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# SUPERCONFORMAL HIGHER-SPIN THEORY IN THE CUBIC APPROXIMATION

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Based on the conformal higher-spin superalgebras $\mathrm{shsc}^{\infty}(4|1)$ constructed previously by us, superconformal higher-spin theory in four-dimensional space-time is constructed in the cubic order. The theory contains an infinite tower of $SU(2,2|1)$-supermultiplets with all conformal higher-spin fields interacting among themselves and with conformal supergravity.

## 1. Introduction

Why conformal higher spins?

A central problem of theoretical physics consists in the unification of all the fundamental interactions including gravity. Such a unified theory must satisfy the following main criteria:

(i) It should be self-consistent at the quantum level;

(ii) It should give an adequate description of low-energy physics.

The finiteness and anomaly free conditions for all classical symmetries of the theory are understood as a self-consistency. An adequate description of low-energy physics includes the description of compactification to the four dimensions, the vanishing of the cosmological constant and the presence of the observed spectrum of particles, including the Glashow–Salam–Weinberg model, QCD and gravity, in the spontaneously broken phase.

The above selection criteria strongly restrict the possible candidates to the role of a unified theory.

What must a unified theory be?

Since Einstein theoretical physics has been a search for the answer to this question. Conventionally this process can be divided into two stages. The first one is a lower-spin stage. This is a search for candidates to the unified theory role among the theories including only lower-spin fields ($s \leqslant 2$). The culmination of this stage was the golden age of supergravity theory. However, within the bounds of supergravity, it did not work well enough to enable one to construct a unified theory, because of a number of obstacles and defects in supergravity.

The first is the non-finiteness in all the supergravity models except the $N=4$ conformal supergravity. For the finiteness of some theory, apparently, one of the two possibilities should take place: either the theory has no conformal invariance, but then an infinite tower of fields should be present, or the theory should be conformally invariant and in this case, generally speaking, the presence of an infinite number of fields is not obligatory. In the former case it seems natural that an infinite tower of fields contains all the higher-spin fields (see ref. [1]).

The $N=4$ conformal supergravity provides an example of the second possibility. This theory with $N=4$ conformal Yang–Mills matter supermultiplet is anomaly free and its finiteness then is a consequence of the superconformal invariance (see ref. [2]). Other supergravity theories do not fit into one of the above possibilities.

The second are phenomenological defects of supergravity consisting in the lack of correspondence between the supergravity spectrum of particles and the observed spectrum. Among the causes of it there is a restriction from above on $N$ that does not allow one to construct extended models with the Grand Unification Group as a gauge subgroup. This restriction is a consequence of the absence of massless higher-spin fields in the supergravity multiplets. For example, the internal group in the $N=4$ extended conformal supergravity is SU(4), which is not sufficient to include the standard $SU(3) \times SU(2) \times U(1)$ model*. Meanwhile in a hypothetical $N=5$ theory, the internal group actually is the GUT group SU(5). But such a theory cannot be constructed off-shell without introducing higher-spin conformal supermultiplets. For the closure of the symmetry algebra one cannot limit oneself to introducing only a finite number of higher-spin fields. Only an infinite tower of all lower and higher-spin symmetries forms an Lie superalgebra for both higher spins ($s>2$) and lower ones ($s \leqslant 2$).

To sum up, we have come to the following: for the finiteness and correspondence with low-energy physics, a unified theory must contain an infinite tower of fields with all higher spins.

In this way we come to the second stage – a higher-spin stage. So our next question is "what should a higher-spin theory be?"

There are two quite different situations here. These are higher-spin theories with or without any mass dimensionful parameters.

The first version is realized in closed string theory [3]. String theory contains an infinite tower of fields with arbitrary high spins, their masses growing with their spins. The mass parameter here is inversely related to the square root of the slope $\alpha'$. In the zero slope limit, $\alpha' \to 0$, masses of all the higher-spin fields ($s>2$) tend to infinity and only the massless lower-spin ($s \leqslant 2$) sector remains as observed. The

* The same problem arises in usual $N$-extended supergravity, where we also have the restriction $N \leqslant 8$.

other limit is the massless limit $\alpha' \to \infty$. In this limit all the higher spins should become massless. However, this limit does not exist as a continuous limiting process for all the momentum in view of the well-known no-go theorem [4, 6]. This theorem states that there exists no gauge invariant interaction among the massless higher-spin fields and the Einstein gravity without cosmological term. Note that the theorem leaves one possibility for the massless limit in string theory: all the higher-spin fields becoming non-interacting in the massless limit. But this possibility might have a physical meaning only for infinite energy. The formal reason for the non-existence of the continuous massless limit is the non-analyticity of the interaction in the mass parameter. A striking example of such non-analyticity of the fundamental fields interaction in string theory is the Born–Infeld tree effective action for the electromagnetic field in the open string theory [5]. The effective lagrangian is [5] $\sqrt{\det(\delta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})}$, and one sees directly that the interaction is non-analytical in the mass parameter (which is $\sim (\alpha')^{-1/2}$). Note that the exact result contains the contributions from all massive higher-spin fields (which are the excitations of the string), because it has been arrived at by performing the exact integration over the whole string $X^{\mu}$.

Nevertheless, the massless limit in the closed string theory may exist. It is connected with the following possibility. It was obtained in ref. [6] that the statement of the no-go theorem of ref. [4] takes place only in the case of the Einstein gravity without cosmological term (as has been formulated above). In the adS background there exists a gauge-invariant interaction of the massless higher-spin fields among themselves and with the adS gravity. The crucial feature of this interaction is its non-analyticity in the cosmological constant $\Lambda$, which does not allow one to pass to a flat limit $\Lambda \to 0$. This is quite analogous to the non-analyticity in the mass of higher spins in string theory and suggests the idea that the slope parameter $\alpha'$ and the cosmological constant $\Lambda$ are related.

It looks natural that there is a massless limit in the closed string theory; the limit however is not a continuous process but rather a phase transition connected with the appearance of the cosmological term. In this case the metric in the ground state of the string theory will have a different vacuum expectation value, the adS background, than it has in the usual closed string theory. Since the massless higher-spin theory in adS has an infinite-dimensional gauge symmetry algebra [7, 8], then the phase transition from it to the string phase is spontaneous breaking of the higher-spin symmetry. Thus we expect that hidden symmetry in string theory is a higher-spin adS gauge symmetry. The massless $adS_4$ higher-spin theory is developed in refs. [6–10]. Let us mention a characteristic feature of the higher-spin interaction. It contains higher derivatives that can be considered as bringing about some effective non-locality, as in strings.

However, there is still a dimensionful parameter in the adS massless higher-spin theory. Meanwhile, in the ultra high-energy domain all mass parameters can become non-essential and therefore any theory can effectively be considered as a conformal field theory [11]. Then all dimensionful parameters of the low-energy theory appear as a result of spontaneous breaking of the original conformally invariant interacting theory. In this way we come to the idea that a conformally invariant phase for the adS higher-spin theory, a conformal higher-spin theory, may exist. Conformal Weyl gravity is an example of a gravity theory without any mass parameters (dimensionful coupling constants). The superconformal higher-spin theory is a generalization of the Weyl supergravity to all higher spins. The conformal higher-spin theory is developed in refs. [12–18] and in the present paper. Such a theory has a symmetry larger than the adS theory. This is a conformal space-time supersymmetry and an infinite tower of conformal higher-spin symmetries [12, 13]. It should be mentioned that the adS higher-spin symmetry algebra is contained as a subalgebra in the conformal higher-spin algebra. Especially interesting is the extended superconformal higher-spin theory based on the $N=5$ higher-spin conformal superalgebra $\mathrm{shsc}^{\infty}(4|5)$ [13], containing SU(5). (It is not impossible that this $N=5$ theory is finite.)

Summing up the above expounded arguments, the following scenario may be suggested. In the ultra high-energy domain a unified theory is effectively described as a conformal higher-spin theory generalizing Weyl gravity. The spontaneous conformal symmetry breaking leads to the massless higher-spin theory in the anti-de Sitter universe generalizing the adS supergravity. Further, the adS higher-spin symmetry breaking leads to the string-like phase with the massive higher spins coupled to the Einstein gravity on the flat background with zero cosmological constant. The above scenario is schematically illustrated in fig. 1.

It should be mentioned that one can look at the above scenario in two different ways. Firstly, it may be treated straightforwardly as a scenario for the fundamental unified theory, i.e. the unified hypothetical lagrangian, or its spontaneously broken

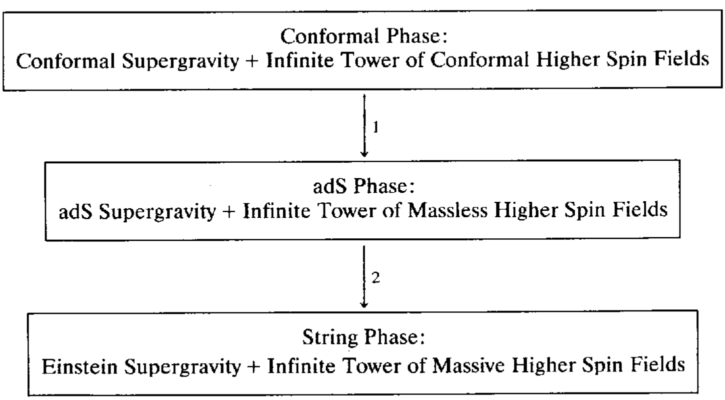


Fig. 1. Phases of Unified Theory. The arrows 1, 2 denote the symmetry breakings (see in the text).

versions, actually describes this picture at all energies. Then, apparently, the scenario should be played in high dimensions. Secondly, it may be considered as a hierarchy of effective theories, each working effectively only in its own energy domain; and their lagrangians, generally speaking, are not connected straightforwardly with each other. Subsequently it also might be considered in four dimensions.

Above we have expounded the philosophy of the massless higher-spin theories. Now we pass to the methods of constructing them. In principle there are two different approaches. The first consists in the investigation of string theory in the ultra high-energy domain when a conformal regime should be realized. The adS phase is some, apparently instable, phase between the conformal and string phases. To analyse it one can use one out of the two powerful methods: the effective action approach, or string field theory.

The other approach to constructing massless higher-spin theories consists in a generalization of the traditional field-theoretical methods used in supergravity. Such an approach is developed in refs. [6–10] for the massless adS higher-spin theory and in refs. [12–15] and in the present paper for the superconformal higher-spin theory. The first stage is constructing the infinite-dimensional higher-spin superalgebras generalizing the supergravity superalgebras. For the $adS_4$ theory this has been done in refs. [7, 8] and for the conformal theory in refs. [12, 13] (for the lower-dimensional higher-spin theories and new superconformal algebras see refs. [16–18]). The second stage is developing gauge invariant dynamics on the basis of these global superalgebras (in terms of their gauge fields and curvatures).

The present paper is devoted to the conformally invariant dynamics. In a preceding paper [14] we obtained a higher-spin generalization of the Weyl gravity in the cubic order approximation. In the present work we extend our previous results to the superconformal case and give a more detailed description of our construction. In sect. 2, we present necessary information about conformal higher spin superalgebra $shsc^{\infty}(4|1)$, its gauge fields and curvatures. In sect. 3, a brief description of free conformally invariant higher-spin dynamics in the spin-tensor formalism is presented. In sect. 4, the linearized conformal higher-spin theory on the geometrical basis of linearized curvatures is described. In sect. 5, the lagrangian of the superconformal interacting higher-spin theory is proposed and its gauge invariance in the cubic order is proved. In sect. 6, we summarize the main results of the paper and briefly discuss some further problems.

## 2. Conformal higher-spin superalgebra $shsc^{\infty}(4|1)$

In ref. [13] we constructed an infinite-dimensional generalization $igl(M|N;\mathbb{C})$ (i means infinite) of the superalgebra $gl(M|N;\mathbb{C})$. This superalgebra is embedded as a subalgebra in $shs(2N|2M;\mathbb{C})$ constructed in refs. [7,8], analogously the embedding $gl(M|N;\mathbb{C}) \subset osp(2N|2M;\mathbb{C})$. Our construction is based on the oscillator realizations method and the symbol operator theory (see refs. [8, 13]). The real form of igl(4|N; ℂ) (exactly its factor-algebra with respect to its centre) is a generalization of the conformal superalgebra in $D = 3 + 1$ SU(2, 2|$N$), denoted by us as shsc$^\infty$(4|$N$) (shsc stands for super higher-spin conformal). The gauge fields and curvatures for this superalgebra generalize the gauge fields and curvatures of $N$-extended conformal supergravity.

In this paper we consider the $N = 1$ case and in the present section we concentrate our attention on the superalgebra shsc$^\infty$ (4|1). We shall sometimes refer to our paper [13] as (I) and to the formulae in (I) as, for instance, (I.7.7).

### 2.1. THE OPERATORIAL REALIZATION

Generating elements to realize the conformal superalgebra can be conveniently chosen as a supertwistor* $Z = (a^\alpha, a_{\dot\beta}, \alpha)$ and a dual supertwistor $\overline{Z} = (\bar a_\alpha, \bar a^{\dot\beta}, \alpha^\dagger)$ with the commutation relations

$$\left[Z^A, \overline{Z}_B\right\} = 2\delta^A_B\,, \tag{2.1a}$$

or

$$\left[a^\alpha, \bar a_\beta\right] = 2\delta^\alpha_\beta\,, \qquad \left[a_{\dot\alpha}, \bar a^{\dot\beta}\right] = 2\delta^{\dot\beta}_{\dot\alpha}\,, \tag{2.1b}$$

$$\{\alpha, \alpha^\dagger\} = 2\,. \tag{2.1c}$$

The Grassmann parity and hermitian conjugation are defined as

$$\varepsilon(\alpha) = \varepsilon(\alpha^\dagger) = 1\,, \qquad \varepsilon(a) = \varepsilon(\bar a) = 0\,, \tag{2.2a}$$

$$(a^\alpha)^\dagger = \bar a^{\dot\alpha}\,, \qquad (\bar a^{\dot\alpha})^\dagger = a^\alpha\,, \qquad (\bar a_\alpha)^\dagger = a_{\dot\alpha}\,, \tag{2.2b}$$

$$(a_{\dot\alpha})^\dagger = \bar a_\alpha\,, \qquad (\alpha)^\dagger = \alpha^\dagger\,, \qquad (\alpha^\dagger)^\dagger = \alpha\,. \tag{2.2c}$$

Below we work only with the Weyl (symmetric) symbols of the operators, denoting both the operators and their symbols by the same letters as in (I).

Let us define a "particle number" operator (or a "superhelicity" operator, as it is called in the twistor theory) with the Weyl symbol

$$T = \overline{Z}_A Z^A = \bar a_\alpha a^\alpha + \bar a^{\dot\beta} a_{\dot\beta} + \alpha^\dagger \alpha\,. \tag{2.3}$$

* For our two-component multispinorial notations see appendix A.

An algebra formed by all quadratic polynomials on the above generating elements is isomorphic to osp(2|8). A subalgebra formed by all quadratic polynomials commuting with "particle number" operator, except $T$ itself, is isomorphic to the conformal superalgebra SU(2,2|1). For the explicit expressions for the superconformal generators see eqs. (I.3.8, 9).

A conformal higher-spin superalgebra shsc$^\infty$(4|1) is defined as the algebra of all arbitrary order polynomials commuting with $T$ except the powers of $T$ themselves (this is a centre of igl(4|1;$\mathbb{C}$), see (I)). In the algebra shsc$^\infty$(4|1) there acts a representation of su(2,2|1). One can get explicit expressions for the generators of this representation as the differential operators according to

$$\tilde{A}(B) = [A, B\}_*, \qquad A \in \mathrm{su}(2,2|1), \quad B \in \mathrm{shsc}^\infty(4|1) \tag{2.4}$$

where $[,\}_*$ is the super-commutator of the Weyl symbols $A$ and $B$ (see (I.3.11)).

In shsc$^\infty$(4|1) there exists an important involutive automorphism $\mathscr{R}$ called as the Weyl reflection or inversion (element of the Weyl group of conformal algebra), which acts on the generating elements and the superconformal generators as follows:

$$\left(a^\alpha, a_{\dot\beta}, \bar{a}_\alpha, \bar{a}^{\dot\beta}, \alpha, \alpha^\dagger\right) \overset{\mathscr{R}}{\to} \left(\bar{a}^\alpha, \bar{a}_{\dot\beta}, a_\alpha, a^{\dot\beta}, \alpha^\dagger, \alpha\right), \tag{2.5}$$

$$\begin{aligned}&\left(P_{\alpha\dot\beta}, K_{\alpha\dot\beta}, M_{\alpha(2)}, M_{\dot\beta(2)}, D, U, Q_\alpha, Q_{\dot\beta}, S_\alpha, S_{\dot\beta}\right)\\ &\overset{\mathscr{R}}{\to} \left(K_{\alpha\dot\beta}, P_{\alpha\dot\beta}, M_{\alpha(2)}, M_{\dot\beta(2)}, -D, -U, S_\alpha, S_{\dot\beta}, Q_\alpha, Q_{\dot\beta}\right).\end{aligned} \tag{2.6}$$

The Weyl reflection automorphism reflects the conformal and chiral weights from $c$ and $u$ to $-c$ and $-u$ ($[D, T^{c,u}]_* = cT^{c,u}$, $[U, T^{c,u}]_* = \frac{3}{2}iuT^{c,u}$, $D$ and $U$ are dilatation and u(1) chiral generators from su(2,2|1) and $T^{c,u}$ is an arbitrary element of shsc$^\infty$(4|1) with defined conformal $c$ and chiral $u$ weights). The reflection automorphism is commuting with hermitian conjugation,

$$\mathscr{R} \circ \dagger = \dagger \circ \mathscr{R}, \tag{2.7}$$

where $\circ$ denotes a composition of the mappings.

### 2.2. SPECTRUM OF THE GAUGE FIELDS

First of all note that shsc$^\infty$(4|1) as a linear space can be decomposed into the direct sum of the subspaces $\oplus_{N=1}^{\infty} L(N)$ formed by polynomials with a fixed degree of homogeneity $2N$ (from the defining property of commutativity with $T$ it follows that all elements have even degrees of homogeneity). We call the subspace

L($N$) the $N$th level. Under the su(2,2|1) representation (2.4) each level decomposes into the direct sum of su(2,2|1) irreducible representation spaces (irrepses)

$$\mathrm{L}(N)=\bigoplus_{s=1}^{N}\mathrm{V}(s)\,. \tag{2.8}$$

In its turn the irreps V($s$) decomposes into the sum of so(4,2) irrepses,

$$\begin{aligned}\mathrm{V}(s)=\mathrm{D}(s,s,0)&\oplus\mathrm{D}\big(s-\tfrac{1}{2},s-\tfrac{1}{2},\tfrac{1}{2}\big)\\&\oplus\mathrm{D}\big(s-\tfrac{1}{2},s-\tfrac{1}{2},-\tfrac{1}{2}\big)\oplus\mathrm{D}(s-1,s-1,0)\,,\end{aligned} \tag{2.9}$$

where $\mathrm{D}(n_1,n_2,n_3)$ is the so(4,2) irreps with the highest weight $(n_1,n_2,n_3)$ under the Cartan subalgebra of so(4,2). Here $n_1$ is the maximal conformal weight in the representation and $(n_2-n_3)/2$ and $(n_2+n_3)/2$ define a Lorentz signature ($(n_2-n_3)$ and $(n_2+n_3)$ are the numbers of dotted and undotted indices, respectively) of the vector with highest conformal weight. Note that the representations $(s-\frac{1}{2},s-\frac{1}{2},\pm\frac{1}{2})$ are mutually conjugated under the Weyl reflection (these are usually called chirally or complex conjugated representations). The dimension of the so(4,2) irreps $\mathrm{D}(\jmath,\jmath,u)$ is equal to

$$d(\jmath,\jmath,u)=\tfrac{1}{12}(2\jmath+3)(\jmath+u+1)(\jmath+u+2)(\jmath-u+1)(\jmath-u+2)\,. \tag{2.10}$$

The first level consists of only one su(2,2|1) irreps V(1) which is the adjoint representation, and under so(4,2) we have

$$\mathrm{V}(1)=\mathrm{D}(1,1,0)\oplus\mathrm{D}\big(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\big)\oplus\mathrm{D}\big(\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\big)\oplus\mathrm{D}(0,0,0)\,. \tag{2.11}$$

The basis in these irrepses can be chosen as follows:

$$\begin{aligned}&\mathrm{D}(1,1,0):\{P,K,M,D\}\,,\qquad \mathrm{D}\big(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\big):\{S_\alpha,Q_{\dot\beta}\}\,,\\&\mathrm{D}\big(\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\big):\{S_{\dot\beta},Q_\alpha\}\,,\mathrm{D}(0,0,0):\{U\}\,.\end{aligned} \tag{2.12}$$

To construct a gauge theory it is necessary to introduce the basis in so(4,2) irrepses connected with the decomposition so(4,2) → so(1,1) ⊕ so(3,1), where so(3,1) is the Lorentz subalgebra and so(1,1) generated by the dilatation generator $D$. In this basis all generators and gauge fields will have the defined Weyl weight and the manifest four-dimensional Lorentz index structure. Such a superconformal basis has been constructed in (I) by standard group-theory methods. The structure constants of shsc$^\infty$(4|1) in this basis have also been calculated therein.

