Λ is discrete

$$\Lambda = \lambda + m, \tag{A2.5}$$

m are integers. If $\lambda = 0$ we have single-valued representations of the group SO(4,2). For other values of λ the representations are multi- or infinite-valued.

Thus *representations of the universal covering group are classified by the values of four numbers* l, j_1, j_2, λ. Taking into account (A2.3), representations (l, j_1, j_2, λ) and $(l', j_1', j_2', \lambda')$ are equivalent provided that

$$l = 4 - l', \qquad j_1 = j_2', \qquad j_2 = j_1', \qquad \lambda' = \lambda.$$

Consider the coordinate realization of the generators of the conformal group. Introduce a discrete basis in the representation space. In the case of scalar representations $\sigma = (l, 0)$ the basis is [35b] the set of eigen vectors of commutating generators of the maximal compact subgroup (A2.4). Let $|\sigma, \lambda; v, m\rangle$ be the basis vector, where $(\lambda + m)$ is the eigen value (A2.5), v is the set of the rest quantum numbers. Now introduce the coordinate basis analogous to (A1.6)

$$|\sigma, \lambda; x\rangle = \sum_{v,m} f^{\sigma,\lambda}_{v,m}(x)\, |\sigma, \lambda; v, m\rangle, \tag{A2.6}$$

where $f^{\sigma,\lambda}_{v,m}(x) = \langle m, v; \lambda, \tilde{\sigma}|\sigma, \lambda; x\rangle$ is the eigen function of the complete set of commutating generators of the group (A2.4) (its analog in a one-dimensional space is the function (A1.7)). The action of the generators of the conformal group is defined in this basis by the equations

$$\begin{aligned}
&P\,|\sigma, \lambda; x\rangle = \mathrm{i}\partial_\mu\,|\sigma, \lambda; x\rangle, \qquad M_{\mu\nu}\,|\sigma, \lambda; x\rangle = [\mathrm{i}(x_\mu\partial_\nu - x_\nu\partial_\mu) + \Sigma_{\mu\nu}]\,|\sigma, \lambda; x\rangle,\\
&D\,|\sigma, \lambda; x\rangle = -\mathrm{i}(d + x_\nu\partial_\nu)\,|\sigma, \lambda; x\rangle,\\
&K_\mu\,|\sigma, \lambda; x\rangle = -\mathrm{i}(2dx_\mu + 2x_\mu x_\nu\partial_\nu - x^2\partial_\mu + 2\mathrm{i}\Sigma_{\mu\nu}x_\nu)\,|\sigma, \lambda; x\rangle.
\end{aligned} \tag{A2.7}$$

Finite transformations of the vectors (A2.6) explicitely depend on the number λ, see section 3, and has comprehensively been examined for the case of the one-dimensional space. Consider, in particular, the transformation $\mathrm{e}^{\mathrm{i}\psi\Lambda}$ at $\psi = 2\pi n$. Acting with this operator on both parts of the decomposition (A2.6), and taking (A2.5) into account, we find

$$\mathrm{e}^{2\pi\mathrm{i}\Lambda n}\,|\sigma, \lambda; x\rangle = \mathrm{e}^{2\pi\mathrm{i}\lambda n}\,|\sigma, \lambda; x\rangle. \tag{A2.8}$$

Consider the discrete series $D_\pm$. Their characteristic feature consists in limiting the spectrum of momenta by the condition [35b, c]

$$p^2 > 0, \qquad \pm p_0 > 0 \qquad \text{in } D_\pm \text{ series} \tag{A2.9}$$

and in the fixed value of the number λ

$$\lambda = \pm d \qquad \text{in } D_{\pm} \text{ series.} \tag{A2.10}$$

As in Appendix 1, it may be shown that the condition (A2.9) holds, when passing to the representations of the universal covering group, i.e. to any real dimension values. It is essential that due to (A2.9) it is these representations that contribute to field states. In this connection we will be interested only in unitary representations, which are necessary for the positivity of the field theory. The condition of unitarity for the discrete series representations of the universal covering group studied in [9, 52, 54] is of the form

$$l > 1 + j, \quad \text{if} \quad j_1 = j, \quad j_2 = 0 \quad \text{or} \quad j_1 = 0, \quad j_2 = j; \qquad l \geqslant 2 + j_1 + j_2, \quad \text{if} \quad j_1, j_2 \neq 0. \tag{A2.11}$$

Introduce the coordinate basis in the series $D_{\pm}$ representation space. Designate basis vectors via $|\sigma_{\pm}, x\rangle$. Their properties are described in detail in Appendix 1. As in that appendix, one more basis $|\tilde{\sigma}_{\pm}, x\rangle$ may be introduced, where *the generators of the conformal group are non-local*, see (A1.18a)

$$|\tilde{\sigma}_{+}, x\rangle = \int \mathrm{d}y \Delta_{\sigma_+}^{-1}(y - x)\, |\sigma_{+}, y\rangle, \tag{A2.12}$$

where $\Delta_{\sigma_+}(x) \sim (-x^2 + \mathrm{i}\varepsilon x_0)^{-l}$, and $\Delta_{\sigma_+}^{-1}(x)$ is determined by the relationship $\int \mathrm{d}y \Delta_{\sigma_+}(x_1 - y) \times \Delta_{\sigma_+}^{-1}(y - x_2) = I(x_1 - x_2)$, $I(x)$ is the operator of projecting to the positive-frequency states (its Fourier-transform is $\theta(p_0)\theta(p^2)$). Similar to (A1.16), each field state may be associated with two different projections to the basis vectors of the irreducible representations space. Accordingly, *there are two invariant three-point* functions

$$Q^{\tilde{\sigma} d_1 d_2}(x|x_1 x_2) = \langle x, \tilde{\sigma}_{+} |\varphi_{d_1}(x_1)\varphi_{d_2}(x_2)\,|0\rangle, \qquad C^{\sigma d_1 d_2}(x x_1 x_2) = \langle x, \sigma_{+} |\varphi_{d_1}(x_1)\varphi_{d_2}(x_2)\,|0\rangle$$

related by the transformation of type (A1.19). An explicit expression for the $Q^{\tilde{\sigma}}$ function was found in [22], and its Fourier transform in x coordinate is given in section 3, see (3.14). This function enters into the partial wave expansion of field states, since the completeness relation in the space of the irreducible representation of the D_{+} series is of the form

$$\int \mathrm{d}x\, |\sigma_{+}, x\rangle\, \langle x, \tilde{\sigma}_{+}| = I. \tag{A2.13}$$

In conclusion it should be noted that due to (A2.10) the transformation of the vectors $|\sigma_{+}, x\rangle$ by $\mathrm{e}^{2\pi \mathrm{i}\Lambda n}$ depends on the dimension, see (A2.8, 10)

$$\mathrm{e}^{2\pi \mathrm{i}\Lambda n}\, |\sigma_{+}, x\rangle = \mathrm{e}^{2\pi \mathrm{i} l n}\, |\sigma_{+}, x\rangle. \tag{A2.14}$$

The same aquation also holds for the basis $|\tilde{\sigma}_{+}, x\rangle$.

Appendix 3. Basic integral relations

In the present appendix the basic relations required for calculations in sections 5–7 have been collected. Relationship (3.1) was obtained in [32] (see also [5], appendix). The derivation of the relations (3.2) and (3.4) is given in [5], that of the rest in [56]:

$$\int \mathrm{d}x_2\,\mathrm{d}x_3 \frac{1}{(\frac{1}{2}x_{12}^2)^{(l_1+l_2-l_3-s)/2}} \frac{1}{(\frac{1}{2}x_{13}^2)^{(l_1+l_3-l_2-s)/2}} \frac{1}{(\frac{1}{2}x_{23}^2)^{(D-l_1+l_4+2s)/2}} [\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_2x_3) - \text{traces}]$$
$$\times \frac{1}{(\frac{1}{2}x_{24}^2)^{(D-l_4-l_2+l_3-s)/2}} \frac{1}{(\frac{1}{2}x_{43}^2)^{(D-l_4+l_2-l_3-s)/2}} [\lambda^{x_4}_{\nu_1\ldots\nu_s}(x_2x_3) - \text{traces}]$$
$$= \tfrac{1}{2}(2\pi)^D 2^{-s} I_{\mu\nu}\delta(x_1 - x_4)\delta_{\sigma_1\tilde{\sigma}_4}, \tag{A3.1}$$

$$\int \mathrm{d}x_4 \frac{1}{(\frac{1}{2}x_{14}^2)^{\delta_1}} \frac{1}{(\frac{1}{2}x_{24}^2)^{\delta_2}} \frac{1}{(\frac{1}{2}x_{34}^2)^{\delta_3}} [\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_4x_2) - \text{traces}]$$
$$= (2\pi)^h \frac{\Gamma(h-\delta_1)\Gamma(h-\delta_2+s)\Gamma(h-\delta_3)}{\Gamma(\delta_1+s)\Gamma(\delta_2)\Gamma(\delta_3)}$$
$$\times \frac{1}{(\frac{1}{2}x_{23}^2)^{h-\delta_1}} \frac{1}{(\frac{1}{2}x_{13}^2)^{h-\delta_2}} \frac{1}{(\frac{1}{2}x_{12}^2)^{h-\delta_3}} [\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_3x_2) - \text{traces}] \tag{A3.2}$$

where $\delta_1 + \delta_2 + \delta_3 = D$;

