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CONFORMAL INVARIANCE IN QUANTUM YANG–MILLS THEORY

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A new (nonlinear) realization of conformal transformations of nonabelian fields is found. These transformations (unlike the usual linear ones) leave the generating functional invariant and permit one to obtain a nontrivial quantum Yang–Mills theory. The gauge field dimension is shown to be canonical while the field tensor has an anomalous dimension. The invariant propagators are found. The possibility of the solution of the confinement problem is discussed.

1. Conformal solution in quantum gauge theories is possible either for fixed values of the charge-zeroes of the function $\beta(g)$, or for $\beta(g) \equiv 0$, as for example in $N = 4$ extended supersymmetric Yang–Mills theory [1] and in $N = 4$ extended conformal supergravity [2]. Study of conformal solutions is of particular interest in connection with the problem of confinement.

It is well known that classical equations of the gauge fields are conformally invariant, but the quantum formulation of the conformal-invariant nontrivial gauge theory is still absent. In fact with the assumption that the vacuum is invariant with respect to special conformal transformations and the gauge field $A_\mu(x)$ is transformed as a conformal vector with the dimension $d_A = 1$

$$A_\mu(x) \to A_\mu(x) + \epsilon_\lambda K_\lambda A_\mu(x), \qquad (1)$$

where

$$K_\lambda A_\mu = (x^2\partial_\lambda - 2x_\lambda x_\tau \partial_\tau - 2x_\lambda) A_\mu + 2x_\mu A_\lambda - 2\delta_{\lambda\mu} x_\tau A_\tau,$$

then, as known, a purely longitudinal expression $\langle 0|A_\mu(x_1)A_\nu(x_2)|0\rangle \sim \partial_\mu\partial_\nu \ln x_{12}^2$ is obtained for the propagator. The reason is that the gauge term in the generating functional in the euclidian theory

$$Z(J) = N^{-1}\int [\mathrm{d}A] \det|\partial\nabla| \exp\Big(\int \mathrm{d}x \times \mathrm{Sp}\left[-(1/4\alpha)(\partial_\mu A_\mu)^2 - \tfrac{1}{8}F_{\mu\nu}^2 + A_\mu J_\mu\right]\Big) \qquad (2)$$

breaks its invariance with respect to (1)

$$S_{\text{gauge}} = \frac{1}{4\alpha}\int \mathrm{d}x\, \mathrm{Sp}(\partial_\mu A_\mu)^2 \to \frac{1}{4\alpha}\int \mathrm{d}x\, \mathrm{Sp}(\partial_\mu A_\mu + 4\epsilon_\lambda A_\lambda)^2 \qquad (3)$$

(the surface terms are omitted on the right-hand side). From (3) it follows that S_{gauge} remains invariant on the subspace of the fields of the form $A_\mu^S = \partial_\mu S(x) S^{-1}(x)$ only, which correspond to a pure gauge. These fields form an invariant with respect to the subspace (1): $A_\mu^S \to A_\mu^{S'}$, where

$$S' = [1 + \epsilon_\lambda(x^2\partial_\lambda - 2x_\lambda x_\tau \partial_\tau)] S.$$

For such fields we find:

$$\delta S_{\text{gauge}} \sim \epsilon_\lambda \int \mathrm{d}x\, \mathrm{Sp}\left[\partial_\mu(\partial_\mu S \cdot S^{-1})\partial_\lambda S \cdot S^{-1}\right] = -\epsilon_\lambda \int \mathrm{d}x\, \mathrm{Sp}(\Box S^{-1}\partial_\lambda S + \partial_\mu S^{-1}\partial_\lambda\partial_\mu S) = 0.$$

Thus, the generating functional (2) is invariant with respect to (1) only in a purely gauge sector, which is the reason of the longitudinality of the propagator. Note that any other gauge does not admit invariants with respect to solution (1).

2. The idea of the proposed approach consists in constructing a new (nonlinear with respect to A_μ) realization of conformal transformations, which keeps the generating functional (2) invariant on the whole space of the fields A_μ and thus allows a nontrivial conformal quantum Yang–Mills theory. This new realization is determined by the structure of the functional (2). In fact, the term $F_{\mu\nu}^2$ is invariant with respect to the direct product of conformal and gauge groups. The gauge term breaks each of these symmetries. We show, however, that a complete change of the product

$$[\mathrm{d}A]\det|\partial\nabla|\exp\left(-\frac{1}{4\alpha}\int \mathrm{d}x\,\mathrm{Sp}\,(\partial_\mu A_\mu)^2\right),$$

under the action of special conformal transformations (1) can be compensated for by a special gauge transformation nonlinearly dependent on the field. *Thus we show that there are combined transformations consisting of conformal and specially selected gauge transformations, that leaves the functional* $Z(J)|_{J=0}$ *and, therefore, the vacuum invariant.* These combined transformations form a nonlinear (with respect to the field A_μ) realization of the conformal group. We determine an explicit form of these transformations for quantum fields and the invariant propagators. In conclusion we present a brief discussion of a new possibility of the solution of the confinement problem, which appears in our approach.