As a result, the gauge field corresponding to $\mathrm{shsc}^{\infty}(4|1)$ has the form*

$$\omega_{\mu}=\sum_{N=1}^{\infty}\sum_{s=1}^{N}\sum_{(\jmath,c,u,l,j)} i^{-\varepsilon_{\jmath}}\omega_{\mu,\alpha(2l),\dot\beta(2j)}^{(N,s,\jmath,c,u)}T^{(N,s,\jmath,c,u)\alpha(2l),\dot\beta(2j)}$$

$$\varepsilon_{\jmath}=0\ (1) \tag{2.13}$$

for $\jmath$ integer (half-integer). Here indices of the fields and generators have the following meaning. The index** $N=1,2,\ldots$ is a number of the level $\mathrm{L}(N)$. The index $s=1,2,\ldots,N$ defines $\mathrm{su}(2,2|1)$-irreps $\mathrm{V}(s)$ (the gauge fields with the fixed $s$ form the $\mathrm{su}(2,2|1)$-supermultiplet with maximal spin $s+1$; 1 due to the additional vector index $\mu$). Thus on the $N$th level there are $N$ conformal gauge supermultiplets with the maximal spins from $N+1$ to 2. The index $\jmath=s-1, s-\frac{1}{2}, s$ defines the conformal multiplet (the gauge fields describe the spin $\jmath+1$). The index $u$ is a chiral weight: $u=0$ for bosons ($\jmath$ integer) and $u=\pm\frac{1}{2}$ for fermions ($\jmath$ half-integer). The index $c=-\jmath,-\jmath+1,\ldots,\jmath$ is the conformal weight of the generators and the indices $l,j=0,\frac{1}{2},1,\ldots$ define a Lorentz signature ($2j$ and $2l$ are the numbers of dotted and undotted indices) and are restricted by

$$l+j\leqslant\jmath\,,\qquad l\geqslant\left|\frac{c-u}{2}\right|,\qquad j\geqslant\left|\frac{c+u}{2}\right|, \tag{2.14}$$

with $l+\frac{1}{2}(c-u)$ and $j+\frac{1}{2}(c+u)$ integers. The above restrictions follow directly from the spectral analysis of $\mathrm{V}(s)$ under the decomposition $\mathrm{so}(4,2)\to\mathrm{so}(3,1)\oplus\mathrm{so}(1,1)$.

It should be mentioned that the structure of the gauge field (2.13) is analogous to the structure of the string field $\Phi[X]$ in string field theory. There is an infinite tower of levels and there are fields with all spins from maximal ($\jmath=N+1$) to minimal ($\jmath=1$) on each of the $N$ levels. In order to work effectively with the infinite tower of levels one can use the following approximation procedure. The number $N$ in eq. (2.13) should be limited by some $N_{\max}$. The original expressions are restored in the limit $N_{\max}\to\infty$.

The Grassmann parity of the fields and generators is defined as

$$\varepsilon\left(\omega_{\ldots}^{(N,s,\jmath,c,u)}\right)=\varepsilon\left(T_{\ldots}^{(N,s,\jmath,c,u)}\right)=\varepsilon_{\jmath}=0\ (1) \tag{2.15}$$

* The hermitian conjugation is

$$\omega_{\mu}^{\dagger}=-\omega_{\mu}\,,\qquad\left(\omega_{\mu,\alpha(2l),\dot\beta(2j)}^{(N,s,\jmath,c,u)}\right)^{\dagger}=\omega_{\mu,\beta(2j),\dot\alpha(2l)}^{(N,s,\jmath,c,-u)}\,.$$

** Our notation here is different from the ones in (I). The level number $N$ equals $n+s$ in the convention of (I).

for $\jmath$ integer (half-integer), and the following relations hold:

$$T^{\jmath}\omega_{\mu}^{\jmath'} = (-1)^{4\jmath\jmath'}\omega_{\mu}^{\jmath'}T^{\jmath}\,, \tag{2.16a}$$

$$\omega_{\mu}^{\jmath}\omega_{\nu}^{\jmath'} = (-1)^{4\jmath\jmath'}\omega_{\nu}^{\jmath'}\omega_{\mu}^{\jmath}\,, \tag{2.16b}$$

i.e. $\omega_{\mu}$ in eq. (2.13) is an element of the second-class Grassmann shell of shsc$^{\infty}(4|1)$ ($i^{-\varepsilon_{\jmath}}$ in eq. (2.13) is introduced for convenience).

Note that the gauge fields of usual conformal supergravity are, in our notations,

$$\begin{aligned}&(e_{\mu\alpha\dot\beta}, \omega_{\mu\alpha(2)}, \omega_{\mu\dot\beta(2)}, b_{\mu}, f_{\mu\alpha\dot\beta}, A_{\mu}, \psi_{\mu\alpha}, \psi_{\mu\dot\beta}, \phi_{\mu\alpha}, \phi_{\mu\dot\beta})\\ &\sim\Big(\omega^{(1,1,1,-1,0)}_{\mu,\alpha,\dot\beta}, \omega^{(1,1,1,0,0)}_{\mu,\alpha(2)}, \omega^{(1,1,1,0,0)}_{\mu,\dot\beta(2)}, \omega^{(1,1,1,0,0)}_{\mu}, \omega^{(1,1,1,1,0)}_{\mu,\alpha,\dot\beta}, \omega^{(1,1,0,0,0)}_{\mu},\\ &\qquad \omega^{(1,1,\frac{1}{2},-\frac{1}{2},\frac{1}{2})}_{\mu,\alpha}, \omega^{(1,1,\frac{1}{2},-\frac{1}{2},-\frac{1}{2})}_{\mu,\dot\beta}, \omega^{(1,1,\frac{1}{2},\frac{1}{2},-\frac{1}{2})}_{\mu,\alpha}, \omega^{(1,1,\frac{1}{2},\frac{1}{2},\frac{1}{2})}_{\mu,\dot\beta}\Big).\end{aligned} \tag{2.17}$$

### 2.3. THE CURVATURES

The curvatures of conformal supergravity in the two-component notation read (see (I.3.9) for the commutation relations of su(2,2|1) in these notations)

$$\begin{aligned}
R^{\alpha\dot\beta}_{\mu\nu}(P) &= \mathscr{D}_{\mu}e^{\alpha\dot\beta}_{\nu} + i\psi^{\alpha}_{\mu}\psi^{\dot\beta}_{\nu} - (\mu\leftrightarrow\nu)\,,\\
R^{\alpha\dot\beta}_{\mu\nu}(K) &= \mathscr{D}_{\mu}f^{\alpha\dot\beta}_{\nu} + i\phi^{\alpha}_{\mu}\phi^{\dot\beta}_{\nu} - (\mu\leftrightarrow\nu)\,,\\
R^{\alpha(2)}_{\mu\nu}(M) &= \partial_{\mu}\omega^{\alpha(2)}_{\nu} + e^{\alpha}_{\mu\dot\beta}f^{\alpha\dot\beta}_{\nu} + \omega_{\mu}{}^{\alpha}{}_{\gamma}\omega_{\nu}{}^{\alpha\gamma}\\
&\quad - \psi^{\alpha}_{\mu}\phi^{\alpha}_{\nu} - (\mu\leftrightarrow\nu)\,,\ \text{h.c.},\\
R_{\mu\nu}(D) &= \partial_{\mu}b_{\nu} + e_{\mu\alpha\dot\beta}f^{\alpha\dot\beta}_{\nu} - \tfrac{1}{2}\psi_{\mu\alpha}\phi^{\alpha}_{\nu} + \tfrac{1}{2}\psi_{\mu\dot\alpha}\phi^{\dot\alpha}_{\nu} - (\mu\leftrightarrow\nu)\,,\\
R^{\alpha}_{\mu\nu}(Q) &= \mathscr{D}_{\mu}\psi^{\alpha}_{\nu} + \tfrac{3}{4}iA_{\mu}\psi^{\alpha}_{\nu} - ie^{\alpha}_{\mu\dot\beta}\phi^{\dot\beta}_{\nu} - (\mu\leftrightarrow\nu)\,,\ \text{h.c.},\\
R^{\alpha}_{\mu\nu}(S) &= \mathscr{D}_{\mu}\phi^{\alpha}_{\nu} - \tfrac{3}{4}iA_{\mu}\phi^{\alpha}_{\nu} - if^{\alpha}_{\mu\dot\beta}\psi^{\dot\beta}_{\nu} - (\mu\leftrightarrow\nu)\,,\ \text{h.c.},\\
R_{\mu\nu}(U) &= \partial_{\mu}A_{\nu} + i\psi_{\mu\alpha}\phi^{\alpha}_{\nu} + i\psi_{\mu\dot\beta}\phi^{\dot\beta}_{\nu} - (\mu\leftrightarrow\nu)\,,
\end{aligned} \tag{2.18}$$

where $\mathscr{D}_{\mu}$ is the covariant derivative with respect to $M$ and $D$ generators. Curvatures of shsc$^{\infty}(4|1)$ are defined as usual:

$$R_{\mu\nu} = \partial_{\mu}\omega_{\nu} - \partial_{\nu}\omega_{\mu} + [\omega_{\mu}, \omega_{\nu}\}_{*}\,, \tag{2.19}$$

and in the components we have (see (I))

$$
\begin{aligned}
R^{(N,s,\jmath,c,u)}_{\mu\nu,\alpha(2l),\beta(2j)} = {} & \partial_\mu \omega^{(N,s,\jmath,c,u)}_{\nu,\alpha(2l),\beta(2j)} - (\mu \leftrightarrow \nu) \\
& + \sum \delta(c'+c''-c)\delta(u'+u''-u)\delta(m-l'-l''+l)\delta(r-l''+l'-l) \\
& \times \delta(t-l'+l''-l)\delta(p-j'-j''+j)\delta(q-j''-j+j') \\
& \times \delta(k-j-j'+j'')\pi(N+N'+N'')i^{\jmath'+\jmath''-\jmath-1} \\
& \times \begin{Bmatrix} N' & s' & \jmath' & c' & u' & l' & j' \\ N'' & s'' & \jmath'' & c'' & u'' & l'' & j'' \\ N & s & \jmath & c & u & l & j \end{Bmatrix} \\
& \times \omega^{(N',s',\jmath',c',u')}_{\mu,\alpha(t)\gamma(m),\beta(k)\dot\delta(p)} \omega^{(N'',s'',\jmath'',c'',u'')\gamma(m)}_{\nu,\alpha(r)}{}_{,\dot\beta(q)}{}^{\dot\delta(p)} .
\end{aligned}
\tag{2.20}
$$

Here the parity function $\pi(n) = 0$ (1) for $n$ even (odd) and the structure coefficients denoted as were given in* (I.7.7). They are expressed through the group-theoretical factors which are well known from angular momentum theory (Clebsch–Gordan coefficients, $9j$-symbols etc.). We will not present here the complicated explicit expression but instead discuss a simple symmetry property of the structure coefficients.

Let us define symmetrized coefficients by the relation

$$
S\begin{Bmatrix} N_1 & s_1 & \jmath_1 & c_1 & u_1 & l_1 & j_1 \\ N_2 & s_2 & \jmath_2 & c_2 & u_2 & l_2 & j_2 \\ N_3 & s_3 & \jmath_3 & c_3 & u_3 & l_3 & j_3 \end{Bmatrix}
= (-1)^{N_3+\jmath_3-l_3-j_3+s_3} \begin{Bmatrix} N_1 & s_1 & \jmath_1 & c_1 & u_1 & l_1 & j_1 \\ N_2 & s_2 & \jmath_2 & c_2 & u_2 & l_2 & j_2 \\ N_3 & s_3 & \jmath_3 & -c_3 & -u_3 & l_3 & j_3 \end{Bmatrix} . \tag{2.21}
$$

* As we have noted above, our notation here is different from the one in (I). In particular, in the structure coefficients $\{\ldots\}$ the new index $N = n + s$ is introduced instead of the old index $n$. Thus there is the following correspondence between the notations:

$$
\omega^{(N,s,\jmath,c,u)}_{\cdots} = {}^{(N-s)}\omega^{(s,\jmath,c,u)}_{\cdots}, \qquad R^{(N,s,\jmath,c,u)}_{\cdots} = {}^{(N-s)}R^{(s,\jmath,c,u)}_{\cdots},
$$

$$
\begin{Bmatrix} N_1 & s_1 & \jmath_1 & c_1 & u_1 & l_1 & j_1 \\ N_2 & s_2 & \jmath_2 & c_2 & u_2 & l_2 & j_2 \\ N_3 & s_3 & \jmath_3 & c_3 & u_3 & l_3 & j_3 \end{Bmatrix} = \begin{bmatrix} N_1 - s_1 & s_1 & \jmath_1 & c_1 & u_1 & l_1 & j_1 \\ N_2 - s_2 & s_2 & \jmath_2 & c_2 & u_2 & l_2 & j_2 \\ N_3 - s_3 & s_3 & \jmath_3 & c_3 & u_3 & l_3 & j_3 \end{bmatrix},
$$

where the r.h.s. are given in terms of the conventions in (I). In eq. (2.20) the summation over all the internal indices $(N', N'', \ldots, j', j'', t, \ldots, p)$ is understood.

These symmetrized coefficients get multiplied with the factor

$$(-1)^{\sum_{i=1}^{3}(N_i+l_i+j_i)+4(\jmath_1\jmath_2+\jmath_2\jmath_3+\jmath_1\jmath_3)} \tag{2.22}$$

under the interchange of any two rows. It can be verified straightforwardly by looking at the expression (I.7.7) and noticing the following symmetry properties for the constituent parts of it. The symmetrized structure coefficients of the conformal superalgebra shsc(3|1) (see ref. [16] and (I.C.4)),

$$S\begin{pmatrix} s_1 & s_2 & s_3 \\ c_1 & c_2 & c_3 \\ l_1 & l_2 & l_3 \end{pmatrix} = (-1)^{s_3-l_3}\begin{pmatrix} s_1 & s_2 & s_3 \\ c_1 & c_2 & -c_3 \\ l_1 & l_2 & l_3 \end{pmatrix}, \tag{2.23}$$

and the symmetrized structure constants of the Clifford algebra $C_2$ (see (I.7.8, 9)),

$$SA^{l_1,l_2,l_3}_{u_1,u_2,u_3} = A^{l_1,l_2,l_3}_{u_1,u_2,-u_3}, \tag{2.24}$$

get multiplied with the factors

$$(-1)^{\sum_{i=1}^{3}(s_i-l_i)} \tag{2.25}$$

and

$$(-1)^{\sum_{i=1}^{3}l_i+4(l_1l_2+l_2l_3+l_1l_3)} \tag{2.26}$$

under the interchange of any two columns respectively.

The above symmetry property expresses the super-anticommutativity of the supercommutator and the existence of an invariant bilinear form on $\mathrm{shsc}^{\infty}(4|1)$, which is written as

$$(A, B) = \mathrm{tr}(A * B), \tag{2.27}$$

where the trace is defined by

$$\mathrm{tr}(A(Z, \bar{Z})) = A(0,0) \tag{2.28}$$

and $A * B$ is the Weyl product of the symbols $A$ and $B$.

With the help of the invariant form from curvatures one can construct a topological invariant of the type $\int_{\mathrm{M}^4} R^{\mathscr{A}} \wedge R^{\mathscr{B}} G_{\mathscr{A}\mathscr{B}}$,

$$I = \sum(-1)^{N-s} i^{2(l+j)} \int_{\mathrm{M}^4} R^{(N,s,\jmath,c,u)}_{\alpha(2l),\dot{\beta}(2j)} \wedge R^{(N,s,\jmath,-c,-u)\alpha(2l),\dot{\beta}(2j)}. \tag{2.29}$$

It would be interesting to find the topological meaning of invariants of the above type which are associated to higher-spin superalgebras. Is there a connection with the classification of the four-dimensional manifolds and theorems about indices of some elliptic operators (in the euclidean compact case)?

Above we have considered the superalgebra shsc$^\infty$(4|1). To construct a bosonic conformal higher-spin theory we used in ref. [14] the conformal higher-spin algebra hsc$^\infty$(4) which is a subalgebra* of shsc$^\infty$(4|1). It contains only su(2,2) but not su(2,2|1) as a finite-dimensional subalgebra. hsc$^\infty$(4) has the same operator realization as shsc$^\infty$(4|1) and only the odd generating elements $\alpha$ and $\alpha^\dagger$ now equal zero. The gauge fields are $\omega^{(N,s,c)}_{\mu,\alpha(2l),\dot\beta(2j)}$, where there are no indices $\jmath$ and $u$ because in the bosonic case chiral weight $u=0$ and $\jmath=s$ (due to the absence of $\alpha$ and $\alpha^\dagger$, each $N$th level is decomposed directly into the sum of spin-$s$ su(2,2) but not of su(2,2|1) irreducible representations and $s=1,\dots,N$). The corresponding curvatures $R^{(N,s,c)}_{\mu\nu,\alpha(2l),\dot\beta(2j)}$ have the same structure as (2.20), where now there are structure coefficients**

$$\begin{Bmatrix} N_1 & s_1 & c_1 & l_1 & j_1 \\ N_2 & s_2 & c_2 & l_2 & j_2 \\ N_3 & s_3 & c_3 & l_3 & j_3 \end{Bmatrix}.$$

These coefficients also have the simple symmetry properties. The symmetrized coefficients

$$S\begin{Bmatrix} N_1 & s_1 & c_1 & l_1 & j_1 \\ N_2 & s_2 & c_2 & l_2 & j_2 \\ N_3 & s_3 & c_3 & l_3 & j_3 \end{Bmatrix} = (-1)^{N_3-l_3-j_3}\begin{Bmatrix} N_1 & s_1 & c_1 & l_1 & j_1 \\ N_2 & s_2 & c_2 & l_2 & j_2 \\ N_3 & s_3 & -c_3 & l_3 & j_3 \end{Bmatrix} \tag{2.30}$$

get multiplied with the factor

$$(-1)^{\Sigma_{i=1}^3 (N_i+l_i+j_i)} \tag{2.31}$$

under the interchange of any two rows.

The topological invariant constructed with the help of the invariant bilinear form for hsc$^\infty$(4) reads

$$I=\sum_{N=1}^{\infty}\sum_{s=1}^{N}\sum_{c,l,j}(-1)^{N-s+l+j}\int_{M^4} R^{(N,s,c)}_{\alpha(2l),\dot\beta(2j)}\wedge R^{(N,s,-c)\alpha(2l),\dot\beta(2j)}. \tag{2.32}$$

* As discussed in (I), the Bose subalgebra of shsc$^\infty$(4|1) is hsc$^\infty$(4) ⊕ hsc$^\infty$(4) ⊕ ( $\bigoplus_{N=1}^{\infty}$u(1)$_N$).

** Here again we have changed the notations as in (2.20).

## 3. "Pure spin" conformally invariant lagrangians

Usually in the tensor formalism the massless states of spin $s$ are represented by a real totally symmetric tensor field $\phi_{\mu(s)}$ in the Bose case (integer $s$) and a Majorana spinor-tensor $\psi_{\mu(s-1/2)}$ in the Fermi case (half-integer $s$). We suppress the four-component spinor indices below.