$$\int \mathrm{d}x_4 \frac{1}{(\frac{1}{2}x_{14}^2)^{\delta_1}} \frac{1}{(\frac{1}{2}x_{24}^2)^{\delta_2}} \frac{1}{(\frac{1}{2}x_{34}^2)^{\delta_3}} \lambda^{x_2}_{\mu}(x_4x_1)[\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_4x_2) - \text{traces}]$$
$$= (2\pi)^h \frac{\Gamma(h-\delta_1)\Gamma(h-\delta_2+s-1)\Gamma(h-\delta_3)}{\Gamma(\delta_1+s)\Gamma(\delta_2+1)\Gamma(\delta_3)} \frac{1}{(\frac{1}{2}x_{12}^2)^{h-\delta_3}} \frac{1}{(\frac{1}{2}x_{13}^2)^{h-\delta_2}} \frac{1}{(\frac{1}{2}x_{23}^2)^{h-\delta_1}}$$
$$\times \Big\{(h-\delta_1)(h-\delta_2+s-1)\lambda^{x_2}_{\mu}(x_3x_1)[\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_3x_2) - \text{traces}] \tag{A3.3}$$
$$+ \tfrac{1}{2}(h-\delta_3)\sum_k \frac{1}{x_{12}^2}[g_{\mu\mu_k}(x_{12})\lambda^{x_1}_{\mu_1\ldots\hat{\mu}_k\ldots\mu_s}(x_3x_2) - \text{traces}]\Big\}$$

where $\delta_1 + \delta_2 + \delta_3 = D$;

$$\int \mathrm{d}x_4 \frac{1}{(\frac{1}{2}x_{14}^2)^{\delta_1}} \frac{1}{(\frac{1}{2}x_{24}^2)^{\delta_2}} \frac{1}{(\frac{1}{2}x_{34}^2)^{\delta_3}} [q^{x_4}_{\mu_1}(x_1|x_2x_3)\ldots q^{x_4}_{\mu_s}(x_1|x_2x_3) - \text{traces}]$$
$$= (2\pi)^h \frac{\Gamma(h-\delta_1)\Gamma(h-\delta_2)\Gamma(h-\delta_3)\Gamma(D-\delta_1+s-1)}{\Gamma(\delta_1+s)\Gamma(\delta_2+s)\Gamma(\delta_3+s)\Gamma(D-\delta_1-1)} \tag{A3.4}$$
$$\times \frac{1}{(\frac{1}{2}x_{23}^2)^{h-\delta_1}} \frac{1}{(\frac{1}{2}x_{13}^2)^{h-\delta_2-s}} \frac{1}{(\frac{1}{2}x_{12}^2)^{h-\delta_3-s}} [\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_2x_3) - \text{traces}],$$

where $\delta_1 + \delta_2 + \delta_3 + s = D, \qquad q^{x_4}_{\mu}(x_1|x_2x_3) = g_{\mu\nu}(x_{14})\lambda^{x_4}_{\nu}(x_2x_3)$;

$$\int \mathrm{d}x_4 \frac{1}{(\frac{1}{2}x_{14}^2)^{\delta_1}} \frac{1}{(\frac{1}{2}x_{24}^2)^{\delta_2}} \frac{1}{(\frac{1}{2}x_{34}^2)^{\delta_3}} \partial^{x_2}_{\mu}[q^{x_4}_{\mu_1}(x_1|x_2x_3)\ldots q^{x_4}_{\mu_s}(x_1|x_2x_3) - \text{traces}]$$
$$= (2\pi)^h \frac{\Gamma(h-\delta_1)\Gamma(h-\delta_2)\Gamma(h-\delta_3)\Gamma(D-\delta_1+s-1)}{\Gamma(\delta_1+s)\Gamma(\delta_2+s)\Gamma(\delta_3+s)\Gamma(D-\delta_1-1)}$$
$$\times \Big\{-2s\frac{(h-\delta_1)(\delta_3+s-1)}{(D-\delta_1-1)(\delta_2+s)}\lambda^{x_2}_{\mu}(x_1x_3)[\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_2x_3) - \text{traces}] +$$

$$+\frac{(\delta_3+s-1)}{(D-\delta_1-1)(h-\delta_2-1)(\delta_2+s)}[\delta_2(\delta_1-1)+s(h-1)]$$
$$\times\frac{1}{x_{12}^2}\left(\sum_{k=1}^{s} g_{\mu\mu_k}(x_{12})\lambda^{x_1}_{\mu_1\ldots\hat{\mu}_k\ldots\mu_s}(x_2x_3)-\text{traces}\right)\Bigg\}\frac{1}{(\frac{1}{2}x_{12}^2)^{h-\delta_3-s}}\frac{1}{(\frac{1}{2}x_{13}^2)^{h-\delta_2-s}}\frac{1}{(\frac{1}{2}x_{23}^2)^{h-\delta_1}}, \quad \text{(A3.5)}$$

where $\delta_1+\delta_2+\delta_3+s=D$;

$$\int \mathrm{d}x_4\frac{1}{(\frac{1}{2}x_{14}^2)^{\delta_1}}\frac{1}{(\frac{1}{2}x_{24}^2)^{\delta_2}}\frac{1}{(\frac{1}{2}x_{34}^2)^{\delta_3}}\lambda^{x_2}_{\mu}(x_4x_3)[q^{x_4}_{\mu_1}(x_1|x_2x_3)\ldots q^{x_4}_{\mu_s}(x_1|x_2x_3)-\text{traces}]$$
$$=(2\pi)^h\frac{\Gamma(h-\delta_1)\Gamma(h-\delta_2)\Gamma(h-\delta_3)\Gamma(D-\delta_1+s-1)}{\Gamma(\delta_1+s)\Gamma(\delta_2+s)\Gamma(\delta_3+s)\Gamma(D-\delta_1-1)}$$
$$\times\Bigg\{\frac{(D-\delta_1-1)(h-\delta_3-s)+s(h-1)}{(D-\delta_1-1)(\delta_2+s)}\lambda^{x_2}_{\mu}(x_1x_3)[\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_2x_3)-\text{traces}]$$
$$+\frac{1}{2}\frac{(h-\delta_1)(\delta_3-\delta_2-1)}{(D-\delta_1-1)(h-\delta_2-1)(\delta_2+s)}\frac{1}{x_{12}^2}\left[\sum_{k=1}^{s} g_{\mu\mu_k}(x_{12})\lambda^{x_1}_{\mu_1\ldots\hat{\mu}_k\ldots\mu_s}(x_2x_3)-\text{traces}\right]\Bigg\}$$
$$\times\frac{1}{(\frac{1}{2}x_{23}^2)^{h-\delta_1}}\frac{1}{(\frac{1}{2}x_{13}^2)^{h-\delta_2-s}}\frac{1}{(\frac{1}{2}x_{12}^2)^{h-\delta_3-s}}, \quad \text{(A3.6)}$$

where $\delta_1+\delta_2+\delta_3+s=D$;

$$\int \mathrm{d}x_4\frac{1}{(\frac{1}{2}x_{14}^2)^{\delta_1}}\frac{1}{(\frac{1}{2}x_{24}^2)^{\delta_2}}\frac{1}{(\frac{1}{2}x_{34}^2)^{\delta_3}}q^{x_4}_{\mu}(x_2|x_1x_3)[\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_4x_3)-\text{traces}]$$
$$=(2\pi)^h\frac{\Gamma(h-\delta_1-1)\Gamma(h-\delta_2)\Gamma(h-\delta_3+s-1)}{\Gamma(\delta_1+s+1)\Gamma(\delta_2+1)\Gamma(\delta_3+1)}\frac{1}{(\frac{1}{2}x_{12}^2)^{h-\delta_3-1}}\frac{1}{(\frac{1}{2}x_{13}^2)^{h-\delta_2}}\frac{1}{(\frac{1}{2}x_{23}^2)^{h-\delta_1-1}}$$
$$\times\Big\{(h-\delta_1-1)[(D-\delta_2-1)(h-\delta_3-1)+s(h-1)]\lambda^{x_2}_{\mu}(x_1x_3)(\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_2x_3)-\text{traces})$$
$$+\tfrac{1}{2}(\delta_3-\delta_1-s)(h-\delta_2)\left[\frac{1}{x_{12}^2}\sum_{k=1}^{s} g_{\mu\mu_k}(x_{12})\lambda^{x_1}_{\mu_1\ldots\hat{\mu}_k\ldots\mu_s}(x_2x_3)-\text{traces}\right]\Big\}, \quad \text{(A3.7)}$$

where $\delta_1+\delta_2+\delta_3=D-1$,

$$q^{x_4}_{\mu}(x_2|x_1x_3)=g_{\mu\nu}(x_{24})\lambda^{x_4}_{\nu}(x_1x_3)=\frac{x_{12}^2}{x_{14}^2}\lambda^{x_2}_{\mu}(x_1x_4)+\frac{x_{23}^2}{x_{34}^2}\lambda^{x_2}_{\mu}(x_4x_3);$$

$$\int \mathrm{d}x_4\frac{1}{(\frac{1}{2}x_{14}^2)^{\delta_1}}\frac{1}{(\frac{1}{2}x_{24}^2)^{\delta_2}}\frac{1}{(\frac{1}{2}x_{34}^2)^{\delta_3}}\frac{1}{x_{24}^2}\left[\sum_{k=1}^{s} g_{\mu\nu}(x_{24})g_{\nu\mu_k}(x_{14})\lambda^{x_1}_{\mu_1\ldots\hat{\mu}_k\ldots\mu_s}(x_4x_3)-\text{traces}\right]$$
$$=(2\pi)^h\frac{\Gamma(h-\delta_1)\Gamma(h-\delta_2)\Gamma(h-\delta_3+s-1)}{\Gamma(\delta_1+s-1)\Gamma(\delta_2)\Gamma(\delta_3)}\frac{1}{(\frac{1}{2}x_{12}^2)^{h-\delta_3-1}}\frac{1}{(\frac{1}{2}x_{13}^2)^{h-\delta_2}}\frac{1}{(\frac{1}{2}x_{23}^2)^{h-\delta_1}}$$
$$\times\Big\{-s\frac{(h-\delta_1)}{(\delta_1+s-1)}\lambda^{x_2}_{\mu}(x_1x_3)[\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_2x_3)-\text{traces}]+ \quad \text{(A3.8)}$$