3. Consider variation of the functional (2) under the action of transformations (1). Let us make in $\det|\partial\nabla|$ the substitution (1) and multiply it by the factor

$$B(\epsilon)=\det\left|1+\epsilon_\lambda\left(x^2\partial_\lambda-2x_\lambda x_\tau\partial_\tau\right)\right|,$$

independent of A_μ. Neglecting the terms $\sim\epsilon_\lambda^2$ we obtain after some simple calculations:

$$\begin{aligned}&\det|\partial\nabla'|B(\epsilon)\\ &\quad=\det|\partial\nabla+\epsilon_\lambda\left(x^2\partial_\lambda-2x_\lambda x_\tau\partial_\tau-4x_\lambda\right)\partial\nabla\\ &\quad+4\epsilon_\lambda\nabla_\lambda|\\ &\quad=\det|\partial\nabla|\exp\left\{\mathrm{Sp}\left[\epsilon_\lambda\left(x^2\partial_\lambda-2x_\lambda x_\tau\partial_\tau-4x_\lambda\right)\right.\right.\\ &\quad\left.\left.+4\epsilon_\lambda\nabla_\lambda(1/\partial\nabla)\right]\right\},\end{aligned}$$

where $\nabla'_\mu=\partial_\mu+A'_\mu$. The first term in the exponential can be omitted, since it does not depend on A_μ. As a result we find, up to insignificant factors,

$$\begin{aligned}&\det|\partial\nabla|\exp\left(-\frac{1}{4\alpha}\int\mathrm{d}x\,\mathrm{Sp}\,(\partial_\mu A_\mu)^2\right)\\ &\quad\to\det|\partial\nabla+4\epsilon_\lambda\nabla_\lambda|\\ &\quad\times\exp\left(-\frac{1}{4\alpha}\int\mathrm{d}x\,\mathrm{Sp}\,(\partial_\mu A_\mu+4\epsilon_\lambda A_\lambda)^2\right).\end{aligned}\qquad(4)$$

If we substitute this into (2) and make an additional gauge transformation

$$A_\mu\to A_\mu-\nabla_\mu(4\epsilon_\lambda(1/\partial\nabla)A_\lambda),$$

we obtain the original expression. Thus we proved that functional (2) at $J=0$ is invariant with respect to nonlinear infinitesimal transformations

$$A_\mu(x)\to A_\mu(x)+\delta A_\mu(x),\qquad(5)$$

where

$$\delta A_\mu(x)=\epsilon_\lambda K_\lambda A_\mu(x)-4\epsilon_\lambda\nabla_\mu\frac{1}{\partial\nabla}A_\lambda(x).$$

The global transformations were considered by us in ref. [3], where their mathematical structure was investigated and the group law was proved.

Note that (5) in the abelian case coincides with the transformation law obtained by us before [4] in a conformal QED.[‡1]

[‡1] The combined transformations including conformal and gauge transformations in QED were also considered in ref. [5].

4. Let us consider the transformation law of quantum (euclidian) fields. It can be changed at the expence of (dimensional) renormalizations and the resolution of uncertainty $0 \times \infty$ in nonlinear terms [6]. Usually this leads to the emergence of anomalous dimensions. It can be easily seen, however, that the dimension of the quantum field A_μ remains a canonical one because of the special structure of the interaction ($L_{\text{int}} = A_\mu j_\mu$), and the dimension d_F of the field tensor $F_{\mu\nu}$ becomes anomalous. For illustration we consider the renormalized equations

$$z_3 F_{\mu\nu} = (z_2 z_3)^{1/2}(\partial_\mu A_\nu - \partial_\nu A_\mu) + z_1[A_\mu, A_\nu], \quad (6)$$

$$(z_2 z_3)^{1/2}\partial_\nu F_{\mu\nu} + z_1[A_\nu, F_{\mu\nu}] = \tilde{z}_1 \partial_\mu \bar{C} C + \alpha^{-1} z_2 \partial_\mu \partial_\nu A_\nu, \quad (7)$$

where z_2, z_3 and z_1 are renormalization constants of the fields A_μ, $F_{\mu\nu}$ and the vertex respectively. The uncertainty in the nonlinear terms is resolved as follows (see details in ref. [6]): the products of the type $z_1[A_\mu(x), A_\nu(x)]$ are defined as the $\epsilon \to 0$ limit of the expression $z_1(\epsilon)[A_\mu(x), A_\nu(x+\epsilon)]$ averaged over the angle of the vector ϵ_μ, and the renormalization constants are considered as power functions: $z_i(\epsilon) \sim (\epsilon^2)^{\delta_i}$. Equating the total dimensions of individual terms in each of eqs. (6), (7) and taking into account that the dimension of the term $z_1[A_\nu, F_{\mu\nu}]$ is equal to three (because it is the current of the Yang–Mills field) we find $d_A = 1$, d_F is arbitrary and $z_2 = $ const. Thus, the renormalization of the field A_μ appears to be finite and its transformation law (5) is transferred without change to the quantum case. Renormalization of the charge and the combination of the constants $z_1(z_2 z_3)^{-1/2}$, which enters the covariant derivative, is also finite.