The simplest conformally invariant higher-derivative actions for these fields were proposed in ref. [2] in the following form:

$$A(s)=\int \mathrm{d}^4x\,\phi_{\mu(s)}\Box^{s}P^{\mu(s)}_{s\,\nu(s)}\phi^{\nu(s)} \tag{3.1}$$

for integer $s>0$, and

$$A(s)=\int \mathrm{d}^4x\psi_{\mu(s-1/2)}\Box^{s-1/2}\not\partial P^{\mu(s-1/2)}_{s\,\nu(s-1/2)}\psi^{\nu(s-1/2)} \tag{3.2}$$

for half-integer $s$. The spin projection operators $P_s$ are totally symmetric, "traceless",

$$\eta_{\mu\mu}P^{\mu(s)}_{s\,\nu(s)}=0\,,\qquad \gamma_\mu P^{\mu(s-1/2)}_{s\,\nu(s-1/2)}=0\,, \tag{3.3}$$

and transverse

$$\partial_\mu P^{\mu(s)}_{s\,\nu(s)}=0\,,\qquad \partial_\mu P^{\mu(s-1/2)}_{s\,\nu(s-1/2)}=0 \tag{3.4}$$

(the same relations hold for lower indices). As a result, the actions are invariant under the following gauge transformations:

$$\delta\phi_{\mu(s)}=\partial_\mu\xi_{\mu(s-1)}-\eta_{\mu\mu}\lambda_{\mu(s-2)}\,, \tag{3.5}$$

$$\delta\psi_{\mu(s-1/2)}=\partial_\mu\varepsilon_{\mu(s-3/2)}-\gamma_\mu\kappa_{\mu(s-3/2)}\,. \tag{3.6}$$

The actions (3.1) and (3.2) can be rewritten in the form

$$A(s)=(-1)^s\int \mathrm{d}^4x\,C_{\mu(s),\nu(s)}C^{\mu(s),\nu(s)} \tag{3.7}$$

for integer $s>0$, and

$$A(s)=(-1)^{s-1/2}\int \mathrm{d}^4x\,\overline{C}^{-}_{\mu(s-1/2),\nu(s-1/2)}C^{+\mu(s-1/2),\nu(s-1/2)} \tag{3.8}$$

for half-integer $s$. Here we have introduced a linearized Weyl tensors (integer $s$)

and spinor-tensors (half-integer $s$) associated to spin $s$,

$$C_{\mu(s),\nu(s)} = \mathscr{P}^{\rho(s),\sigma(s)}_{s\,\mu(s),\nu(s)} \underbrace{\partial_\rho \dots \partial_\rho}_{s} \phi_{\sigma(s)} \tag{3.9}$$

for integer $s > 0$, and

$$C^-_{\mu(s-1/2),\nu(s-1/2)} = \mathscr{P}^{\rho(s-1/2),\sigma(s-1/2)}_{s\,\mu(s-1/2),\nu(s-1/2)} \underbrace{\partial_\rho \dots \partial_\rho}_{s-1/2} \psi_{\sigma(s-1/2)}, \tag{3.10}$$

$$C^+_{\mu(s-1/2),\nu(s-1/2)} = \not{\partial} C^-_{\mu(s-1/2),\nu(s-1/2)} \tag{3.11}$$

for half-integer $s$.

In eqs. (3.9) and (3.10) the $\mathscr{P}^{\dots}_{s\dots}$ are the projectors obeying the following irreducibility conditions:
(i) antisymmetry

$$\mathscr{P}^{\rho(s),\sigma(s)}_{s\,\mu(s-1)\nu,\nu(s)} = 0, \qquad \mathscr{P}^{\rho(s-1/2),\sigma(s-1/2)}_{s\,\mu(s-3/2)\nu,\nu(s-1/2)} = 0 \tag{3.12}$$

(ii) "tracelessness"

$$\eta^{\mu\mu}\mathscr{P}^{\rho(s),\sigma(s)}_{s\,\mu(s),\nu(s)} = 0, \qquad \gamma^{\mu}\mathscr{P}^{\rho(s-1/2),\sigma(s-1/2)}_{s\,\mu(s-1/2),\nu(s-1/2)} = 0 \tag{3.13}$$

(and analogously for upper indices). By virtue of these properties the Weyl (spinor)-tensors obey the corresponding irreducibility properties and are invariant under the gauge transformations (3.5) and (3.6).

In order to pass from eqs. (3.1) and (3.2) to eqs. (3.7) and (3.8) it suffices to integrate by parts and employ the observation that, in view of the properties satisfied by $\mathscr{P}_{s\dots}^{\;\dots}$, the following equality holds:

$$\Box^{s} P^{\nu(s)}_{s\,\mu(s)} = \mathscr{P}^{\rho(s),\nu(s)}_{s\,\sigma(s),\mu(s)} \underbrace{\partial_\rho \dots \partial_\rho}_{s} \underbrace{\partial^\sigma \dots \partial^\sigma}_{s}, \tag{3.14}$$

with the same expression for half-integer $s$.

In the cases of spins $s = 1, 2$ the corresponding Weyl tensors $C_{\mu,\nu}$ and $C_{\mu(2),\nu(2)}$ coincide with the Maxwell tensor and the linearized gravitational Weyl tensor, and the lagrangians coincide with the Maxwell and the linearized Weyl lagrangians respectively. In the case of $s = \frac{1}{2}$ we have $C^- = \psi$ and $C^+ = \not{\partial}\psi$, and $A(\frac{1}{2})$ is the massless Dirac action. For $s = \frac{3}{2}$ we have the linearized Weyl spinor-tensor for the conformal gravitino

$$C^-_{\mu,\nu} = \partial_\mu \psi_\nu - \tfrac{1}{2}\gamma_\mu(\not{\partial}\psi_\nu - \partial_\nu \not{\psi}) - \tfrac{1}{6}\gamma_\mu\gamma_\nu(\not{\partial}\not{\psi} - \partial^\mu\psi_\mu) - (\mu \leftrightarrow \nu), \tag{3.15}$$

$$\not{\psi} = \gamma^\mu \psi_\mu, \qquad \gamma^\mu C^-_{\mu,\nu} = 0, \qquad C^-_{\mu,\nu} = -C^-_{\nu,\mu}, \qquad C^+_{\mu,\nu} = \not{\partial} C^-_{\mu,\nu} \tag{3.16}$$

and $A(\frac{3}{2})$ is the linearized spin-$\frac{3}{2}$ action in conformal supergravity.

The actions (3.1) and (3.2) are invariant under scale transformations of the form

$$x'^{\mu} = e^{-\lambda} x^{\mu}, \phi'_{\mu(s)} = e^{(2-s)\lambda} \phi_{\mu(s)},$$

$$\psi'_{\mu(s-1/2)} = e^{(2-s)\lambda} \psi_{\mu(s-1/2)}. \tag{3.17}$$

The Weyl (spinor)-tensors are transformed as $C' = e^{d\lambda} C$ with the scale dimensions $d = 2$, $\frac{3}{2}$ and $\frac{5}{2}$ for $C_{\mu(s),\nu(s)}$, $C^{-}_{\mu(s-1/2),\nu(s-1/2)}$ and $C^{+}_{\mu(s-1/2),\nu(s-1/2)}$, respectively. The half-integer spin $s$ is described by two Weyl spinor-tensors with different scale dimensions.

In ref. [2] the numbers of off- and on-shell degrees of freedom were calculated. The action (3.1) describes the $2s+1$ off- and $s(s+1)$ on-shell degrees of freedom and the action (3.2) describes $-2(2s+1)$ and $-2(s+\frac{1}{2})^2$ ones, respectively (minus for Fermi). For the $N=1$ conformal $\{s_{\max}\}$-supermultiplets consisting of the spins $s = s_{\max}, s_{\max} - \frac{1}{2}, s_{\max} - 1$, the grading total numbers of degrees of freedom are equal to zero both off- and on-shell. Hence the necessary conditions for on- and off-shell closure of supersymmetry are satisfied.

## 4. Geometrical description for the free conformal higher-spin fields

In this section a linearized conformally invariant higher-spin dynamics will be constructed in terms of the gauge fields and linearized curvatures, and the equivalency of this formulation with the standard formulation in terms of symmetric tensors given above will be shown.

### 4.1. LINEARIZED CURVATURES AND CONFORMAL COHOMOLOGICAL COMPLEX

We shall use the following expansion procedure. The gravitational vierbein is expanded into two parts,

$$\omega^{(1,1,1,-1,0)}_{\mu,\alpha,\dot{\beta}} = \sigma_{\mu\alpha\dot{\beta}} + \tilde{\omega}^{(1,1,1,-1,0)}_{\mu,\alpha,\dot{\beta}}, \tag{4.1}$$

where $\sigma_{\mu\alpha\dot{\beta}}$ is the zeroth-order background part (flat vierbein, see appendix A) and $\tilde{\omega}$ is the first-order dynamical part (the tilde will henceforth be omitted for simplicity). All the other fields are supposed to be of first order.

We describe the conformal spin $(\jmath + 1)$ in terms of the set of gauge fields $\omega^{A}_{\mu}$, where $A$ is the collective index in the so(4,2)-representation space $D(\jmath, \jmath, 0)$ (for integer $\jmath$) or $D(\jmath, \jmath, \frac{1}{2}) \oplus D(\jmath, \jmath, -\frac{1}{2})$ (for half-integer $\jmath$). In our superalgebra shsc$^{\infty}(4|1)$ such representation spaces have an infinite multiplicity. However, considering the linearized case, we shall work only with one set of the spin-$(\jmath + 1)$ gauge fields. So in this section we suppress the indices $N$ and $s$ in the gauge fields notation $\omega^{(N,s,\jmath,c,u)}_{\mu\ldots}$ and simply write this as $\omega^{(\jmath,c,u)}_{\mu\ldots}$.

Disregarding the second-order terms, for the linearized curvatures we have

$$R^{\ell} = \mathscr{D}^{\ell}\omega = \mathrm{d}\omega + \mathscr{P}\omega\,, \tag{4.2}$$

where the operator $\mathscr{P}$ is defined as

$$\mathscr{P}\omega = \left[P^{\alpha\dot{\beta}}, \sigma_{\alpha\dot{\beta}} \wedge \omega\right]_* = \tilde{P}^{\alpha\dot{\beta}}\left(\sigma_{\alpha\dot{\beta}} \wedge \omega\right), \tag{4.3}$$

with $P_{\alpha\dot{\beta}}$ the translation generator's symbol ($\tilde{P}_{\alpha\dot{\beta}}$ is the translation generator in the so(4, 2)-representation [see eq. (2.4)], and $\sigma_{\alpha\dot{\beta}} = \sigma_{\mu\alpha\dot{\beta}}\, \mathrm{d}x^{\mu}$.

In the components eq. (4.2) can be rewritten in the form*

$$\begin{aligned} R^{\ell(s,c,u)}_{\mu\nu,\alpha(n),\dot{\beta}(m)} &= \partial_{\mu}\omega^{(s,c,u)}_{\nu,\alpha(n),\dot{\beta}(m)} + a(s,c,u,n,m)\sigma_{\mu\gamma\dot{\beta}}\omega^{(s,c+1,u)\gamma}_{\nu,\alpha(n),\dot{\beta}(m-1)} \\ &\quad + a(s,c,-u,m,n)\sigma_{\mu\alpha\dot{\delta}}\omega^{(s,c+1,u)\quad\dot{\delta}}_{\nu,\alpha(n-1),\dot{\beta}(m)} \\ &\quad - b(s,c,u,n,m)\sigma_{\mu\gamma\dot{\delta}}\omega^{(s,c+1,u)\gamma\quad\dot{\delta}}_{\nu,\alpha(n)\quad,\dot{\beta}(m)} \\ &\quad - b(s,-c-1,-u,n-1,m-1)\sigma_{\mu\alpha\dot{\beta}}\omega^{(s,c+1,u)}_{\nu,\alpha(n-1),\dot{\beta}(m-1)} - (\mu \leftrightarrow \nu)\,, \end{aligned} \tag{4.4}$$

where the coefficients are

$$a(s,c,u,n,m) = \left[\frac{(n+c-u+2)(m-c-u)(2s+m-n+2)(2s+n-m+4)}{16(n+2)(m+1)}\right]^{1/2}$$

$$b(s,c,u,n,m) = \left[\frac{(n+c-u+2)(m+c+u+2)(2s-n-m)(2s+n+m+6)}{16(n+2)(m+2)}\right]^{1/2}. \tag{4.5}$$

The component equations (4.4) and (4.5) obtained directly from eqs. (4.2) and (4.3), where in eq. (2.13) for the gauge field one should substitute eqs. (I.6.12, 9) for the generators and calculate the commutator of symbols $[P, T, \ldots]_*$ (see appendix B).

Our operator $\mathscr{P}$ has the following properties:

(1) It increases the rank of an arbitrary differential form at unity;

(2) It decreases the conformal weight of an arbitrary differential form at unity;

* In ref. [14] we denoted by $s$ the spin of the gauge fields. There the spin of the generators equalled $s-1$. Here we denote by $s$ the spin of the generators and the spin of the fields is $s+1$.

(3) $\mathscr{P}$ is nilpotent

$$\mathscr{P}^2 = 0; \tag{4.6}$$

(4) $\mathscr{P}$ anticommutes with the usual exterior differential

$$\mathscr{P}\,\mathrm{d} + \mathrm{d}\,\mathscr{P} = 0. \tag{4.7}$$

Due to the properties (3) and (4) we have

$$(\mathscr{D}')^2 = \mathrm{d}^2 + \mathrm{d}\,\mathscr{P} + \mathscr{P}\,\mathrm{d} + \mathscr{P}^2 = 0, \tag{4.8}$$

and the linearized curvatures are invariant under the linearized gauge transformations

$$\delta_g \omega = \mathscr{D}'\mathscr{E} = \mathrm{d}\,\mathscr{E} + \mathscr{P}\mathscr{E}, \tag{4.9}$$

and the Bianchi identities are satisfied

$$\mathscr{D}'R' = \mathrm{d}R' + \mathscr{P}R' \equiv 0. \tag{4.10}$$

Due to the properties (1) and (3) the operator $\mathscr{P}$ is a cohomology operator and it converts a sequence of linear spaces $\Lambda_q$ of differential forms taking their values in the so(4, 2) representation into the cohomological complex:

$$\ldots \to \Lambda_{q-1} \overset{\mathscr{P}}{\to} \Lambda_q \overset{\mathscr{P}}{\to} \ldots, \tag{4.11}$$

where $q = 0, 1, 2, 3, 4$ is the rank of the differential forms.

One can define a local cohomological complex at some fixed point $x$ of the flat four-dimensional space-time. Here $\Lambda_{q,x}$ will be spaces of the rank-$q$ antisymmetric tensors (components of the differential forms at the point $x$). Evidently, this cohomology does not depend on $x$. A cohomology theory defined in such a way is, generally speaking, non-trivial (i.e. cohomological spaces $\mathrm{H}_x^q$ can have nonzero dimensions). We call it a conformal cohomological complex.

Note that the conformal complex is analogous to the deRham complex and $\mathscr{P}$ is the analog of d. In the conformal complex one can introduce an operation generalizing the Hodge star $*$

$$\circledast = \mathscr{R} \circ *, \tag{4.12}$$

where $\mathscr{R}$ is the Weyl reflection automorphism described in sect. 2.

Thanks to the involutivity of $\mathscr{R}$ and

$$*^2 = (-1)^{q+1} \tag{4.13}$$

(in the Minkowsky signature; here $q$ is the rank of differential form), the following relation holds:

$$\circledast^2 = (-1)^{q+1}. \tag{4.14}$$

Now with the help of $\circledast$ one can define an operator $\mathscr{K}$:

$$\mathscr{K} = \circledast \mathscr{P} \circledast, \qquad \mathscr{P} = -\circledast \mathscr{K} \circledast. \tag{4.15}$$

The above defined operator $\mathscr{K}$ has the following properties:
(1) it decreases the rank of an arbitrary differential form at unity;
(2) it increases the conformal weight at unity;
(3) $\mathscr{K}$ is nilpotent

$$\mathscr{K}^2 = 0. \tag{4.16}$$

The operators $\mathscr{P}$ and $\mathscr{K}$ are mutually conjugated under a non-degenerate bilinear form for two arbitrary differential forms with the same rank,

$$\langle A, B \rangle = \int \mathrm{tr}\left(A \wedge \circledast B\right), \tag{4.17}$$

where $A \wedge B$ includes the Weyl product for components of the forms $A$ and $B$ which are Weyl symbols in our construction. The operator $\mathscr{K}$ is an analog of the divergency $\delta$ in the deRham complex on the Riemann manifold. In subsect. 4.2 we are going to apply the above formal construction to the conformally invariant higher-spin dynamics.

### 4.2. CONSTRAINTS FOR THE AUXILIARY FIELDS

In sect. 3 we have seen that the free conformal higher-spin theory can be described in terms of the totally symmetric tensor or the spin-tensor. However, our set of gauge fields is sufficiently broader. The physical spin-($s$ + 1) fields in this set are $\omega^{(s,-s,0)}_{\mu,\alpha(s),\dot\beta(s)}$ (for integer $s$) or $\omega^{(s,-s,1/2)}_{\mu,\alpha(s+1/2),\dot\beta(s-1/2)}$ and $\omega^{(s,-s,-1/2)}_{\mu,\alpha(s-1/2),\dot\beta(s+1/2)}$ (for half-integer $s$). These fields generalize the conformal supergravity (CSG) ones $e_{\mu\alpha\dot\beta}, \psi_{\mu\alpha}, \psi_{\mu\dot\beta}, A_\mu$. All other fields are auxiliary. The auxiliary fields are necessary for us to build the gauge invariant curvatures. But to construct the conformal theory of higher spins we must find such constraints which will allow us to express all the auxiliary fields through the physical ones up to a pure gauge part. A solution, as we will show later, is given by following constraints:

$$\mathscr{K}R^{\prime}=0, \tag{4.18}$$

where the operator $\mathscr{K}$ conjugated to $\mathscr{P}$ from the linearized curvatures is given by eq. (4.15).

In the components the constraints (4.18) can be rewritten in a form

$$\begin{aligned} & a(\jmath,-c-1,-u,n,m)\sigma^{\nu}_{\gamma\dot\beta}R^{\prime(\jmath,c,u)\gamma}_{\nu\mu,\alpha(n),\dot\beta(m-1)}+a(\jmath,-c-1,u,m,n)\sigma^{\nu}_{\alpha\dot\delta}R^{\prime(\jmath,c,u)}_{\nu\mu,\alpha(n-1),\dot\beta(m)}{}^{\dot\delta} \\ & \quad -b(\jmath,-c-1,-u,n,m)\sigma^{\nu}_{\gamma\dot\delta}R^{\prime(\jmath,c,u)\gamma}_{\nu\mu,\alpha(n)}{}_{,\dot\beta(m)}{}^{\dot\delta} \\ & \quad -b(\jmath,c,u,n-1,m-1)\sigma^{\nu}_{\alpha\dot\beta}R^{\prime(\jmath,c,u)}_{\nu\mu,\alpha(n-1),\dot\beta(m-1)}=0, \end{aligned} \tag{4.19}$$

where $a$ and $b$ were given in (4.5) and we have taken into account that

$$\mathscr{R}\left(T^{(N,s,\jmath,c,u)}_{\alpha(n),\dot\beta(m)}\right)=(-1)^{N-l-j-|u|}i^{-2u}T^{(N,s,\jmath,-c,-u)}_{\alpha(n),\dot\beta(m)}, \tag{4.20}$$

as follows from the definition of $\mathscr{R}$ in (2.5) and the definition of the generators in (I).

Let us show that the above constraints make it possible to express all the auxiliary fields. For it firstly introduce some notations. Let $d(\jmath)$ denote a dimension of the so(4,2)-representation space $D(\jmath,\jmath,0)$ (for integer $\jmath$) or $D(\jmath,\jmath,\frac{1}{2})\oplus D(\jmath,\jmath,-\frac{1}{2})$ (for half-integer $\jmath$) (the gauge fields taking their values in this representation space describe the spin-($\jmath+1$), as discussed above). Furthermore, let $d(\jmath,c)$ denote the dimension of the dilatation generator $D$ eigensubspace in this space with eigenvalue (conformal weight) $c$ ($c=-\jmath,\dots,\jmath$). Due to the reflection automorphism $\mathscr{R}$ we have

$$d(\jmath,c)=d(\jmath,-c). \tag{4.21}$$

In this notation the number of constraints in eq. (4.18) for fixed $\jmath$ is equal to

$$4(d(\jmath)-d(\jmath,-\jmath)). \tag{4.22}$$

Here 4 is the number of components of the one-form (see property (1) of the operator $\mathscr{K}$) and we have subtracted $d(\jmath,-\jmath)$ due to property (2) of $\mathscr{K}$. In the component decomposition of $\mathscr{K}R^{l}$ there are no terms with the minimal conformal weight, but the dimension of the minimal conformal weight eigensubspace is $d(\jmath,-\jmath)$.

However, the constraints are not independent due to the nilpotency of $\mathcal{K}$. The number of identities

$$\mathcal{K}(\mathcal{K}R') \equiv 0 \tag{4.23}$$

is equal to

$$d(s) - d(s, -s) - d(s, -s+1), \tag{4.24}$$

where the last two terms have been subtracted again due to property (2) of $\mathcal{K}$.

