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BFV approach to geometric quantization

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Abstract

A gauge-invariant approach to geometric quantization is developed. It yields a complete quantum description for dynamical systems with non-trivial geometry and topology of the phase space. The method is a global version of the gauge-invariant approach to quantization of second-class constraints developed by Batalin, Fradkin and Fradkina (BFF). Physical quantum states and quantum observables are respectively described by covariantly constant sections of the Fock bundle and the bundle of hermitian operators over the phase space with a flat connection defined by the nilpotent BFV–BRST operator. Perturbative calculation of the first non-trivial quantum correction to the Poisson brackets leads to the Chevalley cocycle known in deformation quantization. Consistency conditions lead to a topological quantization condition with metaplectic anomaly.

1. Introduction

Quantization of dynamical systems with curved phase space has been one of the major problems in mathematical physics for more than two decades. In classical mechanics the central object is an associative commutative algebra $\mathscr{F}$ of functions, classical observables, on a symplectic manifold $(\mathscr{M}, \omega)$. The problem of quantization is to construct an associative non-commutative algebra of quantum observables $\mathscr{F}(\hbar)$ satisfying the correspondence principles and to describe its representation by operators in the Hilbert space of quantum states.

The difficulty lies in the non-trivial global geometry of the phase space. No general procedure is known to directly quantize non-linear Poisson brackets. At the classical level, celebrated Dirac brackets [1–4] have established a relation between dynamical systems with curved phase space and physically equivalent

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systems with flat phase space of higher dimension subject to second-class constraints. However, there is still no general way to directly quantize highly non-linear Dirac brackets without introducing some extra gauge degrees of freedom.

A variety of powerful approaches to quantization on symplectic manifolds have been developed. The most-well-studied *geometric quantization* approach [5–19] describes quantum states as sections of a certain line bundle with connection. Although in some cases it does allow to adequately describe quantum physical systems, it has two major drawbacks. First, though it provides a description of quantum states, there is no way to quantize the entire algebra of classical observables. Only a limited subalgebra of functions of the special form can be quantized. Second, it is not directly derived from the basic physical principles. Historically, a set of mathematical rules has been presented that was satisfactory for quantization of some physical systems. Later, in applications to other systems it was discovered that the original rules could lead to physically incorrect results and had to be modified, so a number of corrections were introduced. In particular, the original method failed to produce the correct quantum description for the multidimensional harmonic oscillator (since it did not take into account normal ordering effects, the shift $\frac{1}{2}\hbar$ was missing). That led to the so-called metaplectic correction [9–11,14,15]. However, the problem of quantization of the entire algebra of classical observables (since there is still no way to deal with "curved normal ordering", no higher order quantum corrections can be produced in this approach), as well as the problem of physical substantiation, still remains open in this approach (see Ref. [19] for the latest review). Of course, the first problem is a consequence of the latter.

Another interesting approach was developed by Berezin [20]. It is known as the *Berezin quantization* and is based on the notion of co- and contravariant symbols, an extension of the standard theory of symbols of operators for systems with flat phase space. Although this approach is very effective in the case of homogeneous Kähler manifolds, it requires the explicit knowledge of an overcomplete set of vectors in the space of functions on $\mathcal{M}$ and, when no group action is present, there is no conventional way to obtain them.

The *deformation quantization* approach was formulated in Refs. [21–23] (see Ref. [24] for a review). It studies formal deformations of the algebra of classical observables in the context of Hochschild and Chevalley cohomologies of associative and Lie algebras and requires solving associativity and Jacobi identities. However, it does not provide any information on Hilbert space representations of resulting deformations. In particular, for compact $\mathcal{M}$, topological quantization conditions are missing in this approach, and formal deformations obtained in this way do not admit any hermitian representations for generic (non-integer) values of the deformation parameter.

Thus, a general quantization method yielding both the correct Hilbert space of physical quantum states and the entire algebra of quantum observables represented by operators acting in this space is needed. It is also desirable that such a method would be naturally deduced from the basic physical principles, rather than constitute a set of formal prescriptions.

The present paper is an attempt to solve this problem. A global quantization method that takes into account the geometry and topology of the phase space and yields both the Hilbert space and the algebra of quantum observables is proposed. It is based on the general BFF conversion procedure that allows one to convert arbitrary second-class constraints into the first-class ones.

In a series of papers [25–28], Batalin, Fradkin and Fradkina have extended the general BFV quantization method for dynamical systems with constraints [29–39] to quantization of systems with curved phase space. Their quantization method for curved phase space consists of three steps. First, by doubling the dimension of the initial curved phase space and simultaneously imposing second-class constraints, the original dynamical system is represented by the physically equivalent system with canonical commutation relations and second-class constraints [25,26]. Then, second-class constraints are converted into the first-class ones by introducing new canonical variables, their number being equal to the number of constraints (BFF conversion procedure) [25–28]. Finally, the resulting dynamical system with first-class constraints is quantized according to the standard BFV quantization method [30–39].

The idea of introducing auxiliary gauge degrees of freedom to convert second-class constraints into the first-class ones is very natural from the physical point of view. It dates its roots to the Stuckelberg variables that are introduced to achieve a gauge-invariant description of massive vector fields [40]. Other applications of this idea include a gauge-invariant description of chiral models, restoration of gauge invariance in the action functional of Yang–Mills theory with sources [41] and a gauge-invariant description of anomalies [42]. The BFF conversion procedure [25–28] provides a practical implementation of this idea in the most general setting and has a natural physical and geometrical interpretation in the case of quantization on the curved phase space.

Remarkably, it turns out that in the global quantization context this physical method leads to a beautiful geometric theory. As we will show, the Hilbert space of physical quantum states can be interpreted as a space of covariantly constant sections of the Fock bundle over $\mathscr{M}$ with copies of the Fock space as fibers. The structure group is a group of unitary operators in the Fock space. This bundle has a natural flat connection defined by the nilpotent BFV–BRST operator $\hat{\Omega}$. In the case when the structure group can be reduced to its finite-parametric subgroup isomorphic to $[\mathrm{Sp}(2N;\mathbb{R})\times \mathrm{U}(1)]/\mathbb{Z}_2$, the necessary and sufficient condition for the existence of the Fock bundle reduces to the corrected topological quantization condition with metaplectic anomaly [11,14,15]. Furthermore, the algebra of quantum observables is realized as an algebra of covariantly constant sections of the associated bundle with copies of the algebra of hermitian operators in the Fock space as fibers. The group of unitary operators acts on the fibers by conjugations, and the natural flat connection is defined by the adjoint operator ad $\hat{\Omega}$. Working with the entire infinite-dimensional Fock bundle instead of just a line bundle (which can be regarded as a vacuum bundle; i.e. copies of the Fock vacuum are glued to every point of $\mathscr{M}$), we are able to explicitly perform quantization of the entire algebra of classical observables. In particular, perturbative calculation of the first non-trivial quantum correction to the Poisson brackets on $\mathscr{M}$ leads to a non-trivial Chevalley cocycle discovered by Vey in the deformation quantization context [21,22].

Our approach can be called a gauge theory of geometric quantization. It should be mentioned that there is a formal similarity between the construction of the present paper and the gauge theory of higher spin fields developed in Refs. [43–45]. The key idea is to enlarge the gauge symmetry from the finite-parametric symmetry of (super)gravity to its infinite-dimensional enveloping algebra. The corresponding gauge field contains an infinite tower of higher spin fields.

This paper is organized as follows. Section 2 is a brief reminder on BFV quantization. In Section 3, dynamical systems with curved phase space are re-formulated as systems with second-class constraints. The global definition of second-class constraints requires the definition of the symplectic potential which can be interpreted as a connection on the prequantum line bundle. Topological quantization conditions ensuring the existence of such a bundle, the classification of inequivalent bundles and the global definition of the action functional are briefly discussed. Difficulties of quantization of systems with second-class constraints are addressed and geometric quantization with polarization is interpreted as a Gupta–Bleuler-type quantization of second-class constraints.

In Section 4, according to the general BFF conversion procedure the classical dynamical system with second-class constraints is converted into a system with abelian first-class constraints by introducing auxiliary gauge degrees of freedom that are described by canonical variables with standard canonical Poisson brackets. Master equations for the first-class constraints and for classical observables commuting with them are solved perturbatively up to the third order in auxiliary canonical variables. The converted dynamical system with first-class constraints can be regarded as a gauge theory of the group of canonical transformations $\mathrm{Symp}(\mathbb{R}^{2N})$. (Here $(\mathbb{R}^{2N}, \Lambda)$ is a canonical phase space of auxiliary gauge degrees of freedom.) The master equations for first-class constraints are nothing but zero-curvature conditions for the corresponding gauge field, and classical observables are covariantly constant quantities. Geometrically, classical observables are covariantly constant sections of the $\mathrm{Symp}(\mathbb{R}^{2N})$-bundle over $\mathscr{M}$ with copies of the space of functions on the canonical phase space of gauge degrees of freedom as fibers.

In Sections 5 and 6, BFV quantization of the BFF converted system is performed. In Section 5, a quantization of the algebra of classical observables is performed in terms of Weyl symbols. First non-trivial quantum corrections to the classical expressions for the first-class constraints and observables are found explicitly. A star product formula for functions on an arbitrary symplectic manifold with a symplectic connection is obtained. The first non-trivial correction to the Poisson brackets is computed. It coincides with the Chevalley cocycle known in deformation quantization [21,20]. Geometrically, Weyl symbols of quantum observables are covariantly constant sections of the W-bundle, which is a quantization of the $\mathrm{Symp}(\mathbb{R}^{2N})$-bundle. It has copies of the Weyl algebra $\mathscr{W}_{2N}$ as fibers and is endowed with a flat connection defined by the $*$-adjoint $\mathrm{ad}_* \Omega$ of the Weyl symbol

Ω of the nilpotent BFV–BRST operator $\hat{\Omega}$. A representation of this algebra by hermitian operators in the Hilbert space of quantum states is constructed in Section 6. Physical states are covariantly constant sections of the Fock bundle with flat connection. Fibers of the Fock bundle over $\mathscr{M}$ are copies of the standard Fock space created by acting with creation operators obtained by quantization of the auxiliary gauge degrees of freedom on the Fock vacuum. The structure group is a group of unitary operators in the Fock space, and the flat connection is defined by $\hat{\Omega}$. We consider an important special case of Fock bundles when the structure group can be reduced to its finite-dimensional subgroup isomorphic to $[\mathrm{Sp}(2N;\mathbb{R})\times \mathrm{U}(1)]/\mathbb{Z}_2$ and obtain a topological quantization condition necessary for the existence of such bundles. It coincides with the corrected quantization condition with a metaplectic anomaly known in geometric quantization [11,14,15].

In conclusion, we discuss relations of our results with deformation and geometric quantization. Both of these approaches may be regarded as limited gauge fixed versions of the gauge-invariant approach, each of them separately providing only a partial view of the picture. The gauge-invariant approach based on BFV quantization synthesizes these two directions. We also discuss possible avenues of further development and indicate potential applications of the method to conformal and topological field theories. Our notations and conventions are explained in Appendix A, while some perturbative calculations relevant to the star product are gathered in Appendix B.

2. BFV quantization

Let us consider a classic dynamical system with irreducible first-class constraints $T_\alpha = T_\alpha(q, p)$, $\alpha = 1, 2, \dots, n$,

$$\left[T_\alpha, T_\beta\right]_{\mathrm{PB}} = f^{\gamma}_{\alpha\beta} T_\gamma, \tag{2.1}$$

where the structure functions $f^{\gamma}_{\alpha\beta} = f^{\gamma}_{\alpha\beta}(q, p)$ generally depend on the phase space variables q and p (for simplicity we will consider only Bose degrees of freedom) and

$$\left[q^i, p_j\right]_{\mathrm{PB}} = \delta^i_j, \quad i, j = 1, 2, \dots, 2N. \tag{2.2}$$

According to Dirac [4], a classical observable A is a function of q and p whose Poisson brackets with constraints vanish on the constraints surface defined by $T_\alpha = 0$, i.e.

$$[T_\alpha, A]_{\mathrm{PB}} = V^{A,\beta}_{\alpha} T_\beta \tag{2.3}$$

for some functions $V^{A,\beta}_{\alpha} = V^{A,\beta}_{\alpha}(q, p)$. Obviously, classical observables are defined modulo the equivalence

$$A' \approx A + \lambda^\alpha T_\alpha, \tag{2.4}$$

with arbitrary $\lambda^\alpha = \lambda^\alpha(q, p)$, and the equivalence classes form an associative commutative algebra $\mathscr{F}_{\mathrm{obs}}$ with respect to the standard multiplication of functions.

The problem of quantization is to construct an associative non-commutative algebra $\mathscr{F}_{\rm obs}(\hbar)$ of quantum observables satisfying the correspondence principle and to describe its hermitian representation in the Hilbert space of quantum states. A solution to this problem for systems subject to first-class constraints is provided by the BFV quantization method [30–39].

Starting point of the BFV quantization is an extended phase space $\mathscr{M}_{\rm ext}$ [29–34]. The original phase space variables are complemented with dynamically active lagrangian multipliers to the first-class constraints, λ^α, and their momenta π_α,

$$\left[\lambda^\alpha, \pi_\beta\right]_{\rm PB} = \delta^\alpha_\beta, \tag{2.5}$$

and two sets of anticommuting ghosts and their momenta,

$$\left\{C^\alpha, \bar{\mathscr{P}}_\beta\right\}_{\rm PB} = \delta^\alpha_\beta, \tag{2.6}$$

$$\left\{\mathscr{P}^\alpha, \bar{C}_\beta\right\}_{\rm PB} = \delta^\alpha_\beta. \tag{2.7}$$

The subspace of variables (q, p) and $(C, \bar{\mathscr{P}})$ is called a minimal subspace $\mathscr{M}_{\rm min}$, and the subspace of (λ, π) and $(\mathscr{P}, \bar{C})$ an auxiliary subspace $\mathscr{M}_{\rm aux}$, so that $\mathscr{M}_{\rm ext} = \mathscr{M}_{\rm min} \oplus \mathscr{M}_{\rm aux}$.

An associative supercommutative algebra $\mathscr{F}_{\rm ext}$ of functions on the extended phase space $\mathscr{M}_{\rm ext}$ is quantized in the standard way by substituting the classical Poisson brackets with commutators and anticommutators:

$$\begin{aligned}
&\left[\hat{q}^i, \hat{p}_j\right] = i\hbar\delta^i_j, && (\hat{q}^i)^\dagger = \hat{q}^i, && (\hat{p}_j)^\dagger = \hat{p}_j,\\
&\left[\hat{\lambda}^\alpha, \hat{\pi}_\beta\right] = i\hbar\delta^\alpha_\beta, && (\hat{\lambda}^\alpha)^\dagger = \hat{\lambda}^\alpha, && (\hat{\pi}_\beta)^\dagger = \hat{\pi}_\beta,\\
&\left\{\hat{C}^\alpha, \hat{\bar{\mathscr{P}}}_\beta\right\} = i\hbar\delta^\alpha_\beta, && (\hat{C}^\alpha)^\dagger = \hat{C}^\alpha, && \left(\hat{\bar{\mathscr{P}}}_\beta\right)^\dagger = -\hat{\bar{\mathscr{P}}}_\beta,\\
&\left\{\hat{\mathscr{P}}^\alpha, \hat{\bar{C}}_\beta\right\} = i\hbar\delta^\alpha_\beta, && (\hat{\mathscr{P}}^\alpha)^\dagger = \hat{\mathscr{P}}^\alpha, && \left(\hat{\bar{C}}_\beta\right)^\dagger = -\hat{\bar{C}}_\beta.
\end{aligned} \tag{2.8}$$

The resulting algebra $\mathscr{F}_{\rm ext}(\hbar)$ has a natural grading with respect to the ghost number gh defined as an eigenvalue of the ghost number operator $\hat{\mathscr{G}}$:

$$\begin{aligned}
&\hat{\mathscr{G}} := \tfrac{1}{2}\left(\hat{C}^\alpha\hat{\bar{\mathscr{P}}}_\alpha - \hat{\bar{\mathscr{P}}}_\alpha\hat{C}^\alpha + \hat{\mathscr{P}}^\alpha\hat{\bar{C}}_\alpha - \hat{\bar{C}}_\alpha\hat{\mathscr{P}}^\alpha\right), \quad \hat{\mathscr{G}}^\dagger = \hat{\mathscr{G}},\\
&\left[\hat{\mathscr{G}}, \hat{\mathscr{A}}\right] = i\hbar\ \mathrm{gh}(\hat{\mathscr{A}})\hat{\mathscr{A}},\\
&\mathrm{gh}(\hat{q}) = \mathrm{gh}(\hat{p}) = \mathrm{gh}(\hat{\lambda}) = \mathrm{gh}(\hat{\pi}) = 0,\\
&\mathrm{gh}(\hat{C}) = -\mathrm{gh}(\hat{\bar{\mathscr{P}}}) = \mathrm{gh}(\hat{\mathscr{P}}) = -\mathrm{gh}(\hat{\bar{C}}) = 1.
\end{aligned} \tag{2.9}$$

The heart of the BFV quantization is a nilpotent BFV operator $\hat{\Omega}$ generating BRST transformations in the extended phase space:

$$\hat{\Omega} = \hat{\Omega}_{\min} + \hat{\Omega}_{\text{aux}}, \tag{2.10}$$

$$\hat{\Omega}^2 = 0, \tag{2.11}$$

$$\hat{\Omega}^2_{\min} = \hat{\Omega}^2_{\text{aux}} = \left\{\hat{\Omega}_{\min}, \hat{\Omega}_{\text{aux}}\right\} = 0, \tag{2.12}$$

$$\text{gh}(\hat{\Omega}) = \text{gh}\left(\hat{\Omega}_{\min}\right) = \text{gh}\left(\hat{\Omega}_{\text{aux}}\right) = 1, \tag{2.13}$$

$$\hat{\Omega}^\dagger = \hat{\Omega}, \quad \hat{\Omega}^\dagger_{\min} = \hat{\Omega}_{\min}, \quad \hat{\Omega}^\dagger_{\text{aux}} = \hat{\Omega}_{\text{aux}}, \tag{2.14}$$

$$\hat{\Omega}_{\min} = \hat{C}^\alpha \hat{\mathscr{T}}_\alpha - \sum_{k=1}^{\infty} \frac{1}{k!(k+1)!} \hat{C}^{\alpha_1} \dots \hat{C}^{\alpha_{k+1}} \hat{U}^{\beta_1 \dots \beta_k}_{\alpha_1 \dots \alpha_{k+1}} \hat{\bar{\mathscr{P}}}_{\beta_1} \dots \hat{\bar{\mathscr{P}}}_{\beta_k} \tag{2.15}$$

and

$$\hat{\Omega}_{\text{aux}} = \hat{\mathscr{P}}^\alpha \hat{\pi}_\alpha. \tag{2.16}$$

The structure operators $\hat{U} = U(\hat{q}, \hat{p})$ are found from the nilpotence condition (2.11). (For concreteness, $C\bar{\mathscr{P}}$ order is assumed in the series (2.15).) Operators $\hat{\mathscr{T}}_\alpha = \mathscr{T}_\alpha(\hat{q}, \hat{p})$ are quantum counterparts of the classical constraints T_α. The auxiliary part $\hat{\Omega}_{\text{aux}}$ can be interpreted as a minimal BFV operator for abelian first-class constraints $\pi_\alpha = 0$ with ghosts $(\mathscr{P}, \bar{C})$ (recall that the π_α are conjugate momenta to the lagrangian multipliers λ^α).

The nilpotence condition for $\hat{\Omega}_{\min}$ can be solved recurrently by expanding $\hat{\mathscr{T}}$ and $\hat{U}$ in powers of $\hbar$. The number of non-vanishing terms in the sum in (2.15) is called the rank of the system. For example, if in the classical theory (2.1) $f^\gamma_{\alpha\beta}$ are constants (constraints form a closed Lie algebra), one has a rank-one system

$$\hat{\Omega}_{\min} = \hat{C}^\alpha \hat{\mathscr{T}}_\alpha - \tfrac{1}{2} \hat{C}^\alpha \hat{C}^\beta f^\gamma_{\alpha\beta} \hat{\bar{\mathscr{P}}}_\gamma \tag{2.17}$$

and

$$\left[\hat{\mathscr{T}}_\alpha, \hat{\mathscr{T}}_\beta\right] = i\hbar f^\gamma_{\alpha\beta} \hat{\mathscr{T}}_\gamma. \tag{2.18}$$

Consider now an adjoint operator

$$\text{ad}\,\hat{\Omega} := [\hat{\Omega}, \cdot\}, \quad (\text{ad}\,\hat{\Omega})^2 = 0. \tag{2.19}$$

The algebra $\mathscr{F}_{\text{ext}}(\hbar)$ endowed with ad $\hat{\Omega}$ becomes a graded differential algebra, and one can consider its cohomology

$$H^*\left(\mathscr{F}_{\text{ext}}(\hbar), \text{ad}\,\hat{\Omega}\right) = \frac{\text{Ker}(\text{ad}\,\hat{\Omega})}{\text{Im}(\text{ad}\,\hat{\Omega})}. \tag{2.20}$$

According to the BFV quantization procedure, the algebra of quantum observables $\mathscr{F}_{\rm obs}(\hbar)$ (quantization of Dirac's classical observables) is identified with the zero-ghost-number cohomology of the operator ad $\hat{\Omega}$,

$$\mathscr{F}_{\rm obs}(\hbar) \approx H^0\big(\mathscr{F}_{\rm ext}(\hbar), \text{ad } \hat{\Omega}\big), \tag{2.21}$$

i.e. quantum observables $\hat{\mathscr{A}}$ corresponding to the classical observables A are closed zero-ghost-number operators,

$$\text{ad } \hat{\Omega}(\hat{\mathscr{A}}) = 0, \tag{2.22}$$

modulo exact operators of the form $(1/i\hbar)\{\hat{\Omega}, \hat{\Psi}\}$,

$$\hat{\mathscr{A}}' \approx \hat{\mathscr{A}} + \frac{1}{i\hbar}\text{ad } \hat{\Omega}(\hat{\Psi}), \quad \text{gh}(\hat{\Psi}) = -1. \tag{2.23}$$

Equivalence (2.23) is a quantum generalization of the classical equivalence (2.4).