A substantial result is the appearance of the anomalous dimension d_F of the field $F_{\mu\nu}$ (this is possible in the nonabelian theory only; in conformal QED $d_F = 2$). As a result the infinitesimal transformations of the field $F_{\mu\nu}$ have the form

$$F_{\mu\nu}(x) \to F_{\mu\nu}(x) + \epsilon_\lambda K_\lambda F_{\mu\nu}(x) + [\omega(x), F_{\mu\nu}(x)],$$
$$\omega(x) = 4\epsilon_\lambda (1/\partial \nabla) A_\lambda(x), \quad (8)$$

where

$$K_\lambda F_{\mu\nu} = (x^2 \partial_\lambda - 2x_\lambda x_\tau \partial_\tau - 2d_F) F_{\mu\nu} + 2\mathrm{i} x_\tau \Sigma_{\lambda\tau} F_{\mu\nu},$$

d_F is arbitrary and, as can be shown, for the propagator the following expression is obtained

$$\langle 0|F_{\mu\nu}(x_1) F_{\sigma\tau}(x_2)|0\rangle \sim [g_{\mu\sigma}(x_{12}) g_{\nu\tau}(x_{12}) - g_{\mu\tau}(x_{12}) g_{\nu\sigma}(x_{12})]/(x_{12}^2)^{d_F},$$

where $g_{\mu\nu}(x) = \delta_{\mu\nu} - 2x_\mu x_\nu / x^2$.

Consider the transformation of the ghost fields $C(x)$ and $\bar{C}(x)$. They must lead to a similar change in $\det|\partial \nabla|$, as in (4). It can be easily verified that for this the ghost fields should be conformal scalars with different dimensions: $d_C = 0$, $d_{\bar{C}} = 2$,

$$K_\lambda C(x) = (x^2 \partial_\lambda - 2x_\lambda x_\tau \partial_\tau) C(x),$$
$$K_\lambda \bar{C}(x) = (x^2 \partial_\lambda - 2x_\lambda x_\tau \partial_\tau - 4x_\lambda) \bar{C}(x).$$

For compensation of transformation (4) we should perform the BRST transformation with a parameter being a definite functional of the fields. In this case the compensating terms arise due to transformation of the integration measure in the functional integral [7]. Let us find the required BRST transformation for quantum fields. Comparing the BRST variation $\delta A_\mu = -\nabla_\mu C(x)\epsilon$ where ϵ is a small parameter, with the second term in (5) and taking into account the equality $C(x)\bar{C}(y) = (1/\partial \nabla)\delta(x-y)$, we find: $\epsilon = 4\epsilon_\lambda \int \mathrm{d}y\, \bar{C}^a(y) \times A_\lambda^a(y)$. As a result, for the ghost fields we have the following transformation laws:

$$\delta C(x) = \epsilon_\lambda \{K_\lambda C(x) + 2[C(x), (1/\partial \nabla) A_\lambda(x)]\},$$
$$\delta \bar{C}(x) = \epsilon_\lambda \left(K_\lambda \bar{C}(x) - \frac{4}{\alpha} \int \mathrm{d}y\, \partial_\mu A_\mu(x) A_\lambda^a(y) \bar{C}^a(y) \right).$$

5. Let us find the invariant propagator $D_{\mu\nu}(x_{12}) = \langle 0|A_\mu(x_1) A_\nu(x_2)|0\rangle$. The invariance condition

of $D_{\mu\nu}$ (conformal Ward identity) has the form:

$$\langle 0|\delta A_\mu(x_1)A_\nu(x_2)|0\rangle + \langle 0|A_\mu(x_1)\delta A_\nu(x_2)|0\rangle = 0,$$

where the variation δA_μ is given in (5). The first addend in this equation contains the nonlinear term $D_{\mu\lambda\nu}(x_{12}) = \langle 0|\nabla_\mu(1/\partial\nabla)A_\lambda(x_1)A_\nu(x_2)|0\rangle$ and the second the term $D_{\nu\lambda\mu}(x_{21})$. Let us use the scale invariance and the equality $\partial_\mu^{x_1}D_{\mu\lambda\nu}(x_{12}) = D_{\lambda\nu}(x_{12})$. In transverse gauge we find for their sum (see for details in ref. [3]):

$$D_{\mu\lambda\nu}(x_{12}) + D_{\nu\lambda\mu}(x_{21}) = (\partial_\mu^{x_1}/\Box_{x_1})D_{\lambda\nu}(x_{12}) + (\partial_\nu^{x_2}/\Box_{x_2})D_{\lambda\mu}(x_{12}).$$