In this way we have for the number of independent constraints

$$3(d(s) - d(s, -s)) + d(s, -s+1). \tag{4.25}$$

Surprisingly, this is exactly equal to the number of auxiliary fields minus the number of auxiliary gauge parameters (parameters for such gauge transformations under which only the *auxiliary* fields are transformed and not the *physical* ones).

Indeed, the number of auxiliary fields is equal to the number of all fields minus the number of physical ones, i.e. equal to (4.22). The number of auxiliary gauge parameters is equal to the number of all gauge parameters minus the number of gauge parameters for transformations under which the physical fields are transformed, i.e. equal to (4.24). Then the number of all auxiliary fields minus the number of auxiliary gauge parameters is equal to the number of independent constraints (4.25).

Practically, finding explicit expressions for the auxiliary fields is a very complicated task (as compared to the $adS_4$ case [9], there are four terms in the expression $\mathcal{P}\omega$ in the curvatures (4.4) and in the constraints (4.19)). Here we present only an algorithm to do it and later give some of these expressions that will be useful for us. Denoting the gauge fields with defined conformal weight $c$ as $\omega(s, c)$, the constraints can be rewritten in the form

$$\mathcal{K}\,\mathrm{d}\omega(s, c) + \mathcal{K}\mathcal{P}\omega(s, c+1) = 0. \tag{4.26}$$

These are recurrent relations allowing us to express the fields with $c+1$ in the terms of fields with $c$ up to a pure gauge part $\omega'(s, c+1) = \mathcal{P}\mathcal{E}(s, c+2)$. It permits us to express step by step all fields with $c > -s$ through the derivatives of the physical fields with minimal conformal weight $c = -s$.

It is interesting to mention that the above constraints have the Maxwell-like form $\delta F = 0$, where instead of the usual divergence $\delta$ there is a "divergence" $\mathcal{K}$ from the conformal cohomological complex.

Let us analyse the constraints for spins 2 and $\frac{3}{2}$. For spin 2 ($s = 1$) we have

(1) $c = -1$

$$\sigma_\alpha{}^{\nu\dot\beta} R'^{(1,-1,0)}_{\mu\nu,\alpha\dot\beta} = 0, \qquad \sigma^{\nu\alpha}{}_{\dot\beta} R'^{(1,-1,0)}_{\mu\nu,\alpha\dot\beta} = 0, \qquad \sigma^{\nu\alpha\dot\beta} R'^{(1,-1,0)}_{\mu\nu,\alpha\dot\beta} = 0; \tag{4.27}$$

(2) $c=0$

$$\sigma^{\nu\alpha}{}_{\dot\beta}R'^{(1,0,0)}_{\mu\nu,\alpha(2)} + \sigma^{\nu\dot\beta}_{\alpha}R'^{(1,0,0)}_{\mu\nu,\dot\beta(2)} + \sigma^{\nu}_{\alpha\dot\beta}R'^{(1,0,0)}_{\mu\nu} = 0. \tag{4.28}$$

The constraints (4.27) are equivalent to the linearized zero-torsion condition ($R'^{(1,-1,0)}_{\mu\nu,\alpha\dot\beta} = R'_{\mu\nu\alpha\dot\beta}(P)$, note the correspondence between our notation and usual supergravity notation in eq. (2.17))

$$R'^{(1,-1,0)}_{\mu\nu,\alpha\dot\beta} = 0. \tag{4.29}$$

The constraint (4.28) together with the Bianchi identity (we also have taken into account (4.29)),

$$\epsilon^{\mu\nu\rho\sigma}\left(\sigma_{\nu\alpha}{}^{\dot\beta}R'^{(1,0,0)}_{\rho\sigma,\dot\beta(2)} + \sigma^{\alpha}_{\nu\dot\beta}R'^{(1,0,0)}_{\rho\sigma,\alpha(2)} + \sigma_{\nu\alpha\dot\beta}R'^{(1,0,0)}_{\rho\sigma}\right) = 0, \tag{4.30}$$

is equivalent to the linearized Einstein equations ($R'^{(1,0,0)}_{\mu\nu,\alpha(2)} = R'_{\mu\nu,\alpha(2)}$, h.c.)

$$\epsilon^{\mu\nu\rho\sigma}\sigma_{\nu\alpha}{}^{\dot\beta}R'^{(1,0,0)}_{\rho\sigma,\dot\beta(2)} = 0, \qquad \epsilon^{\mu\nu\rho\sigma}\sigma_{\nu}{}^{\alpha}{}_{\dot\beta}R'^{(1,0,0)}_{\rho\sigma,\alpha(2)} = 0, \tag{4.31}$$

and the relation ($R'^{(1,0,0)}_{\mu\nu} = -R'_{\mu\nu}(D)$)

$$R'^{(1,0,0)}_{\mu\nu} = 0. \tag{4.32}$$

The proof of the equivalence of (4.28), (4.30) and (4.31), (4.32) is given in appendix A as an example of manipulations with two-component multispinors.

For spin $\frac{3}{2}$ ($s=\frac{1}{2}$) we have the following constraints:

$$\sigma^{\nu\alpha}{}_{\dot\beta}R'^{(1/2,-1/2,1/2)}_{\mu\nu,\alpha} = 0, \qquad \sigma^{\nu}{}_{\alpha}{}^{\dot\beta}R'^{(1/2,-1/2,-1/2)}_{\mu\nu,\dot\beta} = 0, \tag{4.33}$$

which are exactly the chirality-duality and tracelessness constraints for the Weyl gravitino curvature ($R^{(1/2,-1/2,\pm 1/2)} \sim R(Q)$),

$$\gamma^{\nu}R'_{\mu\nu}(Q) = 0. \tag{4.34}$$

The constraints (4.29) and (4.30) allow us to express the auxiliary fields $f$ and $w$ through the physical spin-2 field (Weyl graviton) $e_{\mu\alpha\dot\beta}$ up to a pure gauge part (the field $b_\mu$). The constraints (4.33) allow us to express the auxiliary fields $\phi$ through the physical spin-$\frac{3}{2}$ fields $\psi$.

To conclude this subsection, note that all the auxiliary gauge symmetries can be fixed by the following gauge conditions:

$$\mathscr{K}\omega = 0. \tag{4.35}$$

The number of these conditions for fixed $\jmath$ is equal to the number of gauge parameters with the exception of $\varepsilon(\jmath, -\jmath)$ under which only physical fields are transformed. The gauge (4.35) fixes also the symmetry with parameters $\varepsilon(\jmath, -\jmath+1)$ under which both the auxiliary and physical fields are transformed. The residual symmetry in this gauge generalizes the linearized general coordinate transformations and Q-supersymmetry.

In particular for the spins 2 and $\frac{3}{2}$ we have

$$\sigma_{\alpha}^{\mu\dot\beta}\omega_{\mu,\alpha\dot\beta}^{(1,-1,0)}=0\,,\qquad \sigma^{\mu\alpha}{}_{\dot\beta}\omega_{\mu,\alpha\dot\beta}^{(1,-1,0)}=0\,,\qquad \sigma^{\mu\alpha\dot\beta}\omega_{\mu,\alpha\dot\beta}^{(1,-1,0)}=0\,,\tag{4.36a}$$

$$\sigma^{\mu\alpha}{}_{\dot\beta}\omega_{\mu,\alpha(2)}^{(1,0,0)}+\sigma_{\alpha}^{\mu\dot\beta}\omega_{\mu,\dot\beta(2)}^{(1,0,0)}=-\sigma_{\alpha\dot\beta}^{\mu}\omega_{\mu}^{(1,0,0)}\,,\tag{4.36b}$$

$$\sigma^{\mu\alpha}{}_{\dot\beta}\omega_{\mu,\alpha}^{(1/2,-1/2,1/2)}=0\,,\qquad \sigma_{\alpha}^{\mu\dot\beta}\omega_{\mu,\dot\beta}^{(1/2,-1/2,-1/2)}=0\,.\tag{4.36c}$$

The gauge conditions (4.36a) are equivalent to the linearized gauge conditions $(\omega_{\mu,\alpha\dot\beta}^{(1,-1,0)}\sim h_{\mu\alpha\dot\beta})$

$$h_{\mu,a}-h_{a,\mu}=0\,,\qquad h_{\mu,}{}^{\mu}=0\tag{4.37a}$$

which fix the local Lorentz and dilatation symmetries; the gauge condition (4.36b) removes the field $b_\mu(\sim\omega_\mu^{(1,0,0)})$ and fixes K-symmetry, and the gauge (4.36c) is equivalent to $(\psi_\mu\sim\omega_\mu^{(1/2,-1/2,\pm1/2)})$

$$\gamma^\mu\psi_\mu=0\,,\tag{4.37b}$$

which fixes the local S-supersymmetry for spin-$\frac{3}{2}$. Note that the gauge $\mathscr{K}\omega=0$ is formally analogous to the Lorentz gauge $\delta A=0$.

### 4.3. THE SOLUTION OF CONSTRAINTS

Our following task is to find a general solution of the above proposed constraints for the curvatures. Together with the linearized Bianchi identities we have the following system of equations for the curvatures:

$$\begin{cases}\mathscr{K}R^{\mathscr{S}}=0\,, & (4.38\text{a})\\ \mathscr{P}R^{\mathscr{S}}=-\mathrm{d}\,R^{\mathscr{S}}. & (4.38\text{b})\end{cases}$$

Before looking for the solution of system (4.38) in terms of the curvatures, we first consider an auxiliary homogeneous system

$$\begin{cases}\mathscr{K}B=0 & (4.39\text{a})\\ \mathscr{P}B=0 & (4.39\text{b})\end{cases}$$

for two-forms $B$ taking their values in the spin-$\jmath$ so(4, 2) representation space $D(\jmath, \jmath, 0)$ (for integer $\jmath$) or $D(\jmath, \jmath, \frac{1}{2}) \oplus D(\jmath, \jmath, -\frac{1}{2})$ (for half-integer $\jmath$).

Let us calculate a number $N(\jmath)$ of independent components of the general solution of the above system. It is equal to the number of components of the two-form $B$, $6d(\jmath)$, minus the number of independent equations in (4.39). The number of independent equations (4.39a) has been calculated in (4.25). The number of independent equations (4.39b) is the same by $\circledast$-duality. Thus we have

$$\begin{aligned} N(\jmath) &= 6d(\jmath) - 6(d(\jmath) - d(\jmath, -\jmath)) - 2d(\jmath, -\jmath + 1) \\ &= 6d(\jmath, -\jmath) - 2d(\jmath, -\jmath + 1)\,. \end{aligned} \tag{4.40}$$

Using the information about so(4, 2) representations given in sect. 2 (see below eq. (2.11)), we have for the dimensions of the eigensubspaces with conformal weights $c = -\jmath, -\jmath + 1$:

$$d(\jmath, -\jmath) = \begin{cases} (\jmath + 1)^2, & \text{integer } \jmath \\ 2(\jmath + \frac{3}{2})(\jmath + \frac{1}{2}), & \text{half-integer } \jmath \end{cases} \tag{4.41}$$

and

$$d(\jmath, -\jmath + 1) = \begin{cases} 2\jmath(\jmath + 2) + \jmath^2, & \text{integer } \jmath \\ 2(\jmath + \frac{3}{2})(\jmath + \frac{1}{2}) + 2(\jmath + \frac{5}{2})(\jmath - \frac{1}{2}) & \\ \quad + 2(\jmath + \frac{1}{2})(\jmath - \frac{1}{2}), & \text{half-integer } \jmath\,. \end{cases} \tag{4.42}$$

Thus the number of independent components of the general solution of (4.39) reads

$$N(\jmath) = \begin{cases} 2(2\jmath + 3)\,, & \text{integer } \jmath \\ 4(2\jmath + 3)\,, & \text{half-integer } \jmath\,. \end{cases} \tag{4.43}$$

In the half-integer spin case we have obtained a doubled number because the corresponding so(4, 2) representation is a sum of two irrepses.

An explicit expression for the general solution of the homogeneous system has the form

$$\begin{aligned} B^{(\jmath, c, u)}_{\mu\nu, \alpha(n), \dot\beta(m)} &= \delta(n - 2\jmath)\delta(m)\delta(c + u)\sigma_{\mu\nu}{}^{\alpha(2)} C^{(\jmath, c, u)}_{\alpha(2\jmath + 2)} \\ &\quad + \delta(n)\delta(m - 2\jmath)\delta(c - u)\bar\sigma_{\mu\nu}{}^{\dot\beta(2)} \bar C^{(\jmath, c, u)}_{\dot\beta(2\jmath + 2)}\,, \end{aligned} \tag{4.44}$$

where $C$ and $\bar{C}$ are arbitrary, mutually conjugated multispinors

$$\left(C^{(\jmath,c,u)}_{\alpha(2\jmath+2)}\right)^{\dagger}=\bar{C}^{(\jmath,c,-u)}_{\dot{\alpha}(2\jmath+2)}, \tag{4.45}$$

and for $\sigma_{\mu\nu}$ and their properties used below see appendix A.

Due to the $\delta$-functions, the only non-zero components are

$$C^{(\jmath,0,0)}_{\alpha(2\jmath+2)},\qquad \bar{C}^{(\jmath,0,0)}_{\dot{\beta}(2\jmath+2)},\qquad \text{integer } \jmath\,, \tag{4.46a}$$

$$C^{(\jmath,1/2,-1/2)}_{\alpha(2\jmath+2)},\qquad C^{(\jmath,-1/2,1/2)}_{\alpha(2\jmath+2)},\qquad \bar{C}^{(\jmath,1/2,1/2)}_{\dot{\beta}(2\jmath+2)},\qquad \bar{C}^{(\jmath,-1/2,-1/2)}_{\dot{\beta}(2\jmath+2)},\qquad \text{half-integer } \jmath\,. \tag{4.46b}$$

It is easy to see that the number of components of $C$ and $\bar{C}$ is equal to (4.43).

Now let us proceed with solving the "non-homogeneous" system (4.38) for the curvatures. It can be written as

$$\begin{cases}\mathscr{K}R^{\mathscr{C}}(\jmath,c)=0, & c=-\jmath,-\jmath+1,\dots,\jmath-1, \quad (4.47a)\\ \mathscr{P}R^{\mathscr{C}}(\jmath,c)=-\mathrm{d}R^{\mathscr{C}}(\jmath,c-1), & c=-\jmath+1,\dots,\jmath\,. \quad (4.47b)\end{cases}$$

In the r.h.s. of eq. (4.47b) there is a differential on the curvature with $c-1$. Thus eqs. (4.47) for given fixed $c$ form a non-homogeneous linear system for the curvatures with conformal weight $c$. A general solution of the system with fixed $c$ is a sum of the general solution of the homogeneous system and a partial solution of the non-homogeneous system. It can be obtained recurrently by $c$. For simplicity we shall first solve the system in the bosonic case (integer $\jmath$). The first step is $c=-\jmath$ and

$$\mathscr{K}R^{\mathscr{C}}(\jmath,-\jmath)=0, \tag{4.48}$$

and $\mathscr{P}R^{\mathscr{C}}(\jmath,-\jmath)\equiv 0$ because $c=-\jmath$ is a minimal conformal weight. As it follows from the above analysis, it is equivalent to

$$R^{\mathscr{C}}(\jmath,-\jmath)=0 \tag{4.49}$$

(the homogeneous system has no solutions with the non-zero conformal weight, see eqs. (4.44) and (4.46)).

Further, let the curvatures with some fixed $-\jmath\leqslant c'<0$ be equal to zero,

$$R^{\mathscr{C}}(\jmath,c')=0\,. \tag{4.50}$$

Then for $c = c' + 1$ we have a homogeneous system

$$\begin{cases} \mathscr{K} R^{\prime}(\jmath, c'+1) = 0, \\ \mathscr{P} R^{\prime}(\jmath, c'+1) = 0. \end{cases} \tag{4.51}$$

Hence, for $c' < -1$ we have

$$R^{\prime}(\jmath, c'+1) = 0.$$

In this way we have proved by induction that all the curvatures with negative conformal weight are equal to zero

$$R^{\prime}(\jmath, c) = 0, \qquad c < 0. \tag{4.52}$$

It generalizes the zero-torsion condition $R^{\prime}(P) = 0$ for spin 2.

For $c' = -1$ the homogeneous system (4.51) has a non-trivial solution (see eq. (4.44)),

$$R^{\prime(\jmath,0)}_{\mu\nu,\alpha(n),\dot\beta(m)} = \tfrac{1}{4}\delta(m)\delta(n-2\jmath)\sigma_{\mu\nu}{}^{\alpha(2)}C^{(\jmath,0)}_{\alpha(2\jmath+2)}$$

$$+\tfrac{1}{4}\delta(n)\delta(m-2\jmath)\bar{\sigma}^{\dot\beta(2)}_{\mu\nu}\bar{C}^{(\jmath,0)}_{\dot\beta(2\jmath+2)}, \tag{4.53}$$

where $C$ and $\bar{C}$ are the multispinors which represent the spin-($\jmath + 1$) Weyl tensor described in section 3 in the tensor formalism and

$$C^{(\jmath,0)}_{\alpha(2\jmath+2)} = \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}\sigma_{\mu\nu\alpha(2)}R^{\prime(\jmath,0)}_{\rho\sigma,\alpha(2\jmath)}, \tag{4.54a}$$

$$\bar{C}^{(\jmath,0)}_{\dot\beta(2\jmath+2)} = -\frac{i}{4}\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\mu\nu\dot\beta(2)}R^{\prime(\jmath,0)}_{\rho\sigma,\dot\beta(2\jmath)} \tag{4.54b}$$

(we suppress the index $u$ in the quantities $R^{(\jmath,c,u)}$ as $u = 0$ in the bosonic case). Substituting eq. (4.4) for the curvatures into eq. (4.54) we have

$$C^{(\jmath,0)}_{\alpha(2\jmath+2)} = -\partial_{\alpha}{}^{\dot\beta}\omega^{(\jmath,0)}_{\alpha\dot\beta,\alpha(2\jmath)}, \tag{4.55a}$$

$$\bar{C}^{(\jmath,0)}_{\dot\beta(2\jmath+2)} = -\partial^{\alpha}{}_{\dot\beta}\omega^{(\jmath,0)}_{\alpha\dot\beta,\dot\beta(2\jmath)}. \tag{4.55b}$$

Here the following notations are used:

$$\partial_{\alpha\dot\beta} = \sigma^{\mu}_{\alpha\dot\beta}\partial_{\mu}, \qquad \omega_{\alpha\dot\beta,\ldots} = \sigma^{\mu}_{\alpha\dot\beta}\omega_{\mu,\ldots}. \tag{4.56}$$

Note that in eqs. (4.55) the Weyl multispinors have been expressed through the auxiliary fields. Using the constraints (4.52), one can express $C$ and $\overline{C}$ only through the derivatives on the physical fields.

To this end let us write the useful part of the constraints (4.52) in components:

$$\bar{\sigma}^{\mu\nu}_{\dot\beta(2)} R'^{(\jmath,-c)}_{\mu\nu,\alpha(c),\dot\beta(2\jmath-c)} = 0\,,$$

$$\partial^{\gamma}{}_{\dot\beta}\,\omega^{(\jmath,-c)}_{\gamma\dot\beta,\alpha(c),\dot\beta(2\jmath-c)} \sim \omega^{(\jmath,-c+1)}_{\alpha\dot\beta,\alpha(c-1),\dot\beta(2\jmath-c+1)}\,, \tag{4.57a}$$

$$\sigma^{\mu\nu}_{\alpha(2)} R'^{(\jmath,-c)}_{\mu\nu,\alpha(2\jmath-c),\dot\beta(c)} = 0\,,$$

$$\partial_{\alpha}{}^{\dot\rho}\,\omega^{(\jmath,-c)}_{\alpha\dot\rho,\alpha(2\jmath-c),\dot\beta(c)} \sim \omega^{(\jmath,-c+1)}_{\alpha\dot\beta,\alpha(2\jmath-c+1),\dot\beta(c-1)}\,. \tag{4.57b}$$

The above constraints allow us to express the auxiliary fields $\omega^{(\jmath,0)}_{\alpha\dot\beta,\alpha(2\jmath)}$, $\omega^{(\jmath,0)}_{\alpha\dot\beta,\dot\beta(2\jmath)}$ entering the expressions for the Weyl tensors through the physical fields, and as a result we have the Weyl tensors expressed only in terms of the physical fields:

$$C^{(\jmath,0)}_{\alpha(2\jmath+2)} \sim \underbrace{\partial_{\alpha}{}^{\dot\beta}\ldots\partial_{\alpha}{}^{\dot\beta}}_{\jmath+1}\,\omega^{(\jmath,-\jmath)}_{\alpha\dot\beta,\alpha(\jmath),\dot\beta(\jmath)}\,, \tag{4.58a}$$

$$\overline{C}^{(\jmath,0)}_{\dot\beta(2\jmath+2)} \sim \underbrace{\partial^{\alpha}{}_{\dot\beta}\ldots\partial^{\alpha}{}_{\dot\beta}}_{\jmath+1}\,\omega^{(\jmath,-\jmath)}_{\alpha\dot\beta,\alpha(\jmath),\dot\beta(\jmath)}\,. \tag{4.58b}$$

Here the physical field $\omega^{(\jmath,-\jmath)}_{\alpha\dot\beta,\alpha(\jmath),\dot\beta(\jmath)}$ is a multispinor representation for the traceless part of the totally symmetric tensor $\phi_{\mu(\jmath+1)}$ in sect. 3. It should be mentioned that the simple expressions for the Weyl tensors in the two-component multispinor formalism take the place of complicated expressions in the tensor formalism, and the elementary symmetrization–antisymmetrization operations take the place of cumbersome projectors.