Thus, for every classical observable A the BFV quantization procedure gives a receipt to construct the corresponding quantum observable $\hat{\mathscr{A}}$. The practical procedure consists of the following steps. First, one has to find $\hat{\mathscr{A}}_{\rm min}$, a minimal quantum BFV extension of A. One looks for $\hat{\mathscr{A}}_{\rm min}$ in the form of a power series in ghosts:

$$\hat{\mathscr{A}}_{\rm min} = \hat{\mathscr{A}}_0 + \sum_{k=1}^{\infty} \frac{1}{(k!)^2}\hat{C}^{\alpha_1}\ldots\hat{C}^{\alpha_k}\hat{\mathscr{A}}^{\beta_1\ldots\beta_k}_{\alpha_1\ldots\alpha_k}\hat{\bar{\mathscr{P}}}_{\beta_1}\ldots\hat{\bar{\mathscr{P}}}_{\beta_k}, \tag{2.24}$$

where $\hat{\mathscr{A}}_0 = \mathscr{A}_0(\hat{q}, \hat{p})$ and $\hat{\mathscr{A}}^{\beta_1\ldots\beta_k}_{\alpha_1\ldots\alpha_k} = \mathscr{A}^{\beta_1\ldots\beta_k}_{\alpha_1\ldots\alpha_k}(\hat{q}, \hat{p})$ are operators to be found from the equation

$$\left[\hat{\Omega}_{\rm min}, \hat{\mathscr{A}}_{\rm min}\right] = 0. \tag{2.25}$$

As in the case of solving the nilpotence condition for $\hat{\Omega}_{\rm min}$, Eq. (2.25) is solved recurrently in powers of $\hbar$. Correspondence with the classical observable A is given by

$$\lim_{\hbar\to 0}\left(\mathscr{A}_0(\hat{q}, \hat{p})\big|_{\hat{q}\to q, \hat{p}\to p}\right) = A(q, p). \tag{2.26}$$

Finally, the cohomology class of $\hat{\mathscr{A}}_{\rm min}$ is parametrized by a gauge fermion $\hat{\Psi}$:

$$\hat{\mathscr{A}}_{\Psi} = \hat{\mathscr{A}}_{\rm min} + \frac{1}{i\hbar}\{\hat{\Omega}, \hat{\Psi}\}. \tag{2.27}$$

The gauge fermion $\hat{\Psi}$ is Grassmann-odd and has the form

$$\hat{\Psi} = \hat{\bar{C}}_\alpha\hat{\theta}^\alpha + \hat{\chi}^\alpha\hat{\bar{\mathscr{P}}}_\alpha, \quad \text{gh}(\hat{\Psi}) = -1, \quad \hat{\Psi}^\dagger = \hat{\Psi}, \tag{2.28}$$

where the $\hat{\chi}$ are quantum gauge fixing conditions and the $\hat{\theta}$ are usually chosen in the form $\hat{\theta}^\alpha = \hat{\lambda}^\alpha$ (gauge $\lambda^\alpha = 0$ complements the first-class constraints $\pi_\alpha = 0$). In particular, the quantum hamiltonian governing the time evolution of the quantum system is given by

$$\hat{\mathscr{H}}_{\Psi}=\hat{\mathscr{H}}_{\min}+\frac{1}{i\hbar}\{\hat{\Omega},\hat{\Psi}\} \tag{2.29}$$

and is called the unitarizing hamiltonian.

Let us now consider a representation of $\mathscr{F}_{\text{ext}}(\hbar)$ by operators in the Hilbert space $\mathfrak{R}_{\text{ext}}$. It has a natural grading with respect to the ghost number defined by the eigenvalues of the ghost number operator

$$\hat{\mathscr{G}}|\psi\rangle=\hbar\ \mathrm{gh}(|\psi\rangle)|\psi\rangle. \tag{2.30}$$

Endowed with the hermitian nilpotent BFV–BRST operator $\hat{\Omega}$, $\mathfrak{R}_{\text{ext}}$ becomes a cohomology complex. Elements of its cohomology,

$$H^*\left(\mathfrak{R}_{\text{ext}},\hat{\Omega}\right)=\frac{\mathrm{Ker}(\hat{\Omega})}{\mathrm{Im}(\hat{\Omega})}, \tag{2.31}$$

are closed states,

$$\hat{\Omega}|\psi\rangle=0, \tag{2.32a}$$

modulo exact states of the form $\hat{\Omega}|\chi\rangle$, i.e.

$$|\psi'\rangle\approx|\psi\rangle+\hat{\Omega}|\chi\rangle. \tag{2.32b}$$

According to the BFV quantization procedure, the Hilbert space of physical quantum states is identified with the zero-ghost-number cohomology

$$\mathfrak{R}_{\text{phys}}\approx H^0\left(\mathfrak{R}_{\text{ext}},\hat{\Omega}\right). \tag{2.33}$$

Physical states are also called singlets, while exact states are called null states since their scalar products with closed states vanish.

For finite-dimensional systems all representations of $\mathscr{F}_{\text{ext}}(\hbar)$ are equivalent. However, this is not true for infinite-dimensional systems, where representation-specific anomalies can arise. To obtain a correct physical description of the quantum system, one needs to select a physically correct vacuum.

Dirac quantization of systems with first-class constraints [4] can be recovered from the BFV formulation in the following way. Consider a Schrödinger representation

$$\begin{aligned}\hat{p}_j&=-i\hbar\frac{\partial}{\partial q^j}, & \hat{\pi}_\alpha&=-i\hbar\frac{\partial}{\partial\lambda^\alpha},\\ \hat{\bar{\mathscr{P}}}_\alpha&=i\hbar\frac{\partial_l}{\partial C^\alpha}, & \hat{\bar{C}}_\alpha&=i\hbar\frac{\partial_l}{\partial\mathscr{P}^\alpha},\end{aligned} \tag{2.34}$$

and $\hat{q}$, $\hat{\lambda}$, $\hat{C}$ and $\hat{\mathscr{P}}$ are represented by operators of multiplication on the corresponding classical variables. The extended Hilbert space $\mathfrak{R}_{\text{ext}}=\mathfrak{R}_{\min}\oplus\mathfrak{R}_{\text{aux}}$ is defined as a space of complex functions of the variables q, λ, C, $\mathscr{P}$. Then zero-ghost-number states are independent of ghosts C and $\mathscr{P}$, $\psi_0 = \psi_0(q, \lambda)$, and the condition $\hat{\Omega}\psi_0 = 0$ reduces to

$$\hat{\mathscr{T}}_\alpha\left(q, -i\hbar\frac{\partial}{\partial q}\right)\psi_0 = 0, \tag{2.35}$$

and

$$\frac{\partial\psi_0}{\partial\lambda^\alpha} = 0, \tag{2.36}$$

i.e. (2.32a) enforces quantum constraints on the physical states.

The gauge fixed Schrödinger equation can be written in the form

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{\mathscr{H}}_\Psi\psi. \tag{2.37}$$

Together with Eq. (2.32) it completely describes the dynamics of the quantum constrained system in the gauge $\hat{\Psi}$.

In the Heisenberg representation, time evolution is governed by the quantum counterparts of the hamiltonian equations of motion

$$i\hbar\dot{\hat{\Gamma}}^A = \left[\hat{\Gamma}^A, \hat{\mathscr{H}}_\Psi\right], \tag{2.38}$$

where Γ^A is a short-hand notation for all coordinates of the extended phase space $\mathscr{M}_{\text{ext}}$, $\Gamma = (q, p; C, \bar{\mathscr{P}}; \lambda, \pi; \mathscr{P}, \bar{C})$. One obviously has

$$i\hbar\,\partial_t\hat{\Omega} = \left[\hat{\Omega}, \hat{\mathscr{H}}_\Psi\right] = 0. \tag{2.39}$$

Using standard procedures one can pass from the Heisenberg representation to the path integral representation of the quantum dynamical system (see Refs. [38,39]). The resulting path integral expressions for the S-matrix and the generating functional for the Green functions are well defined and free of the difficulties encountered in the heuristic approach to the path integral quantization.

Extended phase space $\mathscr{M}_{\text{ext}}$ was introduced by Fradkin and Vilkovisky in Refs. [30,31]. Therein an idea of quantization in $\mathscr{M}_{\text{ext}}$ was applied to quantization of the gravitational field (in particular, the correct path integral measure and Ward identities were obtained). In a series of papers by Batalin, Fradkin, Fradkina and Vilkovisky [31–34] a consistent quantization method for dynamical systems with constraints (BFV quantization method) was first formulated in the hamiltonian path integral representation. In particular, consistent quantization of relativistic systems with bosonic first-class constraints (expressions (2.16)–(2.17) for Ω, the unitarizing hamiltonian $\mathscr{H}_\Psi$ (2.29), (2.24), the gauge fermion Ψ (2.28), and the Fradkin–Vilkovisky theorem) was first obtained in Refs. [31,32] and further developed in Refs. [33,34], where the nilpotency of Ω was taken as a basis for the BFV quantization method. The Fradkin–Vilkovisky theorem states that physics (S-matrix, amplitudes, etc.) is independent of the choice of a particular gauge, i.e. that physical quantities are defined only on the cohomology classes of Ω. Global symmetry transformations with fermionic parameters leaving a gauge fixed action with ghosts invariant (BRST transformations) were discovered in the lagrangian formulation of Yang–Mills theory by Bechi, Rouet and Stora [46] and Tyutin [47]. It was pointed out in Refs [33,34] that the nilpotent operator Ω introduced in Refs. [31–34] generates BRST transformations in the hamiltonian formulation of gauge theories. In Ref. [34], path integral quantization was formulated for dynamical systems of the most general type – with an open algebra of first-class constrains (structure coefficients depend on q and p), second-class constraints and both Fermi and Bose degrees of freedom.

The BFV quantization method in the operator representation was developed by Batalin and Fradkin in a series of papers [35–39]. The structure of extended phase space in the Fock representation was studied by Kugo and Ojima [48] and Nishijima [49], where the so-called quartet mechanism was introduced. The BFV quantization method was generalized to the case of reducible first-class constraints in Ref. [37]. A detailed survey of the operator BFV quantization is given in Refs. [38,39]. Finally, operator quantization was generalized to the case of second-class constraints in Refs. [25–28], where a general gauge-invariant BFF approach to second-class constraints was formulated. A lagrangian BV path integral representation of the BFV quantization was formulated in a series of papers by Batalin and Vilkovisky [50]. Equivalence of lagrangian and hamiltonian BFV quantization in the path integral representation was demonstrated in Ref. [39].

3. Second-class constraints

3.1. Second-class constraints

Consider a symplectic manifold $\mathcal{M}$ with non-degenerate symplectic structure ω, $d\omega = 0$. In a generic coordinate system x^μ, $\mu = 1, 2, \ldots, 2N$, a dynamical system with the hamiltonian $H = H(x)$ is described by the hamiltonian equations of motion

$$\dot{x}^\mu = [x^\mu, H]^\omega_{\mathrm{PB}}, \tag{3.1}$$

where Poisson brackets are defined by the inverse $\omega^{\mu\nu}$ of the symplectic form $\omega_{\mu\nu}$, $\omega_{\mu\nu}\omega^{\nu\rho} = \delta^\rho_\mu$, so that

$$[x^\mu, x^\nu]^\omega_{\mathrm{PB}} = \omega^{\mu\nu}(x). \tag{3.2}$$

The first step of the BFF procedure to quantize the system (3.1), (3.2) is to get rid of the non-linear Poisson brackets (3.2). One first introduces a new set of variables p_μ, $\mu = 1, 2, \ldots, 2N$, and a new symplectic structure on the enlarged phase space of both variables x and p,

$$[x^\mu, p_\nu]_{\mathrm{PB}} = \delta^\mu_\nu. \tag{3.3}$$

To reduce the number of physical degrees of freedom, one subjects the new variables to the constraints

$$\theta_\mu = p_\mu - V_\mu = 0, \tag{3.4}$$

with such $V_\mu = V_\mu(x)$ that

$$\partial_\mu V_\nu - \partial_\nu V_\mu = \omega_{\mu\nu}(x) \tag{3.5}$$

and

$$[\theta_\mu, \theta_\nu]_{\mathrm{PB}} = \omega_{\mu\nu}. \tag{3.6}$$

The symplectic potential V is defined up to an abelian gauge transformation

$$V'_\mu = V_\mu + \partial_\mu \varphi. \tag{3.7}$$

Since the constraints are second-class we can introduce Dirac brackets

$$\begin{aligned} [x^\mu, x^\nu]_{\mathrm{D}} &= \omega^{\mu\nu}(x), \\ [x^\mu, p_\nu]_{\mathrm{D}} &= \delta^\mu_\nu + \omega^{\mu\rho}\partial_\nu V_\rho, \\ [p_\mu, p_\nu]_{\mathrm{D}} &= \partial_\mu V_\rho \omega^{\rho\sigma} \partial_\nu V_\sigma, \end{aligned} \tag{3.8}$$

and the hamiltonian equations of motion are

$$\dot{x}^\mu = [x^\mu, H]_{\mathrm{D}}, \quad \dot{p}_\mu = [p_\mu, H]_{\mathrm{D}}. \tag{3.9}$$

This hamiltonian system with phase space variables x^μ and p_μ and second-class constraints (3.4) is obviously equivalent to the original dynamical system (3.1) and follows from the lagrangian action

$$S = \int_{t_0}^{t_1} \mathrm{d}t \left[\dot{x}^\mu V_\mu - H(x) \right]. \tag{3.10}$$

Under the gauge transformation of the symplectic potential (3.7) the action changes on the constant

$$S' = S + \varphi(x_1) - \varphi(x_0). \tag{3.11}$$

3.2. Topological action and quantization condition

There is an important subtle point in this consideration. Although Eq. (3.5) can be solved locally in each coordinate patch, there does not exist a global solution for a one-form of symplectic potential V unless ω is exact. Consequently the action (3.10) is well defined only locally (on the paths that entirely lie in a contractible coordinate patch). However, in the Feynman path integral integration is performed over all paths on $\mathscr{M}$, so we need a global definition for the action.

Consider an arbitrary closed path Γ and a two-dimensional surface Σ bounded by Γ, $\partial\Sigma = \Gamma$. Let us define the action as follows:

$$S_\Sigma(\Gamma) = \int_\Sigma \omega - \int_\Gamma \mathrm{d}t\, H. \tag{3.12}$$

For this definition to be consistent, the Feynman phase factor $\exp[iS_\Sigma(\Gamma)/\hbar]$ must be independent of the choice of Σ, i.e. if Σ' is another surface bounded by Γ and $S_{\Sigma'}(\Gamma)$ is the corresponding action, we must have

$$\exp\left(\frac{i}{\hbar}\left[S_\Sigma(\Gamma)-S_{\Sigma'}(\Gamma)\right]\right)=1, \tag{3.13}$$

or, introducing the two-dimensional cycle $\Sigma''=\Sigma-\Sigma'$,

$$\frac{1}{2\pi\hbar}\int_{\Sigma''}\omega\in\mathbb{Z}. \tag{3.14}$$

Since the integration is performed over all paths Γ, the integral of $\omega/2\pi\hbar$ over any two-dimensional cycle must be integer, i.e. $[\omega/2\pi\hbar]\in \mathrm{H}^2(\mathscr{M},\mathbb{Z})$, and one obtains a topological quantization condition of geometric (pre)quantization [6,7,11,19].

3.3. Čech cohomology and Wu–Yang action

Let $\{\mathscr{O}_\alpha\}$ be a contractible open covering of $\mathscr{M}$, i.e. $\mathscr{M}=\bigcup_\alpha\mathscr{O}_\alpha$ and each coordinate patch and every finite intersection $\mathscr{O}_{\alpha_1}\cap\ldots\cap\mathscr{O}_{\alpha_k}$ is diffeomorphic to an open ball in $\mathbb{R}^{2N}$. When ω is not exact, on intersections $\mathscr{O}_\alpha\cap\mathscr{O}_\beta$, transitions from one coordinate patch to the other change the symplectic potential according to abelian gauge transformations

$$V_\alpha=V_\beta+\mathrm{d}\varphi_{\alpha\beta}, \tag{3.15}$$

and there does not exist a global gauge, i.e. a set of local gauges fixing φ_α in (3.7) in each coordinate patch $\mathscr{O}_\alpha$ such that $\varphi_{\alpha\beta}$ could be set to zero for all intersections $\mathscr{O}_\alpha\cap\mathscr{O}_\beta$. In other words, the set of gauge parameters $\{\varphi_{\alpha\beta}\}$ cannot be represented in the form $\varphi_{\alpha\beta}=\varphi_\alpha-\varphi_\beta$ for any set $\{\varphi_\alpha\}$. The problem of the global definition of the action functional when the symplectic potential does not exist as a one-form is the same one one has trying to define the global action for a point particle moving in the field of a Dirac magnetic monopole [51] (the electromagnetic interaction term $\dot{x}^\mu A_\mu$ and the field strength $F_{\mu\nu}$ are counterparts of our $\dot{x}^\mu V_\mu$ and $\omega_{\mu\nu}$).

Consider a path Γ going from the point $x_\alpha\in\mathscr{O}_\alpha$ to the point $x_\beta\in\mathscr{O}_\beta$ through a point $x_{\alpha\beta}\in\mathscr{O}_\alpha\cap\mathscr{O}_\beta$ in the non-empty overlap $\mathscr{O}_\alpha\cap\mathscr{O}_\beta$. Naively one could define the action as (we will omit the topologically trivial term with the hamiltonian while discussing the $\dot{x}V$-term)

$$S(\Gamma)=\int_{x_\alpha}^{x_{\alpha\beta}}V_\alpha+\int_{x_{\alpha\beta}}^{x_\beta}V_\beta. \tag{3.16}$$

The problem is that this definition depends on the choice of $x_{\alpha\beta}$. The correct definition was given by Wu and Yang in their work on Dirac magnetic monopoles [52] (see also Refs. [53,54]),

$$S(\Gamma)=\int_{x_\alpha}^{x_{\alpha\beta}}V_\alpha-\varphi_{\alpha\beta}(x_{\alpha\beta})+\int_{x_{\alpha\beta}}^{x_\beta}V_\beta, \tag{3.17}$$

where $\varphi_{\alpha\beta}(x_{\alpha\beta})$ is the value of the gauge parameter $\varphi_{\alpha\beta}$ (3.15) at the point $x_{\alpha\beta}$. It is easy to see that this definition is independent of the choice of $x_{\alpha\beta}$.

Following Ref. [53], consider now a path Γ going from x_α to x_β through a triple overlap $\mathscr{O}_\alpha \cap \mathscr{O}_\beta \cap \mathscr{O}_\gamma$, and let the points $x_{\alpha\gamma} \in \mathscr{O}_\alpha \cap \mathscr{O}_\gamma$, $x_{\gamma\beta} \in \mathscr{O}_\gamma \cap \mathscr{O}_\beta$ and $x_{\alpha\beta\gamma} \in \mathscr{O}_\alpha \cap \mathscr{O}_\beta \cap \mathscr{O}_\gamma$ lie on the path. With the help of the gauge transformations

$$V_\alpha = V_\beta + \mathrm{d}\varphi_{\alpha\beta}, \quad V_\beta = V_\gamma + \mathrm{d}\varphi_{\beta\gamma}, \quad V_\gamma = V_\alpha + \mathrm{d}\varphi_{\gamma\alpha}, \tag{3.18}$$

the Wu–Yang action

$$S(\Gamma) = \int_{x_\alpha}^{x_{\alpha\beta\gamma}} V_\alpha - \varphi_{\alpha\beta}(x_{\alpha\beta\gamma}) + \int_{x_{\alpha\beta\gamma}}^{x_\beta} V_\beta \tag{3.19}$$

can be rewritten in the form [53]

$$S(\Gamma) = \int_{x_\alpha}^{x_{\alpha\gamma}} V_\alpha - \varphi_{\alpha\gamma}(x_{\alpha\gamma}) + \int_{x_{\alpha\gamma}}^{x_{\gamma\beta}} V_\gamma - \varphi_{\gamma\beta}(x_{\gamma\beta}) + \int_{x_{\gamma\beta}}^{x_\beta} V_\beta$$
$$- \left[\varphi_{\alpha\beta}(x_{\alpha\beta\gamma}) + \varphi_{\beta\gamma}(x_{\alpha\beta\gamma}) + \varphi_{\gamma\alpha}(x_{\alpha\beta\gamma})\right]. \tag{3.20}$$

Furthermore, adding the three equations (3.18) together we obtain

$$\mathrm{d}(\varphi_{\alpha\beta} + \varphi_{\beta\gamma} + \varphi_{\gamma\alpha}) = 0, \tag{3.21}$$

i.e. the two-cocycle $n_{\alpha\beta\gamma} = \varphi_{\alpha\beta} + \varphi_{\beta\gamma} + \varphi_{\gamma\alpha}$ is constant over the entire triple overlap $\mathscr{O}_\alpha \cap \mathscr{O}_\beta \cap \mathscr{O}_\gamma$. The first five terms in the action (3.20) are just the Wu–Yang prescriptions for going from $\mathscr{O}_\alpha$ to $\mathscr{O}_\beta$ through $\mathscr{O}_\gamma$. The entire action (3.20) is the Wu–Yang prescription (3.19) for going from $\mathscr{O}_\alpha$ directly to $\mathscr{O}_\beta$. Thus, the action is ambiguous up to a constant $n_{\alpha\beta\gamma}$. In order for the Feynman phase factor $\exp[iS(\Gamma)/\hbar]$ to be well defined, all $n_{\alpha\beta\gamma}/2\pi\hbar$ must be integers. Thus we again arrive at the topological quantization condition

$$\frac{1}{2\pi\hbar}(\varphi_{\alpha\beta} + \varphi_{\beta\gamma} + \varphi_{\gamma\alpha}) \in \mathbb{Z}. \tag{3.22}$$

The closed two-form ω is a de Rahm representative of the Čech cocycle $\{n_{\alpha\beta\gamma}/2\pi\hbar\} \in \mathrm{H}^2(\mathscr{M}, \mathbb{Z})$, and the quantization conditions (3.14) and (3.22) are equivalent.

3.4. Prequantum bundle and second-class constraints

The set $\{V_\alpha\}$ of one-forms of symplectic potentials in the patches $\mathscr{O}_\alpha$ together with the set of gauge parameters $\{\varphi_{\alpha\beta}\}$ define a line bundle L over $\mathscr{M}$ with transition functions $g_{\alpha\beta} = \exp(-i\varphi_{\alpha\beta}/\hbar)$, connection V and the curvature ω, the Kostant–Souriau prequantum bundle [6,7,11,19]. A section ψ of L is represented by the set $\{\psi_\alpha\}$ of local functions $\psi_\alpha = \psi_\alpha(x)$ defined on $\mathscr{O}_\alpha$, and on the intersections $\mathscr{O}_\alpha \cap \mathscr{O}_\beta$ one has

$$\psi_\alpha = g_{\alpha\beta}\psi_\beta. \tag{3.23}$$

Obviously $g_{\alpha\beta}^{-1} = g_{\beta\alpha}$ and the consistency condition (cocycle identity)

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1 \tag{3.24}$$

is ensured by the quantization condition (3.22) obtained as a consistency condition for the path integral.

Under the substitution $p_\mu \to -i\hbar\partial_\mu$, our second-class constraints $\theta_\mu = p_\mu - V_\mu$ can be interpreted as covariant derivatives $\nabla_\mu^V = \partial_\mu - (\mathrm{i}/\hbar)V_\mu$ acting on the sections of L, and one has

$$\left[\nabla_\mu^V, \nabla_\nu^V\right] = -\frac{i}{\hbar}\omega_{\mu\nu}. \tag{3.25}$$

On the intersections we have (the indices α, β, γ labeling the coordinate patches are not to be confused with the indices μ, $\nu, \ldots$)

$$\nabla_\alpha^V = g_{\alpha\beta} \nabla_\beta^V g_{\alpha\beta}^{-1}, \tag{3.26}$$

where ∇_α^V is a covariant differential on $\mathscr{O}_\alpha$, $\nabla_\alpha^V = d - (\mathrm{i}/\hbar)V_\alpha$, and

$$\nabla_\alpha^V \psi_\alpha = g_{\alpha\beta}\left(\nabla_\beta^V \psi_\beta\right), \tag{3.27}$$

as required. Inequivalent line bundles are classified by the cohomology class of the two-form $c_1(L) = \omega/2\pi\hbar$, the first Chern class of L. The Feynman phase factor $\exp[iS(\Gamma)/\hbar]$ with the Wu–Yang action $S(\Gamma)$ defines parallel translations of sections of L along Γ.

Thus, we have found an operator representation for our constraints. However, since the constraints are second-class and do not commute on the constraint surface, we cannot interpret sections ψ of the prequantum bundle L as wave functions and enforce the quantum constraints $\nabla^V \psi = 0$.