$$+\frac{1}{2}\,\frac{\delta_2(h-\delta_3)+(h-1)(s-1)}{\delta_2(\delta_1+s-1)}\,\frac{1}{x_{12}^2}\sum_{k=1}^{s}\left[g_{\mu\mu_k}(x_{12})\lambda^{x_1}_{\mu_1\ldots\hat{\mu}_k\ldots\mu_s}(x_2x_3)-\text{traces}\right]\Big\},$$

where $\delta_1+\delta_2+\delta_3=D-1$,

$$g_{\mu\nu}(x_{24})g_{\nu\mu_k}(x_{14})=g_{\mu\mu_k}(x_{12})-2x_{12}^2\lambda^{x_2}_{\mu}(x_1x_4)\lambda^{x_1}_{\mu_k}(x_2x_4);$$

$$\int \mathrm{d}x_4\,\frac{1}{(\frac{1}{2}x_{34}^2)^{\delta_1}}\,\frac{1}{(\frac{1}{2}x_{24}^2)^{\delta_2}}\,\frac{1}{(\frac{1}{2}x_{14}^2)^{\delta_3}}\,\lambda^{x_2}_{\mu}(x_3x_4)[\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_3x_4)-\text{traces}] \tag{A3.9}$$

$$=(2\pi)^h\,\frac{\Gamma(h-\delta_1+s)\Gamma(h-\delta_2)\Gamma(h-\delta_3)}{\Gamma(\delta_1)\Gamma(\delta_2)\Gamma(\delta_3+s)}\,\frac{1}{2\delta_2}\,\frac{1}{(\frac{1}{2}x_{12}^2)^{h-\delta_1}}\,\frac{1}{(\frac{1}{2}x_{13}^2)^{h-\delta_2}}\,\frac{1}{(\frac{1}{2}x_{23}^2)^{h-\delta_3}}$$

$$\times\Big\{2(h-\delta_1)\lambda^{x_2}_{\mu}(x_3x_1)[\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_3x_2)-\text{traces}]$$

$$+\frac{1}{x_{12}^2}\Big[\sum_{k=1}^{s}g_{\mu\mu_k}(x_{12})\lambda^{x_1}_{\mu_1\ldots\hat{\mu}_k\ldots\mu_s}(x_3x_2)-\text{traces}\Big]\Big\},$$

where $\delta_1+\delta_2+\delta_3=D$.

Appendix 4. Amputation condition for three-point functions of scalar, vector and tensor

Represent a three-point function $\{A,B\}$ in the form

$$\{A,B\}=AC_1^{\sigma_1\sigma_2d_3}(x_1x_2x_3)+BC_2^{\sigma_1\sigma_2d_3}(x_1x_2x_3), \tag{A4.1}$$

where C_1 and C_2 are determined in (2.10). Designate via $\{\tilde{A}_i,\tilde{B}_i\}$ the function $\{A,B\}$ amputated in the ith leg ($i=1,2,3$)

$$\{\tilde{A}_i,\tilde{B}_i\}=\tilde{A}_i\tilde{C}^{\sigma_1\sigma_2d_3}_{1,i}+\tilde{B}_i\tilde{C}^{\sigma_1\sigma_2d_3}_{2,i},$$

where $\tilde{C}_{1,i}$ and $\tilde{C}_{2,i}$ are obtained from C_1 and C_2 by the substitution $d_i\to D-d_i$. The coefficients $\tilde{A}_i$ and $\tilde{B}_i$ may be expressed via A, B

$$\tilde{A}_i=\alpha_iA+\beta_iB,\qquad \tilde{B}_i=\gamma_iA+\delta_iB, \tag{A4.2}$$

where α_i, β_i, γ_i and δ_i are determined by the relations

$$\int \mathrm{d}x\Delta^{-1}_{\sigma_1}(x_1-x)C_1^{\sigma_1\sigma_2d_3}(xx_2x_3)=\alpha_1C_1^{\tilde{\sigma}_1\sigma_2d_3}+\gamma_1C_2^{\tilde{\sigma}_1\sigma_2d_3} \tag{A4.3a}$$

and analogously in the general case

$$\int\Delta^{-1}_{\sigma_i}C_1=\alpha_i\tilde{C}_{1,i}+\gamma_i\tilde{C}_{2,i},\qquad \int\Delta^{-1}_{\sigma_i}C_2=\beta_i\tilde{C}_{1,i}+\delta_i\tilde{C}_{2,i}. \tag{A4.3}$$

The integrals appeared are calculated using the relationships of Appendix 3. From (A3.5, 6) we have

$$\alpha_1=\Gamma_1\,\frac{(d_1-1)(D-d_1+d_2-d_3-s-1)+s(D-2)}{(d_1-1)(D-d_1+d_2-d_3+s-1)},$$

$$\beta_1 = \Gamma_1 \frac{2s(h - d_1)(d_1 + d_3 - d_2 + s - 1)}{(d_1 - 1)(D - d_1 + d_2 - d_3 + s - 1)},$$

$$\delta_1 = \Gamma_1 \frac{(d_1 + d_3 - d_2 + s - 1)[(D - d_1 - 1)(d_1 + d_2 - d_3 - s - 1) + s(D - 2)]}{(d_1 - 1)(D - d_1 - d_2 + d_3 + s - 1)(D - d_1 + d_2 - d_3 + s - 1)},$$

$$\gamma_1 = \Gamma_1 \frac{2(d_1 - h)(d_3 - d_2)}{(d_1 - 1)(D - d_1 - d_2 + d_3 + s - 1)(D - d_1 + d_2 - d_3 + s - 1)}$$

where

$$\Gamma_1 = \frac{\Gamma((D - d_1 - d_2 + d_3 + s + 1)/2)\Gamma((D - d_1 + d_2 - d_3 + s + 1)/2)}{\Gamma((d_1 - d_2 + d_3 + s + 1)/2)\Gamma((d_1 + d_2 - d_3 + s + 1)/2)}.$$

Using (A3.7, 8), we find that $\alpha_2 \ldots \delta_2$ are obtained from $\alpha_1 \ldots \delta_1$ by the substitution $d_1 \rightleftarrows d_2$. Finally, using (A3.3) we find

$$\alpha_3 = \Gamma_3 \frac{D - d_1 + d_2 - d_3 + s - 1}{d_2 + d_3 - d_1 + s - 1} \cdot \frac{D - d_2 - d_3 + d_1 + s - 1}{d_1 - d_2 + d_3 + s - 1}, \qquad \beta_3 = 0$$

$$\gamma_3 = \Gamma_3 \frac{2(d_3 - h)}{(d_1 - d_2 + d_3 + s - 1)(d_2 + d_3 - d_1 + s - 1)}, \qquad \delta_3 = 1$$

where Γ_3 is obtained from Γ_1 by the substitution $d_1 \rightleftarrows d_3$ and $s \to s - 2$.

Now let us find the three-point function $\{A, B\}$ satisfying the amputation condition. For this it is necessary to find the expressions for the coefficients $A = A(d_1 d_2 d_3)$, $B = B(d_1 d_2 d_3)$ such as the result of amputation of the function $\{A(d_1 d_2 d_3), B(d_1 d_2 d_3)\}$ in the ith leg would coincide with the formal substitution $d_i \to D - d_i$. For example, the condition of amputation in a tensor leg implies that

$$A(D - d_1, d_2, d_3) = \tilde{A}_1(d_1 d_2 d_3), \qquad B(D - d_1, d_2, d_3) = \tilde{B}_1(d_1 d_2 d_3)$$

and analogously in scalar and vector legs. Taking (A4.2) into consideration, we obtain a set of functional equations for A and B. The solution of these equations has been considered in [56]. Let, in particular, a general expression for the function $\{A, B\}$ be given, which satisfies the condition of amputation over both scalar and tensor legs

$$\begin{aligned} \{A, B\} = {} & \Gamma\left(\frac{d_2 + d_3 - d_1 + s + 1}{2}\right)\Gamma\left(\frac{d_3 - d_2 + d_1 + s - 1}{2}\right)\Gamma\left(\frac{d_1 - d_3 + d_2 + s + 1}{2}\right) \\ & \times \Gamma\left(\frac{d_1 + d_2 + d_3 - D + s + 1}{2}\right) \\ & \times \left[K_1(d_1 d_2 d_3)\{d_2 - d_1 - d_3 - s + 1, 1\} + K_2(d_1 d_2 d_3) \frac{(d_2 - d_3)}{(D - d_1 - 1)} \right. \\ & \left. \times \left\{ \frac{(D - d_1 + s - 1)(d_1 + d_3 - d_2 + s - 1)}{(d_2 - d_3)}, 1 \right\} \right]; \end{aligned}$$

where K_1 and K_2 are the arbitrary functions which are invariant under each of the substitutions $d_1 \to D - d_1$ and $d_3 \to D - d_3$.