Thus we have obtained the solution of the constraints for $R(\jmath,c)$ with $c \leqslant 0$. Now let us find a solution for the curvatures with $c > 0$. The homogeneous system has no solutions with $c > 0$, so a solution of the non-homogeneous system is unique. It can be found recurrently for $0 < c \leqslant \jmath$. At the first step $c = 1$ substituting into the r.h.s. of (4.47b) (Bianchi identities) the solution (4.53) for $c = 0$, we have (only equations with nontrivial right-hand sides are written)

$$\sigma^{\mu\nu}_{\alpha(2)} R'^{(\jmath,1)}_{\mu\nu,\alpha(2\jmath-1),\dot\beta} \sim \partial^{\alpha}{}_{\dot\beta}\,C^{(\jmath,0)}_{\alpha(2\jmath+2)}\,, \tag{4.59a}$$

$$\bar{\sigma}^{\mu\nu}_{\dot\beta(2)} R'^{(\jmath,1)}_{\mu\nu,\alpha,\dot\beta(2\jmath-1)} \sim \partial_{\alpha}{}^{\dot\beta}\,\overline{C}^{(\jmath,0)}_{\dot\beta(2\jmath+2)}\,. \tag{4.59b}$$

All the other curvature components satisfy the homogeneous system and hence are equal to zero. Thus the solution for $c=1$ is

$$R'^{(\jmath,1)}_{\mu\nu,\alpha(n),\dot\beta(m)} \sim \delta(m-1)\delta(n-2\jmath+1)\sigma_{\mu\nu}^{\alpha(2)}\partial^{\alpha}{}_{\dot\beta}C^{(\jmath,0)}_{\alpha(2\jmath+2)}$$
$$+\delta(n-1)\delta(m-2\jmath+1)\bar\sigma_{\mu\nu}{}^{\dot\beta(2)}\partial_{\alpha}{}^{\dot\beta}\bar C^{(\jmath,0)}_{\dot\beta(2\jmath+2)}. \tag{4.60}$$

Proceeding analogously, we finally obtain the general solution of the constraints:

$$R'^{(\jmath,c)}_{\mu\nu,\alpha(n),\dot\beta(m)} = \tfrac{1}{4}A(\jmath,c)\theta(c)\Big[\delta(n-2\jmath+c)\delta(m-c)$$
$$\times\sigma_{\mu\nu}{}^{\alpha(2)}\underbrace{\partial^{\alpha}{}_{\dot\beta}\dots\partial^{\alpha}{}_{\dot\beta}}_{c}C^{(\jmath,0)}_{\alpha(2\jmath+2)} + \delta(n-c)\delta(m-2\jmath+c)$$
$$\times\bar\sigma_{\mu\nu}{}^{\dot\beta(2)}\underbrace{\partial_{\alpha}{}^{\dot\beta}\dots\partial_{\alpha}{}^{\dot\beta}}_{c}\bar C^{(\jmath,0)}_{\dot\beta(2\jmath+2)}\Big],$$

$$A(\jmath,c) = (-1)^{c}2^{-c}\sqrt{\frac{(\jmath-c)!}{c!\jmath!}}, \qquad \theta(c)=1\ (0)\ \text{at}\ c\geqslant 0\ (c<0). \tag{4.61}$$

The fermionic case can be examined in the analogous way. Here the Weyl multispinors are

$$C^{(\jmath,-1/2,1/2)}_{\alpha(2\jmath+2)} = \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}\sigma_{\mu\nu\alpha(2)}R'^{(\jmath,-1/2,1/2)}_{\rho\sigma,\sigma(2\jmath)}$$
$$\sim \underbrace{\partial_{\alpha}{}^{\dot\beta}\dots\partial_{\alpha}{}^{\dot\beta}}_{\jmath+1/2}\omega^{(\jmath,-\jmath,1/2)}_{\alpha\dot\beta,\alpha(\jmath+1/2),\dot\beta(\jmath-1/2)}, \tag{4.62a}$$

$$\bar C^{(\jmath,-1/2,-1/2)}_{\dot\beta(2\jmath+2)} = -\frac{i}{4}\epsilon^{\mu\nu\rho\sigma}\bar\sigma_{\mu\nu\dot\beta(2)}R'^{(\jmath,-1/2,-1/2)}_{\rho\sigma,\dot\beta(2\jmath)}$$
$$\sim \underbrace{\partial^{\alpha}{}_{\dot\beta}\dots\partial^{\alpha}{}_{\dot\beta}}_{\jmath+1/2}\omega^{(\jmath,-\jmath,-1/2)}_{\alpha\dot\beta,\alpha(\jmath-1/2),\dot\beta(\jmath+1/2)}, \tag{4.62b}$$

$$C^{(\jmath,1/2,-1/2)}_{\alpha(2\jmath+2)} = \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}\sigma_{\mu\nu\alpha(2)}R'^{(\jmath,1/2,-1/2)}_{\rho\sigma,\alpha(2\jmath)}$$
$$\sim \underbrace{\partial_{\alpha}{}^{\dot\beta}\dots\partial_{\alpha}{}^{\dot\beta}}_{\jmath+3/2}\omega^{(\jmath,-\jmath,-1/2)}_{\alpha\dot\beta,\alpha(\jmath-1/2),\dot\beta(\jmath+1/2)}, \tag{4.63a}$$

$$\bar C^{(\jmath,1/2,1/2)}_{\dot\beta(2\jmath+2)} = -\frac{i}{4}\epsilon^{\mu\nu\rho\sigma}\bar\sigma_{\mu\nu\dot\beta(2)}R'^{(\jmath,1/2,1/2)}_{\rho\sigma,\dot\beta(2\jmath)}$$
$$\sim \underbrace{\partial^{\alpha}{}_{\dot\beta}\dots\partial^{\alpha}{}_{\dot\beta}}_{\jmath+3/2}\omega^{(\jmath,-\jmath,1/2)}_{\alpha\dot\beta,\alpha(\jmath+1/2),\dot\beta(\jmath-1/2)}. \tag{4.63b}$$

Here $\omega^{(s,-s,1/2)}_{\alpha\beta,\alpha(s+1/2),\dot\beta(s-1/2)}$ and $\omega^{(s,-s,-1/2)}_{\alpha\beta,\alpha(s-1/2),\dot\beta(s+1/2)}$ represent the $\gamma$-traceless part of the field $\psi_{\mu(s+1/2)}$. Note that the Weyl multispinors with $c=-\frac{1}{2}$ (4.62a, b) contain $s+\frac{1}{2}$ derivatives on the physical fields and those with $c=+\frac{1}{2}$ (4.63a, b) contain $s+\frac{3}{2}$ derivatives. They represent the Weyl spinor-tensor $C^-$ and $C^+$ respectively (see sect. 3).

The final expression for the curvatures both for the bosonic and the fermionic cases can be written in the form

$$R^{\prime(s,c,u)}_{\mu\nu,\alpha(n),\dot\beta(m)}=\tfrac{1}{4}\theta(c+|u|)\Big[A(s,c,u)\delta(n-2s+c+u)\delta(m-c-u)\sigma_{\mu\nu}{}^{\alpha(2)}$$
$$\times\underbrace{\partial^{\alpha}{}_{\dot\beta}\dots\partial^{\alpha}{}_{\dot\beta}}_{c+u}C^{(s,-u,u)}_{\alpha(2s+2)}+A(s,c,-u)\delta(n-c+u)$$
$$\times\delta(m-2s+c-u)\bar\sigma_{\mu\nu}{}^{\dot\beta(2)}\underbrace{\partial_{\alpha}{}^{\dot\beta}\dots\partial_{\alpha}{}^{\dot\beta}}_{c-u}\bar C^{(s,u,u)}_{\dot\beta(2s+2)}\Big],$$

$$A(s,c,u)=(-1)^{c+u}2^{-c+u}\sqrt{\frac{(s-c)!}{(c+u)!(s-u)!}}\,. \tag{4.64}$$

Eq. (4.61) follows from (4.64) for $s$ an integer and $u=0$.

In this way we have obtained the expression for all curvature components only in terms of derivatives on the Weyl multispinors, which in turn are expressed only through the derivatives on the higher-spin physical fields.

### 4.4. LINEARIZED ACTIONS FOR THE CONFORMAL HIGHER-SPIN FIELDS IN TERMS OF CURVATURES

In subsect. 4.3 we have expressed all the auxiliary fields and curvatures only in terms of physical fields. Our next task is to construct a linearized conformally invariant action which will produce equations of motion for the physical higher-spin fields.

We are going to look for the actions in the MacDowell–Mansuori form, quadratic on the curvatures (see ref. [19]). The dimensionless real action for spin $(s+1)$ must have the general form

$$\mathcal{A}^{\prime}(s)=\sum_{c,u,n,m}A(s,c,u,n,m)\int R^{\prime(s,c,u)}_{\alpha(n),\dot\beta(m)}\wedge R^{\prime(s,-c,-u)\alpha(n),\dot\beta(m)},$$

$$A(s,c,u,n,m)=\bar A(s,c,-u,m,n)(-1)^{n+m}. \tag{4.65}$$

Disregarding the terms containing the curvatures which are equal to zero on the standard constraints (4.18) (or (4.64)) and the full derivative terms, we get the unique actions:
(i) Bosonic case (integer $s$)

$$A'(s) = (-1)^s i\beta_s \int \left[ R'^{(s,0,0)}_{\alpha(2s)} \wedge R'^{(s,0,0)\alpha(2s)} - \text{h.c.} \right]; \tag{4.66}$$

(ii) Fermionic case (half-integer $s$)

$$A'(s) = (-1)^{s-1/2}\beta_s \int \left[ R'^{(s,1/2,-1/2)}_{\alpha(2s)} \wedge R'^{(s,-1/2,1/2)\alpha(2s)} + \text{h.c.} \right], \tag{4.67}$$

where $\beta_s$ are overall, dimensionless normalization constants.

The above actions generalize the actions $\int R'_{ab}(M) \wedge R'_{cd}(M)\epsilon^{abcd}$ and $\int \overline{R}'(Q)\gamma_5 \wedge R^l(S)$ for spins 2 and $\frac{3}{2}$ to arbitrary higher spin. Varying the action (4.66) with respect to the auxiliary fields $\omega^{(s,1,0)}_{\mu,\alpha(2s-1),\dot\beta}$ and $\omega^{(s,1,0)}_{\mu,\alpha,\dot\beta(2s-1)}$, we get the constraints

$$\epsilon^{\mu\nu\rho\sigma}\sigma_\nu{}^{\alpha}{}_{\dot\beta} R'^{(s,0,0)}_{\rho\sigma,\alpha(2s)} = 0, \tag{4.68a}$$

$$\epsilon^{\mu\nu\rho\sigma}\sigma_{\nu\alpha}{}^{\dot\beta} R'^{(s,0,0)}_{\rho\sigma,\dot\beta(2s)} = 0. \tag{4.68b}$$

These constraints are contained in the standard constraints. It states that only non-zero components of $R'(s,0,0)$ are Weyl multispinors. Substituting their solution (4.53) into the action, we can express it simply as $C^2$

$$A'(s) = (-1)^s\beta_s \int d^4x \Big[ C^{(s,0,0)}_{\alpha(2s+2)} C^{(s,0,0)\alpha(2s+2)} + \overline{C}^{(s,0,0)}_{\dot\beta(2s+2)} \overline{C}^{(s,0,0)\dot\beta(2s+2)} \Big]. \tag{4.69}$$

It evidently is a multispinorial representation of the standard $C^2$-action (3.7) for the conformal higher-spin field.

In the fermionic case we get analogously

$$A'(s) = (-1)^{s-1/2} i\beta_s \int d^4x \Big[ C^{(s,1/2,-1/2)}_{\alpha(2s+2)} C^{(s,-1/2,1/2)\alpha(2s+2)} + \overline{C}^{(s,-1/2,-1/2)}_{\dot\beta(2s+2)} \overline{C}^{(s,1/2,1/2)\dot\beta(2s+2)} \Big]. \tag{4.70}$$

In this way we have demonstrated that the free conformally invariant higher-spin actions introduced in ref. [2] can be geometrically formulated in terms of curvatures, similarly to conformal supergravity. An analogous formulation for the higher-spin fields in $adS_4$ was constructed in ref. [9]. Note that constraints in adS also admit cohomological interpretation, as it was shown in ref. [24].

## 5. Cubic interaction in superconformal higher-spin theory

### 5.1. SOME REMARKS ABOUT ACTIONS QUADRATIC ON THE CURVATURES

Let $T_{\mathscr{A}}$ be a basis in some Lie superalgebra g containing the Poincaré or $adS_4$ algebra as a subalgebra, and let $R^{\mathscr{A}}$ be curvature two-forms. In ref. [19] it was proposed to choose an action of the following form:

$$A = \int R^{\mathscr{A}} \wedge R^{\mathscr{B}} Q_{\mathscr{A}\mathscr{B}} + A_{\mathrm{YM}}, \tag{5.1}$$

where $Q_{\mathscr{A}\mathscr{B}}$ is some (super)symmetric bilinear form and $A_{\mathrm{YM}}$ is Yang–Mills action for the vector fields (if there is some internal subalgebra in g).

However there is a question in this approach: How ought one to choose the bilinear form $Q_{\mathscr{A}\mathscr{B}}$? If one chose the invariant bilinear form $G_{\mathscr{A}\mathscr{B}}$ on g, one would get a topological invariant and have no dynamical equations of motion. Hence one should look for another bilinear form, but then the action will no longer be invariant under the original gauge transformations. Meanwhile it may be invariant under some deformed gauge transformations. Often the action (5.1) is not sufficient to construct the theory and should be supplemented with the constraints. Then the deformed gauge transformations may be found from the covariance properties of these constraints.

As g contains a Lorentz subalgebra, the basis $T_{\mathscr{A}}$ can be chosen as $\{T^{\Omega}_{\alpha(n),\dot\beta(m)}\}$, where $\Omega$ is the set of all other indices. Let us split the set into three parts:

$$\begin{aligned}
&\{T_A\}: && T^{\Omega}_{\alpha(n),\dot\beta(m)}, && n>m;\\
&\{T_{\bar A}\}: && T^{\Omega}_{\alpha(n),\dot\beta(m)}, && m>n;\\
&\{T_a\}: && T^{\Omega}_{\alpha(n),\dot\beta(m)}, && n=m.
\end{aligned} \tag{5.2}$$

The invariant bilinear form $G_{\mathscr{A}\mathscr{B}}$ in this basis has a block-diagonal form with only non-zero blocks $G_{AB}, G_{\bar A\bar B}, G_{ab}$. Then the topological invariant is

$$I = \int_{\mathrm{M}^4} \left[ R^A \wedge R^B G_{AB} + R^{\bar A} \wedge R^{\bar B} G_{\bar A\bar B} + R^a \wedge R^b G_{ab} \right]. \tag{5.3}$$

A non-trivial action (or at least its main part) in this basis can be proposed in the following simple form

$$A_0 = i \int \left[ R^A \wedge R^B G_{AB} - R^{\bar A} \wedge R^{\bar B} G_{\bar A\bar B} \right], \tag{5.4}$$

and the Yang–Mills term is

$$A_{\mathrm{YM}} = 2\gamma \int \mathrm{d}^4x \sqrt{-g}\, g^{\mu\rho} g^{\nu\sigma} R^{\hat{a}}_{\mu\nu} R^{\hat{b}}_{\rho\sigma} G_{\hat{a}\hat{b}}$$

$$= \gamma \int R^{\hat{a}} \wedge * \, R^{\hat{b}} G_{\hat{a}\hat{b}}\,, \tag{5.5}$$

where $\{R^{\hat{a}}\}$ is a subset of the set $\{R^a\}$ $(n = m = 0)$ corresponding to the internal subalgebra (if it exists in g), and $g_{\mu\nu} = e^{\alpha\dot{\beta}}_{\mu} e_{\nu\alpha\dot{\beta}}$ is the four-dimensional metric constructed from the vierbein $e^{\alpha\dot{\beta}}_{\mu}$ which is contained among the gauge fields of g.

Let us examine the variation of the action

$$A = A_0 + A_{\mathrm{YM}} \tag{5.6}$$

under the original gauge transformations

$$\delta_g R^A = f_{B\mathscr{A}}{}^A R^B \mathscr{E}^{\mathscr{A}} + f_{\bar{B}\mathscr{A}}{}^A R^{\bar{B}} \mathscr{E}^{\mathscr{A}} + f_{b\mathscr{A}}{}^A R^b \mathscr{E}^{\mathscr{A}}\,,$$

$$\delta_g R^{\bar{A}} = f_{B\mathscr{A}}{}^{\bar{A}} R^B \mathscr{E}^{\mathscr{A}} + f_{\bar{B}\mathscr{A}}{}^{\bar{A}} R^{\bar{B}} \mathscr{E}^{\mathscr{A}} + f_{b\mathscr{A}}{}^{\bar{A}} R^b \mathscr{E}^{\mathscr{A}}\,,$$

$$\delta_g R^a = f_{B\mathscr{A}}{}^a R^B \mathscr{E}^{\mathscr{A}} + f_{\bar{B}\mathscr{A}}{}^a R^{\bar{B}} \mathscr{E}^{\mathscr{A}} + f_{b\mathscr{A}}{}^a R^b \mathscr{E}^{\mathscr{A}}\,. \tag{5.7}$$

It has the form

$$\begin{aligned}\delta_g A_0 = i \int \Big[ & R^A \wedge R^C \mathscr{E}^{\mathscr{A}} \big( f_{C\mathscr{A}}{}^B G_{AB} + (-1)^{\mathscr{E}_A \mathscr{E}_C} f_{A\mathscr{A}}{}^B G_{CB} \big) \\ & - R^{\bar{A}} \wedge R^{\bar{C}} \mathscr{E}^{\mathscr{A}} \big( f_{\bar{C}\mathscr{A}}{}^{\bar{B}} G_{\bar{A}\bar{B}} + (-1)^{\mathscr{E}_{\bar{A}} \mathscr{E}_{\bar{C}}} f_{\bar{A}\mathscr{A}}{}^{\bar{B}} G_{\bar{C}\bar{B}} \big) \\ & + 2R^A \wedge R^{\bar{C}} \mathscr{E}^{\mathscr{A}} \big( f_{\bar{C}\mathscr{A}}{}^B G_{AB} - (-1)^{\mathscr{E}_A \mathscr{E}_{\bar{C}}} f_{A\mathscr{A}}{}^{\bar{B}} G_{\bar{C}\bar{B}} \big) \\ & + 2R^A \wedge R^a \mathscr{E}^{\mathscr{A}} f_{a\mathscr{A}}{}^B G_{AB} - 2R^{\bar{A}} \wedge R^a \mathscr{E}^{\mathscr{A}} f_{a\mathscr{A}}{}^{\bar{B}} G_{\bar{A}\bar{B}} \Big]\,. \end{aligned} \tag{5.8}$$

The first and second terms in eq. (5.8) are identically equal to zero due to the invariance property of $G_{AB}, G_{\bar{A}\bar{B}}$. The remaining terms can be brought to the form

$$\begin{aligned}\delta_g A_0 = 2i \int \Big[ & 2R^A \wedge R^{\bar{C}} \mathscr{E}^{\mathscr{A}} f_{\bar{C}\mathscr{A}}{}^B G_{AB} \\ & + \big( R^A f_{a\mathscr{A}}{}^B G_{AB} - R^{\bar{A}} f_{a\mathscr{A}}{}^{\bar{B}} G_{\bar{A}\bar{B}} \big) \wedge R^a \mathscr{E}^{\mathscr{A}}\,. \end{aligned} \tag{5.9}$$

The variation of the Yang–Mills term is

$$\delta_g A_{\mathrm{YM}} = 2\gamma \int R^{\hat{a}} \wedge * \left( f_{A\mathscr{A}}{}^{\hat{b}} R^{A} \mathscr{E}^{\mathscr{A}} + f_{\bar{A}\mathscr{A}}{}^{\hat{b}} R^{\bar{A}} \mathscr{E}^{\mathscr{A}} + f_{a\mathscr{A}}{}^{\hat{b}} R^{a} \mathscr{E}^{\mathscr{A}} \right) G_{\hat{a}\hat{b}} + \left(\delta_g A\right)_1, \tag{5.10}$$

where $(\delta_g A)_1$ appears due to the variation of $g_{\mu\nu}$ in eq. (5.5). Generally speaking, this term may spoil the gauge invariance. However, in the cubic approximation it reduces to the linearized part of general coordinate transformations (the linearized zero-torsion condition is supposed to be satisfied)

$$\delta g_{\mu\nu} = \left(\partial_\nu \xi^\rho\right) g_{\rho\mu} + \left(\partial_\mu \xi^\rho\right) g_{\rho\nu} + \xi^\rho \partial_\rho g_{\mu\nu}, \tag{5.11}$$

with parameters

$$\xi^\rho e_{\rho\alpha\dot{\beta}} = \mathscr{E}_{\alpha\dot{\beta}} \tag{5.12}$$

($\mathscr{E}_{\alpha\dot{\beta}}$ is contained in the gauge parameters of the algebra g). These variations can be compensated for by deformations in the transformation law for Yang–Mills fields

$$\Delta_1 \omega_\mu^{\hat{a}} = \xi^\nu R_{\nu\mu}^{\hat{a}}, \tag{5.13}$$

which differs from the usual general coordinate transformations by a spin-1 gauge transformation with the parameter $\mathscr{E}^{\hat{a}} = \xi^\mu \omega_\mu^{\hat{a}}$. In the cubic approximation the deformations (5.13) do not contribute to $\delta A_0$ because the vector fields enter $A_0$ only in interactions terms, and $\Delta_1 \omega_\mu^{\hat{a}}$ is of at least second order (see ref. [6]).