3.5. Quantization of second-class constraints (Dirac, conversion and Gupta–Bleuler methods)

Generally, there are two ways to quantize systems with second-class constraints. The original Dirac method [1–4] consists in finding an operator representation for the Dirac brackets of canonical variables q and p and solving the constraints $\theta(\hat{q}, \hat{p}) = 0$. We can enforce the constraints in the Dirac method since the operators $\hat{q}$ and $\hat{p}$ are subject to the commutation relations dictated by the Dirac brackets (not Poisson) and the constraints commute with respect to Dirac brackets. However, the task of finding a direct operator representation of highly non-linear Dirac brackets (Eqs. (3.8) in the case in question) appears to be unsurmountable. Moreover, even at the formal level there is a serious problem with the Jacobi identities when the structure coefficients become operators.

The alternative way of quantizing systems with second-class constraints, the BFF conversion method, was proposed by Batalin, Fradkin and Fradkina in Refs. [25–28]. The BFF conversion essentially consists of two steps. First, one converts second-class constraints into the first-class ones by introducing some extra gauge degrees of freedom, Stuckelberg-type variables, and then quantizes the resulting system with first-class constraints according to the standard BFV quantization method described in the previous section. In this paper will apply BFF conversion to our dynamical system with the hamiltonian H and the second-class constraints $\theta_\mu = p_\mu - V_\mu$.

Let us note that there is also a third approach to quantize second-class constraints. This method is somewhat reminiscent of the Gupta–Bleuler quantization. One selects a commuting subset of N constraints $\theta_{\underline{i}}$, $\underline{i} = 1, 2, \ldots, N$, $[\theta_{\underline{i}}, \theta_{\underline{j}}]_{\mathrm{PB}} = 0$, among the $2N$ second-class constraints θ_μ, quantizes the canonical variables q and p according to the Poisson brackets and then enforces the quantum constraints $\hat{\theta}_{\underline{i}}$ on the physical states as if they were an independent system of first-class constraints, forgetting about the other half $\hat{\theta}_{\bar{i}}$, $\bar{i} = N+1, N+2, \ldots, 2N$. Though in some instances this approach may work, it generally can lead to a physically incorrect quantum theory. There may be difficulties with the consistent separation of second-class constraints into the effective first-class constraints $\theta_{\underline{i}}$ and effective gauge conditions $\theta_{\bar{i}}$. Even if such a separation does exist at the classical level, quantum anomalies can destroy quantum theory constructed in this way. In particular, achieving unitarity in the physical sector can be a problem. Moreover, there is no way to describe the entire algebra of quantum observables in this "on-the-mass-shell" approach. Only functions that commute with $\theta_{\underline{i}}$ are quantizable (if $[A, \theta_{\underline{i}}]_{\mathrm{PB}} = 0$, A can be regarded as an observable of the system with effective first-class constraints $\theta_{\underline{i}}$).

3.6. Classification of topologically inequivalent second-class constraints and prequantum bundles

Let us consider a question of how many topologically inequivalent definitions of second-class constraints lead to the consistent definition of the path integral. Equivalently, we are interested in the moduli space of line bundles with connection and the curvature $-i\omega/\hbar$. From the previous section it is obvious that topologically inequivalent situations are parametrized by the sets of U(1) gauge transformations $\{g_{\alpha\beta}\}$ satisfying the cocycle condition (3.24) modulo trivial (exact) cocycles of the form $\{g_\alpha g_\beta^{-1}\}$ for some $\{g_\alpha\}$, i.e. by the elements of the first Čech cohomology group $\mathrm{H}^1(\mathcal{M}, \mathrm{U}(1))$. Using simple homotopy considerations this group can also be represented as principal homogeneous space of the group of characters of the fundamental group π_1 of $\mathcal{M}$,

$$\mathrm{H}^1(\mathcal{M}, \mathrm{U}(1)) \simeq \mathrm{Hom}(\pi_1, \mathrm{U}(1))/\mathrm{U}(1). \tag{3.28}$$

Indeed, consider two topologically inequivalent definitions $S_1(\Gamma)$ and $S_2(\Gamma)$ of the Wu–Yang action which are based on two inequivalent line bundles L_1 and L_2 with connections $-iV_1/\hbar$ and $-iV_2/\hbar$ with the same curvature $-i\omega/\hbar$. Further, consider the corresponding Feynman phase factors $\exp(iS_1/\hbar)$ and $\exp(iS_2/\hbar)$. Then the quantity $\chi_{12}([\Gamma]) = \exp\{i[S_1(\Gamma) - S_2(\Gamma)]/\hbar\}$ depends only on the homotopy class $[\Gamma]$ of the path Γ, i.e. it is a character of the fundamental group π_1.

Thus, we see that inequivalent characters correspond to inequivalent line bundles and vice versa and, factoring out the overall U(1) factor, Hom(π_1, U(1))/U(1) is a sought after moduli space. (See Refs. [6,54] for more details.)

3.7. Symplectic connection

A torsion-free linear connection compatible with the symplectic structure should satisfy the invariance condition $\nabla_X \omega = 0$ for any vector field X, or in local coordinates

$$\nabla_\mu \omega_{\nu\rho} = \partial_\mu \omega_{\nu\rho} - \Gamma^\sigma_{\mu\nu}\omega_{\sigma\rho} - \Gamma^\sigma_{\mu\rho}\omega_{\nu\sigma} = 0. \tag{3.29}$$

In contrast with the riemannian geometry, this fixes Γ not uniquely, but up to an arbitrary totally symmetric tensor of the third rank. This arbitrariness can be fixed if there are other geometric structures on $\mathscr{M}$ (e.g. polarization, complex structure, riemannian metric or group action) and the covariant derivative is required to be compatible with them. For example, in the physically important case of Kähler manifolds, the symplectic connection is fixed by the requirement of compatibility with the complex structure $J = (J^\nu_\mu)$, $J^2 = -\mathbb{1}$. The compatibility condition

$$\nabla_\mu J^\rho_\nu = 0 \tag{3.30}$$

in the complex coordinates $x^\mu \to (z^{\underline{\alpha}}, \bar{z}^{\bar{\beta}})$, $\underline{\alpha}, \bar{\beta} = 1, \dots, N$, requires that $\Gamma^{\underline{\gamma}}_{\underline{\alpha\beta}}$ and $\Gamma^{\bar{\gamma}}_{\bar{\alpha}\bar{\beta}}$ be the only non-zero components of $\Gamma^\rho_{\mu\nu}$ (see e.g. Ref. [55]) and thus fixes the arbitrariness. In effect, it reduces the holonomy group $\mathrm{U}(N) \subset \mathrm{Sp}(2N; \mathbb{R}) \subset \mathrm{GL}(2N; \mathbb{R})$.

The curvature tensor of the linear symplectic connection

$$R_{\mu\nu\rho}{}^\sigma = \partial_\mu \Gamma^\sigma_{\nu\rho} - \partial_\nu \Gamma^\sigma_{\mu\rho} + \Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\rho} - \Gamma^\sigma_{\nu\lambda}\Gamma^\lambda_{\mu\rho} \tag{3.31}$$

has an important symmetry property,

$$R_{\mu\nu\rho}{}^\sigma \omega^{\rho\lambda} = R_{\mu\nu\rho}{}^\lambda \omega^{\rho\sigma}, \tag{3.32}$$

and satisfies the Bianchi identities (that follow from the symmetry of the connection, $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$)

$$R_{\mu\nu\rho}{}^\sigma + R_{\rho\mu\nu}{}^\sigma + R_{\nu\rho\mu}{}^\sigma = 0. \tag{3.33}$$

In the complex case, the symplectic curvature (3.31) coincides with the Riemann tensor of the Kähler metric, and the only non-zero components are $R_{\underline{\alpha}\bar{\beta}\underline{\gamma}}{}^{\underline{\delta}}$, $R_{\underline{\alpha}\bar{\beta}\bar{\gamma}}{}^{\bar{\delta}}$, $R_{\bar{\alpha}\underline{\beta}\underline{\gamma}}{}^{\underline{\delta}}$ and $R_{\bar{\alpha}\underline{\beta}\bar{\gamma}}{}^{\bar{\delta}}$.

4. Classical BFF conversion

4.1. First-class constraints and classical observables

To convert the second-class constraints θ_μ into the first-class ones, let us introduce additional variables ϕ_a, $a = 1, 2, \dots, 2N$, with the Poisson brackets

$$[\phi_a, \phi_b]_{\mathrm{PB}} = -\Lambda_{ab}, \tag{4.1}$$

where $\Lambda_{ab} = -\Lambda_{ba}$ is a flat symplectic metric with the inverse Λ^{ab}, $\Lambda_{ab}\Lambda^{bc} = \delta^c_a$.

Following Ref. [27] we are looking for abelian first-class constraints in the form

$$\mathcal{T}_\mu = p_\mu - W_\mu(x, \phi), \tag{4.2}$$

$$\left[\mathcal{T}_\mu, \mathcal{T}_\nu\right]_{\mathrm{PB}} = 0, \tag{4.3}$$

with the initial conditions

$$\mathcal{T}_\mu \mid_{\phi=0} = \theta_\mu, \quad \text{or} \quad W_\mu \mid_{\phi=0} = V_\mu, \tag{4.4}$$

where V_μ is the symplectic potential.

Introducing the notation ($\partial_\phi^a := \partial/\partial\phi_a$)

$$\mathcal{R}_{\mu\nu} := \partial_\mu W_\nu - \partial_\nu W_\mu - \partial_\phi^a W_\mu \Lambda_{ab} \partial_\phi^b W_\nu, \tag{4.5}$$

abelian Poisson bracket relations (4.3) can be regarded as zero-curvature equations for $W_\mu = W_\mu(x, \phi)$,

$$\mathcal{R}_{\mu\nu} = 0, \tag{4.6}$$

with the initial conditions (4.4).

Let $A = A(x)$ be a classical observable of the original dynamical system on $\mathcal{M}$. Then the corresponding observable $\mathcal{A} = \mathcal{A}(x, \phi)$ of the converted system with first-class constraints (4.2) must commute with $\mathcal{T}_\mu$,

$$\left[\mathcal{T}_\mu, \mathcal{A}\right]_{\mathrm{PB}} = 0, \tag{4.7}$$

i.e. satisfy the differential equations

$$\mathcal{D}_\mu \mathcal{A} = 0, \tag{4.8}$$

with the initial conditions

$$\mathcal{A}(x, \phi)|_{\phi=0} = A(x), \tag{4.9}$$

where we have introduced the notation

$$\mathcal{D}_\mu := -\left[\mathcal{T}_\mu, \cdot\right]_{\mathrm{PB}} = \partial_\mu - \partial_\phi^a W_\mu \Lambda_{ab} \partial_\phi^b. \tag{4.10}$$

We will call the function $\mathcal{A}$ of x and ϕ satisfying (4.8) and (4.9) the BFF extension of the original classical observable A.

4.2. *Symp*($\mathbb{R}^{2N}$) *gauge theory*

The above construction admits a natural physical interpretation which is already obvious from our notations. Consider a group Symp($\mathbb{R}^{2N}$) of symplectic diffeomorphisms (symplectomorphisms, canonical transformations of the linear symplectic space $\mathbb{R}^{2N}$, $\Lambda = \Lambda^{ab} \mathrm{d}\phi_\alpha \wedge \mathrm{d}\phi_b$) with the Lie algebra Ham($\mathbb{R}^{2N}$) of hamiltonian vector fields on $\mathbb{R}^{2N}$, i.e. vector fields of the form $X_f = [f, \cdot\,]_{\mathrm{PB}} = -\partial_\phi^a f \Lambda_{ab} \partial_\phi^b$ for some hamiltonian function $f = f(\phi)$. Then $[W_\mu(x, \phi), \cdot\,]_{\mathrm{PB}}$ is a Ham($\mathbb{R}^{2N}$)-valued gauge field acting on the functions $\mathcal{A} = \mathcal{A}(x, \phi)$, $\mathcal{D}_\mu$ is the corresponding covariant derivative and $[\mathcal{R}_{\mu\nu}, \cdot\,] = [\mathcal{D}_\mu, \mathcal{D}_\nu]$ is the curvature. The zero-curvature equations (4.6) reduce W_μ to a pure gauge, and (4.8) singles out gauge-invariant observables

$$\delta\mathscr{A}=\xi^\mu\left[\mathscr{T}_\mu,\mathscr{A}\right]_{\mathrm{PB}}=0, \tag{4.11}$$

where $\xi^\mu=\xi^\mu(t)$ are arbitrary time-dependent parameters. The zero-curvature equations (4.6) can then be regarded as Frobenius consistency conditions for (4.8), (4.11).

Eqs. (4.6), (4.8) are obviously invariant under Symp($\mathbb{R}^{2N}$) gauge transformations

$$\delta W_\mu=\mathscr{D}_\mu\epsilon\equiv\partial_\mu\epsilon-\partial^a_\phi W_\mu\,\Lambda_{ab}\partial^b_\phi\epsilon, \tag{4.12a}$$

$$\delta\mathscr{A}=-[\epsilon,\mathscr{A}]_{\mathrm{PB}}, \tag{4.12b}$$

with arbitrary gauge parameters $\epsilon=\epsilon(x,\phi)$. Jacobi identities for three first-class constraints reduce to the Bianchi identities

$$\left[\mathscr{T}_\mu,\left[\mathscr{T}_\nu,\mathscr{T}_\rho\right]\right]+\text{cycle}=-\mathscr{D}_{[\mu}\mathscr{R}_{\nu\rho]}=0. \tag{4.13}$$

Introducing an extended hamiltonian $\mathscr{H}_\lambda=\mathscr{H}+\lambda^\mu\mathscr{T}_\mu$ for our system with first-class constraints $\mathscr{T}_\mu$, $\mathscr{D}_\mu\mathscr{H}_\lambda=0$, the hamiltonian equations of motion read as follows:

$$\dot{x}^\mu=\left[x^\mu,\mathscr{H}_\lambda\right]_{\mathrm{PB}}=\lambda^\mu, \tag{4.14a}$$

$$\dot{p}_\mu=\left[p_\mu,\mathscr{H}_\lambda\right]_{\mathrm{PB}}=-\partial_\mu\mathscr{H}+\lambda^\nu\partial_\mu W_\nu, \tag{4.14b}$$

$$\dot{\phi}_a=[\phi_a,\mathscr{H}_\lambda]_{\mathrm{PB}}=-\Lambda_{ab}\partial^b_\phi\mathscr{H}+\lambda^\mu\Lambda_{ab}\partial^b_\phi W_\mu. \tag{4.14c}$$

They are invariant under the gauge transformations generated by the constraints

$$\delta x^\mu=-\xi^\nu[\mathscr{T}_\nu,x^\mu]_{\mathrm{PB}}=\xi^\mu, \tag{4.15a}$$

$$\delta p_\mu=-\xi^\nu\left[\mathscr{T}_\nu,p_\mu\right]_{\mathrm{PB}}=\xi^\nu\partial_\mu W_\nu, \tag{4.15b}$$

$$\delta\phi_a=-\xi^\nu[\mathscr{T}_\nu,\phi_a]_{\mathrm{PB}}=\xi^\nu\Lambda_{ab}\partial^b_\phi W_\nu, \tag{4.15c}$$

with arbitrary time-dependent gauge parameters $\xi^\mu=\xi^\mu(t)$. It is easy to see that the system (4.15) with constraints $\mathscr{T}_\mu=0$ and the gauge-invariant hamiltonian $\mathscr{H}$, $\mathscr{D}_\mu\mathscr{H}=0$, reduces to the original system (3.1) in the "unitary gauge" $\phi_a=0$.

Let us now establish a relation between diffeomorphisms on $\mathscr{M}$ and Symp($\mathbb{R}^{2N}$)-gauge transformations (4.12). First, consider a diffeomorphism

$$\delta x^\mu=\zeta^\mu \tag{4.16}$$

with an arbitrary parameters $\zeta^\mu=\zeta^\mu(x)$. Gauge fields W_μ and observables $\mathscr{A}$ transform according to

$$\delta W_\mu=\zeta^\nu\partial_\nu W_\mu+\partial_\mu\zeta^\nu W_\nu, \tag{4.17a}$$

$$\delta\mathscr{A}=\zeta^\nu\partial_\nu\mathscr{A}. \tag{4.17b}$$

Taking into account (4.6) and (4.8), we can rewrite this diffeomorphism as follows:

$$\delta W_\mu=\mathscr{D}_\mu(\zeta^\nu W_\nu), \tag{4.18a}$$

$$\delta\mathscr{A}=-\zeta^\nu[W_\nu,\mathscr{A}]_{\mathrm{PB}}. \tag{4.18b}$$

Now, these are precisely Symp($\mathbb{R}^{2N}$)-gauge transformations (4.12) with the gauge parameter

$$\epsilon(x,\phi)=\zeta^{\nu}(x)W_{\nu}(x,\phi). \tag{4.19}$$

Thus, there is an embedding of the general coordinate transformations group Diff($\mathscr{M}$) into the Symp($\mathbb{R}^{2N}$)-gauge group.

4.3. Lagrangian formulation

The hamiltonian theory with first-class constraints described above has a natural lagrangian formulation. Indeed, consider the action functional

$$S=\int \mathrm{d}t\Big[\tfrac{1}{2}\dot{\phi}_a\Lambda^{ab}\phi_b+\dot{x}^{\mu}W_{\mu}(x,\phi)-\mathscr{H}(x,\phi)\Big] \tag{4.20}$$

which leads to the lagrangian equations of motion

$$\frac{\delta S}{\delta x^{\mu}}=\dot{x}^{\nu}\partial_{[\mu}W_{\nu]}-\partial_{\mu}\mathscr{H}-\dot{\phi}_a\partial^a W_{\mu}=0, \tag{4.21a}$$

$$\frac{\delta S}{\delta \phi_a}=\Lambda^{ab}\dot{\phi}_b+\partial^a_{\phi}\mathscr{H}-\dot{x}^{\mu}\partial^a_{\phi}W_{\mu}=0. \tag{4.21b}$$

One notes that the master equations (4.6) and (4.8) are equivalent to the Noether identities

$$\frac{\delta S}{\delta x^{\mu}}+\partial^a_{\phi}W_{\mu}\Lambda_{ab}\frac{\delta S}{\delta\phi_b}=\dot{x}^{\nu}\mathscr{R}_{\mu\nu}-\mathscr{D}_{\mu}\mathscr{H}=0, \tag{4.22}$$

and the corresponding gauge transformations leaving the action invariant are given by

$$\delta x^{\mu}=\xi^{\mu}, \tag{4.23a}$$

$$\delta\phi_a=\partial^a_{\phi}W_{\mu}\,\Lambda_{ab}\xi^{\mu}, \tag{4.23b}$$

with an arbitrary time-dependent parameter $\xi^{\mu}=\xi^{\mu}(t)$.

It is now obvious that the action (4.20) indeed leads to the desired hamiltonian formulation with first class constraints and is physically equivalent to the original action in the "unitary gauge" $\phi_a=0$.

4.4. Perturbative expansion for first-class constraints

To solve the differential equations (4.6), (4.3), let us expand our gauge field W_{μ} in the Taylor series in ϕ_a (see Appendix A for our tensor notations):

$$W_{\mu}=\sum_{k=0}^{\infty}W_{\mu}^{a(k)}T_{a(k)}, \tag{4.24}$$

where the totally symmetric monomials

$$T_{a(k)}:=\frac{1}{k!}\phi_{a_1}\cdots\phi_{a_k},\quad k=1,2,\dots,\infty,\quad T_{a(0)}:=1 \tag{4.25}$$

form a basis in the Lie algebra PB($\mathbb{R}^{2N}$) of smooth functions on $\mathbb{R}^{2N}$ with respect to the Poisson bracket (4.1). Then the zero-curvature equations

$$\mathscr{R}_{\mu\nu} = \sum_{k=0}^{\infty} \mathscr{R}_{\mu\nu}^{a(k)} T_{a(k)} = 0 \tag{4.26}$$

reduce to a series of recurrent relations

$$\mathscr{R}_{\mu\nu}^{a(k)} = \partial_\mu W_\nu^{a(k)} - \partial_\nu W_\mu^{a(k)} - \sum_{p+q=k} \frac{k!}{p!q!} W_\mu^{a(p)b} \Lambda_{bc} W_\nu^{a(q)c} = 0, \tag{4.27}$$

with the initial condition

$$W_\mu^{a(0)} = V_\mu. \tag{4.28}$$

Gauge transformations (4.12a) and Bianchi identities take the form

$$\delta W_\mu^{a(k)} = \partial_\mu \epsilon^{a(k)} - \sum_{p+q=k} \frac{k!}{p!q!} W_\mu^{a(p)b} \Lambda_{bc} \epsilon^{a(q)c}, \tag{4.29a}$$

$$\partial_\mu \mathscr{R}_{\nu\rho}^{a(k)} - \sum_{p+q=k} \frac{k!}{p!q!} W_\mu^{a(p)b} \Lambda_{bc} \mathscr{R}_{\nu\rho}^{a(q)c} + \text{cycle} = 0. \tag{4.29b}$$

4.5. Symplectic gravity

Note that polynomials of at most second order form a subalgebra in PB($\mathbb{R}^{2N}$) which is isomorphic to the semidirect sum sp($2N$; $\mathbb{R}$)$\oplus$H$_{2N}$ of symplectic and Heisenberg Lie algebras:

$$M_{ab} = -\tfrac{1}{2}\phi_a\phi_b, \quad P_a = \phi_a, \quad \mathscr{C} = 1, \tag{4.30a}$$

$$[M_{ab}, M_{cd}]_{\text{PB}} = \tfrac{1}{2}(\Lambda_{bc}M_{ad} + \Lambda_{ac}M_{bd} + \Lambda_{bd}M_{ac} + \Lambda_{ad}M_{bc}), \tag{4.30b}$$

$$[M_{ab}, P_c]_{\text{PB}} = \tfrac{1}{2}(\Lambda_{bc}P_a + \Lambda_{ac}P_b), \tag{4.30c}$$

$$[P_a, P_b]_{\text{PB}} = -\Lambda_{ab}\mathscr{C}, \quad [\mathscr{C}, M_{ab}]_{\text{PB}} = [P_a, \mathscr{C}] = 0. \tag{4.30d}$$

This algebra serves as a counterpart of the euclidean (Poincaré) algebra for symplectic geometry; symplectic generators M_{ab} substituting the (Lorentz) rotation generators and P_a being symplectic translations. More precisely, it is the Lie algebra isp($2N$; $\mathbb{R}$) of hamiltonian vector fields with the hamiltonian functions (4.30a),

$$[M_{ab}, \cdot\,]_{\text{PB}} = \tfrac{1}{2}(\phi_a\Lambda_{bc}\partial_\phi^c + \phi_b\Lambda_{ac}\partial_\phi^c), \quad [P_a, \cdot\,]_{\text{PB}} = \Lambda_{ab}\partial_\phi^b, \tag{4.31}$$

that generates the group ISp($2N$; $\mathbb{R}$) of symplectic affine transformations of ($\mathbb{R}^{2N}$, Λ), rather than the algebra (4.30) itself. The sp($2N$; $\mathbb{R}$)$\oplus$H$_{2N}$ is a central extension of isp($2N$; $\mathbb{R}$) by the constant functions. Since constants commute with everything, there is no hamiltonian vector field associated to $\mathscr{C}$. However, it is the central extension sp($2N$; $\mathbb{R}$)$\oplus$H$_{2N}$ that is important in quantum mechanics.