Appendix 5. Current leg amputation. Calculation of normalization factors for the functions (6.24, 25) and (6.29, 30)

Take in (A4.2) at $i = 2$ the conserved current dimension: $d_2 = D - 1$. For the coefficients $\alpha_2 \ldots \delta_2$ we have (at $s \neq 0$)

$$a(d_1 d_2 d_3)|_{d_2 = D-1} = \begin{bmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{bmatrix}_{d_2 = D-1}$$

$$= \frac{1}{2} \frac{\Gamma((d_3 - d_1 + s + 2)/2)\Gamma((d_1 - d_3 + s)/2)}{\Gamma((D - d_1 + d_3 + s)/2)\Gamma((D + d_1 - d_3 + s)/2)}$$

$$\times \begin{bmatrix} d_1 - d_3\,, & -s(D - 2 - d_1 + d_3 + s) \\ \dfrac{d_1 - d_3}{d_1 - d_3 - s}, & -s\dfrac{(D - 2 - d_1 + d_3 + s)}{(d_1 - d_3 - s)} \end{bmatrix}. \quad \text{(A5.1)}$$

Note that $\det a = 0$, hence the matrix (A5.1) has an eigen vector corresponding to eigen value zero. It can readily be seen, that this vector is associated with the transverse function $\{s(D - 2 - d_1 + d_3 + s)/(d_1 - d_3), 1\}$, see (6.25). Thus, the amputation of the transverse function with respect to the current leg reduces it to zero. This is due to the inverse current propagator, determined by the limit $l \to \tilde{d}_j = 1$ of expressions (2.5) at $s = 1$, proves to be the longitudinal one: $\Delta_{\sigma_j}^{-1}(x) \sim g_{\mu\nu}(x)/x^2 = \frac{1}{2}\partial_\mu \partial_\nu \ln x^2$. From this it follows that the amputation of any function $\{A, B\}$ at $d_2 = D - 1$ with respect to the current argument leads to the universal (up to a factor) result

$$\{\tilde{A}_2, \tilde{B}_2\}|_{d_2 = D-1} \sim \{d_1 - d_3 - s, 1\}. \quad \text{(A5.2)}$$

Consider now the amputation of the functions $\{A, B\}$ for $d_2 = 1$. For this purpose it is necessary to consider the matrix which is inverse to (A5.1). Since $\det a = 0$, matrix a^{-1} is singular, it should be determined as the matrix $a(d_1 d_2 d_3)$ in the limit $d_2 \to 1$. Assuming that $d_2 = 1 + \varepsilon$, we have

$$a^{-1} = \frac{1}{\varepsilon}(h - 1)\frac{\Gamma((D - d_1 + d_3 + s)/2)\Gamma((D - s + d_1 - d_3 + s)/2)}{\Gamma((d_3 - d_1 + s + 2)/2)\Gamma((d_1 - d_3 + s + 2)/2)}$$

$$\times \begin{bmatrix} s\,, & -s(d_1 - d_3 - s) \\ \dfrac{d_1 - d_3}{D - 2 - d_1 + d_3 + s}, & -\dfrac{(d_1 - d_3)(d_1 - d_3 - s)}{D - 2 - d_1 + d_3 + s} \end{bmatrix}, \quad \text{(A5.3)}$$

where $\varepsilon \to 0$. At $\varepsilon \neq 0$ the determinant of this matrix is equal to zero, and the eigen vector with a zero eigen value is the vector (A5.2). Thus, the amputation of the function (A5.2) leads to the indefinite quantity $0 \times \infty$. Resolving it, we will obtain a finite result. The amputation of any other function gives the infinite result: $\{s(D - 2 - d_1 + d_3 + s)/(d_1 - d_3), 1\}/\varepsilon$.

It now follows, that the normalization coefficients of the three-point functions are singular, provided that the amputation condition is met. The internal current leg integrals include an indefinite $0 \times \infty$ factor and require a current dimension regularization. This, however, requires cumbersome calculations. Hence it is appropriate instead of a usual set of three-point functions, satisfying the current argument amputation conditions, to introduce auxiliary functions (6.29, 30), determined by the conditions (6.28). For the calculation of the normalization coefficients of these functions, one uses the relations [56]

$$\int dx_2\, dx_3 C_1^{\sigma\sigma_2 d_3}(xx_2x_3) C_1^{\tilde\sigma'\tilde\sigma_2\tilde d_3}(x'x_2x_3) = \frac{1}{s}\int dx_2\, dx_3 C_1^{\sigma\sigma_2 d_3}(xx_2x_3) C_2^{\tilde\sigma'\tilde\sigma_2\tilde d_3}(x'x_2x_3)$$
$$= (2\pi)^D 2^{-s-1}\tfrac{1}{2}\delta_{\sigma\tilde\sigma'}\delta(x-x'),$$

$$\int dx_2\, dx_3 C_2^{\sigma\sigma_2 d_3}(xx_2x_3) C_2^{\tilde\sigma'\tilde\sigma_2\tilde d_3}(x'x_2x_3) = (2\pi)^D 2^{-s-1} s(D+2s-2)\tfrac{1}{2}\delta_{\sigma\tilde\sigma'}\delta(x-x'),$$

where $C_1^{\sigma_1\sigma_2 d_3}$ and $C_2^{\sigma_1\sigma_2 d_3}$ are the functions entering into (A4.1), $C_1^{\tilde\sigma_1\tilde\sigma_2\tilde d_3}$ and $C_2^{\tilde\sigma_1\tilde\sigma_2\tilde d_3}$ are obtained from them by the substitution $d_i \to D - d_i$, $i = 1, 2, 3$; $\delta_{\sigma\sigma'}$ is determined by the condition $\sum_{\sigma'}\delta_{\sigma\sigma'} f(\sigma') = f(\sigma)$, and $\sum_\sigma$ is given in (5.1a).

Appendix 6. Spinor fields. Yukawa model

All the results concerning scalar and tensor fields may readily be generalized to the case of spinor fields. However, there are several problems which require special consideration. We will restrict ourselves to the discussion of spinor fields $\Psi_d(x)$ interacting with the pseudoscalar field $\Phi_\delta(x)$ (Yukawa model)

$$L_{\text{int}} = \lambda\bar\Psi\gamma_5\Psi\Phi. \tag{A6.1}$$

All calculations given below refer to the practically important case of a four-dimensional space where the coupling constant is dimensionless. We will discuss (using the results of refs. [5,6]) the partial wave expansion of vertices and Green functions including spinor fields, the dynamical equations and the solution of the Ward identity.

The graphical form of dynamical equations for the interaction (A6.1) is given in section 8. Solid and dashed lines in (8.3–8.7) should be associated now with the spinor field Ψ_d and pseudoscalar field Φ_δ, respectively. In principle the results of sections 9–13 and 16 can be applied to any three-linear interaction, including (A6.1) as well. Certain changes arise however due to the more complicated structure of the Green functions for spinor fields. They refer primarily to the form of partial wave expansions and the Ward identities.

The spinor field dimension is limited by the positivity condition, see (A2.11), to

$$d > \tfrac{3}{2}.$$

A6.1. Two-point and three-point Green functions

The two-point function of spinor fields is equal to

$$G_d(x_{12}) = \langle 0|\, T\Psi_d(x_1)\bar\Psi_d(x_2)\,|0\rangle = -\frac{i}{(2\pi)^2}\frac{\Gamma(d+\frac{1}{2})}{\Gamma(\frac{5}{2}-d)}\frac{\hat x_{12}}{(\frac{1}{2}x_{12}^2)^d} \tag{A6.2}$$

where

$$\hat x_{12} = \frac{(x_{12})_\mu}{(x_{12}^2)^{1/2}}\gamma_\mu. \tag{A6.3}$$

Its normalization is determined by the condition

$$G_d^{-1}(x) = G_{4-d}(x). \tag{A6.4}$$

Introduce the auxiliary functions

$$S_d(x) = \Gamma(d+\tfrac{1}{2})\frac{\hat{x}}{(\frac{1}{2}x^2)^d}, \qquad \tilde{\Delta}_\delta(x) = \Gamma(\delta)(\tfrac{1}{2}x^2)^{-\delta}.$$

The general expression for the conformal-invariant three-point function is

$$C^{ld\delta}(x_1x_2x_3) = C_+^{ld\delta}(x_1x_2x_3) + C_-^{ld\delta}(x_1x_2x_3) \tag{A6.5}$$

where

$$\gamma_5 C_\pm \gamma_5 = \pm C_\pm, \tag{A6.6}$$

$$C_+^{ld\delta}(x_1x_2x_3) = \frac{1}{2}\frac{1}{(2\pi)^2}\tilde{N}(ld\delta)\Gamma\left(\frac{l+d+\delta-4}{2}\right)S_{(l-d+\delta)/2}(x_{13})\gamma_5 S_{(d-l+\delta)/2}(x_{32})\tilde{\Delta}_{(l+d-\delta)/2}(x_{12}), \tag{A6.7}$$

$$C_-^{ld\delta}(x_1x_2x_3) = \frac{\mathrm{i}}{2}\frac{1}{(2\pi)^2}\tilde{N}(ld\delta)\Gamma\left(\frac{l+d+\delta-3}{2}\right)S_{(l+d-\delta)/2}(x_{12})\gamma_5\tilde{\Delta}_{(l-d+\delta)/2}(x_{13})\tilde{\Delta}_{(d-l+\delta)/2}(x_{23}), \tag{A6.8}$$

where

$$\tilde{N}(ld\delta) = \left\{\Gamma\left(\frac{5-l+d-\delta}{2}\right)\Gamma\left(\frac{5-d+l-\delta}{2}\right)\Gamma\left(\frac{l+d-\delta}{2}\right)\Gamma\left(\frac{4-l-d+\delta}{2}\right)\right. \tag{A6.9}$$
$$\left.\times\,\Gamma\left(\frac{l+d+\delta-4}{2}\right)\Gamma\left(4-\frac{l+d+\delta}{2}\right)\Gamma\left(\frac{l-d+\delta+1}{2}\right)\Gamma\left(\frac{d-l+\delta+1}{2}\right)\right\}^{-1/2}.$$

The normalization factors* are chosen so that the amputation conditions

$$\int \mathrm{d}x G_l^{-1}(x_1-x)C^{ld\delta}(xx_2x_3) = C^{4-l,d,\delta}(x_1x_2x_3),$$
$$\int \mathrm{d}x C^{ld\delta}(x_1xx_3)G_d^{-1}(x-x_2) = C^{l,4-d,\delta}(x_1x_2x_3); \tag{A6.10}$$

$$\int \mathrm{d}x \Delta_\delta^{-1}(x_3-x)C^{ld\delta}(x_1x_2x) = C^{l,d,4-\delta}(x_1x_2x_3) \tag{A6.11}$$

would be met. Note that each of the two terms in the left-hand side of (A6.5) does not satisfy the conditions (A6.10). For the functions $C_\pm$ we have

$$\int \mathrm{d}x G_l^{-1}(x_1-x)C_\pm^{ld\delta}(xx_2x_3) = C_\mp^{4-l,d,\delta}(x_1x_2x_3),$$
$$\int \mathrm{d}x C_\pm^{ld\delta}(x_1xx_3)G_d^{-1}(x-x_3) = -C_\mp^{l,4-d,\delta}(x_1x_2x_3). \tag{A6.12}$$

* Here the normalization which is used differs from that of refs. [5, 6].