Let us consider remaining terms in eqs. (5.9) and (5.10). If from the standard constraints it follows that the $R^A$-curvatures are antiself-dual and the $R^{\bar{A}}$ self-dual,

$$* R^A = -iR^A, \qquad * R^{\bar{A}} = iR^{\bar{A}}, \tag{5.14}$$

then the first term in eq. (5.9) is identically equal to zero due to the simple identity (A.19). The second term in eq. (5.9) can be divided into two parts. There are curvatures for vector fields $R^{\hat{a}}$ in the first part, and there are curvatures $R^{\check{a}}$($\{R^{\check{a}}\} = \{R^a\}$ except $\{R^{\hat{a}}\}$) in the second part. Due to (anti)self-duality of $(R^A)$ $R^{\bar{A}}$ the first part is canceled out with the first and second terms in eq. (5.10) if and only if

$$\gamma = 1. \tag{5.15}$$

In this way only non-vanishing terms are

$$\delta_g A = 2i\int \left[ R^A f_{\check{a}\mathscr{A}}{}^B G_{AB} - R^{\bar{A}} f_{\check{a}\mathscr{A}}{}^{\bar{B}} G_{\bar{A}\bar{B}} \right]$$

$$\wedge R^{\check{a}} \mathscr{E}^{\mathscr{A}} + 2\int R^{\hat{a}} \wedge * R^{\check{c}} \mathscr{E}^{\mathscr{A}} f_{\check{c}\mathscr{A}}{}^{\hat{b}} G_{\hat{a}\hat{b}} \qquad (5.16)$$

(note that the term $R^{\hat{a}} \wedge * R^{\hat{b}} \mathscr{E}^{\mathscr{A}}$ in $\delta_g A_{\mathrm{YM}}$ is equal to zero due to the symmetry property of $G_{\hat{a}\hat{b}}$).

The terms (5.16) should be compensated for by some deformations $\Delta\omega$. As an example of the above type of action let us call to mind the proof of cubic invariance in the extended massless higher-spin theory in $\mathrm{adS}_4$ from ref. [6].

When the linearized constraints and equations of motion are taken into account, the only non-trivial curvature components are the Weyl multispinors $C$ and $\bar{C}$

$$R'_{\mu\nu, i(k), \alpha(n), \dot{\beta}(m)} = \delta(m) \sigma_{\mu\nu}{}^{\alpha(2)} C_{i(k), \alpha(n+2)}$$

$$+ \delta(n) \bar{\sigma}_{\mu\nu}{}^{\dot{\beta}(2)} \bar{C}_{i(k), \dot{\beta}(m+2)} \qquad (5.17)$$

($i(k)$ is the antisymmetrized set of so($N$)-indices $[i_1 \dots i_k]$). At the linearized level all the curvatures from $R^{\check{a}}$ are equal to zero on the constraints, $R^A$-curvatures are antiself-dual and $R^{\bar{A}}$ are self-dual. Hence all the cancellations considered above take place in this case and $\delta_g A = 0$ without any deformations $\Delta\omega$ (except, of course, $\Delta_1\omega$ in eq. (5.13)). This is true because in the cubic order the term (5.16) equals zero as $R^{l\check{a}} = 0$. The knowledge of non-linear constraints in this case is not essential to prove gauge invariance in the cubic order because the terms $(\delta A/\delta\omega)\Delta\omega$ are either equal to zero on the linearized constraints and equations of motion, or are of at least fourth order (for so-called "extra" fields, i.e. fields which do not enter the quadratic action and which are defined only by the standard constraints). In this way the action of the type of eqs. (5.4)–(5.6) solves the problem of constructing the invariant interaction in the approximation under consideration.

### 5.2. THE CUBIC INVARIANT ACTION IN CONFORMAL HIGHER-SPIN THEORY (CASE OF INTEGER SPINS)

In this subsection we present an action obtained previously by us in ref. [14] which generalizes the Weyl gravity action, and prove its gauge invariance in the cubic order with some simplifications. Our construction for the purely bosonic case is based on the conformal higher-spin algebra $\mathrm{hsc}^{\infty}(4)$.

Making the splitting (5.2), one can write down an action of the type of (5.4) (the topological invariant for hsc$^\infty$(4) was given in eq. (2.32))

$$A=\beta\sum_{N=1}^{\infty}\sum_{s=1}^{N}\sum_{c=-s}^{s}\sum_{n,m}(-1)^{N-s}i^{n+m+1}\mathscr{E}(n-m)$$

$$\times\int R^{(N,s,c)}_{\alpha(n),\dot\beta(m)}\wedge R^{(N,s,-c)\alpha(n),\dot\beta(m)}, \tag{5.18}$$

where

$$\mathscr{E}(n-m)=\begin{cases}1, & n>m\\ 0, & n=m\\ -1, & n<m.\end{cases} \tag{5.19}$$

The above action is dimensionless, real, parity conserving and does not contain any dimensionful parameters. There only is a dimensionless real constant $\beta$ (overall normalization), and the correct normalization of the conformal gravity action (terms $-i\beta\int[R^{(1,1,0)}_{\alpha(2)}\wedge R^{(1,1,0)\alpha(2)}-\text{c.c.}]$) follows by setting $\beta=-1/8\alpha^2$, where $\alpha$ is the coupling constant of Weyl gravity.

Let us now consider the action (5.18) perturbatively using our expansion procedure. In second order, taking into account the linearized constraints (see eq. (4.61))

$$R'^{(N,s,c)}_{\mu\nu,\alpha(n),\dot\beta(m)}=\tfrac{1}{4}\theta(c)\delta(m-c)\delta(n-2s+c)\sigma^{\alpha(2)}_{\mu\nu}R'^{(N,s,c)}_{\alpha(n+2),\dot\beta(m)}$$

$$+\tfrac{1}{4}\theta(c)\delta(m-2s+c)\delta(n-c)\bar\sigma^{\dot\beta(2)}_{\mu\nu}\bar R'^{(N,s,c)}_{\alpha(n),\dot\beta(m+2)}, \tag{5.20}$$

$$R'^{(N,s,c)}_{\alpha(2s-c+2),\dot\beta(c)}=A(s,c)\underbrace{\partial^{\alpha}{}_{\dot\beta}\ldots\partial^{\alpha}{}_{\dot\beta}}_{c}C^{(N,s,0)}_{\alpha(2s+2)}, \tag{5.21a}$$

$$\bar R'^{(N,s,c)}_{\alpha(c),\dot\beta(2s-c+2)}=A(s,c)\underbrace{\partial_{\alpha}{}^{\dot\beta}\ldots\partial_{\alpha}{}^{\dot\beta}}_{c}\bar C^{(N,s,0)}_{\dot\beta(2s+2)},\quad c=0,1,\ldots,s, \tag{5.21b}$$

the action is brought to the sum of linearized actions $C^2$ for free conformal higher-spin fields, as in eq. (4.69).

In the cubic order the above action describes a cubic interaction in conformal higher-spin theory. Let us prove its gauge invariance. First, the curvatures $R'^{A}$ are antiself-dual, $R'^{\bar A}$ are self-dual, and among the curvatures $R'^{a}$ the only non-zero ones are $R'^{(N,s,s)}_{\alpha(s),\dot\beta(s)}$ (denoted by $R'^{\dot a}$). Thus in this case after all the cancellations in $\delta_g A$ stipulated by the symmetry properties of $G_{AB}, G_{\bar A\bar B}$ and (anti)self-duality, as considered in subsect. 5.1, only the following term remains non-compensated:

$$\delta_g A = 2i\int\left[R'^{A} f_{\tilde{a}\mathscr{A}}{}^{B} G_{AB} - R'^{\bar{A}} f_{\tilde{a}\mathscr{A}}{}^{\bar{B}} G_{\bar{A}\bar{B}}\right] \wedge R'^{\tilde{a}} \mathscr{E}^{\mathscr{A}}$$

$$= 2i\int\left[R'^{A} \mathscr{E}^{\mathscr{A}} f_{A\mathscr{A}}{}^{\tilde{\bar{b}}} - R'^{\bar{A}} f_{\bar{A}\mathscr{A}}{}^{\tilde{\bar{b}}} \mathscr{E}^{\mathscr{A}}\right] \wedge R'^{\tilde{a}} G_{\tilde{a}\tilde{\bar{b}}}$$

$$= 2\int \mathrm{d}^4x\left[R'^{A}_{\alpha(2)} \mathscr{E}^{\mathscr{A}} f_{A\mathscr{A}}{}^{\tilde{\bar{b}}} G_{\tilde{a}\tilde{\bar{b}}} R'^{\alpha(2)\tilde{a}} + \bar{R}'^{\bar{A}}_{\dot{\beta}(2)} \mathscr{E}^{\mathscr{A}} f_{\bar{A}\mathscr{A}}{}^{\tilde{\bar{b}}} G_{\tilde{a}\tilde{\bar{b}}} \bar{R}'^{\dot{\beta}(2)\tilde{a}}\right],$$

$$R'^{A} = \tfrac{1}{4}\sigma^{\alpha(2)} R'^{A}_{\alpha(2)}, \qquad R'^{\bar{A}} = \tfrac{1}{4}\bar{\sigma}^{\dot{\beta}(2)} \bar{R}'^{\bar{A}}_{\dot{\beta}(2)},$$

$$R'^{a} = \tfrac{1}{4}\sigma^{\alpha(2)} R'^{a}_{\alpha(2)} + \tfrac{1}{4}\bar{\sigma}^{\dot{\beta}(2)} \bar{R}'^{a}_{\dot{\beta}(2)}, \tag{5.22}$$

where we have used the invariance property of $G$ and (anti)self-duality of $(R^{A})$ $R^{\bar{A}}$. The index $\tilde{\bar{a}}$ means $\omega_{\mu}{}^{\tilde{\bar{a}}} = \omega^{(N,s,-s)}_{\mu,\alpha(s),\dot{\beta}(s)}$ (among the coefficients of the bilinear form $G_{\tilde{a}\mathrm{A}}$, the only non-zero ones are $G_{\tilde{a}\tilde{\bar{b}}} = G_{(N,s,s)\alpha(s),\dot{\beta}(s);(N,s,-s)\gamma(s),\dot{\rho}(s)}$, see eq. (2.32)). Let us note that (5.22) can be brought to the form

$$\delta_g A = 2\int \mathrm{d}^4x\left[\left(\delta_g R^{\tilde{\bar{b}}}_{\alpha(2)}\right) R'^{\alpha(2)\tilde{a}} G_{\tilde{\bar{b}}\tilde{a}} + \left(\delta_g R^{\tilde{\bar{b}}}_{\dot{\beta}(2)}\right) \bar{R}'^{\dot{\beta}(2)\tilde{a}} G_{\tilde{\bar{b}}\tilde{a}}\right]$$

$$= -\frac{1}{4\alpha^2}\sum_{N,s}(-1)^{N}\int \mathrm{d}^4x\left[\left(\delta_g R^{(N,s,-s)}_{\alpha(2),\alpha(s)\dot{\beta}(s)}\right) R'^{(N,s,s)\alpha(s+2),\dot{\beta}(s)} + \text{h.c.}\right], \tag{5.23}$$

because the homogeneous gauge transformations for $R^{\tilde{\bar{a}}}$ are

$$\delta_g R^{\tilde{\bar{a}}}_{\alpha(2)} = f_{A\mathscr{A}}{}^{\tilde{\bar{a}}} R^{A}_{\alpha(2)} \mathscr{E}^{\mathscr{A}} + f_{\tilde{b}\mathscr{A}}{}^{\tilde{\bar{a}}} R^{\tilde{b}}_{\alpha(2)} \mathscr{E}^{\mathscr{A}}, \text{ and h.c.} \tag{5.24}$$

Substituting (5.24) into (5.23), we arrive at (5.22), because the term

$$f_{\tilde{b}\mathscr{A}}{}^{\tilde{\bar{b}}} R^{\tilde{b}}_{\alpha(2)} R^{\alpha(2),\tilde{a}} G_{\tilde{\bar{b}}\tilde{a}}$$

and complex conjugate one are equal to zero due to the invariance property of $G$.

The remaining terms in (5.23) should be compensated for by some deformations $\Delta\omega$. Let us consider the structure of the deformation $\Delta A \sim (\delta A/\delta\omega)\,\Delta\omega$. First of all note that since $\Delta\omega$ are of at least second order (remember that the fields $\omega$ and gauge parameters $\mathscr{E}$ are supposed to be of first order), $\delta A/\delta\omega$ should be taken as linearized $\delta A^{l}/\delta\omega$ in the approximation under consideration. In this way we have

$$\Delta A = -\frac{i}{2\alpha^2}\sum_{N,s}(-1)^N\sqrt{s}\int d^4x\epsilon^{\mu\nu\rho\sigma}R'^{(N,s,s)}_{\mu\nu,\alpha(s),\dot\beta(s)}$$

$$\times\left[\sigma_{\rho\alpha}{}^{\dot\beta}\Delta\omega^{(N,s,1-s)\alpha(s+1),\dot\beta(s-1)}_{\sigma,} - \sigma_{\rho}{}^{\alpha}{}_{\dot\beta}\Delta\omega^{(N,s,1-s)\alpha(s-1),\dot\beta(s+1)}_{\sigma,}\right]$$

$$= -\frac{1}{2\alpha^2}\sum_{N,s}(-1)^N\sqrt{s}\int d^4x\left[R'^{(N,s,s)}_{\alpha(s+2),\dot\beta(s)}\Delta\omega^{(N,s,1-s)\alpha\dot\beta,\alpha(s+1),\dot\beta(s-1)}\right.$$

$$\left.+R'^{(N,s,s)}_{\alpha(s),\dot\beta(s+2)}\Delta\omega^{(N,s,1-s)\alpha\dot\beta,\alpha(s-1),\dot\beta(s+1)}\right], \quad (5.25)$$

where we have varied the action, integrated by parts, and used the linearized Bianchi identities and constraints. All the other fields (except $\omega^{(N,s,1-s)}$) either do not enter the linearized action $A'$ (more exactly, enter only the full derivative term) or $\delta A'/\delta\omega$ equal zero on the linearized constraints. Thus in the approximation under consideration the only essential deformations are $\Delta\omega^{(N,s,1-s)}$. Note that $\Delta\omega$ enter eq. (5.25) multiplied with the curvatures $R'^{(N,s,s)}$. The term (5.23) which we need to cancel out also has such a structure ($R'^{\bar a} = R'^{(N,s,s)}$). Hence we get explicit expressions for the deformations $\Delta\omega$ compensating $\delta_g A$:

$$\Delta\omega^{(N,s,-s+1)}_{\alpha\dot\beta,\alpha(s+1),\dot\beta(s-1)} = -\frac{1}{2\sqrt{s}}\delta_g R^{(N,s,-s)}_{\alpha(2),\alpha(s),\dot\beta(s)}, \quad (5.26a)$$

$$\Delta\omega^{(N,s,-s+1)}_{\alpha\dot\beta,\alpha(s-1),\dot\beta(s+1)} = -\frac{1}{2\sqrt{s}}\delta_g R^{(N,s,-s)}_{\dot\beta(2),\alpha(s),\dot\beta(s)} \quad (5.26b)$$

(note that in the approximation under consideration the curvatures $R$ on the r.h.s. of the expression $\delta_g R = fR'\mathscr{E}$ must be taken as linearized). Further we should verify that these deformations are compatible with the constraints.

At the linearized level the fields $\omega^{(N,s,-s+1)}_{\alpha\beta\dots}$ have been expressed through the physical fields $\omega^{(N,s,-s)}_{\alpha\dot\beta\dots}$ with the help of the constraints

$$R'^{(N,s,-s)}_{\mu\nu,\alpha(s),\dot\beta(s)} = 0. \quad (5.27)$$

It looks natural to suppose that in the second order the same constraints remain in force

$$R^{(N,s,-s)}_{\mu\nu,\alpha(s),\dot\beta(s)} = 0, \quad (5.28)$$

where $R$ is no longer linearized but of second order. Straightforward verification gives that the constraints (5.28) are invariant under the deformed gauge transformations

$$\Delta R^{(N,s,-s)}_{\alpha(2),\alpha(s),\dot\beta(s)} + \delta_g R^{(N,s,-s)}_{\alpha(2),\alpha(s),\dot\beta(s)} = 2\sqrt{s}\,\Delta\omega^{(N,s,-s+1)}_{\alpha\beta,\alpha(s+1),\dot\beta(s-1)} + \delta_g R^{(N,s,-s)}_{\alpha(2),\alpha(s),\dot\beta(s)} = 0 \tag{5.29}$$

and analogously for the complex conjugate expression. The deformations found generalize the deformations

$$\Delta\omega^{ab}_{\mu} \sim \mathscr{E}^{\nu}(P) R^{ab}_{\mu\nu}(\mathrm{M}) \tag{5.30}$$

in conformal gravity.

In this way we have completely proved the gauge invariance of the cubic interaction in conformal higher-spin theory. The following characteristic feature of this proof as compared to the one for $\mathrm{adS}_4$ theory should be mentioned. When establishing the invariance in the $\mathrm{adS}_4$ theory, only linearized constraints were essential. In the conformal theory, on the other hand, the second-order constraints (5.28) are essential along with the linearized ones.

The action (5.18) in the cubic order can be brought to the $C'C^{(2)}$-form analogous to the linearized case $(C')^2$, where $C'$ are the linearized Weyl tensors considered in sect. 4 and $C^{(2)}$ are second-order Weyl tensors. To see this, one should note that in the cubic order $R \wedge R \sim R' \wedge R$, substitute the solution of constraints $R' \sim \partial^c C$ and integrate by parts. The second-order Weyl tensors are found in a unique way from the requirement of gauge invariance of the action. Thus the action is defined in a unique way.