Now let us consider an $sp(2N;\mathbb{R})\oplus H_{2N}$-valued gauge field

$$W_\mu^{sp\oplus H} = V_\mu \mathscr{C} + h_\mu^a P_a + \Delta_\mu^{ab} M_{ab}. \tag{4.32}$$

Here $\Delta_\mu^{ab} = \Delta_\mu^{ba}$ is a symplectic connection (counterpart of the Lorentz connection), h_μ^a is a symplectic vielbein and V_μ is an abelian gauge field associated to the central element $\mathscr{C}$.

Curvature components read as follows:

$$F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - h_\mu^a \Lambda_{ab} h_\nu^b, \tag{4.33a}$$

$$R_{\mu\nu}^a = \partial_\mu h_\nu^a - \partial_\nu h_\mu^a + \Delta_\mu^{ab}\Lambda_{bc} h_\nu^c - \Delta_\nu^{ab}\Lambda_{bc} h_\mu^c, \tag{4.33b}$$

$$R_{\mu\nu}^{ab} = \partial_\mu \Delta_\nu^{ab} - \partial_\nu \Delta_\mu^{ab} + \Delta_\mu^{ac}\Lambda_{cd}\Delta_\nu^{db} - \Delta_\nu^{ac}\Lambda_{cd}\Delta_\mu^{db}. \tag{4.33c}$$

Choosing V_μ to be the symplectic potential (3.5) for the curved symplectic metric $\omega_{\mu\nu}$ on $\mathscr{M}$ and imposing a constraint

$$F_{\mu\nu} = 0, \tag{4.34}$$

h_μ^a indeed becomes the symplectic vielbein

$$\omega_{\mu\nu} = h_\mu^a \Lambda_{ab} h_\nu^b. \tag{4.35}$$

Then Δ_μ^{ab} is the symplectic connection on $\mathscr{M}$. Symplectic covariant derivatives of a tensor with tangent indexes are defined by

$$D_\mu^{sp} A^{a_1\ldots a_k} := \partial_\mu A^{a_1\ldots a_k} + \Delta_\mu^{a_1 b}\Lambda_{bc} A^{c a_2 \ldots a_k} + \ldots, \tag{4.36}$$

or for a totally symmetric tensor

$$D_\mu^{sp} A^{a(k)} = \partial_\mu A^{a(k)} + k\Delta_\mu^{ab}\Lambda_{bc} A^{ca(k-1)}. \tag{4.37}$$

Vanishing of the total covariant derivative of the vielbein with respect to both the lower curved index and upper tangent index gives the relation between symplectic and linear connections,

$$\partial_\mu h_\nu^a + \Delta_\mu^{ab}\Lambda_{bc} h_\nu^c - \Gamma_{\mu\nu}^\rho h_\rho^a = 0. \tag{4.38}$$

If the linear torsion $\Gamma_{\mu\nu}^\rho - \Gamma_{\nu\mu}^\rho$ is equal to zero, the symplectic torsion $R_{\mu\nu}^a$ also vanishes

$$R_{\mu\nu}^a = D_\mu^{sp} h_\nu^a - D_\nu^{sp} h_\mu^a = h_\rho^a\left(\Gamma_{\mu\nu}^\rho - \Gamma_{\nu\mu}^\rho\right) = 0. \tag{4.39}$$

Similar to the Lorentz connection in gravity that can be found from the zero-torsion condition, Δ_μ^{ab} can also be found from Eq. (4.39). However, in the symplectic case it is fixed by (4.39) only up to the totally symmetric tensor ϵ^{abc} that can be added to the connection,

$$\Delta_\mu'^{ab} = \Delta_\mu^{ab} + h_\mu^d \Lambda_{dc}\epsilon^{abc}. \tag{4.40}$$

When there is an additional geometric structure on $\mathscr{M}$ (complex structure, polarization, riemannian metric or group action), this arbitrariness can be fixed globally by the requirement of invariance of the corresponding structure, as we discussed in the previous section. For example, if there is a complex structure compatible with the connection, the only non-trivial components of the connection are those associated with the subgroup $U(N) \subset Sp(2N; \mathbb{R})$.

Similar to the situation with the Lorentz connection in gravity (see Ref. [56]), relation (4.38) leads to the following relation between the linear and symplectic curvatures:

$$R^{ab}_{\mu\nu}(\Delta) = R_{\mu\nu\rho}{}^{\sigma}(\Gamma) h^a_\sigma h^\rho_c \Lambda^{cb}, \tag{4.41}$$

where h^μ_a is an inverse vielbein,

$$h^a_\mu h^\nu_a = \delta^\nu_\mu, \quad h^a_\mu h^\mu_b = \delta^a_b. \tag{4.42}$$

Symplectic curvature satisfies the Bianchi identities

$$h^b_\rho \Lambda_{bc} R^{ac}_{\mu\nu} + h^b_\mu \Lambda_{bc} R^{ac}_{\nu\rho} + h^b_\nu \Lambda_{bc} R^{ac}_{\rho\mu} = 0, \tag{4.43}$$

which follow from the zero-torsion condition (4.39).

We will also often use the following relation between the Yang–Mills symplectic and linear covariant derivatives $D^{\rm sp}_\mu$ and ∇_μ (see (4.37), (4.38)):

$$D^{\rm sp}_\mu A^{a(k)} = D^{\rm sp}_\mu\left(h^a_\nu \dots h^a_\nu A^{\nu(k)}\right) = h^a_\nu \dots h^a_\nu \nabla_\mu A^{\nu(k)}. \tag{4.44}$$

4.6. Higher spin fields

Now we are ready to return to the zero-curvature equations (4.27). The first step is $k = 0$:

$$\mathscr{R}^{a(0)}_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - W^a_\mu \Lambda_{ab} W^b_\nu = 0, \tag{4.45}$$

i.e. $W^a_\mu = h^a_\mu$ is a symplectic vielbein (see (4.33a), (4.35)). Next, for $k = 1$, the equations take the form of the zero-torsion conditions (4.39):

$$\mathscr{R}^a_{\mu\nu} = \partial_\mu h^a_\nu - \partial_\nu h^a_\mu - \left(W^{ab}_\mu \Lambda_{bc} h^c_\nu - W^{ab}_\nu \Lambda_{bc} h^c_\mu\right) = 0, \tag{4.46}$$

and, consequently, $W^{a(2)}_\mu = -\Delta^{a(2)}_\mu$. Further, for $k = 2$ we have

$$\mathscr{R}^{a(2)}_{\mu\nu} = -R^{a(2)}_{\mu\nu} - \left[h^b_\mu \Lambda_{bc} W^{a(2)c}_\nu - (\mu \leftrightarrow \nu)\right] = 0, \tag{4.47}$$

where $R^{a(2)}_{\mu\nu}$ is a $sp(2N; \mathbb{R})$ curvature tensor (4.33c). The field $W^{a(3)}_\mu$ is determined by this relation up to an arbitrary totally symmetric tensor of the fourth rank $\epsilon^{a(4)}$ (gauge parameter). To fix this gauge arbitrariness one can choose the gauge fixing condition

$$\Lambda^{ab} h^\mu_b W^{a(3)}_\mu = 0, \quad \text{i.e.} \quad \epsilon^{a(4)} = 0. \tag{4.48}$$

Then, using Bianchi identities (4.43), one obtains (see Appendix B)

$$W^{a(3)}_\mu = \tfrac{3}{4} \Lambda^{ab} h^\nu_b R^{a(2)}_{\mu\nu} = -\tfrac{3}{4} h^{a(3)}_{\rho(3)}\left(\omega^{\rho\nu}\omega^{\rho\sigma} R_{\mu\nu\sigma}{}^{\rho}\right). \tag{4.49}$$

Thus, the "symplectic spin-three" gauge field is expressed in terms of the symplectic curvature up to a pure gauge. Continuing this recurrent procedure, one can express all "higher spin" fields in terms of symplectic curvature and its covariant derivatives. $W_\mu^{a(k)}$ is found from Eq. (4.27) up to an arbitrary totally symmetric tensor $\epsilon^{a(k+1)}$, gauge parameter of the gauge transformations (4.29a), and the gauge can be uniquely fixed by the generalization of (4.48),

$$\Lambda^{ab}h_b^\mu W_\mu^{a(k)}=0. \tag{4.50}$$

To summarize, the sought after first-class constraints have the following form:

$$\mathscr{T}_\mu = p_\mu - V_\mu - h_\mu^a\phi_a + \tfrac{1}{2}\Delta_\mu^{ab}\phi_a\phi_b - \tfrac{3}{4}\Lambda^{ab}h_b^\nu R_{\mu\nu}^{a(2)}\phi_{a_1}\phi_{a_2}\phi_{a_3} + \dots \,. \tag{4.51}$$

4.7. Perturbative solution for classical observables

Our next task is to solve Eqs. (4.8) for classical observables $\mathscr{A}=\mathscr{A}(x,\phi)$. Expanding $\mathscr{A}$ in the powers of ϕ,

$$\mathscr{A}(x,\phi)=\sum_{k=0}^{\infty}\mathscr{A}^{a(k)}T_{a(k)}, \tag{4.52}$$

one reduces them to the following set of recurrent relations:

$$\partial_\mu\mathscr{A}^{a(k)} - \sum_{p+q=k}\frac{k!}{p!q!}W_\mu^{a(p)b}\Lambda_{bc}\mathscr{A}^{a(q)c}=0, \tag{4.53}$$

subject to the initial condition

$$\mathscr{A}^{a(0)}=A. \tag{4.54}$$

The first equation with $k=0$,

$$\partial_\mu A - h_\mu^b\Lambda_{bc}\mathscr{A}^c=0, \tag{4.55}$$

gives

$$\mathscr{A}^a=\Lambda^{ab}h_b^\mu\partial_\mu A. \tag{4.56}$$

Next, for $k=1$ one has

$$D_\mu^{\mathrm{sp}}\mathscr{A}^a - h_\mu^b\Lambda_{bc}\mathscr{A}^{ac}=0 \tag{4.57}$$

and, using Eqs. (4.44) and the consistency condition

$$\Lambda^{cb}h_b^\mu D_\mu^{\mathrm{sp}}\mathscr{A}^a=\Lambda^{ab}h_b^\mu D_\mu^{\mathrm{sp}}\mathscr{A}^c \tag{4.58}$$

which follows from the zero-torsion condition (4.39), one finds

$$\mathscr{A}^{a(2)}=\Lambda^{a(2),b(2)}h_{b(2)}^{\mu(2)}\nabla_{\mu(2)}A. \tag{4.59}$$

Further, the equation for $k=2$,

$$D_\mu^{\mathrm{sp}}\mathscr{A}^{a(2)} - h_\mu^b\Lambda_{bc}\mathscr{A}^{a(2)c} - W_\mu^{a(2)b}\Lambda_{bc}\mathscr{A}^c=0, \tag{4.60}$$

has a unique solution (see (B.2), (B.3))

$$\mathscr{A}^{a(3)}=\Lambda^{a(3),b(3)}h_{b(3)}^{\mu(3)}\left(\nabla_{\mu(3)}A - \tfrac{1}{4}R_{\nu\mu\mu}{}^{\sigma}\omega_{\sigma\mu}\omega^{\nu\rho}\partial_\rho A\right). \tag{4.61}$$

This recurrent procedure can be continued to find higher order terms. As a result, for every classical observable A one obtains a unique BFF-extension $\mathscr{A}$,

$$\mathscr{A} = A + \Phi^{\mu}\partial_{\mu}A + \tfrac{1}{2}\Phi^{\mu_2}\Phi^{\mu_2}\nabla_{\mu_1\mu_2}A + \tfrac{1}{6}\Phi^{\mu_1}\Phi^{\mu_2}\Phi^{\mu_3}\left(\nabla_{\mu_1\mu_2\mu_3}A - \tfrac{1}{4}R_{\nu\mu_1\mu_2}{}^{\sigma}\omega_{\sigma\mu_3}\omega^{\nu\rho}\partial_{\rho}A\right) + \dots, \tag{4.62a}$$

where we have introduced the notation

$$\Phi^{\mu} := \phi_a \Lambda^{ab} h^{\mu}_{b}. \tag{4.62b}$$

In particular, the BFF-extension $\mathscr{X}^{\mu}$ of the phase space coordinates themselves has a normal-coordinate-type expansion

$$\mathscr{X}^{\mu} = x^{\mu} + \Phi^{\mu} + \tfrac{1}{2}\Phi^{\nu_1}\Phi^{\nu_2}\Gamma^{\mu}_{\nu_1\nu_2} + \tfrac{1}{6}\Phi^{\nu_1}\Phi^{\nu_2}\Phi^{\nu_3}\left(-\partial_{\nu_1}\Gamma^{\mu}_{\nu_2\nu_3} + 2\Gamma^{\lambda}_{\nu_1\nu_2}\Gamma^{\mu}_{\lambda\nu_3} - \tfrac{1}{4}R_{\rho\nu_1\nu_2}{}^{\sigma}\omega_{\sigma\nu_3}\omega^{\rho\mu}\right) + \dots . \tag{4.63}$$

Then the BFF-extension $\mathscr{X}^{\mu}$ of any other classical observable is a function $\mathscr{A}(x, \phi) = A(\mathscr{X}(x, \phi))$.

It is easy to see that the flat Poisson bracket (4.1) of two BFF-extensions $\mathscr{A}$ and $\mathscr{B}$ of A and B indeed coincide with the BFF-extension of the curved Poisson bracket (3.2) of the original observables A and B,

$$\left([\mathscr{A}, \mathscr{B}]_{\text{PB}}\right)\big|_{\phi=0} = -\mathscr{A}^{a}\Lambda_{ab}\mathscr{B}^{b} = \partial_{\mu}A\,\omega^{\mu\nu}\partial_{\nu}B. \tag{4.64}$$

Combining the results (4.51) and (4.62), the lagrangian (4.20) takes the form

$$\begin{aligned}\mathscr{L} = {} & \tfrac{1}{2}\dot{\phi}^{a}\Lambda_{ab}\phi^{b} + \dot{x}^{\mu}V_{\mu} - H(x) + \phi_a\left(h^{a}_{\mu}\dot{x}^{\mu} - \Lambda^{ab}h^{\mu}_{b}\partial_{\mu}H\right) \\ & - \tfrac{1}{2}\phi_a\phi_b\left(\Delta^{ab}_{\mu}\dot{x}^{\mu} + \Lambda^{a(2),b(2)}h^{\mu(2)}_{b(2)}\nabla_{\mu(2)}H\right) + \tfrac{1}{6}\phi_{a_1}\phi_{a_2}\phi_{a_3}\Big[\tfrac{3}{4}R^{a(2)}_{\mu\nu}\dot{x}^{\mu}h^{\nu}_{b}\Lambda^{ab} \\ & - \Lambda^{a(3),b(3)}\left(\nabla_{\mu(3)}H - \tfrac{1}{4}R_{\nu\mu\mu}{}^{\sigma}\omega_{\sigma\mu}\omega^{\nu\rho}\partial_{\rho}H\right)\Big] + \dots . \end{aligned} \tag{4.65}$$

4.8. Pure gauge solution and parallel transport of observables

Consider a coordinate neighborhood $\mathscr{O}$ of the point $x_0 \in \mathscr{O}$ and a path Γ going from x_0 to another point $x_1 \in \mathscr{O}$, and let $W_{\mu} = W_{\mu}(x, \phi)$ be a solution(4.51) of the zero-curvature equations. Then parallel transport of a classical observable $\mathscr{A}$ from x_0 to x_1 along the path Γ is defined by the parallel translation operator

$$G(x_1, x_0) = \text{P}\exp\left(-\int_{x_0}^{x_1}[W, \cdot]_{\text{PB}}\right) = \text{P}\exp\left(\int_{t_0}^{t_1}\mathrm{d}t\,\dot{x}^{\mu}\partial^{a}_{\phi}W_{\mu}(x(t), \phi)\Lambda_{ab}\partial^{b}_{\phi}\right), \tag{4.66a}$$

$$[G(x_1, x_0)]^{-1} = G(x_0, x_1). \tag{4.66b}$$

Due to flatness of W it satisfies the equation

$$\partial_\phi^a W_\mu \Lambda_{ab} \partial_\phi^b = \partial_\mu G(x, x_0)[G(x, x_0)]^{-1}, \tag{4.67}$$

i.e. it provides a pure gauge solution to the zero-curvature equations.

Then the master equations for classical observables have a general solution

$$\mathscr{A}(x, \phi) = G(x, x_0)\mathscr{A}_0(\phi), \tag{4.68}$$

with the initial condition $\mathscr{A}_0(\phi) = \mathscr{A}(x_0, \phi)$. Thus it is enough to know the value of $\mathscr{A}$ at the initial point x_0 to be able to reconstruct the function $\mathscr{A}(x, \phi)$ on the entire coordinate neighborhood $\mathscr{O}$. If we change the initial point x_0 to x_0', the initial condition in (4.68) changes to $\mathscr{A}_0' = G(x_0', x_0)\mathscr{A}_0$, and we again have $\mathscr{A}(x, \phi) = G(x, x_0')\mathscr{A}_0'(\phi)$.

4.9. Flat phase space

Let us now consider the simplest example, the flat phase space $\mathscr{M} = \mathbb{R}^{2N}$ with canonical coordinates x^a and canonical Poisson brackets

$$[x^a, x^b]_{\mathrm{PB}}^{\Lambda} = \Lambda^{ab}. \tag{4.69}$$

Introducing p_a and ϕ_a and a new Poisson bracket on the enlarged $6N$-dimensional space of x, p and ϕ,

$$[x^a, p_b]_{\mathrm{PB}} = \delta_b^a, \tag{4.70a}$$

$$[\phi_a, \phi_b]_{\mathrm{PB}} = -\Lambda_{ab}, \tag{4.70b}$$

first-class constraints take the form

$$\mathscr{T}_a = p_a - (V_a + \phi_a) = 0, \quad V_a = \tfrac{1}{2}\Lambda_{ab} x^b. \tag{4.71}$$

The BFF-extension $\mathscr{A}$ of the original observable A satisfies the equation

$$\frac{\partial \mathscr{A}}{\partial x^a} - \Lambda_{ab}\frac{\partial \mathscr{A}}{\partial \phi_b} = 0, \tag{4.72}$$

and an obvious solution is given by

$$\mathscr{X}^a = x^a + \Lambda^{ab}\phi_b, \tag{4.73a}$$

$$\mathscr{A}(x, \phi) = \mathscr{A}(\mathscr{X}(x, \phi)) = \sum_{k=0}^{\infty} \frac{1}{k!}\phi_{a_1}\dots\phi_{a_k}\Lambda^{a(k),b(k)}\frac{\partial^k A}{\partial x^{b_1}\dots\partial x^{b_k}}. \tag{4.73b}$$

Selecting the path $x^a(t) = tx^a$ going from the coordinate origin $x^a(0) = 0$ to some point $x^a(1) = x^a$, the parallel transport operator (4.66) takes the form

$$G(x, 0) = \exp\left(x^a \Lambda_{ab}\frac{\partial}{\partial \phi_b}\right) \tag{4.74}$$

and

$$\mathscr{A}(x, \phi) = G(x, 0)\mathscr{A}_0(\phi), \tag{4.75}$$

i.e. it acts as a Symp($\mathbb{R}^{2N}$)-gauge transformation (4.12b) with the gauge parameter $\epsilon = x^a \phi_a$.

4.10. Fiber bundle interpretation

Consider a bundle $\mathscr{E}$ over $\mathscr{M}$ with fibers $F \simeq C^\infty(\mathbb{R}^{2N})$ and the structure group $G \simeq \text{Symp}(\mathbb{R}^{2N})$. The first-class constraints $\mathscr{T}_\mu$ define a flat connection on $\mathscr{E}$,

$$X(W) = [W, \cdot]_{\text{PB}} = -\partial^a_\phi W \Lambda_{ab} \partial^b_\phi, \quad W = \mathrm{d}x^\mu W_\mu, \tag{4.76}$$

and classical observables commuting with the constraints are constant sections of $\mathscr{E}$. Thus, to every classical observable A on $\mathscr{M}$, the BFF conversion procedure associates a covariantly constant section $\mathscr{A}$ of the Symp($\mathbb{R}^{2N}$)-bundle $\mathscr{E}$ with the flat connection defined by the first-class constraints.

Let $\{\mathscr{O}_\alpha\}$ be a contractible open covering of $\mathscr{M}$, and $\{X(W_\alpha)\}$ and $\{\mathscr{A}_\alpha\}$ are trivializations of the flat connection $X(W)$ and a covariantly constant section $\mathscr{A}$, i.e. $W_\alpha = \mathrm{d}x^\mu W_{\alpha,\mu}$ and $\mathscr{A}_\alpha$ are solutions of Eqs. (4.6) and (4.8) on $\mathscr{O}_\alpha$ that can be represented in the pure gauge form

$$X(W_\alpha) = -\mathrm{d}G_\alpha G_\alpha^{-1}, \tag{4.77}$$

$$\mathscr{A}_\alpha = G_\alpha \mathscr{A}_{0,\alpha}, \tag{4.78}$$

with Symp($\mathbb{R}^{2N}$)-gauge transformations

$$G_\alpha(x_\alpha, x_{0,\alpha}) = \mathrm{P}\exp\left(-\int_{x_{0,\alpha}}^{x_\alpha} X(W_\alpha)\right) \tag{4.79a}$$

and initial conditions

$$\mathscr{A}_{0,\alpha} = \mathscr{A}_\alpha(x_{0,\alpha}, \phi). \tag{4.79b}$$

Here we have selected the origins $x_{0,\alpha} \in \mathscr{O}_\alpha$ of the coordinate patches $\mathscr{O}_\alpha$ (vertices of the triangulation of $\mathscr{M}$ associated to the covering $\{\mathscr{O}_\alpha\}$), and x_α are generic points of $\mathscr{O}_\alpha$.

On the intersections $\mathscr{O}_\alpha \cap \mathscr{O}_\beta$ we have Symp($\mathbb{R}^{2N}$)-gauge transformations $G_{\alpha\beta} = G_{\alpha\beta}(x_{\alpha\beta})$, $x_{\alpha\beta} \in \mathscr{O}_\alpha \cap \mathscr{O}_\beta$, such that

$$G_{\alpha\beta}^{-1} = G_{\beta\alpha}, \quad G_{\alpha\alpha} = \mathbb{1}, \tag{4.80}$$

$$\mathscr{A}_\alpha = G_{\alpha\beta}\mathscr{A}_\beta, \tag{4.81a}$$

$$X(W_\alpha) = G_{\alpha\beta} X(W_\beta) G_{\alpha\beta}^{-1} - \mathrm{d}G_{\alpha\beta}\, G_{\alpha\beta}^{-1}. \tag{4.81b}$$

On non-empty triple overlaps $\mathscr{O}_\alpha \cap \mathscr{O}_\beta \cap \mathscr{O}_\gamma$ they must satisfy the cocycle identities

$$G_{\alpha\beta} G_{\beta\gamma} G_{\gamma\alpha} = \mathbb{1} \tag{4.82}$$

that ensure the consistency of (4.81).

Substituting pure gauge solutions (4.77) and (4.78) in Eqs. (4.81), we find that the combinations $[G_\alpha(x_{\alpha\beta}, x_{0,\alpha})]^{-1}G_{\alpha\beta}(x_{\alpha\beta})G_\beta(x_{\alpha\beta}, x_{0,\beta})$ are constant over the entire overlap $\mathscr{O}_\alpha \cap \mathscr{O}_\beta$,

$$d\left(G_\alpha^{-1}G_{\alpha\beta}G_\beta\right) = 0. \tag{4.83}$$

As a result, general solution for $G_{\alpha\beta}$ can be written in the form

$$G_{\alpha\beta} = G_\alpha G_{0,\alpha\beta} G_\beta^{-1}, \tag{4.84}$$

where the initial conditions $G_{0,\alpha\beta}$ are independent of x.