The explicit expression (A6.9) is obtained from the orthogonality relation, see below. For calculations in (A6.10–12) we have used (A3.2) at $s = 0$ and the relation [40]

$$\int dx_4 S_{\delta_1}(x_{14}) S_{\delta_2}(x_{42}) \tilde{\Delta}_{\delta_3}(x_{34}) = (2\pi)^2 S_{2-\delta_2}(x_{13}) S_{2-\delta_1}(x_{32}) \tilde{\Delta}_{2-\delta_3}(x_{12}) \tag{A6.12a}$$

where $\delta_1 + \delta_2 + \delta_3 = 4$.

Consider the function with the current-dimension vector. Its general γ_5-invariant expression is

$$\begin{aligned} C_\mu^{ld\sigma_j}(x_1x_2x_3) = & \frac{1}{\sqrt{2}}[1 + F(l,d)] S_{3/2+(l-d)/2}(x_{13}) \gamma_\mu S_{3/2-(l-d)/2}(x_{32}) \tilde{\Delta}_{(l+d-3)/2}(x_{12}) \\ & + 2\left[1 - \left(\frac{l-d}{2}\right)^2\right](l+d-3)^{-1}[F(l,d) - 1] S_{(l+d-2)/2}(x_{12}) \tilde{\Delta}_{(l-d+2)/2}(x_{13}) \tilde{\Delta}_{(d-l+2)/2}(x_{23}) \lambda_\mu^{x_3}(x_1x_2) \end{aligned} \tag{A6.13}$$

and may be represented as a sum of two terms $C_\mu = C_{\mu,1} + C_{\mu,2}$, where $C_{\mu,2}$ is the transverse function: $\partial_\mu^{x_3} C_{\mu,2}(x_1x_2x_3) = 0$. The function $C_{\mu,2}$ consists of terms proportional to $F(l,d)$. Calculating the divergence from (A6.13), we have

$$\partial_\mu^{x_3} C_\mu^{ld\sigma_j}(x_1x_2x_3) = -(l-d) S_{(l+d-4)/2}(x_{12}) \tilde{\Delta}_{(4+l-d)/2}(x_{13}) \tilde{\Delta}_{(4+d-l)/2}(x_{23}). \tag{A6.14}$$

Consider a three-point Green functions. Since interaction (A6.1) is γ_5 invariant, we have

$$G(x_1x_2x_3) = \langle 0|\, T\Psi_{d_1}(x_1)\bar{\Psi}_{d_2}(x_2)\Phi_\delta(x_3)\, |0\rangle = g C_+^{d_1d_2\delta}(x_1x_2x_3), \tag{A6.15}$$

where g is the coupling constant

$$\begin{aligned} G_\mu(x_1x_2x_3) &= \langle 0|\, T\Psi_d(x_1)\bar{\Psi}_d(x_2) j_\mu(x_3)\, |0\rangle \\ &= g_j\left\{\frac{1}{\sqrt{2}}(1 + f_j) S_{3/2}(x_{13}) \gamma_\mu S_{3/2}(x_{32}) \tilde{\Delta}_{d-3/2}(x_{12}) \right. \\ &\qquad \left. + (d - \tfrac{3}{2})^{-1}(f_j - 1) S_{d-1}(x_{12}) \tilde{\Delta}_1(x_{13}) \tilde{\Delta}_1(x_{23}) \lambda_\mu^{x_3}(x_1x_2)\right\} \end{aligned} \tag{A6.16}$$

where g_j and f_j are the coupling constants. The constant g_j is determined from the Ward identity

$$\partial_\mu^{x_3} G_\mu(x_1x_2x_3) = -ie[\delta(x_1 - x_3) - \delta(x_2 - x_3)] G_d(x_{12}) \tag{A6.17}$$

and it is equal to

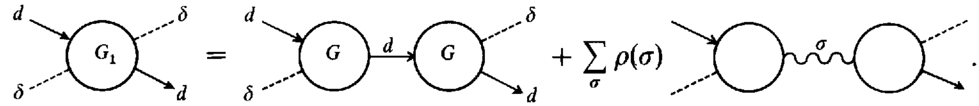
$$g_j(d) = \tfrac{1}{2}(2\pi)^{-4} e \Gamma(d + \tfrac{1}{2})\{\Gamma(d - \tfrac{3}{2})\Gamma(\tfrac{5}{2} - d)\}^{-1}. \tag{A6.18}$$

A6.2. Partial wave expansion

The derivation of a partial wave expansion given in sections 3–5 is completely applicable in the case of spinor fields. The only change refers to the first term in (4.6). Consider, e.g., the Green function

$$G_1(x_1x_2x_3x_4) = \langle 0|\, T\Psi_d(x_1)\bar{\Psi}_d(x_2)\Phi_\delta(x_3)\Phi_\delta(x_4)\, |0\rangle. \tag{A6.19}$$

Write down its partial wave expansion in the form



(A6.20)

In contrast to (4.6), the internal line in the first term is associated with the inverse propagator of the spinor field. According to section 4 this term is present in (A6.20) only in the case when the spinor field dimension falls into the interval $\frac{3}{2} < d < 2$. This term is absent if $d > 2$. Consider now the second term. In contrast with (5.1, 2) it is contributed by the spin-tensor representations

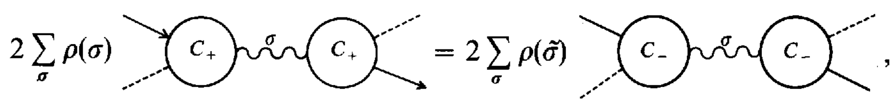
$$\sum_{\sigma} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \mu_j(l)\, dl \qquad \text{(A6.21)}$$

where $j = (j_1, j_2) \oplus (j_2, j_1)$, $|j_1 - j_2| = \frac{1}{2}$. We will restrict ourselves to the discussion of the spinor contribution $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ to (A6.20). The three-point functions (A6.5) entering into (A6.20) include a γ_5 noninvariant part.

Now change to the γ_5 invariant Green function

$$G_1 \to \tfrac{1}{2}(G_1 + \gamma_5 G_1 \gamma_5). \qquad \text{(A6.22)}$$

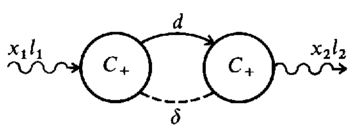
As it follows from (A6.6), (A6.20) involves only terms of the type $C^l_+ G_l^{-1} C^l_+$ and $C^l_- G_l^{-1} C^l_-$. From (A6.12) and (A6.22) it follows that these terms are equal. Thus the expansion (A6.20) of the γ_5 invariant function G_1 may be written in the form



(A6.23)

where $\rho(\sigma) \neq \rho(\tilde{\sigma})$.

The orthogonality relation for the function C_+ is



$$= \tfrac{1}{2}\delta^{(1/2)}_{l_1 l_2} G_d(x_1 - x_2), \qquad \text{(A6.24)}$$

where $\delta^{(1/2)}_{l_1 l_2}$ is determined by the condition

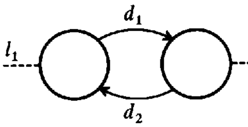
$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} dl_1 \mu_{1/2}(l_1) f(l_1) \delta_{l_1 l_2} = f(l_2).$$

In the normalization (A6.7–9) the function $\mu_{1/2}(l)$ is equal to

$$\mu_{1/2}(l) = \tfrac{1}{4}(2\pi)^{-2} \frac{\Gamma(l + \frac{1}{2})\Gamma(\frac{9}{2} - l)}{\Gamma(\frac{5}{2} - l)\Gamma(l - \frac{3}{2})}. \qquad \text{(A6.25)}$$

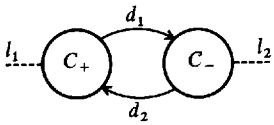
Relation (A6.12a) was used for its calculations. The orthogonality relation for the functions C_- may be obtained from (A6.24) using (A6.12). The orthogonality relation for the scalar leg for the

the functions (A6.5) is of the form



$$= \tfrac{1}{2}\{\delta^{(0)}_{l_1 l_2}\Delta_{l_2}(x_{12}) + \delta^{(0)}_{l_1,4-l_2}\delta(x_1 - x_2)\}.$$

Note that, due to the summation over the spinor indices of the internal lines



$$= 0.$$

A6.3. Ward identity

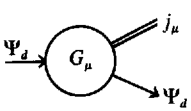
Consider the Green function $(d, d_1 > 2)$

$$G_\mu(x_1x_2x_3x_4) = \langle 0|\, T\Psi_{d_1}(x_1)\bar{\Psi}_d(x_2)\Phi_\delta(x_3)j_\mu(x_4)\,|0\rangle \tag{A6.26}$$

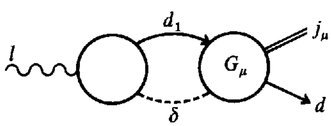
and find the function $\rho_{j_\mu}(\sigma)$ corresponding to the spinor contribution in (A6.26). For this purpose one substitutes the expansion of (A6.26) into the Ward identity. Using the orthogonality relation (A6.24), we find [5, 6]

$$\begin{aligned}\rho_{j_\mu}(l) = &-\tfrac{1}{4}(2\pi)^{-2}eg\tilde{N}(d_1 d\delta)\tilde{N}(ld\delta)\cdot\frac{1}{l-d}\\ &\times\Gamma\left(\frac{8-d_1-d-\delta}{2}\right)\Gamma\left(\frac{4-d-d_1+\delta}{2}\right)\Gamma\left(\frac{5-d_1+d-\delta}{2}\right)\Gamma\left(\frac{d-d_1+\delta+1}{2}\right)\\ &\times\frac{\Gamma((l+d_1+\delta-4)/2)\Gamma((l+d_1-\delta)/2)\Gamma((d_1-l+\delta+1)/2)\Gamma((5-l+d_1-\delta)/2)}{\Gamma((4-l+d)/2)\Gamma((4+l-d)/2)\Gamma((9-l-d)/2)}.\end{aligned} \tag{A6.27}$$

The dynamical equation for Green function (A6.26) is of the form



$$= \Lambda \operatorname*{res}_{l=d} \tag{A6.28}$$



where $g\Lambda = -2\mu_{1/2}(d)$. (A6.29)

Substituting the Ward identity solution into it, we find

$$\Lambda(d)\operatorname*{res}_{l=d}\rho_{j_\mu}(l) = g_j(d), \qquad F(d,d) = f_j.$$

Taking into account (A6.29) the former relation is fulfilled identically and the latter relates the value of the transverse part of the expansion of (A6.27) at $l = d$ to the transverse coupling constant. Note that in the case of scalar fields the equation analogous to (A6.28) is fulfilled identically, since it does not comprise a transverse part.