As a result the invariance condition becomes a linear integro-differential equation. Its solution is the transverse function (the first term in (9)). In the generalized α-gauge some additional limitations on the nonlinear term $D_{\mu\lambda\nu}$ are obtained and

$$D_{\mu\nu}(x_{12}) = A(\delta_{\mu\nu} - \partial_\mu\partial_\nu/\Box)/x_{12}^2 + B(\partial_\mu\partial_\nu/\Box)/x_{12}^2, \tag{9}$$

where A and B are some constants (from the Ward identity follows $B = \alpha$).

Similarly for the Green function $\langle 0|F_{\mu\nu}(x_1)A_\tau(x_2)|0\rangle$ taking into account (5) and (8) we have the following solution:

$$\langle 0|F_{\mu\nu}(x_1)A_\tau(x_2)|0\rangle = C(\delta_{\mu\tau}\partial_\nu^{x_1} - \delta_{\nu\tau}\partial_\mu^{x_1})/(x_{12}^2)^{d_F/2},$$

where C is a certain constant to be calculated from the bootstrap equations and for the nonlinear terms particular limitations arise which are not given here. Analogously for the ghost fields we find

$$\langle 0|\nabla_\mu C(x_1)\bar{C}(x_2)|0\rangle = (\partial_\mu/\Box)\delta(x_{12}).$$

6. In conclusion we indicate the attractive possibility of solving the confinement problem in conformal theory [‡2]. If there is an IR stable fixed point, the conformal theory then corresponds to IR asymptotics. We can expect that with particular values of the dimension d_F of the field $F_{\mu\nu}$ it is the latter which makes the main contribution to the long-distance interaction and leads to its growth (real values of d_F can be calculated from the conformal bootstrap equations).

There is another more tempting possibility. Note that the fields $F_{\mu\nu}$ and A_μ in the conformal theory are independent objects: $F_{\mu\nu}$ has an anomalous dimension, and A_μ has a canonical one. This is a nonperturbative effect. It seems possible that $F_{\mu\nu}$ determines the interaction in the IR region while the field A_μ makes the main contribution to the UV region and leads to a decrease of the effective interaction at short distances (this is possible due to a nontrivial dependence of the conformal vertex Γ_{AAA} on the momenta), i.e. both asymptotics are described in the scope of the conformal theory.

For realization of this program it is necessary to find explicit expressions for the conformal vertices and to clarify what value of d_F implies the growth of the effective interaction in the IR region. Then we should verify if this value satisfies the conformal bootstrap equations. A solution of these questions will be presented in subsequent works. For conformal QED an explicit expression for the vertex and the discussion of the bootstrap program is given in ref. [4].

‡2 See also ref. [8], where another possibility was discussed.

References

[1] L.V. Avdeev, O.V. Tarasov and A.A. Vladimirov, Phys. Lett. 96B (1980) 94;
V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Nucl. Phys. B229 (1983) 391.

[2] E.S. Fradkin and A.A. Tseytlin, preprint N 185, Lebedev Physical Institute (Moscow, 1983); Phys. Lett. 134B (1984) 307.

[3] M.Ya. Palchik and E.S. Fradkin, Dokl. Acad. Nauk SSSR, to be published;
M.Ya. Palchik, preprint N 215, Institute of Automation and Electrometry of the Academy of Sciences of the USSR (Novosibirsk, 1983).

[4] M.Ya. Palchik, J. Phys. A16 (1983) 1523;
A.A. Kozhevnikov, M.Ya. Palchik and A.A. Pomeransky, Yad. Fiz. 37 (1983) 481;
E.S. Fradkin, A.A. Kozhevnikov, M.Ya. Palchik and A.A. Pomeransky, Commun. Math. Phys. 91 (1983) 529.
[5] M. Baker and K. Johnson, Physica 96A (1979) 120;
S. Adler, Phys. Rev. D8 (1973) 2400.
[6] E.S. Fradkin and M. Ya. Palchik, Phys. Rev. C44 (1978) 249.
[7] I.A. Batalin and G.A. Vilkovisky, Phys. Lett. 69B (1977) 309;
E.S. Fradkin and T.E. Fradkina, Phys. Lett. 72B (1978) 343.
[8] A.N. Vasiliev, M.M. Perekalin and Yu.M. Pismak, Teor. Mat. Fiz. 55 (1983) 323.