### 5.3. THE CUBIC INVARIANT ACTION IN SUPERCONFORMAL HIGHER-SPIN THEORY

The action based on the conformal higher-spin superalgebra $\mathrm{shsc}^{\infty}(4|1)$ of the type (5.4) can be written in the form [for the topological invariant for $\mathrm{shsc}^{\infty}(4|1)$ see eq. (2.29)]

$$A_0 = \frac{-1}{8\alpha^2} \sum_{N=1}^{\infty} \sum_{s=1}^{N} \sum_{\jmath,c,u,n,m} (-1)^{N-s} i^{n+m+1} \mathscr{E}(n-m)$$

$$\times \int R^{(N,s,\jmath,c,u)}_{\alpha(n),\dot\beta(m)} \wedge R^{(N,s,\jmath,-c,-u)\alpha(n),\dot\beta(m)} \tag{5.31}$$

and the Yang–Mills term for vector fields* $\omega_\mu^{(N,1,0,0,0)}$ reads

$$A_{\mathrm{YM}} = \frac{-1}{4\alpha^2} \sum_{N=1}^{\infty} (-1)^{N-1} \int \mathrm{d}^4x \sqrt{-g}\, g^{\mu\rho} g^{\nu\sigma} R_{\mu\nu}^{(N,1,0,0,0)} R_{\rho\sigma}^{(N,1,0,0,0)} . \quad (5.32)$$

The action $A = A_0 + A_{\mathrm{YM}}$, as in the purely bosonic theory, does not contain any dimensionful parameters. The metric in eq. (5.32) is defined as

$$g_{\mu\nu} = \tfrac{1}{2} \omega_{\mu,\alpha,\dot\beta}^{(1,1,1,-1,0)} \omega_{\nu,}^{(1,1,1,-1,0)\alpha,\dot\beta} ,$$

$$g^{\mu\nu} = (g_{\mu\nu})^{-1}, \qquad g = \det(g_{\mu\nu}) . \quad (5.33)$$

At the linearized level the solution of constraints have the form

$$R'^{(N,s,\jmath,c,u)}_{\mu\nu,\alpha(n),\dot\beta(m)} = \tfrac{1}{4}\theta(c+|u|)\delta(m-c-u)\delta(n-2\jmath+c+u)\sigma_{\mu\nu}{}^{\alpha(2)} R^{(N,s,\jmath,c,u)}_{\alpha(n+2),\dot\beta(m)}$$

$$+ \tfrac{1}{4}\theta(c+|u|)\delta(m-2\jmath+c-u)\delta(n-c+u)\bar\sigma_{\mu\nu}{}^{\dot\beta(2)} \bar R^{(N,s,\jmath,c,u)}_{\alpha(n),\dot\beta(m+2)} ,$$

$$(5.34)$$

$$R^{(N,s,\jmath,c,u)}_{\alpha(2\jmath-c-u+2),\dot\beta(c+u)} = A(\jmath,c,u) \underbrace{\partial^{\alpha}{}_{\dot\beta} \ldots \partial^{\alpha}{}_{\dot\beta}}_{c+u} C^{(N,s,\jmath,-u,u)}_{\alpha(2\jmath+2)} , \quad (5.35a)$$

$$\bar R^{(N,s,\jmath,c,u)}_{\alpha(c-u),\dot\beta(2\jmath-c+u+2)} = A(\jmath,c,-u) \underbrace{\partial_{\alpha}{}^{\dot\beta} \ldots \partial_{\alpha}{}^{\dot\beta}}_{c-u} \bar C^{(N,s,\jmath,u,u)}_{\dot\beta(2\jmath+2)} , \quad (5.35b)$$

and the action is brought to the sum of free $C^2$-actions.

Let us consider cubic invariance of the action. First, due to the gauge variation of the metric in the Yang–Mills action, which is a sum of the linearized general coordinate transformations and the linearized Weyl transformation in the approximation under consideration,

$$\delta g_{\mu\nu} = \partial_\mu \mathscr{E}_\nu + \partial_\nu \mathscr{E}_\mu + \lambda \eta_{\mu\nu} , \quad (5.36)$$

the compensating deformations for the vector fields are

$$\Delta_1 \omega_\mu^{(N,1,0,0,0)} = \mathscr{E}^\nu R'^{(N,1,0,0,0)}_{\mu\nu} \qquad \left( \mathscr{E}_\mu = \sigma_\mu{}^{\alpha\dot\beta} \mathscr{E}^{(1,1,1,-1,0)}_{\alpha\dot\beta} \right) \quad (5.37)$$

* As considered in (I), in the $N=1$ theory this infinite tower of vector fields is abelian, but in the $N$-extended theory it forms a subalgebra in the $\widehat{SU}(N)$ Kac–Moody algebra with the generators $T_A^n$, $n \geqslant 0$. The coupling constant for vector fields here is the same as the one for the gravitational interaction, $e^2 = \alpha^2$.

(note that under the Weyl transformation $\delta_W g_{\mu\nu} = \lambda g_{\mu\nu}$ we have $\delta_W(\sqrt{-g}\, g^{\mu\nu} g^{\rho\sigma}) = 0$).

The curvatures $(R^A)R^{\bar{A}}$ are (anti)self-dual except following ones

$$R'^{(N,s,s-1/2,s-1/2,1/2)}_{\mu\nu,\alpha(s-1),\dot\beta(s)} = \tfrac{1}{4}\sigma_{\mu\nu}{}^{\alpha(2)} R^{(N,s,s-1/2,s-1/2,1/2)}_{\alpha(s+1),\dot\beta(s)} + \tfrac{1}{4}\bar\sigma_{\mu\nu}{}^{\dot\beta(2)} \bar{R}^{(N,s,s-1/2,s-1/2,1/2)}_{\alpha(s-1),\dot\beta(s+2)}, \tag{5.38a}$$

$$R'^{(N,s,s-1/2,s-1/2,-1/2)}_{\mu\nu,\alpha(s),\dot\beta(s-1)} = \tfrac{1}{4}\sigma_{\mu\nu}{}^{\alpha(2)} R^{(N,s,s-1/2,s-1/2,-1/2)}_{\alpha(s+2),\dot\beta(s-1)} + \tfrac{1}{4}\bar\sigma_{\mu\nu}{}^{\dot\beta(2)} \bar{R}^{(N,s,s-1/2,s-1/2,-1/2)}_{\alpha(s),\dot\beta(s+1)}, \tag{5.38b}$$

which contain both self-dual and antiself-dual parts. So the first term in eq. (5.9) is equal to zero except the terms with the above curvatures.

Transforming the terms in eqs. (5.9) and (5.10) with the help of invariance property of $G$ and (anti)self-duality of curvatures, we finally get, analogously to (5.23) in the purely bosonic case,

$$\delta_g A = -\frac{1}{4\alpha^2} \sum_{\substack{N,s,\jmath,u \\ (\jmath>0)}} (-1)^{N-s} \int d^4x \Big[ i^{2\jmath} \big( \delta_g R^{(N,s,\jmath,-\jmath,u)}_{\alpha(2),\alpha(\jmath+u),\dot\beta(\jmath-u)} \big) \times R^{l(N,s,\jmath,\jmath,u)\alpha(2+\jmath+u),\dot\beta(\jmath-u)} + \text{h.c.} \Big]. \tag{5.39}$$

To compensate these terms, let us try and find deformations $\Delta\omega$. The deformation $\Delta A$ reads (analogously to eq. (5.25))

$$\Delta A = -\frac{1}{2\alpha^2} \sum_{N,s,\jmath,u} (-1)^{N-s} \int d^4x \Big[ i^{2\jmath} \sqrt{\jmath-u}\, R'^{(N,s,\jmath,\jmath,u)}_{\alpha(\jmath+2+u),\dot\beta(\jmath-u)} \times \Delta\omega^{(N,s,\jmath,-\jmath+1,-u)\alpha\dot\beta,\alpha(\jmath+u+1),\dot\beta(\jmath-u-1)} + \text{h.c.} \Big]. \tag{5.40}$$

In the superconformal case the only essential deformations in our approximation are

$$\Delta\omega^{(N,s,\jmath,-\jmath+1,u)}_{\alpha\dot\beta,\alpha(\jmath+u+1),\dot\beta(\jmath-u-1)} = \frac{-1}{2\sqrt{\jmath-u}} \delta_g R^{(N,s,\jmath,-\jmath,u)}_{\alpha(2),\alpha(\jmath+u),\dot\beta(\jmath-u)} \tag{5.41a}$$

and

$$\Delta\omega^{(N,s,\jmath,-\jmath+1,u)}_{\alpha\dot\beta,\alpha(\jmath+u-1),\dot\beta(\jmath-u+1)} = \frac{-1}{2\sqrt{\jmath-u}} \delta_g R^{(N,s,\jmath,-\jmath,u)}_{\dot\beta(2),\alpha(\jmath+u),\dot\beta(\jmath-u)}. \tag{5.41b}$$

At the linearized level the fields $\omega^{(N,s,\jmath,-\jmath+1,u)}_{\cdot \quad \cdot\cdot}$ are expressed through the physical fields $\omega^{(N,s,\jmath,-\jmath+1,u)}_{\cdot \quad \cdot\cdot}$ with the help of the constraints

$$R'^{(N,s,\jmath,-\jmath,u)}_{\alpha(2),\alpha(\jmath+u),\beta(\jmath-u)} = 0, \qquad R'^{(N,s,\jmath,-\jmath,u)}_{\beta(2),\alpha(\jmath+u),\beta(\jmath-u)} = 0. \qquad (5.42\text{a,b})$$

We suppose, as in the bosonic case, the second-order constraints to be satisfied,

$$R^{(N,s,\jmath,-\jmath,u)}_{\alpha(2),\alpha(\jmath+u),\dot\beta(\jmath-u)} = 0, \qquad R^{(N,s,\jmath,-\jmath,u)}_{\beta(2),\alpha(\jmath+u),\dot\beta(\jmath-u)} = 0. \qquad (5.43\text{a,b})$$

Then it can be easily verified that the constraints above are invariant under the deformed gauge transformations $\delta_g \omega + \Delta\omega$. In this way it is completely proved the gauge invariance of the higher-spin superconformal interaction described by action (5.31). The above deformations generalize the deformations under P and Q symmetries in conformal supergravity.

However, it is known [20, 21] that in the conformal supergravity action there is a term

$$\int R(\mathrm{D}) \wedge R(\mathrm{U}) \qquad (5.44)$$

along with the terms $R(\mathrm{M}) \wedge R(\mathrm{M})$, $\overline{R}(\mathrm{Q}) \wedge \gamma_5 R(\mathrm{S})$ and $R(\mathrm{U}) \wedge * R(\mathrm{U})$ appearing in eqs. (5.31) and (5.32) when $N = s = 1$. But in our approach we have not obtained this term and neither its higher-spin generalization. Nevertheless, gauge invariance is proved without such terms in the approximation under consideration. As a matter of fact, the nondiagonal term (5.44) (we call it "non-diagonal" because it is built out of two curvatures associated with the different so(4, 2) representations; $R(\mathrm{D})$ with the adjoint representation, $R(\mathrm{U})$ with the trivial representation) is necessary in the supergravity action to provide the gauge invariance in the fourth order. In cubic order in the action there is one non-defined parameter which can be fixed in such a way that the coefficient in front of the term (5.44) becomes equal to zero. This coefficient is uniquely defined only in fourth order. However, in the present paper we have dealt only with the cubic approximation and such "non-minimal" terms are not essential for our purposes. Note also that in conformal supergravity adding the term (5.44) in the action with the above-defined coefficient is sufficient to provide complete gauge invariance in all orders; on the other hand, adding these terms in higher-spin superconformal theory is no longer sufficient. Along with them one should find a non-linear generalization for all conventional constraints and, possibly, include in the action some higher-order terms on the curvatures. The situation here differs drastically from both the $\mathrm{adS}_4$ higher-spin theory and the bosonic conformal higher-spin theory, where in cubic order the action is uniquely determined without any non-minimal terms.

To conclude this section the following feature of the above action should be mentioned. Kinetic terms for fields with the same spin-($\jmath + 1$) (including vector fields) enter in the action both with positive and negative signs due to the factor $(-1)^{N-s}$. Thus half of the fields have a "wrong" sign. However, the action contains higher derivatives in the kinetic terms for spins $\geqslant \frac{3}{2}$ and hence has no perturbative unitarity even without these "wrong" signs. This is analogous to the situation in the hypothetical $N > 4$ superconformal theories (see ref. [2]), where the kinetic term for the U(1)-vector field has a multiplier $(N-4)/4N$ changing sign when $N > 4$. If the generators $T_{\dots}^{(N,s,\jmath,c,u)}$ with even $(N-s)$ furnish a subalgebra in shsc$^\infty(4|1)$, one might restrict oneself to considering only this subalgebra, so that the "wrong" signs do not appear. However, we have no proof of the existence of such a subalgebra yet. Anyway it does not give a solution of the non-unitarity problem in conformal theory stipulated by higher derivatives (see for a discussion of this problem ref. [2]).

## 6. Conclusion and summary

Here we briefly sum up the main results of this paper and point out a number of problems that need further study.

We have shown that there exists a gauge invariant cubic interaction among bosonic and fermionic conformal higher-spin fields incorporating conformal supergravity. This result opens up the possibility of constructing a self-consistent interacting conformally invariant higher-spin theory. Together with previous results about higher-spin interaction in $adS_4$ [6] it gives hope to solve the longstanding higher-spin problem which would be a considerable step towards a unified theory, as discussed in sect. 1. It seems natural that in our construction there is an infinite number of fields of each spin. It is completely analogous to string field theory. Each level contains all spins from maximal to minimal (spin 1). Such a structure of levels also looks natural from the point of view of spontaneous symmetry breaking. Only the first level (spins $\leqslant 2$) might remain massless; the other higher levels should become massive.

However regarding the infinite multiplicity of spins in the gauge invariant conformal higher-spin interacting theory one should keep the following circumstances in mind. Right from the start we have dealt with the superalgebra shsc$^\infty(4|1)$. In principle it is not impossible for the invariant interaction to be based on another superalgebra containing each spin with a finite multiplicity. In ref. [13] we constructed a whole family of such superalgebras shsc$_\rho^{(n)}(4|N)$, where $n = 1,2,\dots$ is the multiplicity of SU$(2,2|N)$-supermultiplet with the fixed maximal spin in the algebra and $\rho \in \mathbb{R}$ some numerical parameter. They are factor-algebras of the original superalgebra shsc$^\infty(4|1)$. However all those superalgebras seemingly may not be localized. To build a cubic gauge invariant interaction as in sect. 5 it is necessary that some invariant bilinear form exists on the algebra. But shsc$_\rho^{(n)}(4|1)$ apparently does not possess any invariant bilinear form (the structure constants of factoralgebras have no simple symmetry properties which differ from shsc$^\infty(4|1)$).

Meanwhile it cannot be excluded that any other superalgebras with a finite multiplicity of spins exist*. Note that the situation here differs drastically from $adS_4$ higher-spin theory, where the superalgebra shs$(4|N)$ contains each spin a finite number of times.

Now for the readers' convenience we shall briefly reproduce the main steps of our consideration.

In sect. 2, we described the conformal higher-spin superalgebra shsc$^\infty(4|1)$, its operatorial realization, the spectrum of gauge fields, and its curvatures. Some simple symmetry properties of the structure coefficients providing the existence of invariant bilinear form were discussed. The important involutive automorphism $\mathscr{R}$ (Weyl reflection in the so(4, 2)-representations) was introduced.

In sect. 3, the usual conformally invariant description of higher-spin fields in terms of symmetric tensors (bosonic case) and spinor-tensors (fermionic case) has been presented. We introduced higher-spin linearized Weyl tensors $C_s$ and spinor-tensors $C_s^+, C_s^-$ generalizing the gravitational linearized Weyl tensor and rewrite gauge and conformal invariant high derivative actions $\phi_s \Box^s P_s \phi_s$ (integer spin $s$) and $\bar{\psi}_s \Box^{s-1/2} \not{\partial} P_s \psi_s$ (half-integer spin $s$) in the form $C_s^2$ and $\bar{C}_s^- C_s^+$.

In sect. 4, the geometrical description of free conformal higher-spin dynamics is presented. The linearized curvatures $R^\ell = \mathrm{d}\omega + \mathscr{P}\omega$ are constructed with the help of the nilpotent operator $\mathscr{P}\omega = [P^{\alpha\dot{\beta}}, \sigma_{\alpha\dot{\beta}} \wedge \omega]_*$ acting on the differential forms taking their values in shsc$^\infty(4|1)$. With the help of the generalization $\circledast = * \circ \mathscr{R}$ of the Hodge star $*$ including Weyl reflection $\mathscr{R}$, the nilpotent operator $\mathscr{K} = \circledast \mathscr{P} \circledast$ conjugated with $\mathscr{P}$ under some natural scalar product $\int \mathrm{tr}(A \wedge \circledast B)$ was introduced. The operators $\mathscr{P}$ and $\mathscr{K}$ converted the sequence of linear spaces of $q$-forms into the conformal cohomological complex which is analogous to the deRham complex on the Riemann manifold. The linearized conventional constraints which allow us to express all auxiliary fields through the physical ones up to a pure gauge part then were proposed in the simple form $\mathscr{K}R^\ell = 0$. The general solution of these constraints in terms of the curvatures was obtained. It turned out that all curvatures can be expressed through (derivatives on) the Weyl multispinors representing the Weyl tensors and spinor-tensors. The Weyl tensors in this context are non-trivial cohomological classes (harmonic forms) for the conformal cohomological complex. The linearized actions quadratic on the curvatures $R^\ell \wedge R^\ell$ both for integer and half-integer spins were brought to the $C^2$-form as in the tensor formalism of sect. 3, after the conventional constraints had been taken into account. In this way the equivalence of our geometrical formulation and the usual formulation in the symmetric tensor formalism was established.

In sect. 5, the special form of MacDowell–Mansouri actions $i\int[R^A \wedge R^B G_{AB} - R^{\bar{A}} \wedge R^{\bar{B}} G_{\bar{A}\bar{B}}]$ was considered, where $R^A$ $(R^{\bar{A}})$ is the set of curvatures $R^{\Omega}_{\alpha(n), \dot{\beta}(m)}$

* In fact such algebras do exist [e.g. certain factor-algebras of the universal enveloping algebra U(so(4, 2))].

with $n>m$ ($n<m$) (the curvatures $R^a$ with $n=m$ do not enter the action), and $G_{AB}, G_{\bar{A}\bar{B}}$ are the blocks of the invariant bilinear form in the algebra. The cubic invariant action for conformal higher-spin theory (Bose case) then is chosen in the above form. The proof of cubic gauge invariance has the following steps. The general structure of the gauge variation in cubic order is $R' \wedge R'\mathscr{E}$. Firstly, due to the symmetry properties of $G_{AB}$ and $G_{\bar{A}\bar{B}}$, the terms $R^A \wedge R^B\mathscr{E}$ and the complex conjugates $R^{\bar{A}} \wedge R^{\bar{B}}\mathscr{E}$ are cancelled separately. Secondly, taking into account the self-duality of $R'^{\bar{A}}$ and the antiself-duality of $R'^A$ as follows from the linearized conventional constraints, the terms $R'^A \wedge R'^{\bar{B}}\mathscr{E}$ vanish identically. Finally, among the remaining terms $R'^a \wedge R^A\mathscr{E}$ and $R'^a \wedge R^{\bar{A}}\mathscr{E}$ only the terms with $R'^a = R'^{(N,s,s)}_{\alpha(s),\dot{\beta}(s)}$ are non-zero due to the constraints. They must be compensated for by some deformations $\Delta\omega$ in the gauge transformation law for auxiliary fields. Obtaining $\Delta\omega$ from the requirement $\delta_g A + \Delta\omega = 0$, one should verify that these deformations are compatible with the conventional constraints. In this way we find the second-order constraints for the part of auxiliary fields (only for those which get the deformations $\Delta\omega$). The action found in such a way is unique.

In the superconformal case the Yang–Mills term for the vector fields must be added along with the above-considered action. The proof of invariance here is analogous to the purely bosonic case with some technical complications stipulated by the increased number of terms.

There are a number of problems that require a further study. The first is to expand the construction presented here to all orders in the interaction; in particular, to find a non-linear version of the standard constraints. Another problem is to construct $N$-extended theories. These theories may be based on the $N$-extended conformal higher-spin superalgebras shsc$^\infty(4|N)$ constructed in ref. [13]. We hope to return to this problem in a subsequent publication. To transfer the theory presented here to higher dimensions is also an important task. The higher-dimensional higher-spin theories might be based on the superalgebras shs$(M|N)$ (as proposed in ref. [22]) or some real forms of igl$(M|N;\mathbb{C})$ (see ref. [13]).

## Appendix A

We adopt the notations and conventions of refs. [6–9]. The Greek indices $\mu,\nu,\rho,\sigma = 0,1,2,3$ are the indices of components of differential forms. The flat Minkowsky metric is $\eta_{\mu\nu}$ $(+,-,-,-)$. The two-component dotted and undotted spinorial indices $\dot{\alpha},\dot{\beta},\ldots;\alpha,\beta,\ldots$ take on the values 1 and 2. They are raised and lowered by means of the symplectic metric $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$, $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$, $\varepsilon_{12} = \varepsilon^{12} = 1$ as

$$A^\alpha = \varepsilon^{\alpha\beta}A_\beta, \qquad A_\alpha = \varepsilon_{\beta\alpha}A^\beta \tag{A.1}$$

and analogously for dotted indices.