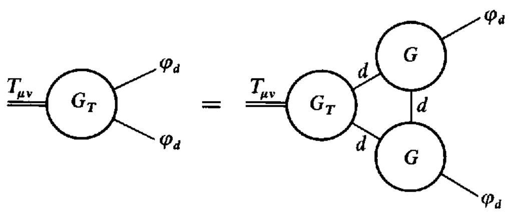
A6.4. Exact relation between fundamental field's dimensions in the Yukawa model

In conclusion let us write the relation, connecting the dimensions d and δ in the case of Yukawa interaction. Similarly to section 16 one gets (for D-dimensional space)

$$\frac{\Gamma((D-\delta+1)/2)\Gamma(\delta-h)}{\Gamma((\delta+1)/2)\Gamma(D-\delta)} = D\cdot\frac{\Gamma((D-2d+\delta)/2)\Gamma(d-h+\frac{1}{2})}{\Gamma((2d-\delta)/2)\Gamma(D-d+\frac{1}{2})}.$$

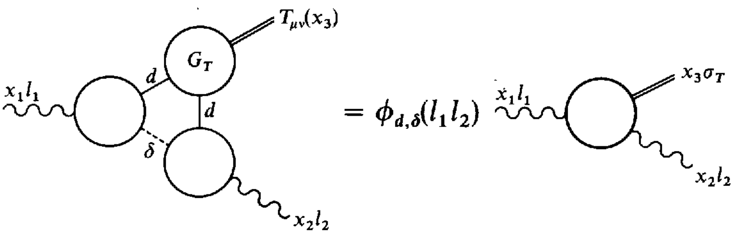
Appendix 7. Dependence between coupling constant and field dimension in 3-vertices approximation

Let us illustrate the above bootstrap program by an example of the theory $\lambda\Phi^3$ in the D-dimentional space. Such a theory comprises two unknown constants: a field dimension d and a coupling constant g. In 3-vertices approximation we have two equations, see (8.10). The Green function equation containing energy-momentum tensor will be considered as the second one



(A7.1)

Let us now find the solution of this equation [36]. The integral in the right side represents a conformal invariant 3-point function. For its calculation we consider a more general integral



(A7.2)

where $\sigma_T = (D, 2)$ are the quantum numbers of the energy-momentum tensor. The normalization of the conformal invariant functions entering into (A7.2) has been chosen as in section 2. The Green function $G_T(x_1x_2x_3) = \langle 0|\, T\Phi_d(x_1)\Phi_d(x_2)T_{\mu\nu}(x_3)\,|0\rangle$ has been normalized by the Ward identity [41]

$$-\partial_\mu^{x_3}G^T_{\mu\nu}(x_1x_2x_3) = \left[\delta(x_1-x_3)\partial_\nu^{x_1}+\delta(x_2-x_3)\partial_\nu^{x_2}-\frac{d}{D}\,\partial_\nu^{x_3}\delta(x_1-x_3)-\frac{d}{D}\,\partial_\nu^{x_3}\delta(x_2-x_3)\right]G_d(x_{12}), \tag{A7.3}$$

and equal

$$G^T_{\mu\nu}(x_1x_2x_3) = -\frac{2^{d-h}}{(2\pi)^h}\,\frac{1}{D-1}\,\frac{\Gamma(d+1)\Gamma(h+1)}{\Gamma(h-d)}\,\frac{1}{N(\sigma_T dd)}\,C^{\sigma_T dd}_{\mu\nu}(x_3x_1x_2). \tag{A7.4}$$

We now find the coefficient $\phi_{d,\delta}(l_1 l_2)$ entering into (A7.2). For this purpose one differentiates (A7.2) with respect to x_3. By utilizing the Ward identity (A7.3) we obtain on the left-hand side

$$\int dy[\partial_\nu^{x_3} C^{l_1 d\delta}(x_1 x_3 y)] C^{\tilde{d}\tilde{\delta} l_2}(x_3 y x_2) - \frac{D-d}{D}\partial_\nu^{x_3}\int dy C^{l_1 d\delta}(x_1 x_3 y) C^{\tilde{d}\tilde{\delta} l_2}(x_3 y x_2)$$
$$- \int dy C^{l_1 d\delta}(x_1 x_3 y)\partial_\nu^{x_3} C^{\tilde{d}\tilde{\delta} l_2}(x_3 y x_2) + \frac{d}{D}\partial_\nu^{x_3}\int dy C^{l_1 \tilde{d}\tilde{\delta}}(x_1 x_3 y) C^{d\delta l_2}(x_3 y x_2). \tag{A7.5}$$

Using the relations of Appendix 3 and the following one

$$\left\{D\left(D - \frac{d+\delta+l_2}{2}\right)\left(h - \frac{d-\delta+l_2}{2}\right) + \left(h + \frac{l_1 - l_2}{2}\right)(d-D)\left(D - \frac{l_1+l_2}{2}\right)\right\}\frac{(x_{13})_\nu}{x_{13}^2}$$
$$+ \left\{D\left(D - \frac{d+\delta+l_2}{2}\right)\left(h - \frac{\delta-d+l_1}{2}\right) + \left(h - \frac{l_1-l_2}{2}\right)(d-D)\left(D - \frac{l_1+l_2}{2}\right)\right.$$
$$\left. + D\left(D - \frac{l_1+l_2}{2}\right)\left(\frac{\delta-d+l_2}{2}\right)\right\}\frac{(x_{23})_\mu}{x_{23}^2}$$
$$= \frac{D}{2}\left\{\frac{\delta-D}{2}(D-d-\delta) - \tfrac{1}{2}l_1(D-l_1) + \frac{1}{2D}d(l_1-l_2)(D-l_1-l_2)\right\}\lambda_\nu^{x_3}(x_1 x_2)$$

we find

$$\int dy C^{l_1 d\delta}(x_1 x_3 y)\partial_\nu^{x_3} C^{\tilde{d}\tilde{\delta} l_2}(x_3 y x_2) - \frac{D-d}{D}\partial_\nu^{x_3}\int dy C^{l_1 d\delta}(x_1 x_3 y) C^{\tilde{d}\tilde{\delta} l_2}(x_3 y x_2)$$
$$= (2\pi)^{-h} N_0(l_1 d\delta) N_0(l_2 d\delta)\frac{\Gamma((l_1+l_2-D)/2)}{\Gamma((2D-l_1-l_2+2)/2)} F_{d\delta}(l_1 l_2)$$
$$\times\left\{\tfrac{1}{2}(\delta-d)(D-d-\delta) - \tfrac{1}{2}l_1(D-l_1) + \frac{d}{2D}(l_1-l_2)(D-l_1-l_2)\right\}$$
$$\times (\tfrac{1}{2}x_{12}^2)^{(D-l_1-l_2)/2}(\tfrac{1}{2}x_{13}^2)^{-(D+l_1-l_2)/2}(\tfrac{1}{2}x_{23}^2)^{-(D-l_1+l_2)/2}, \tag{A7.6}$$

where

$$F_{d,\delta}(l_1 l_2) = \Gamma\left(\frac{l_1+d+\delta-D}{2}\right)\Gamma\left(\frac{d+\delta-l_1}{2}\right)\Gamma\left(\frac{l_1+d-\delta}{2}\right)\Gamma\left(\frac{D-l_1+d-\delta}{2}\right)$$
$$\times\Gamma\left(\frac{D-d-\delta+l_2}{2}\right)\Gamma\left(\frac{2D-d-\delta-l_2}{2}\right)\Gamma\left(\frac{\delta-d+l_2}{2}\right)\Gamma\left(\frac{D-d+\delta-l_2}{2}\right),$$

$$N_0(ld\delta) = \left\{\Gamma\left(\frac{l+d+\delta-D}{2}\right)\Gamma\left(\frac{l+d-\delta}{2}\right)\Gamma\left(\frac{l-d+\delta}{2}\right)\Gamma\left(\frac{d-l+\delta}{2}\right)\Gamma\left(\frac{2D-l-d-\delta}{2}\right)\right.$$
$$\left.\times\Gamma\left(\frac{D-l+d-\delta}{2}\right)\Gamma\left(\frac{D+l-d-\delta}{2}\right)\Gamma\left(\frac{D-l-d+\delta}{2}\right)\right\}^{-1/2}.$$

The right-hand side of (A7.2) is equal to

$$\partial_\mu^{x_3} C_{\mu\nu}^{\sigma_T l_1 l_2}(x_3 x_1 x_2) = \frac{1}{(2\pi)^h}\frac{D-1}{D}(l_1 - l_2)N(\sigma_T l_1 l_2) \times \frac{1}{(\frac{1}{2}x_{12}^2)^{(l_1+l_2-D)/2}}\frac{1}{(\frac{1}{2}x_{13}^2)^{(D+l_1-l_2)/2}}\frac{1}{(\frac{1}{2}x_{23}^2)^{(D-l_1+l_2)/2}}. \tag{A7.7}$$