Thus we have reduced the problem of constructing the Symp($\mathbb{R}^{2N}$)-bundle $\mathscr{E}$ to finding a set $\{G_{0,\alpha\beta}\}$ of canonical transformations of the flat phase space, $G_{0,\alpha\beta} \in$ Symp($\mathbb{R}^{2N}$), associated to the covering $\{\mathscr{O}_\alpha\}$ and satisfying the cocycle identities

$$G_{0,\alpha\beta}G_{0,\beta\gamma} = G_{0,\alpha\gamma}. \tag{4.85}$$

Substituting the solutions (4.78) for $\mathscr{A}_\alpha$ and $\mathscr{A}_\beta$ in Eqs. (4.81) and taking into account (4.84), one sees that the canonical transformations $G_{0,\alpha\beta}$ relate initial conditions $\mathscr{A}_{0,\alpha}$ and $\mathscr{A}_{0,\beta}$ for coordinate patches $\mathscr{O}_\alpha$ and $\mathscr{O}_\beta$ that have non-empty overlaps $\mathscr{O}_\alpha \cap \mathscr{O}_\beta$,

$$\mathscr{A}_{0,\alpha} = G_{0,\alpha\beta}\mathscr{A}_{0,\beta}. \tag{4.86}$$

Inequivalent bundles correspond to inequivalent solutions of the cocycle identties (4.85). Two solutions $\{G_{0,\alpha\beta}\}$ and $\{G'_{0,\alpha\beta}\}$ are said to be equivalent if there exists a set of gauge transformations $\{G_{0,\alpha}\}$, $G_{0,\alpha} \in$ Symp($\mathbb{R}^{2N}$), such that $G'_{0,\alpha\beta} = G_{0,\alpha}G_{0,\alpha\beta}G_{0,\beta}^{-1}$. From (4.78), (4.79), (4.84) and (4.86) it can be easily deduced that $\{G_{0,\alpha}\}$ are gauge transformations defining parallel transports from the initial triangulation vertices $x_{0,\alpha}$ to some new vertices $x'_{0,\alpha}$ where the initial conditions $\mathscr{A}_{0,\alpha}$ are taken. In particular, trivial cocycles are those that can be represented in the form $\{G_{0,\alpha}G_{0,\beta}^{-1}\}$ for some $\{G_{0,\alpha}\}$. Then all $G_{0,\alpha\beta}$ can be reduced to the identity transformations, and the corresponding bundle is just a direct product $\mathscr{M} \times C^\infty(\mathbb{R}^{2N})$. A set of gauge equivalence classes of $\{G_{0,\alpha\beta}\}$ satisfying the cocycle identities is called the first Čech cohomology with coefficients in Symp($\mathbb{R}^{2N}$), $H^1(\mathscr{M}$, Symp($\mathbb{R}^{2N}$)). It is a non-abelian generalization of the Čech cohomology $H^1(\mathscr{M}$, U(1)), where instead of U(1) we now have the entire infinite-dimensional group of canonical transformations of $\mathbb{R}^{2N}$.

On the other hand, the $G_{0,\alpha\beta}$ subject to the cocycle identities generate a representation of the fundamental group π_1 of $\mathscr{M}$ by canonical transformations. (It can be easily seen by considering a triangulation of $\mathscr{M}$, i.e. a polyhedron with the vertices $x_{0,\alpha}$ and edges $l_{\alpha\beta} = [x_{0,\alpha}, x_{0,\beta}]$ associated to the covering $\{\mathscr{O}_\alpha\}$, and noting that $G_{0,\alpha\beta} = [G_\alpha(x_{\alpha\beta}, x_{0,\alpha})]^{-1}G_{\alpha\beta}(x_{\alpha\beta})G_\beta(x_{\alpha\beta}, x_{0,\beta})$ define parallel transports along the edges $l_{\alpha\beta}$. Then every loop on $\mathscr{M}$ is represented by the homotopically equivalent edge loop on the polyhedron and, further, by the associated product of $G_{0,\alpha\beta}$.) Consequently, topologically inequivalent Symp($\mathbb{R}^{2N}$)-bundles with a flat connection are parametrized by the conjugacy classes of such representations $\pi_1 \to$ Symp($\mathbb{R}^{2N}$), i.e. the moduli space of topologically inequivalent classical BFF conversions coincides with Hom(π_1, Symp($\mathbb{R}^{2N}$))/Symp($\mathbb{R}^{2N}$). It is beyond the scope of the present paper to investigate this general setting. Instead, we will consider an important particular case when the structure group can be reduced to the subgroup of linear canonical transformations $\mathrm{Sp}(2N;\mathbb{R}) \subset \mathrm{Symp}(\mathbb{R}^{2N})$. That is, we assume that there exists such a set $\{G_{0,\alpha}\}$, $G_{0,\alpha} \in \mathrm{Symp}(\mathbb{R}^{2N})$, that our $G_{0,\alpha\beta}$ can be represented in the form

$$G_{0,\alpha\beta} = G_{0,\alpha} F_{0,\alpha\beta} G_{0,\beta}^{-1}, \quad F_{0,\alpha\beta} \in \mathrm{Sp}(2N;\mathbb{R}), \tag{4.87}$$

i.e. all the transition operators $G_{\alpha\beta}$ can be reduced to the linear canonical transformations $F_{0,\alpha\beta}$. They must satisfy reduced cocycle identities,

$$F_{0,\alpha\beta} F_{0,\beta\gamma} = F_{0,\alpha\gamma}. \tag{4.88}$$

The linear canonical transformations $F_{0,\alpha\beta}$ are defined by quadratic hamiltonian functions $f_{\alpha\beta}$,

$$F_{0,\alpha\beta} = \exp\left(-\partial_\phi^a f_{\alpha\beta} \Lambda_{ab} \partial_\phi^b\right), \tag{4.89a}$$

$$f_{\alpha\beta} = \tfrac{1}{2}(s_{\alpha\beta})_b^{\ a} \Lambda^{bc} \phi_a \phi_c, \tag{4.89b}$$

and Eqs. (4.86) for $\mathscr{A}'_{0,\alpha} = G_{0,\alpha}\mathscr{A}_{0,\alpha}$ reduce to

$$\mathscr{A}'_{0,\alpha}(\phi) = F_{0,\alpha\beta}\mathscr{A}'_{0,\beta} = \mathscr{A}'_{0,\beta}(S_{\alpha\beta}\phi), \tag{4.90}$$

where

$$(S_{\alpha\beta}\phi)_a = (S_{\alpha\beta})_a^{\ b}\phi_b, \quad (S_{\alpha\beta})_a^{\ b} = (\exp s_{\alpha\beta})_a^{\ b} \tag{4.91}$$

are symplectic $2N \times 2N$-matrices that must satisfy the matrix cocycle identities

$$S_{\alpha\beta} S_{\beta\gamma} = S_{\alpha\gamma} \tag{4.92}$$

in order for (4.88) to be satisfied. Inequivalent solutions of (4.92) define inequivalent symplectic vector bundles over $\mathscr{M}$. It corresponds to topologically inequivalent global choices of the vielbein and the symplectic connection (see Refs. [55,57] for the geometry of fiber bundles).

5. Quantization of BFF-converted system I: Weyl symbols

5.1. *Quantization of first-class constraints*

Converted system with the first-class constraints $\mathscr{T}_\mu = p_\mu - W_\mu$ can now be quantized according to the standard BFV quantization procedure outlined in Section 2. x^μ, p_μ and ϕ_a now become operators,

$$[\hat{x}^\mu, \hat{p}_\nu] = i\hbar\delta_\nu^\mu, \tag{5.1a}$$

$$\left[\hat{\phi}_a, \hat{\phi}_b\right] = -i\hbar\Lambda_{ab}, \tag{5.1b}$$

and they are supplemented with anticommuting ghosts and lagrangian multipliers and their momenta,

$$\left\{\hat{C}^{\mu}, \hat{\bar{\mathscr{P}}}_{\nu}\right\} = i\hbar\delta^{\mu}_{\nu}, \tag{5.2a}$$

$$\left[\hat{\lambda}^{\mu}, \hat{\pi}_{\nu}\right] = i\hbar\delta^{\mu}_{\nu}, \tag{5.2b}$$

$$\left\{\hat{\mathscr{P}}^{\mu}, \hat{\bar{C}}_{\nu}\right\} = i\hbar\delta^{\mu}_{\nu}. \tag{5.2c}$$

The quantum BFV–BRST operator has a standard form:

$$\hat{\Omega} = \hat{\Omega}_{\text{min}} + \hat{\Omega}_{\text{aux}}, \tag{5.3a}$$

$$\hat{\Omega}_{\text{min}} = \hat{C}^{\mu}\hat{\mathscr{T}}_{\mu}, \quad \hat{\Omega}_{\text{aux}} = \hat{\mathscr{P}}^{\mu}\hat{\pi}_{\mu}, \tag{5.3b}$$

and the operators $\hat{\mathscr{T}}_{\mu}$ are to be found from the nilpotency condition

$$\hat{\Omega}^2 = 0, \tag{5.4}$$

or, equivalently,

$$\left[\hat{\mathscr{T}}_{\mu}, \hat{\mathscr{T}}_{\nu}\right] = 0. \tag{5.5}$$

Let us introduce Weyl symbols ϕ_a of the operators $\hat{\phi}_a$. A composition of two Weyl (symmetric) symbols $\mathscr{A}$ and $\mathscr{B}$ of operators $\hat{\mathscr{A}}$ and $\hat{\mathscr{B}}$ is given by the Weyl–Moyal multiplication formula

$$\mathscr{A} * \mathscr{B} = \mathscr{A} \exp\left(-\tfrac{1}{2}i\hbar\overleftarrow{\partial}^{a}_{\phi}\Lambda_{ab}\overrightarrow{\partial}^{b}_{\phi}\right)\mathscr{B}, \tag{5.6}$$

and the commutator is given by the sin-bracket,

$$[\mathscr{A}, \mathscr{B}]_* = -2i\mathscr{A} \sin\left(\tfrac{1}{2}\hbar\overleftarrow{\partial}^{a}_{\phi}\Lambda_{ab}\overrightarrow{\partial}^{b}_{\phi}\right)\mathscr{B}. \tag{5.7}$$

Symbols $\mathscr{A} = \mathscr{A}(\phi)$ with the composition law (5.6) form an associative non-commutative Weyl algebra $\mathscr{W}_{2N}$, a quantization of the algebra of smooth functions $C^{\infty}(\mathbb{R}^{2N})$. The group of its inner automorphisms $\text{Aut}(\mathscr{W}_{2N})$ is a quantization of the group of canonical transformations $\text{Symp}(\mathbb{R}^{2N})$. Its Lie algebra $\text{Der}(\mathscr{W}_{2N})$ of inner derivations of $\mathscr{W}_{2N}$, i.e. the algebra of operators of the form $[\mathscr{A}, \cdot\,]_*$, is isomorphic to the factor-algebra $[\mathscr{W}_{2N}]/\{1\}$ of the commutator version of $\mathscr{W}_{2N}$ by its center spanned by the unity. It is a quantization of the Lie algebra of hamiltonian vector fields, $\text{Ham}(\mathbb{R}^{2N})$.

Weyl quantization of the converted system is obtained by substituting the commutative multiplication of functions and Poisson brackets with the $*$-product and sin-brackets. So the quantum version (5.5) of the zero-curvature equations takes the form

$$\mathscr{R}_{\mu\nu} = \partial_{\mu}W_{\nu} - \partial_{\nu}W_{\mu} + \frac{1}{i\hbar}\left[W_{\mu}, W_{\nu}\right]_* = 0, \tag{5.8}$$

or in components

$$\mathscr{R}^{a(k)}_{\mu\nu} = \partial_\mu W^{a(k)}_\nu - \partial_\nu W^{a(k)}_\mu - \sum_{n=0}^{\infty} (-1)^n \left(\tfrac{1}{2}\hbar\right)^{2n} \times \sum_{p+q=k} \frac{k!}{p!q!(2n+1)!} W^{a(p)b(2n+1)}_\mu \Lambda_{b(2n+1),c(2n+1)} W^{a(q)c(2n+1)}_\nu = 0. \tag{5.9}$$

For the sake of simplicity in this section we denote Weyl symbols W_μ, $\mathscr{R}_{\mu\nu}$ and $\mathscr{A}$ of quantum gauge fields $\hat{W}_\mu$, curvature $\hat{\mathscr{R}}$ and observables $\hat{\mathscr{A}}$ by the same letters as their classical counterparts. Thereafter, when dealing with the classical quantities we will use (0) to indicate that the quantity is independent of $\hbar$, e.g. $\overset{(0)}{W}_\mu$, $\overset{(0)}{\mathscr{A}}$, etc.

Quantum zero-curvature equations are invariant under the quantum gauge transformations

$$\delta W_\mu = \partial_\mu \epsilon + \frac{1}{i\hbar}\left[W_\mu, \epsilon\right]_*, \quad \epsilon = \epsilon(x, \phi), \tag{5.10}$$

and the Bianchi identities are given by

$$\partial_\mu \mathscr{R}_{\rho\sigma} + \frac{1}{i\hbar}\left[W_\mu, \mathscr{R}_{\rho\sigma}\right]_* + \text{cycle} = 0. \tag{5.11}$$

To fix the gauge arbitrariness one can choose the gauge (4.50) where $W^{a(k)}_\mu$ are now $\hbar$-dependent coefficients of the Weyl symbol rather than classical quantities.

Master equations (5.9) can be solved perturbatively expanding in the powers of $\hbar$,

$$W_\mu = \sum_{k=0}^{\infty} W^{a(k)}_\mu T_{a(k)}, \quad W^{a(k)}_\mu = \sum_{n=0}^{\infty} \hbar^n \overset{(n)}{W}{}^{a(k)}_\mu, \tag{5.12}$$

where $\overset{(0)}{W}$ are classical gauge fields of the previous section.

As a result one gets the following recurrent relations:

$$\overset{(m)}{\mathscr{R}}{}^{a(k)}_{\mu\nu} = \partial_\mu \overset{(m)}{W}{}^{a(k)}_\nu - \partial_\nu \overset{(m)}{W}{}^{a(k)}_\mu - \sum_{n,t,s,p,q} \delta(p+q-k)\delta(2n+t+s-m) \times \frac{(-1)^n k!}{4^n p!q!(2n+1)!} \overset{(s)}{W}{}^{a(p)b(2n+1)}_\mu \Lambda_{b(2n+1),c(2n+1)} \overset{(t)}{W}{}^{a(q)c(2n+1)}_\nu = 0, \tag{5.13}$$

where $m, k = 0, 1, 2, \ldots, \infty$, $\delta(n) := 0$ (1) for $n \neq 0$ $(n = 0)$.

Quantum corrections $\overset{(n)}{W}{}^{a(k)}_\mu$, $n > 0$, can now be found recurrently. First, we note that all quantum corrections of odd orders vanish in the gauge (4.50). (Since $W^{a(k)}_\mu$

are now coefficients of the Weyl symbol W_μ and contain all quantum corrections, Eq. (4.50) implies that $\Lambda^{ab}h^\mu_b \overset{(n)}{W}{}^{a(k)}_\mu = 0$ for all $n = 0, 1, 2, \ldots$.) It follows from the fact that there are only odd powers of $\hbar$ in the decomposition of the sin-bracket and the initial condition

$$W_\mu(x, \phi, \hbar)\big|_{\phi=0} = V_\mu, \tag{5.14}$$

which is equivalent to (4.3) together with

$$\overset{(n)}{W}{}^{a(0)}_\mu = 0, \quad n > 0. \tag{5.15}$$

Thus, in the Weyl ordering, first non-trivial quantum corrections are of second order. In particular, for $k = 0$ and $m = 2$ one obtains the equation for a second-order quantum correction to the vielbein

$$h^a_\mu \Lambda_{ab} \overset{(2)}{W}{}^b_\nu + \overset{(2)}{W}{}^a_\mu \Lambda_{ab} h^b_\nu - \tfrac{1}{24} \overset{(0)}{W}{}^{a(3)}_\mu \Lambda_{a(3),b(3)} \overset{(0)}{W}{}^{b(3)}_\nu = 0, \tag{5.16}$$

or

$$\overset{(2)}{W}{}^a_\mu = -\tfrac{1}{48} \Lambda^{ad} h^\nu_d \overset{(0)}{W}{}^{b(3)}_\mu \Lambda_{b(3),c(3)} \overset{(0)}{W}{}^{c(3)}_\nu . \tag{5.17}$$

Substituting (4.49) in the above equation, one finds (see Appendix B for calculations)

$$\overset{(2)}{W}{}^a_\mu = -\tfrac{1}{2} \Lambda^{ab} h^\nu_b \overset{(2)}{\omega}{}_{\mu\nu}, \tag{5.18a}$$

where

$$\overset{(2)}{\omega}{}_{\mu\nu} = \tfrac{1}{128}\left(R_{\mu\rho\lambda}{}^{\delta} R_{\nu\sigma\delta}{}^{\lambda} \omega^{\rho\sigma} + 2 R_{\mu\lambda\rho}{}^{\delta} R_{\nu\delta\sigma}{}^{\lambda} \omega^{\rho\sigma}\right), \tag{5.18b}$$

and $R_{\mu\nu\rho}{}^{\sigma}$ is the curvature tensor (3.31).

To summarize, we have found all the terms in the quantum first-class constraints with combined orders in $\hat\phi$ and $\hbar$ not higher than three:

$$\frac{i}{\hbar}\hat{\mathcal{T}}_\mu = \partial_\mu - \frac{i}{\hbar}\left[V_\mu + \left(h^a_\mu + \hbar^2 \overset{(2)}{W}{}^a_\mu\right)\hat\phi_a - \tfrac{1}{2}\Delta^{ab}_\mu \hat\phi_a \hat\phi_b + \tfrac{1}{6} \overset{(0)}{W}{}^{abc}_\mu \hat\phi_a \hat\phi_b \hat\phi_c + \ldots\right]. \tag{5.19}$$

Continuing this recurrent procedure one can express all quantum corrections to symplectic higher spin fields in terms of the curvature (3.31) and its covariant derivatives.

5.2. *Algebra of quantum observables*

Our next task is to describe the algebra of quantum observables. We have converted our original system on $\mathcal{M}$ into the physically equivalent system with first-class constraints. Given a classical observable A on $\mathcal{M}$, the corresponding quantum observable is now described by the quantum observable of the system with first-class constraints, i.e. it is a BFV–BRST cohomology class with a representative (2.27) in the gauge Ψ.

Since the constraints are abelian and, consequently, $\hat{\mathscr{A}}_{\min}$ is independent of ghosts, Eq. (2.25) reduces to the master equation for Weyl symbols of classical observables,

$$\mathscr{D}_\mu \mathscr{A}_{\min} := \frac{i}{\hbar}\left[\mathscr{T}_\mu, \mathscr{A}_{\min}\right]_* = \partial_\mu \mathscr{A}_{\min} - \frac{i}{\hbar}\left[W_\mu, \mathscr{A}_{\min}\right]_* = 0, \tag{5.20}$$

and $\mathscr{A}_{\min} = \mathscr{A}_{\min}(x, \phi)$ is subject to the initial condition (in the Weyl ordering)

$$\mathscr{A}_{\min}|_{\phi=0} = A. \tag{5.21}$$

To solve Eqs. (5.20), let us expand $\mathscr{A}_{\min}$ in the power series in ϕ and $\hbar$,

$$\mathscr{A} = \sum_{k=0}^{\infty} \mathscr{A}^{a(k)} T_{a(k)}, \quad \mathscr{A}^{a(k)} = \sum_{n=0}^{\infty} \hbar^n \overset{(n)}{\mathscr{A}}{}^{a(k)}. \tag{5.22}$$

The initial condition (5.21) reads as follows:

$$\overset{(0)}{\mathscr{A}}{}^{a(0)} = A \quad \text{and} \quad \overset{(n)}{\mathscr{A}}{}^{a(0)} = 0, \quad n > 0. \tag{5.23}$$

Then Eqs. (5.20) reduce to the following recurrent relations:

$$D_\mu^{\rm sp} \overset{(m)}{\mathscr{A}}{}^{a(k)} - h_\mu^b \Lambda_{bc} \overset{(m)}{\mathscr{A}}{}^{a(k)c} - \sum_{n,s,t,p,q} \delta(p+q-k)\delta(2n+t+s-m)$$
$$\times \frac{(-1)^n k!}{4^n p! q! (2n+1)!} \overset{(s)}{W}{}_\mu^{a(p)b(2n+1)} \Lambda_{b(2n+1),c(2n+1)} \overset{(t)}{\mathscr{A}}{}^{a(q)c(2n+1)} = 0. \tag{5.24}$$

These relations allow us to find quantum corrections to the classical BFF-extension (4.62). We will call corresponding operators and their symbols quantum BFF-extensions. First, we note that all quantum corrections of odd orders vanish identically:

$$\overset{(m)}{\mathscr{A}}{}^{a(k)} = 0 \quad \text{for all } n = 1, 3, \ldots, \tag{5.25}$$

and the first non-trivial corrections are of second order. In particular, substituting the solutions (4.49) and (5.18) in the equation with $m = 2$ and $k = 0$,

$$h_\mu^a \Lambda_{ab} \overset{(2)}{\mathscr{A}}{}^b + \overset{(2)}{W}{}_\mu^a \Lambda_{ab} \overset{(0)}{\mathscr{A}}{}^b - \tfrac{1}{24} \overset{(0)}{W}{}_\mu^{a(3)} \Lambda_{a(3),b(3)} \overset{(0)}{\mathscr{A}}{}^{b(3)} = 0, \tag{5.26}$$

we find a second-order quantum correction (see Appendix B for calculations),

$$\overset{(2)}{\mathscr{A}}{}^a = -\Lambda^{ab} h_b^\mu \left(\tfrac{1}{32} R_{\mu\nu\sigma}{}^\rho \omega^{\nu\rho} \omega^{\sigma\rho} \nabla_{\rho(3)} A + \tfrac{5}{6} \overset{(2)}{\omega}{}_{\mu\nu} \omega^{\nu\rho} \partial_\rho A \right). \tag{5.27}$$

To summarize, we have found all the terms in the expansion of the symmetrically ordered BFF-extended quantum observable with combined orders in ϕ and $\hbar$ not higher than three:

$$\mathscr{A} = A + \left(\overset{(0)}{\mathscr{A}}{}^{a} + \hbar^2 \overset{(2)}{\mathscr{A}}{}^{a}\right)\hat{\phi}_a + \tfrac{1}{2}\overset{(0)}{\mathscr{A}}{}^{ab}\hat{\phi}_a\hat{\phi}_b + \tfrac{1}{6}\overset{(0)}{\mathscr{A}}{}^{abc}\hat{\phi}_a\hat{\phi}_b\hat{\phi}_c + \dots \tag{5.28}$$

This recurrent procedure can be continued to find higher order terms. As a result, for every classical observable A on the initial phase space, the quantum conversion procedure provides an effective means to construct the corresponding quantum observable $\mathscr{A}$ on the extended phase space. The Weyl symbol $\mathscr{A}$ is given by

$$\mathscr{A}(x, \phi) = QA(x), \tag{5.29}$$

where a quantum conversion operator Q has the following structure:

$$Q = 1 + \sum_{k=1}^{\infty} \frac{1}{k!}\phi_{a_1}\dots\phi_{a_k}Q^{a(k)},$$
$$Q^{a(k)} = (-1)^k \Lambda^{a(k),b(k)} h^{\mu(k)}_{b(k)} \sum_{m=0}^{\infty}\sum_{l=1}^{k} \hbar^m \overset{(m)}{Q}{}^{\nu(l)}_{\mu(k)} \nabla_{\nu(l)}, \tag{5.30}$$

where coefficients $Q(m, k, l)$ (shorthand for $\overset{(m)}{Q}{}^{\nu(l)}_{\mu(k)}$) are functions of the curvature tensor and its covariant derivatives

$$Q(2n+1, k, l) = 0, \quad \text{for all } k \text{ and } l, \tag{5.31}$$

$$Q(2n, k, l) = \sum_{k=0}^{\infty} \sum_{p_1, \dots, p_k} Q_{\{p_i\}} R^{p_0} (\nabla R)^{p_1} \dots (\nabla^k R)^{p_k}. \tag{5.32}$$

If we introduce scale dimensions according to the rules

$$[R] = -2, \quad [\nabla] = -1, \quad [\hbar] = 2, \quad [\phi] = 1, \tag{5.33}$$

then Q should be dimensionless, $[Q] = 0$, and the summation in (5.32) is restricted to those $\{p_i\}$ satisfying the condition

$$2p_0 + (2p_1 + 1) + (2p_2 + 1) + \dots = 4n + k - l. \tag{5.34}$$

If $H = H(x)$ is the initial hamiltonian, the time evolution of quantum observables $\hat{\mathscr{A}}_\Psi$ is governed by the Heisenberg equation

$$i\hbar\partial_t \hat{\mathscr{A}}_\Psi = \left[\hat{\mathscr{A}}_\Psi, \hat{\mathscr{H}}_\Psi\right]. \tag{5.35}$$

The simplest gauge choice is a unitary gauge

$$\hat{\phi}_a = 0, \quad \hat{\Psi} = \hat{\bar{C}}_\mu \hat{\lambda}^\mu + \hat{\phi}_a h^a_\mu \omega^{\mu\nu} \hat{\bar{\mathscr{P}}}_\nu. \tag{5.36}$$

5.3. Star product and deformation quantization from the BFV quantization

In the previous section we described quantum observables on $\mathscr{M}$ as BFV–BRST cohomology classes calculated for the system subject to first-class constraints constructed via the BFF conversion procedure. Here we will show how an asymptotic expansion of the Weyl–Moyal star product formula for functions on an arbitrary symplectic manifold with connection and curvature, which constitutes a basis for deformation quantization of Refs. [21–24], can be naturally derived in this approach.