A symmetrization is implied separately for any set of upper or lower dotted or undotted spinorial indices denoted by the same letters. The usual summation convention is understood for each pair of a lower and upper index denoted by the same letter. The number of indices is indicated in parentheses (except for a single index). After the symmetrization with the indices is carried out, the maximal possible number of upper and lower indices denoted by the same letter should be contracted. For instance,

$$A_{\alpha(n)}=\frac{1}{n!}\left(A_{\alpha_1\dots\alpha_n}+(n!-1)\text{ permutations of }\alpha_1\dots\alpha_n\right), \tag{A.2}$$

$$C_{\alpha(n-m)}=A_{\alpha(n)}B^{\alpha(m)}=A_{(\beta_1\dots\beta_m\alpha_{m+1}\dots\alpha_n)}B^{\beta_1\dots\beta_m},\ n\geqslant m, \tag{A.3}$$

where brackets denote full symmetrization. The same conventions are used in sect. 3 for the vector indices $\mu,\nu,\rho,\sigma$.

The flat vierbein is

$$\sigma_\mu^{\alpha\dot\beta}=(I,\sigma_1,\sigma_2,\sigma_3)^{\alpha\dot\beta}, \tag{A.4}$$

where $I$ is the unit matrix and $\sigma_{1,2,3}$ are the Pauli matrices. The flat vierbein satisfies the following properties:

$$\sigma_{\mu\alpha\dot\beta}\sigma^{\mu\gamma\dot\delta}=2\delta_\alpha^\gamma\delta_{\dot\beta}^{\dot\delta},\qquad \sigma_{\mu\alpha\dot\beta}\sigma^{\nu\alpha\dot\beta}=2\delta_\mu^\nu. \tag{A.5), (A.6}$$

The vierbein one-form is

$$\sigma_{\alpha\dot\beta}=\sigma_{\mu\alpha\dot\beta}\,\mathrm{d}x^\mu, \tag{A.7}$$

and for its exterior product we have

$$2\sigma^{\alpha\dot\beta}\wedge\sigma^{\gamma\dot\delta}=\varepsilon^{\alpha\gamma}\bar\sigma^{\dot\beta\dot\delta}+\varepsilon^{\dot\beta\dot\delta}\sigma^{\alpha\gamma}, \tag{A.8}$$

where

$$\sigma^{\alpha\gamma}=\sigma_{\mu\nu}^{\alpha\gamma}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu,\qquad \bar\sigma^{\dot\beta\dot\delta}=\bar\sigma_{\mu\nu}^{\dot\beta\dot\delta}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \tag{A.9}$$

and

$$\sigma_{\mu\nu}{}^{\alpha\gamma}=\sigma_\mu{}^\alpha{}_{\dot\beta}\sigma_\nu{}^{\gamma\dot\beta},\qquad \bar\sigma_{\mu\nu}{}^{\dot\beta\dot\delta}=\overline{\left(\sigma_{\mu\nu}{}^{\beta\delta}\right)}=\sigma_{\mu\gamma}{}^{\dot\beta}\sigma_\nu{}^{\gamma\dot\delta}. \tag{A.10}$$

The two-forms $\sigma^{\alpha(2)}$ and $\bar\sigma^{\dot\beta(2)}$ ($\sigma^{\alpha\gamma}=\sigma^{\gamma\alpha},\bar\sigma^{\dot\beta\dot\delta}=\bar\sigma^{\dot\delta\dot\beta}$) are antiself-dual and self-dual respectively,

$$*\,\sigma^{\alpha(2)}=-i\sigma^{\alpha(2)},\qquad *\,\bar\sigma^{\dot\beta(2)}=i\bar\sigma^{\dot\beta(2)}, \tag{A.11}$$

or in the components

$$\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}{}^{\alpha(2)} = -i\sigma^{\mu\nu\alpha(2)}\,, \tag{A.12a}$$

$$\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma}{}^{\dot\beta(2)} = i\bar{\sigma}^{\mu\nu\dot\beta(2)}\,. \tag{A.12b}$$

The quantities $\sigma_{\mu\nu}{}^{\alpha(2)}$ and $\bar{\sigma}_{\mu\nu}{}^{\dot\beta(2)}$ are, as a matter of fact, projectors on the antiself-dual part and self-dual part of the arbitrary two-form $R$ ('t Hooft tensors)

$$R_{\mu\nu} = \tfrac{1}{4}\sigma_{\mu\nu}{}^{\alpha(2)}R_{\alpha(2)} + \tfrac{1}{4}\sigma_{\mu\nu}^{\dot\beta(2)}\bar{R}_{\dot\beta(2)}\,, \tag{A.13}$$

$$R_{\alpha(2)} = \tfrac{1}{4}i\epsilon^{\mu\nu\rho\sigma}\sigma_{\mu\nu\alpha(2)}R_{\rho\sigma}\,, \qquad \bar{R}_{\dot\beta(2)} = \overline{(R_{\beta(2)})}\,. \tag{A.14}$$

In this way any two-form is decomposed into the Lorentz irreducible components. If $R$ is (anti)self-dual, then ($\bar{R}_{\dot\alpha(2)} = 0$) $R_{\beta(2)} = 0$.

Evidently, the following relations hold:

$$\int R \wedge R = i\int \mathrm{d}^4x\left(\bar{R}_{\dot\beta(2)}\bar{R}^{\dot\beta(2)} - R_{\alpha(2)}R^{\alpha(2)}\right), \tag{A.15}$$

$$\int R \wedge * R = -\int \mathrm{d}^4x\left(R_{\alpha(2)}R^{\alpha(2)} + \bar{R}_{\dot\beta(2)}\bar{R}^{\dot\beta(2)}\right), \tag{A.16}$$

due to the identities

$$\sigma^{\alpha(2)} \wedge \sigma_{\gamma(2)} = -16i\delta^{\alpha}_{\gamma}\delta^{\alpha}_{\gamma}\,\mathrm{d}^4x\,, \tag{A.17a}$$

$$\bar{\sigma}^{\dot\beta(2)} \wedge \bar{\sigma}_{\dot\delta(2)} = 16i\delta^{\dot\beta}_{\dot\delta}\delta^{\dot\beta}_{\dot\delta}\,\mathrm{d}^4x\,, \tag{A.18a}$$

$$\sigma_{\alpha(2)} \wedge \bar{\sigma}_{\dot\beta(2)} \equiv 0\,, \tag{A.19a}$$

or in the components

$$\epsilon^{\mu\nu\rho\sigma}\sigma_{\mu\nu}{}^{\alpha(2)}\sigma_{\rho\sigma\gamma(2)} = -16i\delta^{\alpha}_{\gamma}\delta^{\alpha}_{\gamma}\,, \tag{A.17b}$$

$$\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\mu\nu}{}^{\dot\beta(2)}\bar{\sigma}_{\rho\sigma\dot\delta(2)} = 16i\delta^{\dot\beta}_{\dot\delta}\delta^{\dot\beta}_{\dot\delta}\,, \tag{A.18b}$$

$$\epsilon^{\mu\nu\rho\sigma}\sigma_{\mu\nu}{}^{\alpha(2)}\bar{\sigma}_{\rho\sigma}{}^{\dot\beta(2)} \equiv 0\,. \tag{A.19b}$$

As a simple example of the manipulations with the two-component spinorial indices we now show the equivalence of the constraints (4.28),

$$\varphi_{\mu\alpha\dot\beta} = \sigma^{\nu\alpha}{}_{\dot\beta}R'_{\mu\nu,\alpha(2)}(M) + \sigma^{\nu}{}_{\alpha}{}^{\dot\beta}R'_{\mu\nu,\dot\beta(2)}(M) - \sigma^{\nu}_{\alpha\dot\beta}R'_{\mu\nu}(D) = 0 \tag{A.20}$$

to the linearized Einstein equations in conformal gravity,

$$\lambda^{\mu}_{\alpha\dot{\beta}} = \epsilon^{\mu\nu\rho\sigma}\Big(\sigma_{\nu\alpha}{}^{\dot{\beta}} R'_{\rho\sigma,\dot{\beta}(2)}(M) - \sigma_{\nu}{}^{\alpha}{}_{\dot{\beta}} R'_{\rho\sigma,\alpha(2)}(M)\Big) = 0\,, \tag{A.21}$$

and find their general solution. The curvatures can be decomposed into the Lorentz irreducible components as follows:

$$R'_{\mu\nu,\alpha(2)}(M) = \sigma_{\mu\nu}{}^{\alpha(2)} C_{\alpha(4)} + \sigma_{\mu\nu\alpha}{}^{\gamma} R_{\alpha\gamma} + \sigma_{\mu\nu\alpha(2)} R + \bar{\sigma}_{\mu\nu}{}^{\dot{\beta}(2)} R_{\dot{\beta}(2),\alpha(2)}\,, \tag{A.22a}$$

$$R'_{\mu\nu,\dot{\beta}(2)}(M) = \bar{\sigma}_{\mu\nu}{}^{\dot{\beta}(2)} \bar{C}_{\dot{\beta}(4)} + \bar{\sigma}_{\mu\nu\dot{\beta}}{}^{\dot{\delta}} \bar{R}_{\dot{\beta}\dot{\delta}} + \bar{\sigma}_{\mu\nu\dot{\beta}(2)} \bar{R} + \sigma_{\mu\nu}{}^{\alpha(2)} \bar{R}_{\alpha(2),\dot{\beta}(2)}\,, \tag{A.22b}$$

$$R'_{\mu\nu}(D) = \sigma_{\mu\nu}{}^{\alpha(2)} R_{\alpha(2)}(D) + \bar{\sigma}_{\mu\nu}{}^{\dot{\beta}(2)} \bar{R}_{\dot{\beta}(2)}(D)\,. \tag{A.22c}$$

Due to the Bianchi identities (4.30) ($R(P) = 0$)

$$\chi^{\mu}_{\alpha\dot{\beta}} = \epsilon^{\mu\nu\rho\sigma}\Big(\sigma_{\nu\alpha}{}^{\dot{\beta}} R'_{\rho\sigma\dot{\beta}(2)}(M) + \sigma_{\nu}{}^{\alpha}{}_{\dot{\beta}} R'_{\rho\sigma\alpha(2)}(M) + \sigma_{\nu\alpha\dot{\beta}} R'_{\rho\sigma}(D)\Big) = 0\,, \tag{A.23}$$

we have

$$\chi_{\alpha\dot{\beta},\alpha\dot{\beta}} \sim \bar{R}_{\alpha(2),\dot{\beta}(2)} - R_{\dot{\beta}(2),\alpha(2)} = 0\,,$$

$$\chi_{\alpha}{}^{\dot{\beta}}{}_{,\alpha\dot{\beta}} \sim R_{\alpha(2)} - \tfrac{2}{3} R_{\alpha(2)}(D) = 0\,,$$

$$\chi^{\alpha}{}_{\dot{\beta},\alpha\dot{\beta}} \sim \bar{R}_{\dot{\beta}(2)} - \tfrac{2}{3} \bar{R}_{\dot{\beta}(2)}(D) = 0\,,$$

$$\chi^{\alpha\dot{\beta}}{}_{,\alpha\dot{\beta}} \sim R - \bar{R} = 0\,. \tag{A.24}$$

The constraints (A.20) in irreducible components are brought to the form

$$\varphi_{\alpha\dot{\beta},\alpha\dot{\beta}} \sim \bar{R}_{\alpha(2),\dot{\beta}(2)} + R_{\dot{\beta}(2),\alpha(2)} = 0\,,$$

$$\varphi_{\alpha}{}^{\dot{\beta}}{}_{,\alpha\dot{\beta}} \sim R_{\alpha(2)} + 2R_{\alpha(2)}(D) = 0\,,$$

$$\varphi^{\alpha}{}_{\dot{\beta},\alpha\dot{\beta}} \sim \bar{R}_{\dot{\beta}(2)} + 2\bar{R}_{\dot{\beta}(2)}(D) = 0\,,$$

$$\varphi^{\alpha\dot{\beta}}{}_{,\alpha\dot{\beta}} \sim R + \bar{R} = 0\,. \tag{A.25}$$

Combining eqs. (A.24) and (A.25), we see that

$$\overline{R}_{\alpha(2),\dot\beta(2)} = R_{\dot\beta(2),\alpha(2)} = 0, \qquad R_{\alpha(2)} = R_{\alpha(2)}(D) = 0,$$
$$\overline{R}_{\dot\beta(2)} = \overline{R}_{\dot\beta(2)}(D) = 0, \qquad R = \overline{R} = 0 \tag{A.26}$$

and the quantities $C$ and $\overline{C}$ remain arbitrary.

A general solution of the constraints and the Bianchi identities (it is a homogeneous linear system for the irreducible curvature components) has the form

$$R'_{\mu\nu,\alpha(2)}(M) = \sigma_{\mu\nu}{}^{\alpha(2)} C_{\alpha(4)}, \qquad R'_{\mu\nu,\dot\beta(2)}(M) = \bar\sigma_{\mu\nu}{}^{\dot\beta(2)} \overline{C}_{\dot\beta(2)},$$
$$R'_{\mu\nu}(D) = 0, \tag{A.27}$$

i.e. only the Weyl tensor is different from zero. The Einstein equations in irreducible components read as follows:

$$\lambda_{\alpha\dot\beta,\alpha\dot\beta} \sim \overline{R}_{\alpha(2),\dot\beta(2)} + R_{\dot\beta(2),\alpha(2)} = 0,$$
$$\lambda_\alpha{}^{\dot\beta}{}_{,\alpha\dot\beta} \sim R_{\alpha(2)} = 0, \qquad \lambda^\alpha{}_{\dot\beta,\alpha\dot\beta} = \overline{R}_{\dot\beta(2)} = 0,$$
$$\lambda^{\alpha\dot\beta}{}_{,\alpha\dot\beta} \sim R + \overline{R} = 0, \tag{A.28}$$

and is equivalent to (A.25) modulo the Bianchi identities (A.24) and has the general solution (A.27).

In principle, any linear system of equations for the quantities with the vector and spinor indices can be solved analogously by passing to the Lorentz irreducible multispinorial components; all quantities of the type of $\gamma$-matrices and complicated projection operators then disappear. The two-component spinorial algebra with applications is explained in detail in ref. [23].

## Appendix B

In this appendix we deduce the expression (4.4) with (4.5) for the linearized curvatures in components. For simplicity we shall consider only the bosonic case (integer $\jmath$ and $u = 0$). To write down eqs. (4.4) and (4.5) we need to know an explicit expression for the translation generator in the so(4,2)-representation $(\jmath, \jmath, 0)$ in the conformal basis

$$\tilde{P}_{\gamma\dot\delta}\left(T^{(\jmath,c)}_{\alpha(n),\dot\beta(m)}\right) = \left[P_{\gamma\dot\delta}, T^{(\jmath,c)}_{\alpha(n),\dot\beta(m)}\right]_* . \tag{B.1}$$

To calculate it we shall take advantage of the oscillator representation for $T$ in terms of the generating elements introduced in sect. 2*

$$T^{(s,c)}_{\alpha(2l),\dot\beta(2j)} = i^{c-s-1}B(s,c,l,j) \times \bar a_{\alpha(l+c/2)} a_{\alpha(l-c/2)} \bar a_{\dot\beta(j-c/2)} a_{\dot\beta(j+c/2)} \times \sum_{s_1+s_2=s} (-1)^{s_2-c/2} \frac{\left(\bar a_\gamma a^\gamma\right)^{s_1-l}\left(\bar a^{\dot\delta} a_{\dot\delta}\right)^{s_2-j}}{(s_1+l+1)!(s_1-l)!(s_2+j+1)!(s_2-j)!}, \tag{B.2}$$

where the normalization constant is chosen so that the invariant bilinear form is given as in eq. (2.32)

$$B(s,c,l,j) = \left[\frac{(2l+1)!(2j+1)!(s+l+j+2)!(s-l-j)!}{(l-c/2)!(l+c/2)!(j-c/2)!(j+c/2)!}\frac{(s+l-j+1)!(s-l+j+1)!}{(2s+2)!}\right]^{1/2}. \tag{B.3}$$

Eq. (B.2) is found in (I) (see (I.4.6, 7), where we suppose $N=s$ ($n=0$) and omit $N$ in the notation $T(N,s,c)$, since $n=N-s>0$ gives the same answer for the commutator due to commutativity of $P$ and $(\bar a_\alpha a^\alpha + \bar a^{\dot\beta} a_{\dot\beta})$). The operator $\tilde P$ then is a differential operator (see (I.3.11))

$$\tilde P_{\alpha\dot\beta} = i\left(a_\alpha \frac{\partial}{\partial a^{\dot\beta}} - \bar a_{\dot\beta}\frac{\partial}{\partial \bar a^\alpha}\right). \tag{B.4}$$

In this way our task is reduced to the calculation of the action of (B.4) on the generators (B.2). To do it the following relations will be useful for us:

$$a_\gamma \bar a_{\alpha(l+c/2)} a_{\alpha(l-c/2)} = \bar a_{(\alpha(l+c/2)} a_{\alpha(l-c/2)} a_{\gamma)} + \varepsilon_{\alpha\gamma}\frac{(l+c/2)}{(2l+1)}\bar a_{\alpha(l+c/2-1)} a_{\alpha(l-c/2)}\left(\bar a_\beta a^\beta\right), \tag{B.5}$$

* $\bar a_{\alpha(l+c/2)} = \underbrace{\bar a_\alpha \ldots \bar a_\alpha}_{l+c/2 \text{ times}}$, etc.

where $(\dots)$ denote the symmetrization over all indices and

$$\frac{\partial}{\partial \bar{a}^{\gamma}}\left(\bar{a}_{\alpha(l+c/2)} a_{\alpha(l-c/2)}\left(\bar{a}_{\beta} a^{\beta}\right)^{\jmath_1-l}\right)$$

$$= (l+c/2)\varepsilon_{\alpha\gamma}\bar{a}_{\alpha(l+c/2-1)} a_{\alpha(l-c/2)}\left(\bar{a}_{\beta} a^{\beta}\right)^{\jmath_1-l}$$

$$+(\jmath_1-l)\bar{a}_{\alpha(l+c/2)} a_{\alpha(l-c/2)} a_{\gamma}\left(\bar{a}_{\beta} a^{\beta}\right)^{\jmath_1-l-1}$$

$$= (\jmath_1-l)\bar{a}_{(\alpha(l+c/2)} a_{\alpha(l-c/2)} a_{\gamma)}\left(\bar{a}_{\beta} a^{\beta}\right)^{\jmath_1-l-1}$$

$$+\frac{(l+c/2)(\jmath_1+l+1)}{(2l+1)}\varepsilon_{\alpha\gamma}\bar{a}_{\alpha(l+c/2-1)} a_{\alpha(l-c/2)}\left(\bar{a}_{\beta} a^{\beta}\right)^{\jmath_1-l}. \quad \text{(B.6)}$$

To prove (B.5) one should contract the l.h.s. and r.h.s. with $\varepsilon^{\alpha\gamma}$ and take into account that

$$\delta^{\alpha}_{\alpha}\varphi_{\alpha(n)} = \frac{n+2}{n+1}\varphi_{\alpha(n)}. \quad \text{(B.7)}$$

Calculating (B.1) with the help of eqs. (B.5) and (B.6), we have after reduction of the homogeneous terms the following result:

$$\left[P_{\gamma\dot{\delta}}, T^{(\jmath,c+1)}_{\alpha(n),\dot{\beta}(m)}\right]_* = a(\jmath,c,n,m)\varepsilon_{\gamma\alpha}T^{(\jmath,c)}_{\alpha(n-1),\dot{\beta}(m)\dot{\delta}} + a(\jmath,c,m,n)\varepsilon_{\dot{\delta}\dot{\beta}}T^{(\jmath,c)}_{\alpha(n)\gamma,\dot{\beta}(m-1)}$$

$$- b(\jmath,c,n,m)\varepsilon_{\gamma\alpha}\varepsilon_{\dot{\delta}\dot{\beta}}T^{(\jmath,c)}_{\alpha(n-1),\dot{\beta}(m-1)}$$

$$- b(\jmath,-c-1,n-1,m-1)T^{(\jmath,c)}_{\alpha(n)\gamma,\dot{\beta}(m)\dot{\delta}}, \quad \text{(B.8)}$$

where the coefficients $a$ and $b$ are given in eq. (4.5) ($u=0$). The expression (4.4) now can be simply deduced from (B.8).

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