Using (A7.5)–(A7.7) and (A7.2) we find

$$\phi_{d,\delta}(l_1 l_2) = -\frac{8D}{D-1}N_0(l_1 d\delta)N_0(l_2 d\delta)N^{-1}(\sigma_T l_1 l_2)\frac{\Gamma((l_1+l_2-D+2)/2)}{\Gamma((2D-l_1-l_2+2)/2)}\frac{1}{(l_1-l_2)(D-l_1-l_2)}$$
$$\times\left\{\Gamma\left(\frac{D+l_1-l_2+2}{2}\right)\Gamma\left(\frac{D-l_1+l_2+2}{2}\right)\Gamma\left(\frac{2D-l_1-l_2+2}{2}\right)\Gamma\left(\frac{l_1+l_2+2}{2}\right)\right\}^{-1}$$
$$\times\left\{\tfrac{1}{2}(F_{d,\delta} - F_{D-d,\delta})(\delta - d)(D-d-\delta) - \tfrac{1}{2}(F_{d,\delta} - F_{D-d,\delta})l_1(D-l_1)\right.$$
$$\left. + \tfrac{1}{2}\left[\frac{d}{D}F_{d,\delta} - \frac{D-d}{D}F_{D-d,\delta}\right](l_1 - l_2)(D - l_1 - l_2)\right\}. \tag{A7.8}$$

This is now used in (A7.2) in the limit $l_1 = l_2 = d$, $\delta = d$. From (A7.1, 2), (A7.4) and (A7.8), we find

$$g^2\tilde{\phi}_{d,d} = \frac{4}{D-1}(2\pi)^h(d-h)\Gamma(h+1)\Gamma(d+1)\Gamma(D-d+1),$$

where g is the coupling constant and the function $\tilde{\phi}_{d,\delta}$ is equal to

$$\tilde{\phi}_{d,\delta} = \lim_{l_1\to l_2=d}\phi_{d,\delta}(l_1 l_2) = \frac{4D}{D-1}N^{-1}(\sigma_T dd)\frac{\Gamma(d-h+1)}{\Gamma(D-d+1)}\frac{1}{h-d}\left\{\frac{1}{h}(h-d)^2 + \tfrac{1}{2}[2d(D-d) - \delta(D-\delta)]\right.$$
$$\left.\times\left.\frac{\mathrm{d}}{\mathrm{d}l}\right|_{l=d}\ln\frac{\Gamma((l+d+\delta-D)/2)\Gamma((d-l+\delta)/2)\Gamma((l+d-\delta)/2)\Gamma((D-l+d-\delta)/2)}{\Gamma((D-d+l-\delta)/2)\Gamma(D-(d+l+\delta)/2)\Gamma((\delta-d+l)/2)\Gamma((D-d-l+\delta)/2)}\right\}.$$

Assuming $d = \delta$, for the coupling constant we find

$$g^2 = 8\mu(d)\left\{\frac{4}{h}\frac{(h-d)^2}{d(D-d)} + Q(d)\right\}^{-1} \tag{A7.9}$$

where $\mu(d)$ is function (5.8),

$$Q(d) = \Psi(\tfrac{3}{2}d - h) + \Psi(D - \tfrac{3}{2}d) - \Psi(d/2) - \Psi(h - d/2),$$

$$\Psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x).$$

It should be noted that the normalization factor $N(d,d,d)$ entering into the definition of the function $C^{ddd}(x_1x_2x_3)$ becomes imaginary in the vicinity of $d \sim h-1$. This results in an imaginary coupling constant g. Therefore it will be convenient to include the factor $N(d,d,d)$ into the coupling constant. Let us then define a new coupling constant through the relation

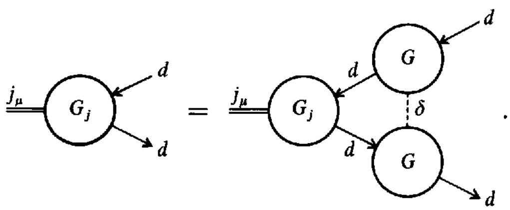
$$G^2(d) = N^2(d,d,d)g^2,$$

$$G(x_1x_2x_3) = G\frac{1}{(2\pi)^h}(\tfrac{1}{2}x_{12}^2)^{-d/2}(\tfrac{1}{2}x_{13}^2)^{-d/2}(\tfrac{1}{2}x_{23}^2)^{-d/2}.$$

From (A7.9) we find

$$G^2(d) = 8\mu(d)\frac{\Gamma((3d-D)/2)\Gamma^3(d/2)}{\Gamma((2D-3d)/2)\Gamma^3((D-d)/2)}\left[\frac{4}{h}\,\frac{(h-d)^2}{d(D-d)} + Q(d)\right]^{-1}. \tag{A7.10}$$

In conclusion let us give the dependence of the coupling constant on the dimensions appearing in the theory $\lambda\varphi^+\varphi\chi$. As an example of the bootstrap equation consider an equation for current-containing Green function. In a 3 vertices approximation we have



(A7.11)

This equation is solved similar to (A7.1), by using Ward identity. We find for the coupling constant $g(d,\delta)$

$$g^2(d,\delta) = 4\mu(d)\left\{\Psi\left(d-h+\frac{\delta}{2}\right)+\Psi\left(h-d+\frac{\delta}{2}\right)+\Psi\left(d-\frac{\delta}{2}\right)\right.$$
$$\left.+\Psi\left(D-d-\frac{\delta}{2}\right)-2\left[\Psi\left(\frac{\delta}{2}\right)+\Psi\left(h-\frac{\delta}{2}\right)\right]\right\}^{-1}. \tag{A7.12}$$

References*

[1] K. Wilson, Phys. Rev. D3 (1971) 1818;
K. Wilson, J. Kogut, Physics Reports 12C (1974) 75;
A.Z. Patashinski and V.L. Pokrovski, Fluctuation Theory of Phase Transitions (Nauka, Moscow, 1975);
A.Z. Patashinski and V.L. Pokrovski, Usp. Fis. Nauk 121 (1977) 55;
R. Brout, Physics Reports 10C (1974) 1;
B. Schroer, Scaling Laws and Scale Invariance in the Theory of Phase Transition and in Relativistic Quantum Field Theory, Lectures given at IV Simposio Brasileiro de Fisica Teorica, Rio de Janeiro, 1972; preprint Freie Univ., Berlin, 1972;
B. Schroer, Phys. Rev. B8 (1973) 4200;
B. Schroer, Revista Brasileira de Fisica, vol. 4 (1974) 323.

[2] A.A. Migdal, 4-Dimensional Soluble Models of Conformal Field Theory, preprint Landau Inst. for Theor. Phys., Chernogolovka, 1972.

[3] G. Mack, Renormalization and Invariance in Quantum Field Theory, ed. E.R. Caianiello (Plenum Press, New York, 1974) pp. 123–157; J. de Physique 34, Suppl. No. 10 (1973) 99. See also [32b].

* For convenience of the readers the Russian-written preprints quoted below are supplied with brief summeries here.