Consider two functions $A=A(x)$ and $B=B(x)$ on $\mathscr{M}$, and let $\mathscr{A}=\mathscr{A}(x, \phi)$ and $\mathscr{B}=\mathscr{B}(x, \phi)$ be the Weyl symbols of their quantum BFF-extensions. Then an associative and non-commutative $*_\Gamma$-product on $\mathscr{M}$ is naturally defined by

$$A *_\Gamma B = (\mathscr{A} * \mathscr{B})|_{\phi=0}. \tag{5.37}$$

Substituting expressions (5.29) for $\mathscr{A}$ and $\mathscr{B}$, calculating the Weyl–Moyal multiplication and setting $\phi_a = 0$ (fixing a gauge), we finally arrive at the asymptotic expansion (see Appendix B for calculations)

$$\begin{aligned} A *_\Gamma B = {} & AB + \tfrac{1}{2} i\hbar \partial_\mu A\, \omega^{\mu\nu} \partial_\nu B - \tfrac{1}{8}\hbar^2 \nabla_{\mu(2)} A \omega^{\mu(2),\nu(2)} \nabla_{\nu(2)} B \\ & - \tfrac{1}{48} i\hbar^3 \mathscr{L}_{\mu(3)} A \omega^{\mu(3),\nu(3)} \mathscr{L}_{\nu(3)} B + \dots, \end{aligned} \tag{5.38}$$

where the operators $\mathscr{L}$ are defined by

$$\mathscr{L}_{\mu(3)} = \nabla_{\mu(3)} - R_{\rho\mu\mu}{}^{\lambda} \omega_{\lambda\mu} \omega^{\rho\sigma} \partial_\sigma. \tag{5.39}$$

One sees that the well-known Weyl–Moyal star product for flat connections,

$$A * B = \sum_{n=0}^{\infty} \left(\tfrac{1}{2} i\hbar\right)^n \frac{1}{n!} \nabla_{\mu(n)} A \omega^{\mu(n),\nu(n)} \nabla_{\nu(n)} B, \tag{5.40}$$

is modified to take into account curvature effects (in the notation $*_\Gamma$, Γ stands for the connection Γ).

The $*_\Gamma$-product (5.38) satisfies the correspondence principles

$$\lim_{\hbar \to 0} (A *_\Gamma B) = AB, \tag{5.41a}$$

$$\lim_{\hbar \to 0} \left(\frac{1}{i\hbar} [A, B]_* \right) = [A, B]_{\mathrm{PB}} = \partial_\mu A\, \omega^{\mu\nu} \partial_\nu B, \tag{5.41b}$$

i.e. it defines a quantum deformation of the associative commutative algebra of functions of $\mathscr{M}$, and its commutator version defines a quantum deformation of the Poisson brackets algebra. The deformations of associative and Lie algebras are governed by the Hochschild and Chevalley cohomologies, respectively. Remarkably, the deformation given by (5.38), (5.39) is precisely the deformation discovered by Vey [22,21] (see also Refs. [23,24]). Indeed, in Darboux coordinates, operator $\mathscr{L}$ can be rewritten in the form

$$\mathscr{L}_{\mu(3)} = \partial_{\mu(3)} - 3\Gamma^{\rho}_{\mu\mu} \partial_{\rho\mu} - \partial_\nu \Gamma^{\sigma}_{\mu\mu} \omega_{\sigma\mu} \omega^{\nu\rho} \partial_\rho, \tag{5.42}$$

and the third order term in (5.38) coincides with the Chevalley cocycle S^3_Γ of Refs. [21–24],

$$S^3_\Gamma(A, B) = \mathscr{L}_{\mu(3)} A \omega^{\mu(3),\nu(3)} \mathscr{L}_{\nu(3)} B. \tag{5.43}$$

The operator $\mathscr{L}$ has a simple geometric interpretation. In Darboux coordinates let us introduce quantities $\Gamma_{\mu\nu\rho} = \omega_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho}$ which are totally symmetric. Then $\mathscr{L}$ can be expressed through the Lie derivative of the connection Γ along the hamiltonian vector field $X_A = [A, \cdot\,]_{\mathrm{PB}}$ with the hamiltonian A [22],

$$\mathscr{L}_{\mu(3)}A = \partial_{\mu(3)}A - \left(\pounds_{X_A}\Gamma\right)_{\mu(3)}. \tag{5.44}$$

Thus, BFV quantization of the BFF-converted system with first-class constraints provides straightforward means to recurrently compute manifest expressions for the Hochshild and Chevalley cocycles of all orders, and deformation quantization can be derived from BFV quantization in the gauge $\phi = 0$.

5.4. *Pure gauge solution, parallel transport of quantum observables and W-bundle*

All the considerations of Sections 4.8–4.10 are quantized directly by substituting Poisson brackets and commutative multiplication of functions with the sin-brackets and $*$-product. In particular, parallel transport of Weyl symbols of quantum observables is defined by the quantization of (4.66) (recall that we denote Weyl symbols and classical quantities by the same letters),

$$\begin{aligned} G(x_1, x_0) &= \mathrm{P}\exp\left(\frac{i}{\hbar}\int_{x_0}^{x_1}\mathrm{ad}_*\,W\right) \\ &= \mathrm{P}\exp\left(\frac{2}{\hbar}\int_{t_0}^{t_1}\mathrm{d}t\,\dot{x}^{\mu}(t)W_{\mu}(x(t),\phi)\,\sin\left(\tfrac{1}{2}\hbar\overleftarrow{\partial}{}^{a}_{\phi}\Lambda_{ab}\overrightarrow{\partial}{}^{b}_{\phi}\right)\right). \end{aligned} \tag{5.45}$$

Then the master equations (5.20) for Weyl symbols of classical observables have a general solution,

$$\mathscr{A}(x,\phi) = G(x, x_0)\mathscr{A}_0(\phi), \tag{5.46}$$

with the initial condition $\mathscr{A}_0(\phi) = \mathscr{A}(x_0, \phi)$.

Let us introduce a Weyl symbol,

$$\begin{aligned} \mathscr{U}(x_1, x_0) &= \mathrm{P}\exp_*\left(\frac{i}{\hbar}\int_{x_0}^{x_1}W\right) \\ &= 1 + \frac{i}{\hbar}\int_{t_0}^{t_1}\mathrm{d}t\,\dot{x}^{\mu}(t)W_{\mu}(x(t),\phi) \\ &\quad - \frac{1}{\hbar^2}\iint_{\Delta_+}\mathrm{d}t\,\mathrm{d}t'\,\dot{x}^{\mu}(t)\dot{x}^{\nu}(t')W_{\mu}(x(t),\phi) * W_{\nu}(x(t'),\phi) \\ &\quad + \dots, \end{aligned} \tag{5.47}$$

$$\Delta_+ = \{t_0 \leqslant t \leqslant t_1,\ t_0 \leqslant t' \leqslant t_1,\ t \geqslant t'\},$$

$$[\mathscr{U}(x_1, x_0)]^{\dagger} = [\mathscr{U}(x_1, x_0)]^{-1} = \mathscr{U}(x_0, x_1). \tag{5.48}$$

Then pure gauge solutions can be written in the form

$$\frac{i}{\hbar}W_\mu = \left[\partial_\mu \mathscr{U}(x, x_0)\right] * \left[\mathscr{U}(x, x_0)\right]^\dagger, \tag{5.49}$$

$$\mathscr{A}(x, \phi) = \mathscr{U}(x, x_0) * \mathscr{A}_0(\phi) * \left[\mathscr{U}(x, x_0)\right]^\dagger. \tag{5.50}$$

Now quantization of the Symp($\mathbb{R}^{2N}$)-bundle $\mathscr{E}$ is straightforward. Consider a bundle $\mathscr{W}\mathscr{M}$ (which we will call the W-bundle) over $\mathscr{M}$ with fibers $\mathrm{F} \simeq \mathscr{W}_{2N}$ and the structure group Aut($\mathscr{W}_{2N}$). The symbol Ω (Weyl symbol of the BFV–BRST operator $\hat{\Omega}$) defines a flat connection on $\mathscr{W}\mathscr{M}$,

$$\frac{i}{\hbar}\mathrm{ad}_* \ \Omega = \mathrm{d} - \frac{i}{\hbar}\mathrm{ad}_* \ W, \quad C^\mu \leftrightarrow \mathrm{d}x^\mu, \tag{5.51}$$

and the Weyl symbols of quantum observables are covariantly constant sections of $\mathscr{W}\mathscr{M}$. Thus, for every classical observable $A(x)$ on $\mathscr{M}$ BFV quantization yields a section $\mathscr{A}$ of the W-bundle with the flat connection ad Ω.

Let $\{\mathscr{O}_\alpha\}$ be the covering of $\mathscr{M}$ as in Section 4.10 and $\{W_\alpha\}$ and $\{\mathscr{A}_\alpha\}$ are trivializations of the flat connection and a quantum observable $\mathscr{A}$, i.e.

$$\frac{i}{\hbar}W_\alpha = (\mathrm{d}\mathscr{U}_\alpha) * \mathscr{U}_\alpha^{-1}, \tag{5.52}$$

$$\mathscr{A}_\alpha = \mathscr{U}_\alpha * \mathscr{A}_{0,\alpha} * \mathscr{U}_\alpha^{-1}, \tag{5.53}$$

$$\mathscr{U}_\alpha = \mathrm{P}\exp\left(\frac{i}{\hbar}\int_{x_{0,\alpha}}^{x_\alpha} W_\alpha\right). \tag{5.54}$$

Then on intersections $\mathscr{O}_\alpha \cap \mathscr{O}_\beta$ we have

$$\mathscr{A}_\alpha = \mathscr{U}_{\alpha\beta} * \mathscr{A}_\beta * \mathscr{U}_{\alpha\beta}^{-1}, \tag{5.55}$$

$$W_\alpha = \mathscr{U}_{\alpha\beta} * W_\beta * \mathscr{U}_{\alpha\beta}^{-1} + i\hbar(\mathrm{d}\mathscr{U}_{\alpha\beta}) * \mathscr{U}_{\alpha\beta}^{-1}, \tag{5.56}$$

$$\Omega_\alpha = \mathscr{U}_{\alpha\beta} * \Omega_\beta * \mathscr{U}_{\alpha\beta}^{-1}, \tag{5.57}$$

and on triple intersections

$$\mathscr{U}_{\alpha\beta} * \mathscr{U}_{\beta\gamma} * \mathscr{U}_{\gamma\alpha} = \mathbb{1}. \tag{5.58}$$

However, note that the structure group of the W-bundle is Aut($\mathscr{W}_{2N}$) with the Lie algebra Der($\mathscr{W}_{2N}$) $\simeq \mathscr{W}_{2N}/\{\mathbb{1}\}$, rather than the Weyl algebra itself. Indeed, the symplectic potential term $V_\mu \mathbb{1}$ in the gauge field W_μ falls out of the connection (5.51) since it belongs to the center $\{\mathbb{1}\}$ of $\mathscr{W}_{2N}$. As a result, the overall ϕ-independent phase factor of (5.54) is inessential because it falls out of the right hand sides of Eqs. (5.55) and (5.57). Thus, in order for the W-bundle to exist, the cocycle identity (5.58) must hold only up to an arbitrary ϕ-independent phase, and the overall phase of $\mathscr{U}_{\alpha\beta}$ remains undetermined.

Substituting (5.53) in Eqs. (5.55) we find similar to (4.84) that $\mathscr{U}_{\alpha\beta}$ can be reduced to x-independent initial conditions,

$$\mathscr{U}_{\alpha\beta} = \mathscr{U}_\alpha * \mathscr{U}_{0,\alpha\beta} * \mathscr{U}_\beta^{-1} \tag{5.59}$$

and

$$\mathscr{A}_{0,\alpha} = \mathscr{U}_{0,\alpha\beta} * \mathscr{A}_{0,\beta} * \mathscr{U}_{0,\alpha\beta}^{-1}. \tag{5.60}$$

The moduli space of inequivalent W-bundles is $H^1(\mathscr{M}, \mathrm{Aut}(\mathscr{W}_{2N})) \simeq \mathrm{Hom}(\pi_1, \mathrm{Aut}(\mathscr{W}_{2N}))/\mathrm{Aut}(\mathscr{W}_{2N})$, which is a quantization of $H^1(\mathscr{M}, \mathrm{Symp}(\mathbb{R}^{2N}))$. Again we will limit ourselves to linear canonical transformations, i.e. to the W-bundles with the structure group reducible to $\mathrm{Sp}(2N; \mathbb{R})$,

$$\mathscr{U}_{0,\alpha\beta} = \mathscr{U}_{0,\alpha} * u_{0,\alpha\beta} * \mathscr{U}_{0,\beta}^{-1}, \quad u_{0\alpha\beta} \in \mathrm{Sp}(2N, \mathbb{R}). \tag{5.61}$$

Since for quadratic hamiltonians Poisson brackets coincide with the $*$-commutator, the $u_{0,\alpha\beta}$ are given by

$$u_{0,\alpha\beta} = \exp\left(\frac{i}{\hbar} f_{\alpha\beta}\right), \tag{5.62}$$

where the $f_{\alpha\beta}$ are classical quadratic hamiltonians (4.90b), and for $\mathscr{A}'_{0,\alpha} = \mathscr{U}_{0,\alpha}^{-1} * \mathscr{A}_{0,\alpha} * \mathscr{U}_{0,\alpha}$ we have

$$\mathscr{A}'_{0,\alpha} = u_{0,\alpha\beta} * \mathscr{A}'_{0,\beta} * u_{0,\alpha\beta}^{-1} = \mathscr{A}'_{0,\beta}(S_{\alpha\beta}\phi) \tag{5.63}$$

(see (4.91), (4.92)). Thus, $\mathrm{Symp}(\mathbb{R}^{2N})$-bundles are quantized directly yielding W-bundles without any additional topological quantization conditions. However, in quantum mechanics we are interested not in the algebra of quantum observables by itself, but rather in its hermitian representation in the Hilbert space of physical quantum states $\mathfrak{R}_{\mathrm{phys}}$. While discussing quantum observables, we have neglected the overall ϕ-independent phase factor of the transition operators $\mathscr{U}_{\alpha\beta}$ as well as the ϕ-independent terms $V_\alpha \mathbb{1}$ of W_α. Working with the representation in $\mathfrak{R}_{\mathrm{phys}}$ we will no longer be able to neglect those terms. Moreover, from the preliminary considerations of Section 3 we can anticipate their critical importance for the existence of such a representation.

6. BFV quantization of BFF-converted system II: Hilbert space of quantum states

6.1. Fock representation, Wick ordering and physical states

In the previous section we have constructed the algebra of quantum observables $\mathscr{F}_{\mathrm{obs}}(\hbar)$ in terms of Weyl symbols. Our next task is to construct its representation by hermitian operators acting in the Hilbert space of physical quantum states $\mathfrak{R}_{\mathrm{phys}}$. According to the BFV procedure, $\mathfrak{R}_{\mathrm{phys}}$ of the quantum system with first-class constraints is defined as the zero-ghost-number cohomology space (2.33). Then the algebra of quantum observables defined by the cohomology (2.21) is naturally represented by operators in $\mathfrak{R}_{\mathrm{phys}}$ (2.33).

First, we have to define an extended Hilbert space $\mathfrak{R}_{\mathrm{ext}}$ for our converted system. Let us separate operators $\hat{\phi}_a$ into creation and annihilation parts,

$$\left[\hat{a}^i, \hat{a}_j^\dagger\right] = -\hbar\delta_j^i, \quad \left(\hat{a}^i\right)^\dagger = \hat{a}_i^\dagger, \quad i, j = 1, 2, \dots, N. \tag{6.1}$$

Then the quantum states of $\mathfrak{R}_{\text{ext}}$ are defined by

$$|\psi\rangle = \psi(x, \hat{a}^{\dagger}, \lambda, C, \mathscr{P})\,|0\rangle, \tag{6.2}$$

where the vacuum vector $|0\rangle$ is annihilated by all $\hat{a}^i$,

$$\hat{a}^i\,|0\rangle = 0, \quad i = 1, 2, \ldots, N, \tag{6.3}$$

and

$$\hat{x}^{\mu} = x^{\mu}, \quad \hat{p}_{\mu} = -i\hbar\frac{\partial}{\partial x^{\mu}}, \tag{6.4a}$$

$$\hat{\lambda}^{\mu} = \lambda^{\mu}, \quad \hat{\pi}_{\mu} = -i\hbar\frac{\partial}{\partial \lambda^{\mu}}, \tag{6.4b}$$

$$\hat{C}^{\mu} = C^{\mu}, \quad \hat{\bar{\mathscr{P}}}_{\mu} = i\hbar\frac{\partial_l}{\partial C^{\mu}}, \tag{6.4c}$$

$$\hat{\mathscr{P}}^{\mu} = \mathscr{P}^{\mu}, \quad \hat{\bar{C}}_{\mu} = i\hbar\frac{\partial_l}{\partial \mathscr{P}^{\mu}}. \tag{6.4d}$$

We have selected Fock representation for $\hat{\phi}_a$ and the Schrödinger representation for all the other operators (6.4).

The BFV–BRST operator $\hat{\Omega}$ is now given by

$$\hat{\Omega} = C^{\mu}\hat{\mathscr{T}}_{\mu} - i\hbar\mathscr{P}^{\mu}\frac{\partial}{\partial\lambda^{\mu}} \tag{6.5}$$

with the quantum constraints

$$\frac{i}{\hbar}\hat{\mathscr{T}}_{\mu} = \partial_{\mu} + \frac{1}{i\hbar}\hat{W}_{\mu}. \tag{6.6}$$

The gauge field

$$\begin{aligned}\hat{W} &= W_{\mu}(x, \hat{a}^{\dagger}, \hat{a}) \\ &= \sum_{p,q}\frac{1}{p!q!}W_{\mu,i(q),}{}^{j(p)}\hat{a}^{\dagger}_{j_1}\ldots\hat{a}^{\dagger}_{j_p}\hat{a}^{i_1}\ldots\hat{a}^{i_q}\end{aligned} \tag{6.7}$$

is now a Wick ordered operator which is directly obtained from the Weyl symbol $W_{\mu} = W_{\mu}(x;\, a^{\dagger}, a)$ (5.12) by first passing to the Wick symbol according to the standard rule

$$W_{\mu}^{\text{Wick}} = \exp\left(-\frac{\hbar}{2}\frac{\partial^2}{\partial a_i^{\dagger}\,\partial a^i}\right)W_{\mu}^{\text{Weyl}}, \tag{6.8}$$

and then substituting variables $a^{\dagger}$ and a with the operators $\hat{a}^{\dagger}$ and $\hat{a}$ in the Wick order.

Then Eq. (2.32) for physical states,

$$\hat{\Omega}\,|\psi_{\text{phys}}\rangle = 0, \tag{6.9}$$

is equivalent to

$$\hat{\mathcal{T}}_\mu \mid \psi_{\text{phys}} \rangle = 0 \quad \text{and} \quad \frac{\partial}{\partial \lambda^\mu} \mid \psi_{\text{phys}} \rangle = 0, \tag{6.10}$$

and the zero-ghost-number condition singles out states that are independent of ghosts C and $\mathcal{P}$.

Thus, the physical states $\mid \psi_{\text{phys}} \rangle$ depend non-trivially on $3N$ variables x^μ and $a_i^\dagger$,

$$\mid \psi_{\text{phys}} \rangle = \sum_{k=0}^{\infty} \frac{1}{k!} \psi^{i(k)} \hat{a}_{i_1}^\dagger \dots \hat{a}_{i_k}^\dagger \mid 0 \rangle, \tag{6.11}$$

and satisfy the quantum constraints equations

$$\left(\partial_\mu + \frac{1}{i\hbar} W_\mu(x;\, \hat{a}^\dagger,\, \hat{a}) \right) \mid \psi_{\text{phys}} \rangle = 0 \tag{6.12}$$

that further reduce the number of independent variables to only N. Indeed, the dimension of the original phase space $\mathcal{M}$ is $2N$, and physical states must depend on half of the phase space variables.

6.2. Pure gauge solution and parallel transport of physical states

In the coordinate neighborhood $\mathcal{O}$ of the point $x_0 \in \mathcal{O}$ parallel transport of physical states along the path Γ going from x_0 to $x_1 \in \mathcal{O}$ is defined by the unitary operator

$$\hat{\mathcal{U}}(x_1,\, x_0) = \text{P} \exp\left(\frac{i}{\hbar} \int_{x_0}^{x_1} \hat{W} \right), \tag{6.13}$$

$$\left[\hat{\mathcal{U}}(x_1,\, x_0) \right]^\dagger = \hat{\mathcal{U}}(x_0,\, x_1), \tag{6.14}$$

and the general solution of quantum constraints can be written in the form

$$\mid \psi_{\text{phys}} \rangle = \hat{\mathcal{U}}(x,\, x_0) \mid \psi_{\text{phys},0} \rangle, \tag{6.15}$$

with the x-independent initial condition

$$\mid \psi_{\text{phys},0} \rangle = \psi_{\text{phys},0}(\hat{a}^\dagger) \mid 0 \rangle. \tag{6.16}$$

Thus, it is enough to know the value of the physical wave function at the origin x_0 to be able to reconstruct the entire function on $\mathcal{O}$.