[4] M.Ya. Palchik and E.S. Fradkin, Short Comm. on Phys. 4 (Lebedev Phys. Inst., Moscow, 1974) p. 35; Preprint N 18, Inst. of Automation and Electrometry, 1974.
[5] M.Ya. Palchik and E.S. Fradkin, Introduction to the Conformally Invariant Theory of Quantum Fields, lecture talk given at Intern. School on Important Problems in the Elementary Particles Physics, Sochi, October 1974; published in preprint JINR 2-8874, Dubna, 1975 (Engl. translation in preprints 180, 181 of Lebedev Phys. Inst., Moscow, 1975).
[6] E.S. Fradkin and M.Ya. Palchik, Nucl. Phys. B99 (1975) 317.
[7] V.K. Dobrev, V.B. Petkova, S.G. Petrova and I.T. Todorov, Phys. Rev. D13 (1976) 887. See also [32b].
[8] E.S. Fradkin and M.Ya. Palchik, preprint Bern University (Bern, 1976); preprints N 50, 51, Inst. of Automation and Electrometry, Novosibirks, 1977; Nucl. Phys. B126 (1977) 477 (abbreviated version).
[9] M.Ya. Palchik, preprint N 11, Inst. of Automation and Electrometry, Novosibirks, 1973 (unpublished). This paper shows that to describe fields with anomalous dimensions (in the four-dimensional Minkowsky space) one should use representations of the universal covering group (as a consequence of spectrality). The unitarity condition for the representations $(j, 0)$ and $(0, j)$ is investigated and the limitation on the dimension (for $D = 4$): $d > 1 + j$ is obtained. Partial wave expansions of the states $\varphi\varphi|0\rangle$ and of the Wightman functions, see (3.13) and (3.29) (without the analysis of the Q-functions) are obtained. Transformational properties of generalized free fields under the transformations from the maximal compact subgroup, see relations (3.22) and (3.26), are in vestigated. Part of these results is published in refs. [21, 22].
[10] W. Rühl, Comm. Math. Phys. 34 (1973) 149.
[11] I.A. Swieca and A.H. Völkel, Comm. Math. Phys. 29 (1973) 319.
[12] B.G. Konopelchenko and M.Ya. Palchik, Yadernaya Fizika (Sov. J. of Nucl. Phys.) 19 (1974) 203; preprint N 10, Inst. of Automation and Electrometry, 1973.
[13] B. Schroer and I.A. Swieca, Phys. Rev. D10 (1974) 480.
[14] D.H. Mayer, J. Math. Phys. 18 (1977) 456.
[15] T.H. Go, Comm. Math. Phys. 41 (1975) 157.
[16] W. Rühl, Acta Phys. Austriaca, Suppl. XIV (1975) 643;
J. Rupsch, W. Rühl and B.C. Yunn, Ann. Phys. 89 (1975) 115;
W. Rühl and B.C. Yunn, J. Math. Phys. 17 (1976) 1521.
[17] M. Lüscher and G. Mack, Comm. Math. Phys. 41 (1975) 203.
[17a] G. Mack, Comm. Math. Phys. 53 (1977) 155; preprint DESY, 1976.
[18] B. Schroer, I.A. Swieca and A.H. Völkel, Phys. Rev. D11 (1975) 1509.
[19] B. Schroer, Conformal Invariance in Minkowskian Quantum Field Theory and Globel Operator Expansions, Talk given at the Intern. on Mathematical Problems in Theoretical Physics, Kyoto, Japan, 1975; Preprint, Institute für Theoretische Physik, Freie Universität, Berlin, 1975.
[20] W. Rühl and B.C. Yunn, Comm. Math. Phys. 48 (1976) 215.
[21] M.Ya. Palchik, Phys. Lett. 66B (1977) 259.
[22] M.Ya. Palchik, preprint N 55, Inst. of Automation and Electrometry, Novosibirks, 1977.
[23] G. Mack and A. Salam, Ann. Phys. 53 (1969) 174.
[24] P. Carruthers, Phys. Reports 1C (1971) 1;
R. Jackiw, Phys. Today 825(1), (1972) (see also Usp. Fiz. Nauk 109 (1973) 743);
V.M. Dubovik, Usp. Fiz. Nauk 109 (1973) 756.
[25] I.T. Todorov, Conformal-Invariant Field Theory with Anomalous Dimensions, preprint TH 1697-CERN, 1973.
[26] S. Fubini, Nuovo Cim. A34 (1976) 521.
[27] B.G. Konopelchenko and M.Ya. Palchik, preprint 90-72, Inst. of Nucl. Phys., Novosibirks, 1972 (unpublished). Single valued unitary and nonunitary representations of the conformal group in the two-dimensional Minkowsky space are investigated. The partial wave expansion of the states $\varphi\varphi|0\rangle$ is obtained. The requirement that the representations be single-valued is shown to lead to quantization of the dimensions of fields.
[28] S. Ferrara, R. Gatto, A.F. Grillo and G. Parisi, Nucl. Phys. 49B (1972) 77.
[29] S. Ferrara, A.F. Grillo and R. Gatto, Ann. Phys. 76 (1973) 161.
[30] S. Ferrara, A.F. Grillo and G. Parisi, Lett. Nuovo Cim. 4 (1972) 115.
[31] A.M. Polyakov, Zh. Eksp. Teor. Fiz. 66 (1974) 23.
[32] V. Dobrev, G. Mack, V.B. Petkova, S.G. Petrova and I.T. Todorov, a) JINR, E2-7977, Dubna, 1974; Repts. Math. Phys. 9 (1976) 219;
b) Lecture notes in Physics, Vol. 63 (Springer-Verlag, 1977).
[33] S. Ferrara, A.F. Grillo, R. Gatto and G. Parisi, Nuovo Cim. 19A (1974) 667;
S. Ferrara, A.F. Grillo and R. Gatto, Nuovo Cim. 26A (1975) 226.
[34] V. de Alfaro, S. Fubini and G. Furlan, preprint, TH 2115-CERN, 1976.
[34a] V. Kurak, PUC Rio de Janeiro preprint, September, 1976.
[35] (a) A.N. Leznov and I.A. Fedoseev, Theor. Math. Phys. 5 (1970) 181.
(b) T. Yao, J. Math. Phys. 8 (1967) 1931; 9 (1968) 1615; 12 (1971) 315.
(c) G. Mack, Comm. Math. Phys. 55 (1977) 1; Preprint, Hamburg, 1975.

[36] M.Ya. Palchik and E.S. Fradkin, Short Communications on Physiks, N 10, Lebedev Phys. Inst., Moscow, 1975; Proc. Symp. on Deep Inelastic Processes, Suhumi, 1975, published in: JINR, Dubna, 1977.
[36a] M.Ya. Palchik, Preprint N 74, Institute of Automation and Electrometry, Novosibirsk, 1978.
[37] E.S. Fradkin, M.Ya. Palchik and V.N. Zaikin, Phys. Lett. B57 (1975) 364.
[38] G. Parisi and L. Pelity, Lett. Nuovo Cim. 2 (1970) 627;
M. D'Eramo, L. Pelity and G. Parisi, Lett. Nuovo Cim. 2 (1971) 878.
[39] (a) A.M. Palyakov, Zh. ETF Pis. Red. 12 (1970) 538 (Engl. transl. JETP Lett. 12 (1970) 381);
(b) A.A. Migdal, Phys. Lett. 37B (1971) 98.
[40] K. Symanzik, Lett. Nuovo Cim. 3 (1972) 734.
[41] G. Mack and K. Symanzik, Comm. Math. Phys. 27 (1972) 247.
[42] G. Mack and I.T. Todorov, Phys. Rev. D8 (1973) 1764.
[43] G. Parisi, Lett. Nuovo Cim. 4 (1972) 777.
[44] E. Schreier, Phys. Rev. D3 (1971) 980.
[45] A.A. Migdal, Phys. Lett. 37B (1971) 386.
[45a] R.P. Zaikov, Bulg. J. Phys. 2 (1975) 89; preprint JINR, E2-10250, Dubna, 1976.
[46] R. Nobili, Nuovo Cim. 13A (1973) 129.
[47] J. Schwinger, Phys. Rev. 115 (1959) 728;
E.S. Fradkin, Dokl. Acad. Nauk SSSR (1959) 311; Thesis (1960) in [60] p. 96;
T. Nakano, Progr. Theor. Phys. 21 (1959) 241, 61.
[48] H.A. Kastrup, Phys. Rev. 142 (1966) 1060; 143 (1966) 1041.
[49] F. Gursey and S. Orfanidis, Phys. Rev. D7 (1973) 2414.
[50] B.G. Konopelchenko and M.Ya. Palchik, Docl. Acad. Nauk (SSSR) 214 (1974) 1052.
[51] M. Hortacsu, R. Seiler and B. Schroer, Phys. Rev. D5 (1972) 2519.
[52] G.B. Konopelchenko and M. Ya. Palchik, preprint 98-73, Inst. of Nucl. Phys., Novosibirsk, 1973 (unpublished). The unitarity condition for the representations (j_1, j_2) of the universal covering group in the four-dimensional Minkowsky space is investigated. Restrictions (3.16) and (A2.11) on the field dimensions are obtained. Generalized free fields with arbitrary spin are considered.
[53] I. Segal, Bull. Am. Math. Soc. 77 (1971) 958.
[54] W. Rühl, Comm. Math. Phys. 30 (1973) 287.
[55] S. Ferrara, R. Gatto and A. Grillo, Phys. Rev. D9 (1974) 3564.
[56] M.Ya. Palchik and A.Ya. Politchuk, preprint N 35, Inst. of Automation and Electrometry, Novosibirsk, 1976 (unpublished).
[57] E.S. Fradkin, Zh. Eksp. Teor. Fiz. 29 (1955) 258;
Y. Takahashi, Nuovo Cim. 6 (1957) 370.
[58] E.S. Fradkin and M.Ya. Palchik, Nuovo Cim. 34A (1976) 438.
[59] V.N. Zaikin, M.Ya. Palchik and E.S. Fradkin, Proc. Intern. Symp. on Deep Inelastic Processes, Sukhumi, 1975, published in: JINR, Dubna);
V.N. Zaikin, Short Comm. on Phys. N 6, 15, Lebedev Phys. Inst. Moscow, 1977.
[60] E.S. Fradkin, Zh. Eksp. Teor. Fiz. 26 (1954); 29 (1955) 121; Thesis (1960), in: Proc. Lebedev Inst., vol. 19 (Nauka, Moscow, 1965).
[61] K. Symanzik, in: Lectures in High Energy Physics, ed. B. Jakšić (Gordon and Breach, New York, 1965) pp. 485–517.
[62] D.J. Gross and J. Wess, Phys. Rev. D2 (1970) 753.
[63] M. Gell-Mann and F.E. Low, Phys. Rev. 95 (1954) 1300.
[64] G.G. Callan, Phys. Rev. D2 (1970) 1541;
K. Symanzik, Comm. Math. Phys. 18 (1970) 227.
[65] B. Schroer, Nuovo Cim. Lett. 2 (1971) 867.
[66] I. Lukierski, preprint Bielefeld University, 1976.
[67] J.C. Le Guillou and J. Zinn-Justin, Phys. Rev. Lett. 39 (1977) 95.
[68] K. Johnson, Nuovo Cim. 20 (1961) 773.
[69] A. Wightman, Problems of Relativistic Dynamics of Quantum Fields (Nauka, Moscow, 1968).
[70] G.F. Dell'Antonio, Y. Frishman and D. Zwanziger, Phys. Rev. D6 (1972) 988.
[71] I.M. Gelfand and G.E. Shilov, Generalized Functions, Vol. 1. Generalized Functions (State Publishing House of Physico-Mathematical Literature, Moscow, 1975).
[72] V. Bargmann, Ann. Math. 48 (1949) 568;
N.Ya. Vilenkin, Special Functions and Theory of Group Representation (Nauka, Moscow, 1965);
I.M. Gelfand, M.I. Graev and N.Ya. Vilenkin, Generalized Functions, Vol. 5. Integral Geometry and Its Associated Problems of Representation Theory (State Publishing Haus of Phys.-Math. Literature, Moskow, 1962).
[73] I.S. Gradshteyn and I.M. Ryzhik, Tables of Integrals, Series and Products (Nauka, Moskow, 1971).
[74] B.G. Konopelchenko, Physics of Elementary Particles and Atom Nuclear (JINR, USSR) 8 (1977) 135.