Similarly, for the quantum observables which are now hermitian operators,

$$\hat{\mathcal{A}} = \mathcal{A}(x;\, \hat{a}^\dagger,\, \hat{a}), \quad \hat{\mathcal{A}}^\dagger = \hat{\mathcal{A}}, \tag{6.17}$$

that can be directly obtained from the Weyl symbols (5.22), a general solution to Eq. (2.25) is given in the form

$$\hat{\mathcal{A}} = \hat{\mathcal{U}} \hat{\mathcal{A}}_0 \hat{\mathcal{U}}^{-1}, \tag{6.18}$$

with the x-independent initial condition $\hat{\mathscr{A}}_0 = \hat{\mathscr{A}}_0(\hat{a}^\dagger, \hat{a})$. It is easy to see that due to (6.15) and (6.20) the initial condition $(\hat{\mathscr{A}}\,|\,\psi_{\mathrm{phys}}\rangle)_0$ for the physical state $\hat{\mathscr{A}}\,|\,\psi_{\mathrm{phys}}\rangle$ is given by $\hat{\mathscr{A}}_0\,|\,\psi_{\mathrm{phys},0}\rangle$.

6.3. Global description. Bundle of symplectic spinors and metaplectic anomaly

Let $\mathfrak{R}_0$ be the Fock space spanned by the monomials

$$\hat{a}^\dagger_{i_1}\dots\hat{a}^\dagger_{i_k}\,|0\rangle, \quad k = 0, 1, \dots, \infty, \tag{6.19}$$

$\mathscr{G}$ the group of unitary operators on $\mathfrak{R}_0$, $\mathscr{F}_0$ the associative algebra of hermitian operators on $\mathfrak{R}_0$, and $\mathscr{G}' \simeq \mathrm{Aut}(\mathscr{F}_0)$ the group of inner automorphisms of $\mathscr{F}_0$, i.e. conjugations of the form $\hat{\mathscr{U}}\hat{\mathscr{A}}\hat{\mathscr{U}}^{-1}$, $\hat{\mathscr{U}} \in \mathscr{G}$. Multiplications on the phase $e^{i\varphi}$ form a center of $\mathscr{G}$, and $\mathscr{G}'$ is isomorphic to the quotient $\mathscr{G}/\mathrm{U}(1)$. Consider a Fock bundle $\mathfrak{R}\mathscr{M}$ over $\mathscr{M}$ with fibers $\mathrm{F} \simeq \mathfrak{R}_0$ and the structure group $\mathrm{G} \simeq \mathscr{G}$. Then $\hat{\Omega}$ defines a flat connection in $\mathfrak{R}\mathscr{M}$, and physical states are covariantly constant sections of the Fock bundle with respect to this connection. Further, consider an associated bundle $\mathscr{F}\mathscr{M}$ over $\mathscr{M}$ with fibers $\mathrm{F} \simeq \mathscr{F}_0$ and a structure group $\mathscr{G}'$. Then ad $\hat{\Omega}$ defines a flat connection on $\mathscr{F}\mathscr{M}$, and quantum observables are covariantly constant sections with respect to this connection.

Let $\{\mathscr{O}_\alpha\}$ be a contractible open covering of $\mathscr{M}$, and $\{\hat{W}_\alpha\}$, $\{|\,\psi_\alpha\rangle\}$ and $\{\mathscr{A}_\alpha\}$ are trivializations of the flat connection W, a constant section $|\,\psi\rangle$ of $\mathfrak{R}\mathscr{M}$, and a constant section $\mathscr{A}$ of $\mathscr{W}\mathscr{M}$, i.e. $\hat{W}_\alpha = \hat{W}_{\alpha,\mu}\mathrm{d}x^\mu$, $|\,\psi_\alpha\rangle$ and $\hat{\mathscr{A}}_\alpha$ are pure gauge solutions on $\mathscr{O}_\alpha$,

$$\frac{i}{\hbar}\hat{W}_\alpha = \mathrm{d}\hat{\mathscr{U}}_\alpha\,\hat{\mathscr{U}}_\alpha^{-1}, \tag{6.20a}$$

$$|\,\psi_\alpha\rangle = \hat{\mathscr{U}}_\alpha\,|\,\psi_{0,\alpha}\rangle, \tag{6.20b}$$

$$\hat{\mathscr{A}}_\alpha = \hat{\mathscr{U}}_\alpha\hat{\mathscr{A}}_{0,\alpha}\hat{\mathscr{U}}_\alpha^{-1}, \tag{6.20c}$$

for some finite $\mathscr{G}$-gauge transformations

$$\hat{\mathscr{U}}_\alpha(x_\alpha, x_{0,\alpha}) = \mathrm{P}\exp\left(\frac{i}{\hbar}\int_{x_{0,\alpha}}^{x_\alpha}\hat{W}_\alpha\right), \quad x_{0,\alpha} \in \mathscr{O}_\alpha, \quad x_\alpha \in \mathscr{O}_\alpha, \tag{6.21}$$

and initial conditions $|\,\psi_{0,\alpha}\rangle \in \mathfrak{R}_0$ and $\hat{\mathscr{A}}_{0,\alpha} \in \mathscr{F}_0$. Then on the intersections $\mathscr{O}_\alpha \cap \mathscr{O}_\beta$ we have $\mathscr{G}$-gauge transformations

$$|\,\psi_\alpha\rangle = \hat{\mathscr{U}}_{\alpha\beta}\,|\,\psi_\beta\rangle, \tag{6.22a}$$

$$\hat{\mathscr{A}}_\alpha = \hat{\mathscr{U}}_{\alpha\beta}\hat{\mathscr{A}}_\beta\hat{\mathscr{U}}_{\alpha\beta}^{-1}, \tag{6.22b}$$

$$\hat{W}_\alpha = \hat{\mathscr{U}}_{\alpha\beta}\hat{W}_\beta\hat{\mathscr{U}}_{\alpha\beta}^{-1} + i\hbar\,\mathrm{d}\hat{\mathscr{U}}_{\alpha\beta}\hat{\mathscr{U}}_{\alpha\beta}^{-1}, \tag{6.22c}$$

and on the triple intersections $\mathscr{O}_\alpha \cap \mathscr{O}_\beta \cap \mathscr{O}_\gamma$ the cocycle identity must hold,

$$\hat{\mathscr{U}}_{\alpha\beta}\hat{\mathscr{U}}_{\beta\gamma}\hat{\mathscr{U}}_{\gamma\alpha} = \mathbb{1}. \tag{6.23}$$

Similar to the considerations of Sections 4.10 and 5.4 we find that the $\hat{\mathscr{U}}_{\alpha\beta}$ reduce to x-independent initial conditions

$$\hat{\mathscr{U}}_{\alpha\beta} = \hat{\mathscr{U}}_{\alpha}\hat{\mathscr{U}}_{0,\alpha\beta}\hat{\mathscr{U}}_{\beta}^{-1}, \tag{6.24}$$

which must satisfy the x-independent cocycle identities

$$\hat{\mathscr{U}}_{0,\alpha\beta}\hat{\mathscr{U}}_{0,\beta\gamma}\hat{\mathscr{U}}_{0,\gamma\alpha} = \mathbb{1}. \tag{6.25}$$

The moduli space of Fock bundles with a flat connection is thus a set $\mathrm{Hom}(\pi_1, \mathscr{G})/\mathscr{G}$ of conjugacy classes of unitary representations of the fundamental group π_1 of $\mathscr{M}$ by unitary operators in the Fock space. In this paper we will restrict ourselves to the Fock bundles with transition operators $\hat{\mathscr{U}}_{0,\alpha\beta}$ admitting a reduction to the finite-dimensional subgroup $\mathscr{G}_{\mathrm{f}} \subset \mathscr{G}$ with the Lie algebra $\mathrm{g}_0 \simeq \mathrm{u}(1) \oplus \mathrm{sp}(2N; \mathbb{R})$.

First, let us consider the topological structure of $\mathscr{G}$. Let $\hat{b}$ be a hermitian operator on $\mathfrak{R}_0$, and $\hat{\mathscr{U}} = \exp(i\hat{b}/\hbar)$ is the corresponding unitary operator. In this way the algebra $[i\mathscr{F}_0]$ of (anti-)hermitian operators can be roughly identified with the Lie algebra of $\mathscr{G}$. Operators $\hat{b} = \varphi\mathbb{1}$ proportional to the identity operator generate an U(1) subgroup of $\mathscr{G}$. Next, quadratic operators of the form

$$\hat{f} = \tfrac{1}{2}\left[f^{ij}\hat{a}_i^{\dagger}\hat{a}_j^{\dagger} + \tfrac{1}{2}f_j^{i}\left(\hat{a}_i^{\dagger}\hat{a}^j + \hat{a}^j\hat{a}_i^{\dagger}\right) + f_{ij}\hat{a}^i\hat{a}^j \right] \tag{6.26}$$

form a symplectic subalgebra $\mathrm{sp}(2N; \mathbb{R})$. The correspondingf $N(N+1)$-parametric subgroup of unitary operators of the form

$$\hat{\mathscr{U}} = \exp\left(\frac{i}{\hbar}\hat{f}\right) \tag{6.27}$$

is called the metaplectic group $\mathrm{Mp}(2N; \mathbb{R})$. Topologically it is a double covering of the symplectic group $\mathrm{Sp}(2N; \mathbb{R})$. It is a quantization of classical linear canonical transformations that was studied by Fock [59], Weil [60], Shale [61], Segal [62], Bargmann [63], Berezin [64] and others. The representation of $\mathrm{Mp}(2N; \mathbb{R})$ in the Fock space has a rich history both in physics and in mathematics and has many different names: Fock representation, metaplectic representation, symplectic spinor representation, Shale–Weil representation, etc. (see Refs. [11,65] for reviews). Curiously, at the infinitesimal level it was first manifestly constructed by Majorana [66], who, exploiting the embedding of the Lorentz group $\mathrm{SO}(3,1) \subset \mathrm{Sp}(4; \mathbb{R})$, proposed a new relativistic wave equation of the Dirac type based on symplectic spinors. Instead of Dirac's spinors, he considered a wave function ψ that transformed under the metaplectic representation, and proposed an equation $(L^{\mu}\partial_{\mu} - \mu)\psi = 0$, where the L^{μ} are four generators complementary to the Lorentz generators $M^{\mu\nu}$ in $\mathrm{sp}(4; \mathbb{R})$. Both L^{μ} and $M^{\mu\nu}$ were manifestly constructed as infinite matrices (see Ref. [67] for a review). Another interesting occurrence of symplectic spinors in physics is the Dirac singleton [68]. Dirac exploited an isomorphism $\mathrm{Sp}(4; \mathbb{R}) \simeq \mathrm{SO}(3,2)$ to construct two remarkable representations of the anti-de Sitter group later called Di and Rac [69]. The crucial feature of singletons is that their square root decomposes into the direct sum of all the unitary massless representations of the anti-de Sitter group [69,70].

Returning to our consideration, it is easy to see that operators (6.26), (6.27) indeed realize the double covering of Sp(2N; $\mathbb{R}$) by noting that the Wick normal form of $\frac{1}{2}f_j^i(\hat{a}_i^\dagger\hat{a}^j + \hat{a}^j\hat{a}_i^\dagger)$ is given by (note the minus sign in (6.1))

$$\tfrac{1}{2}f_j^i\left(\hat{a}_i^\dagger\hat{a}^j + \hat{a}^j\hat{a}_i^\dagger\right) = f_j^i\hat{a}_i^\dagger\hat{a}^j - \tfrac{1}{2}\hbar f_i^i. \tag{6.28}$$

Furthermore, since

$$f_j^i = -i(\ln F)_j^i, \tag{6.29}$$

where F is a unitary matrix, we note that

$$\hat{U}(F)\,|0\rangle = \exp\left(\frac{i}{\hbar}\left(\tfrac{1}{2}i\hbar \operatorname{tr}\ln F\right)\right)|0\rangle = (\det F)^{-1/2}\,|0\rangle, \tag{6.30}$$

and as a result of the two-valuedness of the square root we obtain

$$\mathrm{Sp}(2N;\mathbb{R}) \simeq \mathrm{Mp}(2N;\mathbb{R})/\mathbb{Z}_2. \tag{6.31}$$

The metaplectic representation of Mp(2N; $\mathbb{R}$) in $\mathfrak{R}_0$ has two irreducible components $\mathfrak{R}_0 = \mathfrak{R}_{\mathrm{even}} \oplus \mathfrak{R}_{\mathrm{odd}}$; $\mathfrak{R}_{\mathrm{even}}(\mathfrak{R}_{\mathrm{odd}})$ is spanned by the monomials (6.19) of even (odd) degrees. With respect to the maximal compact subgroup U(N) it is decomposed into the infinite direct sum of irreducible representations $\mathfrak{R}_0 = \oplus_{k=0}^{\infty}\mathfrak{R}_{0,k}$, where the $\mathfrak{R}_{0,k}$ are irreducible representations with the basis (6.19). In particular, the vacuum vector $|0\rangle$ forms a one-dimensional representation $F \to (\det F)^{-1/2}$ of U(N) × (6.32) (more precisely, of its double covering MU(N) ⊂ Mp(2N; $\mathbb{R}$) sometimes called the meta-unitary group [15]). Then the representation of U(N) in $\mathfrak{R}_{0,k}$ is given by

$$\hat{U}(F)\hat{a}_{i_1}^\dagger\ldots\hat{a}_{i_k}^\dagger\,|0\rangle = (\det F)^{-1/2}F_{i_1}^{j_1}\ldots F_{i_k}^{j_k}\hat{a}_{j_1}^\dagger\ldots\hat{a}_{j_k}^\dagger\,|0\rangle. \tag{6.32}$$

However, it is not the group Mp(2N; $\mathbb{R}$) itself that interests us, but rather a group $\mathscr{G}_{\mathrm{f}} \simeq [\mathrm{Sp}(2N;\mathbb{R})\times \mathrm{U}(1)]/\mathbb{Z}_2$ of operators of the form (with $\hat{f}$ given by (6.26))

$$\hat{U} = \exp\left(\frac{i}{\hbar}(\varphi + \hat{f})\right). \tag{6.33}$$

Since we are now dealing with states rather than observables, transition operators may also contain multiplications on the overall phase factor. The maximal compact subgroup $\mathscr{G}_{\mathrm{f}}^{\mathrm{c}} \subset \mathscr{G}_{\mathrm{f}}$ is isomorphic to $[\mathrm{U}(1)\times\mathrm{U}(N)]/\mathbb{Z}_2$.

Let us now return to the Fock bundle with flat connection. We are looking for solutions to the cocycle condition (6.25) in the form

$$\hat{u}_{0,\alpha\beta} = \exp\left(\frac{i}{\hbar}\left[\varphi_{\alpha\beta} + \tfrac{1}{2}(f_{\alpha\beta})_j^i\left(\hat{a}_i^\dagger\hat{a}^j + \hat{a}^j\hat{a}_i^\dagger\right)\right]\right), \tag{6.34}$$

i.e. for representations $\pi_1 \to \mathscr{G}_{\mathrm{f}}^{\mathrm{c}} \simeq [\mathrm{U}(1)\times\mathrm{U}(N)]/\mathbb{Z}_2$. (We can always reduce the structure group $\mathscr{G}_{\mathrm{f}}$ to its maximal compact subgroup.)

The vacuum expectation value of the cocycle condition (6.23),

$$\langle 0|\hat{\mathscr{U}}_{0,\alpha\beta}\hat{\mathscr{U}}_{0,\beta\gamma}\hat{\mathscr{U}}_{0,\gamma\alpha}|0\rangle$$
$$= \exp\left(\frac{i}{\hbar}\left\{\left[\phi_{\alpha\beta} - \tfrac{1}{2}\hbar(f_{\alpha\beta})^i_i\right] + \left[\phi_{\beta\gamma} - \tfrac{1}{2}\hbar(f_{\beta\gamma})^i_i\right] + \left[\phi_{\gamma\alpha} - \tfrac{1}{2}\hbar(f_{\gamma\alpha})^i_i\right]\right\}\right)$$
$$= 1, \tag{6.35}$$

is equivalent to a topological quantization condition

$$\frac{1}{2\pi\hbar}\left(n_{\alpha\beta\gamma} - \tfrac{1}{2}\hbar\mu_{\alpha\beta\gamma}\right) \in \mathbb{Z}, \tag{6.36}$$

where

$$n_{\alpha\beta\gamma} = \varphi_{\alpha\beta} + \varphi_{\beta\gamma} + \varphi_{\gamma\alpha},$$
$$\mu_{\alpha\beta\gamma} = (f_{\alpha\beta})^i_i + (f_{\beta\gamma})^i_i + (f_{\gamma\alpha})^i_i. \tag{6.37}$$

The rest of (6.23) is equivalent to the cocycle condition for the SU(N) vector bundle that always exists. Eq. (6.36) is a modification of the original quantization condition (3.22). As a result of normal ordering there has appeared a quantum correction $-\frac{1}{2}\hbar\mu_{\alpha\beta\gamma}$ that we will call a metaplectic anomaly.

The metaplectic anomaly has a natural geometric interpretation. Recall from Section 3 that $\{n_{\alpha\beta\gamma}/2\pi\hbar\}$ is a Čech cocycle with the de Rahm representative $\omega/2\pi\hbar$. Its integrality guarantees the existence of a line bundle L with connection (prequantum bundle) with the first Chern class $c_1(L) = \omega/2\pi\hbar$. Suppose that (3.22) is indeed satisfied, i.e. the Kostant–Souriau prequantum bundle L does exist. Then from our corrected quantization condition (6.36) we find that the metaplectic anomaly must also satisfy a cocycle condition

$$\frac{1}{4\pi}\mu_{\alpha\beta\gamma} \in \mathbb{Z}. \tag{6.38}$$

If (6.38) is satisfied, we can construct another line bundle that we will denote by $K^{-1/2}$. It is called a bundle of half-forms [11] or pure symplectic spinors [9] and is a (minus) square root of the determinant line bundle K. Determinant bundle K has a structure group U(N) acting on the one-dimensional fibers according to the determinant representation $F \to \det F$, $F \in \mathrm{U}(N)$. K is obtained from the bundle of U(N)-frames over $\mathscr{M}$ by taking determinants of the unitary frames at every point of $\mathscr{M}$ as fibers of K. Such a bundle always exists since any simplectic manifold possesses an almost complex structure. The square root bundle $K^{-1/2}$ is constructed by gluing the Fock vacuum $|0\rangle$ to every point x of $\mathscr{M}$. The first Chern class of the determinant bundle is given by

$$c_1(K) = \frac{1}{2\pi}\operatorname{tr} R = \frac{1}{2\pi}R^i_i, \tag{6.39}$$

where R^i_j is a curvature two-form of the bundle of unitary frames. Then the first Chern class of $K^{-1/2}$ is

$$c_1(K^{-1/2}) = -\tfrac{1}{2}c_1(K) = -\frac{1}{4\pi}R^i_i. \tag{6.40}$$

Since the first Chern class is a de Rahm representative of the Čech cocycle $\{\mu_{\alpha\beta\gamma}/4\pi\}$, and it must be integer, we find that for a topological condition (6.38) to be satisfied $c_1(K^{-1/2})$ must be integer and, consequently, $c_1(K)$ must be even. Only in this case does there exist a bundle of symplectic spinors [9]. This is similar to the situation with Dirac spinors on riemannian manifolds. In order for a manifold to admit spinors, its second Stiffel–Whitney class must vanish. The second Stiffel–Whitney class is a real SO(N) counterpart of the class $c_1(K^{-1/2})$ (mod 2) (it is also called the Maslov class [70,11,14]).

However, our assumption of the existence of $K^{-1/2}$, i.e. that the condition (3.22) holds, is actually superfluous. All we need is the entire quantization condition (6.36). Bundles L and $K^{-1/2}$ may not exist separately. All we need is the existence of the line bundle $LK^{-1/2}$ with connection

$$-\frac{i}{\hbar}(V - \tfrac{1}{2}\hbar\Delta^i_i), \tag{6.41}$$

where Δ^i_j is a U(N)-connection. The Chern class is

$$c_1(LK^{-1/2}) = c_1(L) - \tfrac{1}{2}c_1(K) = \frac{1}{2\pi\hbar}\omega - \frac{1}{4\pi}R^i_i. \tag{6.42}$$

The integral form of the quantization condition (6.36) is

$$\frac{1}{2\pi\hbar}\int_\Sigma \omega - \frac{1}{4\pi}\int_\Sigma R^i_i \in \mathbb{Z} \tag{6.43}$$

for any two-dimensional cycle Σ.

To summarize, in order for the Fock bundle with the structure group reducible to the finite dimensional subgroup $\mathcal{G}^c_f$ to exist, the necessary and sufficient condition is given by (6.43). Inequivalent bundles are classified by symplectic vector bundles and line bundles $LK^{-1/2}$. Note that we have obtained the correct quantization condition including the metaplectic anomaly directly from BFV quantization of the system with first class constraints.

It should be mentioned that the U(N)-connection Δ^i_j and curvature R^i_j in (6.39)–(6.43) differ from the torsion-free symplectic connection Δ^{ab}_μ ($\Gamma^\rho_{\mu\nu}$) and curvature $R^{ab}_{\mu\nu}$ ($R^\sigma_{\mu\nu\rho}$) defined in Sections 3.7 and 4.5 when the manifold $\mathcal{M}$ is not complex (kählerian).

Let J^ν_μ be an almost complex structure (that obviously always exists on any symplectic manifold). According to the Newlander–Nirenberg theorem [53] $\mathcal{M}$ is complex if and only if J has no torsion, i.e. if the torsion tensor defined by

$$N^\rho_{\mu\nu} = 2(J^\sigma_\mu\partial_\sigma J^\rho_\nu - J^\sigma_\nu\partial_\sigma J^\rho_\mu - J^\rho_\sigma\partial_\mu J^\sigma_\nu + J^\rho_\sigma\partial_\nu J^\sigma_\mu) \tag{6.44}$$

vanishes identically. Then $\mathscr{M}$ is kählerian and ω is its Kähler form. The torsion-free symplectic connection is now required to respect the complex structure J that fixes its arbitrariness (recall that Γ is defined by (3.29) only up to the totally symmetric tensor of the third rank that falls out of the LHS of (3.29)).

However, it is known (see Ch. IX of Ref. [55]) that an almost complex manifold admits a torsion-free almost complex connection if and only if J has no torsion. That is, when N does not vanish, our torsion-free symplectic connection Γ is not compatible with any almost complex structure. Nevertheless, it is possible to construct another connection (see Theorem 3.4, Ch. IX of Ref. [55])

$$\tilde{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} - Q^{\rho}_{\mu\nu}, \tag{6.45}$$

where

$$Q^{\rho}_{\mu\nu} := \tfrac{1}{4}\left(J^{\sigma}_{\nu}\nabla_{\sigma}J^{\rho}_{\mu} + J^{\rho}_{\sigma}\nabla_{\nu}J^{\sigma}_{\mu} + 2J^{\rho}_{\sigma}\nabla_{\mu}J^{\sigma}_{\nu}\right) \tag{6.46}$$

(the covariant derivatives ∇_{μ} in (6.46) are calculated with respect to the original connection Γ). This new connection is compatible with the almost complex structure J, and its torsion is proportional to the torsion of the almost complex structure

$$T^{\rho}_{\mu\nu} = \tilde{\Gamma}^{\rho}_{\mu\nu} - \tilde{\Gamma}^{\rho}_{\nu\mu} = \tfrac{1}{4}N^{\rho}_{\mu\nu}. \tag{6.47}$$

It is this almost complex U(N)-connection (6.45) that should be taken in place of the original torsion-free connection in Eqs. (6.39)–(6.43) in the non-Kähler case (R^{j}_{i} in Eqs. (6.39)–(6.43) is the corresponding U(N)-curvature).

7. Conclusion

To summarize, we have presented a gauge-invariant approach to geometric quantization that yields a complete quantum description of dynamical systems with non-trivial geometry and topology of the phase space. We have seen that the gauge-invariant approach incorporates geometric and deformation quantization approaches in a unified theory. Both the problem of quantization of the entire algebra of classical observables in terms of operators in geometric quantization, as well as the problem of finding Hilbert space representations of formal deformations arising in deformation quantization can be solved in the gauge-invariant approach.

In this paper we have considered the most general case of arbitrary symplectic manifolds. In applications some additional geometric structures are often given on the phase space, and the quantization process should preserve those structures. Especially important are dynamical systems with symmetry. Suppose the original classical system on $(\mathscr{M}, \omega)$ has a symmetry group G with generators J^{A} with the Poisson brackets $[J^{A}, J^{B}]^{\omega}_{\mathrm{PB}} = f^{AB}_{C}J^{C}$. At the quantum level, operators $\hat{\mathscr{J}}^{A}$ acting on sections of the Fock bundle must commute with $\hat{\Omega}$ and satisfy the same commutation relations $[\hat{\mathscr{J}}^{A}, \hat{\mathscr{J}}^{B}] = i\hbar f^{AB}_{C}\hat{\mathscr{J}}^{C}$ (up to a possible anomaly in the infinite-dimensional case). Then covariantly constant sections of the Fock bundle (zero-ghost-number cohomology (2.23)) form a representation of (central extension of) G. This point of view is close to the Borel–Weil–Bott theory of group representations and the orbit method [8,16,72]. All types of symmetry groups may be considered in this general setting (compact and non-compact, finite-dimensional and infinite-dimensional). It is a very interesting problem to investigate this gauge-invariant approach to group representations. We will return to it in a subsequent publication [73].

Applications to infinite-dimensional dynamical systems with symmetry groups are especially important for quantum field theory applications. It is particularly tempting to conjecture that two-dimensional conformal field theory may be understood in these terms. Instead of the $\mathbb{Z}_2$ metaplectic anomaly in the quantization condition (6.47) which (mod 2) can only take two values – 0 and $\frac{1}{2}$ – we will have a full fledged anomaly. Indeed, the phase space is now infinite-dimensional and the infinite-dimensional group of linear canonical transformations will acquire a central extension by the entire circle [65,72]. At the infinitesimal level it will be given by the central extension of the infinite-dimensional symplectic algebra, and, as a consequence, the conformal algebra generating classical conformal symmetry will also acquire a central extension. One can speculate that better understanding of this geometric picture of two-dimensional conformal field theory will lead to some advances in conformal field theory in more than two dimensions and, in particular, will shed some light on the geometric structure behind anomalous Ward identities in $D > 2$ [74].

Another interesting potential application is to three-dimensional topological field theory developed by Witten [75,76]. Geometric quantization of Chern–Simons theory with a group G on the three-dimensional manifold $\mathscr{M} = \mathbb{R} \times \Sigma$, where Σ is a riemannian surface, leads to quantization of the moduli space of flat connections $\mathscr{T} = \mathrm{Hom}(\pi_1(\Sigma), \mathrm{G})/\mathrm{G}$ which has a natural symplectic structure [75–77]. In essence, one can say that Chern–Simons theory is a non-minimal conversion of a system on $\mathscr{T}$. The first-class constraints of the Chern–Simons theory reduce the number of physical degrees of freedom. On the other hand, the gauge-invariant approach to geometric quantization based on the conversion procedure considered in this paper generally provides a minimal gauge-invariant extension that is sufficient to perform operator quantization. One could introduce more auxiliary gauge degrees of freedom and subject them to additional first-class constraints. The resulting system would have the same physical contents. Chern–Simons theory is such an example. On the other hand, if one would start with the moduli space $\mathscr{T}$, rather than with Chern–Simons theory, and wanted to quantize it, our conversion approach would provide a minimal gauge-invariant formulation, the physical states given by covariantly constant sections of the Fock bundle over $\mathscr{T}$ with the flat connection defined by the BFV–BRST operator $\hat{\Omega}$, and quantum observables given by covariantly constant sections of the associated bundle of hermitian operators over $\mathscr{T}$ with the flat connection ad $\hat{\Omega}$. This theory would describe the physical contents of the quantum Chern–Simons theory and could provide an operator formulation to the results obtained in topological field theory by means of the path integral approach.

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Appendix A

Notations and conventions

Throughout the paper we use compact tensor notations introduced in Ref. [43] on higher spin theory. Lower (upper) indices denoted by the same letter are authomatically symmetrized and instead of writing $\mathscr{A}^{a_1 \cdots a_k}$ we simply write $\mathscr{A}^{a(k)}$, indicating in brackets the number of indices to be symmetrized. Summation over lower and upper indices denoted by the same letter is understood as usual. (Note that there is no summation over the lower (or upper) indices denoted by the same letter, e.g. $A^{aa} = \frac{1}{2}(\mathscr{A}^{a_1 a_2} + \mathscr{A}^{a_2 a_1})$ but $\mathscr{A}^a_a = \Sigma_1^N \mathscr{A}^a_a$.) Summation over upper and lower indices should be performed only after the symmetrization over all lower and upper indices denoted by the same letter is carried out separately. We also use the following short-hand notations:

$$\nabla_{\mu(n)} := \frac{1}{n!} \nabla_{(\mu_1} \cdots \nabla_{\mu_n)}, \quad \text{e.g.} \quad \nabla_{\mu(2)} = \tfrac{1}{2}\left(\nabla_{\mu_1}\nabla_{\mu_2} + \nabla_{\mu_2}\nabla_{\mu_1}\right), \tag{A.1}$$

$$\Lambda_{a(n),b(n)} := \mathrm{Sym}\left(\Lambda_{a_1 b_1} \cdots \Lambda_{a_n b_n}\right), \tag{A.2}$$

$$\omega^{\mu(n),\nu(n)} := \mathrm{Sym}\left(\omega^{\mu_1 \nu_1} \dots \omega^{\mu_n \nu_n}\right), \tag{A.3}$$

$$h^{a(n)}_{\mu(n)} := \mathrm{Sym}\left(h^{a_1}_{\mu_1} \dots h^{a_n}_{\mu_n}\right), \tag{A.4}$$

where Sym stands for the (weighted) symmetrization. To illustrate, the short-hand notation in Eq. (5.13a) should be read as follows:

$$W^{a(p)b(m)}_{\mu} \Lambda_{b(m),c(m)} W^{a(q)c(m)}_{\nu}$$
$$= \mathrm{Sym}\left(W^{a_1 \dots a_p b_1 \dots b_m}_{\mu} \Lambda_{b_1 c_1} \cdots \Lambda_{b_m c_m} W^{a_{p+1} \cdots a_{p+q} c_1 \cdots c_m}_{\nu}\right), \tag{A.5}$$

where symmetrization is performed in the $p+q$ upper indices $a_1, \dots, a_{p+q}$ after the summation over upper and lower indices $b_1, \dots, b_m$ and $c_1, \dots, c_m$ is carried out. When performing calculations (such as in Appendix B) one has to be careful with combinatorial factors appearing as a result of symmetrization.

Appendix B

B.1. Eq. (4.49)

Substituting (4.49) in (4.47) and using Bianchi identities (4.43) one finds that (4.49) is indeed a solution,

$$R^{a(2)}_{\mu\nu} + \tfrac{1}{4}\left[h^b_\mu \Lambda_{bc}\left(\Lambda^{cd} h^\rho_d R^{a(2)}_{\nu\rho} + 2\Lambda^{ad} h^\rho_d R^{ac}_{\nu\rho}\right) - (\mu \leftrightarrow \nu)\right]$$
$$= R^{a(2)}_{\mu\nu} + \tfrac{1}{2} R^{a(2)}_{\mu\nu} + \tfrac{1}{2}\Lambda^{ad} h^\rho_d \left(h^b_\mu \Lambda_{bc} R^{ac}_{\nu\rho} + h^b_\nu \Lambda_{bc} R^{ac}_{\rho\mu}\right)$$
$$= \tfrac{1}{2} R^{a(2)}_{\mu\nu} + \tfrac{1}{2}\Lambda^{ad} h^\rho_d \left(-h^b_\rho \Lambda_{bc} R^{ac}_{\mu\nu}\right) = 0. \tag{B.1}$$

B.2. Eq. (4.61)

First, let us rewrite Eq. (4.60) in the form

$$\overset{(0)}{\mathscr{A}}{}^{a(3)} = \Lambda^{ab} h^\mu_b D^{\rm sp}_\mu \overset{(0)}{\mathscr{A}}{}^{a(2)} - \Lambda^{ab} h^\mu_b \overset{(0)}{W}{}^{a(2)c}_\mu \Lambda_{cd} \overset{(0)}{\mathscr{A}}{}^{d}. \tag{B.2}$$

Due to Eqs. (4.59) and (4.38) the first term reduces to the first term in the RHS of Eq. (4.61). Substituting (4.49) and (4.56) into the second term one obtains

$$-\Lambda^{ab} h^\mu_b \overset{(0)}{W}{}^{a(2)c}_\mu \Lambda_{cd} \overset{(0)}{\mathscr{A}}{}^{d}$$
$$= \tfrac{1}{4}\Lambda^{ab} h^\mu_b \left(\Lambda^{cd} h^\nu_d R^{a(2)}_{\mu\nu} + 2\Lambda^{ad} h^\nu_d R^{ac}_{\mu\nu}\right)\Lambda_{ce}\Lambda^{ef} h^\rho_f \partial_\rho A$$
$$= \tfrac{1}{4}\Lambda^{ab} h^\mu_b \omega^{\rho\nu} R^{a(2)}_{\mu\nu} \partial_\rho A$$
$$= \tfrac{1}{4}\Lambda^{ab} h^\mu_b \left(R_{\mu\nu\lambda}{}^{\sigma} h^a_\sigma h^\lambda_c \Lambda^{ca}\right)\omega^{\rho\nu}\partial_\rho A$$
$$= -\tfrac{1}{4}\Lambda^{a(3),b(3)} h^{\mu(3)}_{b(3)} \left(R_{\nu\mu\mu}{}^{\sigma} \omega_{\sigma\mu} \omega^{\nu\rho} \partial_\rho A\right). \tag{B.3}$$

B.3. Eq. (5.18)

Substituting (4.49) into Eq. (5.17) one finds

$$\overset{(2)}{\omega}{}_{\mu\nu} = \tfrac{1}{24} \overset{(0)}{W}{}^{a(3)}_\mu \Lambda_{a(3),b(3)} \overset{(0)}{W}{}^{b(3)}_\nu$$
$$= \tfrac{3}{128}\omega^{\rho(2),\delta(2)} R_{\mu\delta\delta}{}^{\rho} \omega_{\rho(3),\sigma(3)} \omega^{\sigma(2),\lambda(2)} R_{\nu\lambda\lambda}{}^{\sigma}$$
$$= \tfrac{3}{128} R_{\mu\sigma\sigma}{}^{\rho} \omega_{\rho\sigma} \omega^{\sigma(2),\lambda(2)} R_{\nu\lambda\lambda}{}^{\sigma}$$
$$= \tfrac{1}{256}\Big(R_{\mu\sigma_1\sigma_2}{}^{\rho}\omega_{\rho\sigma_3} + R_{\mu\sigma_2\sigma_3}{}^{\rho}\omega_{\rho\sigma_1}$$
$$+ R_{\mu\sigma_3\sigma_1}{}^{\rho}\omega_{\rho\sigma_2} + R_{\mu\sigma_1\sigma_3}{}^{\rho}\omega_{\rho\sigma_2} + R_{\mu\sigma_3\sigma_2}{}^{\rho}\omega_{\rho\sigma_1}$$
$$+ R_{\mu\sigma_2\sigma_1}{}^{\rho}\omega_{\rho\sigma_3}\Big)\omega^{\sigma_1\lambda_1}\omega^{\sigma_2\lambda_2} R_{\nu\lambda_1\lambda_2}{}^{\sigma_3}$$
$$= \tfrac{1}{128}\Big(R_{\mu\sigma_1\sigma_2}{}^{\lambda_2} R_{\nu\lambda_1\lambda_2}{}^{\sigma_2}\omega^{\sigma_1\lambda_1} + 2R_{\mu\sigma_1\sigma_2}{}^{\lambda_1} R_{\nu\lambda_1\lambda_2}{}^{\sigma_1}\omega^{\sigma_2\lambda_2}\Big). \tag{B.4}$$

Eqs. (3.32) and (3.33) are used to reduce six types of R^2 terms to the two terms in the final expression.

B.4. Eq. (5.27)

From Eq. (5.26) one has

$$\overset{(2)}{\mathscr{A}}{}^{a} = \Lambda^{ad} h_d^{\mu} \left(\tfrac{1}{24} \overset{(0)}{W}{}^{b(3)}_{\mu} \Lambda_{b(3),c(3)} \overset{(0)}{\mathscr{A}}{}^{c(3)} - \overset{(2)}{W}{}^{b}_{\mu} \Lambda_{bc} \overset{(0)}{\mathscr{A}}{}^{c} \right). \tag{B.5}$$

Substituting (4.56) and (5.18) in the first term one obtains

$$\overset{(2)}{W}{}^{b}_{\mu} \Lambda_{bc} \overset{(0)}{\mathscr{A}}{}^{c} = \tfrac{1}{2} \overset{(2)}{\omega}{}_{\mu\nu} \omega^{\nu\rho} \partial_{\rho} A. \tag{B.6}$$

By means of Eqs. (4.48) and (4.61) the second term is reduced to

$$\begin{aligned}\overset{(2)}{W}{}^{b(3)}_{\mu} \Lambda_{b(3),c(3)} \overset{(0)}{\mathscr{A}}{}^{c(3)} &= -\tfrac{3}{4} \omega^{\rho\nu} \omega^{\rho\sigma} R_{\mu\nu\sigma}{}^{\rho} \\ &\quad \times \left(\nabla_{\rho(3)} A - \tfrac{1}{4} R_{\nu\rho\rho}{}^{\sigma} \omega_{\sigma\rho} \omega^{\nu\lambda} \partial_{\lambda} A \right) \\ &= -\tfrac{3}{4} \omega^{\rho\nu} \omega^{\rho\sigma} R_{\mu\nu\sigma}{}^{\rho} \nabla_{\rho(3)} A - 8 \overset{(2)}{\omega}{}_{\mu\nu} \omega^{\nu\rho} \partial_{\rho} A.\end{aligned}$$

Combining (B.6) and (B.7) gives Eq. (5.27).

B.5. Eqs. (5.38), (5.39)

From (5.37) one has

$$\begin{aligned} A *_{\Gamma} B &= (\mathscr{A} * \mathscr{B})\big|_{\phi=0} \\ &= AB - \tfrac{1}{2} i\hbar \overset{(0)}{\mathscr{A}}{}^{a} \Lambda_{ab} \overset{(0)}{\mathscr{B}}{}^{b} - \tfrac{1}{8} \hbar^2 \overset{(0)}{\mathscr{A}}{}^{a(2)} \Lambda_{a(2),b(2)} \overset{(0)}{\mathscr{B}}{}^{b(2)} \\ &\quad + \tfrac{1}{48} i\hbar^3 \left[\overset{(0)}{\mathscr{A}}{}^{a(3)} \Lambda_{a(3),b(3)} \overset{(0)}{\mathscr{B}}{}^{b(3)} - 24 \left(\overset{(2)}{\mathscr{A}}{}^{a} \Lambda_{ab} \overset{(0)}{\mathscr{B}}{}^{b} + \overset{(0)}{\mathscr{A}}{}^{a} \Lambda_{ab} \overset{(2)}{\mathscr{B}}{}^{b} \right) \right] + \dots . \end{aligned} \tag{B.8}$$

The first three terms are obvious. Let us consider in more detail the $\hbar^3$ term that contains non-trivial curvature dependence. Substituting (4.61) in the first term one obtains

$$\begin{aligned} &- \overset{(0)}{\mathscr{A}}{}^{a(3)} \Lambda_{a(3),b(3)} \overset{(0)}{\mathscr{B}}{}^{b(3)} \\ &= \left(\nabla_{\mu(3)} A - \tfrac{1}{4} R_{\rho\mu\mu}{}^{\lambda} \omega_{\lambda\mu} \omega^{\rho\sigma} \partial_{\sigma} A \right) \omega^{\mu(3),\nu(3)} \left(\nabla_{\nu(3)} B - \tfrac{1}{4} R_{\delta\nu\nu}{}^{\kappa} \omega_{\kappa\nu} \omega^{\delta\pi} \partial_{\pi} B \right) \\ &= \nabla_{\mu(3)} A \omega^{\mu(3),\nu(3)} \nabla_{\nu(3)} B + \tfrac{1}{4} \left[\nabla_{\mu(3)} A \omega^{\mu(3),\nu(3)} R_{\rho\nu\nu}{}^{\mu} \omega^{\rho\sigma} \partial_{\sigma} B - (A \leftrightarrow B) \right] \\ &\quad - \tfrac{8}{3} \partial_{\mu} A \overset{(2)}{\omega}{}^{\mu\nu} \partial_{\nu} B, \end{aligned} \tag{B.9}$$

where

$$\overset{(2)}{\omega}{}^{\mu\nu} := \omega^{\mu\rho}\omega^{\nu\sigma}\overset{(2)}{\omega}{}_{\rho\sigma}, \tag{B.10}$$

and $\overset{(2)}{\omega}{}_{\rho\sigma}$ is given by (5.18b).

The second term reduces as follows:

$$\begin{aligned}
&\overset{(2)}{\mathscr{A}}{}^a\Lambda_{ab}\overset{(0)}{\mathscr{B}}{}^b + \overset{(0)}{\mathscr{A}}{}^a\Lambda_{ab}\overset{(2)}{\mathscr{B}}{}^b \\
&\quad = -\Lambda^{ac}h^\mu_c\left(\tfrac{1}{32}R_{\mu\sigma\sigma}{}^\rho\omega^{\sigma(2),\rho(2)}\nabla_{\rho(3)}A + \tfrac{5}{6}\overset{(2)}{\omega}{}_{\mu\sigma}\omega^{\sigma\rho}\partial_\rho A\right)\Lambda_{ab}\Lambda^{bd}h^\nu_d\partial_\nu B \\
&\qquad -(A \leftrightarrow B) \\
&\quad = \tfrac{1}{32}R_{\mu\sigma\sigma}{}^\rho\omega^{\sigma(2),\rho(2)}\nabla_{\rho(3)}A\,\omega^{\mu\nu}\partial_\nu B - (A \leftrightarrow B) - \tfrac{5}{3}\partial_\mu A\,\overset{(2)}{\omega}{}^{\mu\nu}\partial_\nu B.
\end{aligned} \tag{B.11}$$

Combining (B.9) and (B.11) one finally arrives at

$$\begin{aligned}
&-\overset{(0)}{\mathscr{A}}{}^{a(3)}\Lambda_{a(3),b(3)}\overset{(0)}{\mathscr{B}}{}^{b(3)} + 24\left(\overset{(2)}{\mathscr{A}}{}^a\Lambda_{ab}\overset{(0)}{\mathscr{B}}{}^b + \overset{(0)}{\mathscr{A}}{}^a\Lambda_{ab}\overset{(2)}{\mathscr{B}}{}^b\right) \\
&\quad = \nabla_{\mu(3)}A\,\omega^{\mu(3),\nu(3)}\nabla_{\nu(3)}B + \left[\nabla_{\mu(3)}A\,\omega^{\mu(2),\nu(2)}R_{\rho\nu\nu}{}^\mu\omega^{\rho\sigma}\partial_\sigma B - (A \leftrightarrow B)\right] \\
&\qquad - \tfrac{128}{3}\partial_\mu A\,\overset{(2)}{\omega}{}^{\mu\nu}\partial_\nu B \\
&\quad = \left(\nabla_{\mu(3)}A - R_{\rho\mu\mu}{}^\lambda\omega_{\lambda\mu}\omega^{\rho\sigma}\partial_\sigma A\right)\omega^{\mu(3),\nu(3)}\left(\nabla_{\nu(3)}B - R_{\delta\nu\nu}{}^\kappa\omega_{\kappa\nu}\omega^{\delta\pi}\partial_\pi B\right)
\end{aligned} \tag{B.12}$$

that proves (5.38), (5.39).

B.6. Eq. (5.42)

To prove that our operator $\mathscr{L}_{\mu(3)}$ (5.39) indeed coincides with the operator (5.42) of Refs. [21,22], let us consider Darboux coordinates with the canonical symplectic metric $\omega^{\mu\nu}$. The symplectic connection reduces to the totally symmetric tensor $\Gamma_{\mu(3)}$, $\Gamma^\rho_{\mu\nu} = \omega^{\rho\lambda}\Gamma_{\mu\nu\lambda}$, which, modulo gauge transformations, defines its topological class. Then $\mathscr{L}_{\mu(3)}$ can be reduced as follows:

$$\begin{aligned}
\mathscr{L}_{\mu(3)}A &= \nabla_{\mu(3)}A - R_{\rho\mu\mu}{}^\lambda\omega_{\lambda\mu}\omega^{\rho\sigma}\partial_\sigma A \\
&= \partial_{\mu(3)}A - 3\Gamma^\rho_{\mu\mu}\partial_{\mu\rho}A + \left(-\partial_\mu\Gamma^\rho_{\mu\mu} + \partial_\mu\Gamma^\sigma_{\nu\mu}\,\omega_{\sigma\mu}\omega^{\nu\rho}\right)\partial_\rho A \\
&\quad - \partial_\nu\Gamma^\sigma_{\mu\mu}\,\omega_{\sigma\mu}\omega^{\nu\rho}\partial_\rho A \\
&\quad + \left[2\Gamma^\sigma_{\mu\mu}\Gamma^\rho_{\sigma\mu} - \left(\Gamma^\sigma_{\nu\lambda}\Gamma^\lambda_{\mu\mu} - \Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\mu}\right)\omega_{\sigma\mu}\omega^{\nu\rho}\right]\partial_\rho A \\
&= \partial_{\mu(3)}A - 3\Gamma^\rho_{\mu\mu}\partial_{\mu\rho}A - \partial_\nu\Gamma^\sigma_{\mu\mu}\omega_{\sigma\mu}\omega^{\nu\rho}\partial_\rho A.
\end{aligned} \tag{B.13}$$

Finally, it is easy to see that the two last terms are nothing but the Lie derivative of $\Gamma_{\mu(3)}$ along $X_A = \partial_\mu A\,\omega^{\mu\nu}\partial_\nu$,

$$\begin{aligned}
\left(\pounds_{X_A}\Gamma\right)_{\mu(3)} &= \partial_\nu A\,\omega^{\nu\rho}\partial_\rho\Gamma_{\mu(3)} + 3\partial_\mu\partial_\nu A\,\omega^{\nu\rho}\Gamma_{\rho\mu(2)} \\
&= \partial_\rho\Gamma^\lambda_{\mu(2)}\,\omega_{\lambda\mu}\omega^{\rho\nu}\partial_\nu A + 3\Gamma^\nu_{\mu(2)}\partial_{\mu\nu}A.
\end{aligned} \tag{B.14}$$

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